

Endogenous Interference in Randomized Experiments

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Abstract

This paper investigates the identification and inference of treatment effects in randomized controlled trials with social interactions. Two key network features introduce endogeneity: (1) latent variables influencing both network formation and outcomes, and (2) treatment-induced changes to the network structure that mediate treatment effects. I first define parameters in a post-treatment network framework, distinguishing direct effects from indirect effects mediated by network changes, and provide a causal interpretation of coefficients in a linear outcome model. To address endogeneity, I propose a shift-share instrumental-variable strategy and establish its consistency and asymptotic normality in sparse networks, but it fails in denser settings. In dense networks, I introduce a denoised SSIV estimator based on eigendecomposition to restore consistency. Finally, I revisit [Prina \(2015\)](#) as an empirical illustration, showing that treatment can influence outcomes both directly and through network structure changes.

Keywords: Causal inference, endogeneity, interference, mediation analysis, peer effect, random graph, shift-share instrument variable.

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1 Introduction

Investigating spillover effects among units is a central task in many social science problems, but identifying them can be difficult without randomized experiments. Recent studies combine peer effects with randomized controlled trials (RCTs) to improve identification in economics and the social sciences (Sobel, 2006; Banerjee et al., 2013; Cai et al., 2015; Paluck et al., 2016). These studies examine how interventions spread through social networks, potentially amplifying or weakening their effects. This phenomenon is known as interference, violating the “stable unit treatment value assumption” (SUTVA).

The peer effects literature typically assumes networks are exogenous and static, meaning unobserved factors do not affect both network formation and outcomes and the network remains unchanged after intervention. However, even in randomized settings these assumptions can fail, as latent traits may influence both social ties and outcomes and interventions may alter the network structure. Several studies highlight these concerns. Prina (2015) found that offering savings accounts in Nepalese villages significantly changed social networks. Banerjee et al. (2023) showed that introducing formal financial institutions in Indian villages weakened social ties by reducing reliance on informal networks. Carrell et al. (2013) found that a group formation experiment meant to help low-ability students backfired, as it led them to form stronger ties with similarly low-performing peers, reinforcing negative outcomes. Motivated by these concerns, this paper studies the identification and inference of causal effects in RCTs with interference, explicitly accounting for both latent confounding and treatment-induced changes in network structure.

Latent variables complicate identification and can bias causal estimates. Moreover, when treatment affects network structure, it becomes difficult to separate the direct effect of treatment from indirect effects mediated through the network. Yet distinguishing these channels is crucial for uncovering intervention mechanisms and designing effective policies. Although post-intervention networks capture how treatment and latent traits jointly shape networks, using post-intervention networks introduces endogeneity even in randomized settings. Consequently, researchers often rely on pre-intervention network data to estimate treatment effects – either by assuming a static network, or even when network changes are acknowledged, using the pre-intervention network to control for unobserved confounders (Carter et al., 2021). However, this

approach can be problematic: if outcomes depend on the post-intervention network, then using only pre-intervention data introduces measurement error. This paper proposes a novel approach that leverages panel network data. Specifically, it fits outcome regressions using post-intervention networks while constructing instruments from pre-intervention networks to correct for endogeneity and achieve consistent estimation.

A common approach to capturing interference is effective treatment, or exposure mapping, which maps units, treatment assignments, and network structures into some low-dimensional exposure levels (Manski, 2013; Aronow and Samii, 2017). In my setting, to account for treatment-induced changes in the network, I adopt the mediation analysis framework (Pearl, 2001; Heckman and Pinto, 2015), treating exposure mapping as a mediator to distinguish between the direct effect of treatment and the indirect effect mediated by network changes. Within this framework, I define causal parameters that separately capture these effects. Under a linear outcome model that includes both treatment and the network mediator, and allowing for flexible, additive unobserved confounding, I show that the coefficients have clear causal interpretations.

For estimation and inference, I focus on a specific mediator: the fraction of treated friends. This yields a linear model that is a special case of the linear-in-means (LIM) framework (Manski, 1993), capturing contextual peer effects, or the mean impact of peers’ treatments. To address endogeneity, I use shift-share (or “Bartik”) instrumental variables (SSIV) (Bartik, 1991), which combine numtiple shocks with exposure shares. I extend this approach to network settings by constructing the instrument as a combination of the random shock (others’ treatment assignments) and the non-exogenous exposure (pre-intervention network), following Borusyak and Hull (2023). I examine the conditions for the validity of this IV, with particular attention to the relevance condition under varying levels of network sparsity. I show that in denser networks, the SSIV loses validity because overlapping friendships reduce cross-unit variation. As a result, IV estimators based on SSIV are consistent only in relatively sparse networks. To address this limitation, I propose a denoised SSIV estimator that uses eigendecomposition of the pre-intervention network to extract latent traits and then projects them out, following Li and Wager (2022). This modification restores consistency even in denser networks. I then establish the asymptotic normality of both the SSIV and denoised SSIV estimators and provide consistent variance estimators that account for cross-unit dependence induced by the network, enabling valid inference. While the paper focuses on a particular form of peer effects, the analysis

of the relevance condition and the denoising procedure provides broader insights into the general application of SSIV.

This paper contributes to three strands of the literature. First, it builds on peer effects research, following the seminal work of [Manski \(1993\)](#), which examines how individual outcomes are influenced by peers’ behaviors, actions, or characteristics. The literature outlines four broad strategies for identifying peer effects: random peers, random shocks, structural endogeneity, and panel data ([Bramoullé et al., 2020](#)). This paper contributes by developing estimation and inference tools for social interactions with treatment-induced network changes under general endogeneity, leveraging randomized treatments and panel network data to construct valid instrumental variables across diverse sparsity regimes. Alternative approaches to network endogeneity include the matching method ([Auerbach, 2022](#)), the control-function method ([Johnsson and Moon, 2021](#)), and the parametric framework of [Griffith \(2022\)](#), which enables the identification of counterfactual treatment effects while accounting for network-structure changes. All of these methods assume dense networks for validity. The work most closely addressing dynamic network evolution is [Comola and Prina \(2021\)](#). They allow for endogenous peer effects but assume conditional exogeneity of pre- and post-intervention networks, addressing different endogeneity sources. They use “lagged” partner characteristics as instruments and focus on deriving identification conditions rather than asymptotic properties.

Second, I contribute to the shift-share IV literature by demonstrating SSIV’s relevance condition across a broad range of network sparsity regimes. Shift-share designs are widely used in many contexts, including labor, public policy, development, macroeconomics and international trade. There are two main approaches to identification in the SSIV literature: one relies on exogeneity of the share ([Bartik, 1991](#); [Goldsmith-Pinkham et al., 2020](#)), and the other on exogeneity of the shift ([Adão et al., 2019](#); [Borusyak et al., 2022](#); [Borusyak and Hull, 2023](#)). However, these papers impose a crucial relevance condition: most observations must be exposed to only a small number of shocks that influence treatment. In this paper, I identify the network-sparsity regimes that guarantee the SSIV relevance condition and introduce a denoising procedure to restore consistency of IV estimates when that condition is violated. I also connect to [Li and Wager \(2022\)](#), noting that their estimator for indirect effects essentially implements an SSIV strategy to address endogeneity from unobserved confounding.

Third, I contribute to the interference literature by accounting for stochastic and treatment-induced networks. The existing literature on estimating treatment effects under interference primarily follows a design-based approach (Hudgens and Halloran, 2008; Abadie et al., 2020). It makes no assumptions about outcome models and network formation, and inference is based on random treatment assignment. Given the stochastic nature of networks, it is natural to question how this affects inference. Recent work models network formation as realizations from an (unknown) graphon (Auerbach, 2022; Li and Wager, 2022), but does not consider how treatment-induced changes in network structure further influence causal effects. To address this gap, I introduce a mediation framework that explicitly accounts for the treatment-induced changes of networks, thereby disentangling direct treatment effects from indirect, network-mediated channels.

The remainder of the paper proceeds as follows. In Section 2, I introduce the framework, define the parameters of interest, and provide the identification results. In Section 3, I use SSIV for estimation and establish its asymptotic properties under relatively sparse networks. I also propose a denoised SSIV to handle cases where the network becomes denser. Section 4 presents the results of the Monte Carlo simulations. Section 5 illustrates the results in an empirical application based on the RCT in Prina (2015). Section 6 concludes. The appendix of the paper collects all of the proofs, as well as some intermediate results.

Notation I use $O(), O_{\mathbb{P}}(), o_{\mathbb{P}}(), \asymp, \succ, \prec, \gtrsim, \lesssim$ as follows: $a_n = O(b_n)$ if $|a_n| \leq Cb_n$ for n large enough; $X_n = O_{\mathbb{P}}(b_n)$, if for any $\delta > 0$, there exists $M, N > 0$, s.t. $\mathbb{P}[|X_n| \geq Mb_n] \leq \delta$ for any $n > N$; $X_n = o_{\mathbb{P}}(b_n)$, if $\lim \mathbb{P}[|X_n| \geq \varepsilon b_n] \rightarrow 0$ for any $\varepsilon > 0$; $a_n \asymp b_n$ if there exists $k_1, k_2 > 0$ and n_0 , s.t. for all $n > n_0$, $k_1 a_n \leq b_n \leq k_2 a_n$; $a_n \succ b_n$ if $\lim a_n/b_n = \infty$; $a_n \prec b_n$ if $\lim a_n/b_n = 0$; $a_n \gtrsim b_n$ if $b_n = O(a_n)$; $a_n \lesssim b_n$ if $a_n = O(b_n)$. Let 1_n and 0_n denote n -dimensional vectors of ones and zeros, respectively. Let $\|\cdot\|_{\text{op}}$ denote the operator norm. Define $\mathbb{E}_{w_i}(\cdot)$ as the expectation taken over the marginal distribution of w_i . Let X_{-i} represent the set $\{X_j\}_{j \neq i}$.

2 Setup

2.1 Framework and Notation

Consider an RCT with n participants. For each $i \in \{1, \dots, n\}$, let $Y_i \in \mathbb{R}$ denote the outcome, $T_i \in \{0, 1\}$ the treatment assignment with $T_i \sim \text{Bernoulli}(\pi)$ for some $0 < \pi < 1$, and w_i the unobserved covariates. We consider a single large network that permits arbitrary interference. The network is represented by an adjacency matrix $A = \{A_{ij}\}_{i,j=1}^n$, where $A_{ij} \in \{0, 1\}$ indicates whether units i and j are connected. The matrix is undirected, unweighted, and with no self-links. We distinguish between two network structures: A^{pre} (before the intervention) and A^{post} (after the intervention), both of which are observed by the researchers.

Figure 1 illustrates the causal mechanism considered in this paper. I assume that the post-intervention network A^{post} , influenced by the treatment vector, plays a critical role in determining the outcome. In particular, the exposure mapping that converts $\{T_i\}_{i=1}^n$ and A^{post} into low-dimensional statistics serves as a mediator, denoted M_i , through which treatments indirectly affect the outcome. Specifically, T_i has a direct effect on Y_i and an indirect effect via changes in M_i , while the treatments of other units T_{-i} affect Y_i solely through M_i . Moreover, A^{post} is shaped by both the treatments and the latent variable w_i , which may also influence the outcome, thereby confounding the mediator-outcome relationship.

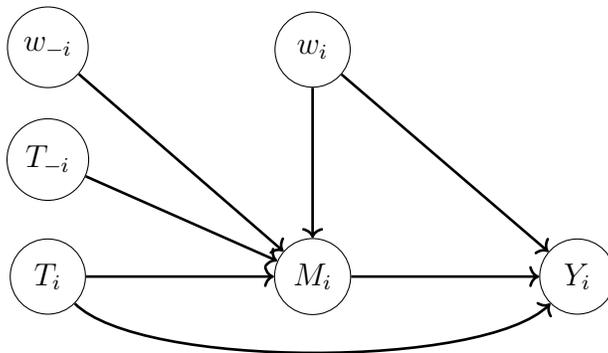


Figure 1: Causal mechanism.

Potential Value Notation To define the causal parameters of interest, I introduce potential value notation for both the mediator and the outcome, following the

mediation analysis literature (Pearl, 2001; Heckman and Pinto, 2015). First, consider a hypothetical intervention on the treatment vector t , and define the potential mediators for unit i as

$$\{M_i(t) : t \in \{0, 1\}^n\}.$$

Note that $M_i(t)$ depends on t in two ways: directly, and indirectly through the network structure A^{post} , which is itself affected by t . Consider a hypothetical intervention on both t_i and m_i . Define the potential outcomes for unit i , corresponding to interventions on t_i and on the mediator $m_i = M_i(t_i, t_{-i})$, as

$$\{Y_i(t_i, m_i) : t_i \in \{0, 1\}, m_i \in \mathcal{M}\},$$

where \mathcal{M} is the set of all possible mediator values. This notation indicates that the potential outcome $Y_i(t_i, m_i)$ depends both on unit i 's own treatment and on the network changes induced by the intervention (as captured by M_i), with the treatment assignments of others influencing Y_i only through M_i . I also define the nested potential outcomes corresponding to an intervention on t_i and $m_i = M_i(t'_i, t_{-i})$ as

$$\{Y_i(t_i, M_i(t'_i, t_{-i})) : t_i \in \{0, 1\}, t'_i \in \{0, 1\}, t_{-i} \in \{0, 1\}^{n-1}\}.$$

Here, $Y_i(t_i, M_i(t'_i, t_{-i}))$ denotes the hypothetical outcome when unit i receives treatment level t_i while its mediator takes the potential value $M_i(t'_i, t_{-i})$, corresponding to treatment t'_i for unit i and t_{-i} for the remaining units. Allowing t_i and t'_i to differ enables us to disentangle the direct and indirect channels through which the treatment affects the outcome. This definition contrasts with the interference framework in Hudgens and Halloran (2008), which considers

$$\{Y_i(t_i, t_{-i}) : t_i \in \{0, 1\}, t_{-i} \in \{0, 1\}^{n-1}\}.$$

The observed M_i and Y_i correspond to their respective potential values as follows:

$$M_i = M_i(t_i, t_{-i}) \quad \text{if } T_i = t_i \text{ and } T_{-i} = t_{-i},$$

and

$$Y_i = Y_i(t_i, m_i) \quad \text{if } T_i = t_i \text{ and } M_i = m_i.$$

Data-Generating Process for Networks I assume that the relationship between any two units i and j —both before and after the intervention—is governed by the following random graph models:

$$A_{ij}^{\text{pre}} = 1\left\{\eta_{ij} \leq q_n^{\text{pre}} g^{\text{pre}}(w_i, w_j)\right\} 1\{j \neq i\}, \quad (2.1)$$

$$A_{ij}^{\text{post}} = 1\left\{\eta_{ij} \leq q_n^{\text{post}} g^{\text{post}}(w_i, w_j, T_i, T_j)\right\} 1\{j \neq i\}, \quad (2.2)$$

where $\{w_i\}_{i=1}^n$ are i.i.d. latent variables and $\{\eta_{ij}\}_{i,j=1}^n$ is a symmetric matrix of unobserved scalar disturbances, with independent upper-diagonal entries. I allow the sparsity parameters q_n^{pre} and q_n^{post} to differ for generality. The pre-intervention graphon g^{pre} depends solely on the latent variables w_i and w_j , whereas the post-intervention graphon g^{post} also incorporates the treatment assignments T_i and T_j . Since neither the idiosyncratic term η_{ij} nor the latent variables w_i vary over time, this framework can also handle cases where the network remains unchanged after the intervention (i.e., $g^{\text{pre}} = g^{\text{post}}$).¹ Changes in the network are therefore primarily attributable to the intervention, while exogenous shocks unrelated to the intervention are reflected in the differences between g^{pre} and g^{post} . Thus, the network formation mechanisms in (2.1) and (2.2) capture the role of latent variables and allow for evolution over time with intervention-induced changes.

The distribution of η_{ij} is not separately identified from the graphons and sparsity parameters, so it is typically normalized to follow a standard uniform distribution. Consequently, $q_n^\diamond g^\diamond$ represents the probability that a given pair is connected, for $\diamond \in \{\text{pre}, \text{post}\}$. The graphon-based network models in (2.1) and (2.2), which take the form of an inhomogeneous Erdős–Rényi graph, is motivated by the Aldous–Hoover Theorem on exchangeable arrays (Aldous, 1981; Hoover, 1979) and has attracted considerable attention in econometrics (Graham, 2020; Parise and Ozdaglar, 2023). Recent studies have further employed this model to account for randomness in network formation (Auerbach, 2022; Cai, 2022; Li and Wager, 2022). The sparsity parameters q_n^\diamond , for $\diamond \in \{\text{pre}, \text{post}\}$, serve as theoretical tools that may cause the connection probability to vanish as $n \rightarrow \infty$. The goal of this paper is to derive theoretical results under a broad range of sparsity regimes, encompassing the following three cases: (a)

¹The assumption that η_{ij} is time-invariant can be relaxed. For example, one may treat the pre- and post-intervention η_{ij} as independent draws from the same distribution or as draws from correlated distributions. In such cases, the SSIV constructed based on A^{pre} might be weaker because additional exogenous shocks can reduce its relevance.

$q_n^\diamond \asymp n^{-1}$ (bounded degree graph); (b) $\lim_{n \rightarrow \infty} q_n^\diamond = 0$ and $\lim_{n \rightarrow \infty} nq_n^\diamond = \infty$ (sparse graph); (c) $q_n^\diamond \asymp 1$ (dense graph).

Assumption 1 collects key aspects of the network model for reference.

Assumption 1. *The latent variables $\{w_i\}_{i=1}^n$ are i.i.d. and independent of $\{\eta_{ij}\}_{i,j=1}^n$, where $\eta_{ij} \stackrel{i.i.d.}{\sim} U[0, 1]$ for $j > i$ and $\eta_{ij} = \eta_{ji}$. The networks are randomly generated according to (2.1) and (2.2), where g^{pre} is a symmetric measurable function of w_i and w_j , and g^{post} is a symmetric measurable function of (T_i, w_i) and (T_j, w_j) , both mapping into $[0, 1]$. The sparsity parameters satisfy $n^{-1} \preceq q_n^\diamond \preceq 1$, for $\diamond \in \{\text{pre}, \text{post}\}$.*

2.2 Parameters of Interest and Identification

Now I define the causal parameters of interest in my setting.

Definition 1. *For $t \in \{0, 1\}$, define*

(1) *the average total effect ToE of unit i 's treatment as*

$$\text{ToE} = \mathbb{E}[Y_i(1, M_i(1, T_{-i})) - Y_i(0, M_i(0, T_{-i}))]; \quad (2.3)$$

(2) *the average direct effect DE(t) of unit i 's treatment as*

$$\text{DE}(t) = \mathbb{E}[Y_i(1, M_i(t, T_{-i})) - Y_i(0, M_i(t, T_{-i}))]; \quad (2.4)$$

(3) *the average indirect effect IE(t) of unit i 's treatment as*

$$\text{IE}(t) = \mathbb{E}[Y_i(t, M_i(1, T_{-i})) - Y_i(t, M_i(0, T_{-i}))]; \quad (2.5)$$

(4) *the spillover effect SE(t) as*

$$\text{SE}(t) = \mathbb{E}[Y_i(t, M_i(t_i = t, t_{-i} = 1_{n-1})) - Y_i(t, M_i(t_i = t, t_{-i} = 0_{n-1}))]. \quad (2.6)$$

The average *total effect* (ToE) captures the overall change in unit i 's outcome when its treatment status changes. I decompose ToE into the average *direct effect* (DE(t)) which measures the impact of changing T_i while holding the mediator at the level it would have under $T_i = t$, and the average *indirect effect* (IE(t)) which

measures the change in the outcome that operates solely through the treatment's impact on the mediator when T_i remains fixed at t . The *spillover effect* ($SE(t)$) captures the effect on unit i 's outcome when all other units are switched from control to treatment, keeping unit i 's treatment fixed at t ². In all these definitions, the expectation is taken over every source of randomness, including potential outcomes and the treatment assignments of others.

The mediation formula relies on the following assumption.

Assumption 2. For all $i = 1, \dots, n$,

(a) $M_i \perp\!\!\!\perp Y_i(t_i, m_i) \mid w_i$ for all $t_i \in \{0, 1\}$ and $m_i \in \mathcal{M}$;

(b) $M_i(t'_i, t_{-i}) \perp\!\!\!\perp Y_i(t_i, m_i) \mid w_i$ for all $t_i \in \{0, 1\}$, $t'_i \in \{0, 1\}$, $t_{-i} \in \{0, 1\}^{n-1}$ and $m_i \in \mathcal{M}$.

Assumption 2 is standard in the mediation-analysis literature. Assumption 2(a) assumes that the variable w_i captures all the confounding factors between the mediator and the outcome. Assumption 2(b) assumes the cross-world independence between the potential outcomes and potential mediators. Furthermore, randomized treatments rule out both treatment-outcome and treatment-mediator confounding.

Theorem 2.1 expresses the parameters of interest in terms of the observed data up to the unknown distribution of w_i .

Theorem 2.1. Under Assumption 2,

(1) $ToE = \mathbb{E}(Y_i|T_i = 1) - \mathbb{E}(Y_i|T_i = 0)$;

(2) $DE(t) = \mathbb{E}_{w_i} \{ \mathbb{E} [\mathbb{E} (Y_i|M_i, T_i = 1, w_i) - \mathbb{E} (Y_i|M_i, T_i = 0, w_i) \mid T_i = t, w_i] \}$;

(3) $IE(t) = \mathbb{E}_{w_i} \{ \mathbb{E} [\mathbb{E} (Y_i|M_i, T_i = t, w_i) \mid T_i = 1, w_i] - \mathbb{E} [\mathbb{E} (Y_i|M_i, T_i = t, w_i) \mid T_i = 0, w_i] \}$;

(4) $SE(t) = \mathbb{E}_{w_i} \left\{ \begin{array}{l} \mathbb{E} [\mathbb{E} (Y_i|M_i, T_i = t, w_i) \mid T_i = t, T_{-i} = 1_{n-1}, w_i] \\ - \mathbb{E} [\mathbb{E} (Y_i|M_i, T_i = t, w_i) \mid T_i = t, T_{-i} = 0_{n-1}, w_i] \end{array} \right\}$.

If the parameter of interest is solely ToE, it can be identified, and consistent estimators can be obtained by regressing the outcome on the treatment indicator with an intercept, or equivalently, by using the difference-in-means estimator, provided the

²The $SE(t)$ in (2.6) is one way to define spillover effects; alternatively, spillover effects may be defined as the change in unit i 's outcome due to differences between any two distinct intervention programs via variations in the treatment of others.

treatment is randomly assigned. However, if the focus is on distinguishing between the direct and indirect effects, or when the spillover effect is of interest, these causal effects are identified only up to the unknown distribution of w_i .

Following the classical Baron–Kenny framework (Baron and Kenny, 1986), I assume a linear, additively separable potential outcome in treatment and mediator (Assumption 3), which implies Assumptions 2(a)–(b). In contrast, I do not assume that the mediator M_i is linear in T_i and w_i .

Assumption 3. *Assume the (partially) linear model for the potential outcome:*

$$Y_i(t_i, m_i) = \beta_0 + \beta_1 t_i + \beta_2 m_i + \lambda(w_i) + \varepsilon_i, \quad (2.7)$$

where $\lambda(\cdot)$ is an unknown measurable function with $\mathbb{E}(\lambda(w_i)) = 0$. Additionally, assume that $\{\varepsilon_i\}_{i=1}^n$ are i.i.d., $\mathbb{E}(\varepsilon_i^2) < \infty$, and $\mathbb{E}(\varepsilon_i | T_i, T_{-i}, A^{post}) = 0$.

Under Assumption 3, the observed outcome also follows the linear model:

$$Y_i = \beta_0 + \beta_1 T_i + \beta_2 M_i + \lambda(w_i) + \varepsilon_i.$$

Let u_i denote the error term in the outcome model: $u_i = \lambda(w_i) + \varepsilon_i$.

Although the framework in Section 2 accommodates general forms of mediators, Section 3 focuses on estimation and inference with the specific mediator

$$M_i = \frac{\sum_{j=1}^n A_{ij}^{post} T_j}{\sum_{j=1}^n A_{ij}^{post}}, \quad (2.8)$$

following the convention that $0/0 = 0$. This specification reflects a widely used form of the anonymous interference assumption in the literature (Hudgens and Halloran, 2008). With this mediator, the linear outcome model in Assumption 3 becomes a special case of the linear-in-means model (Manski, 2013). While this paper focuses on a specific form of peer effects, the discussion of the SSIV relevance condition provides insights applicable to other settings. Readers may view this paper as an illustrative example of using SSIV to identify and estimate peer effects.

Remark 2.1. *Auerbach (2022) studies a partially linear model $Y_i = \beta^\top X_i + \lambda(w_i) + \varepsilon_i$, where the latent variable w_i also drives network formation, as in (2.1) with $q_n^{pre} = 1$. They use a matching approach based on network data A_{ij} to identify and estimate*

β and $\lambda(w_i)$, with X_i corresponding to $(1, T_i, M_i)$ in this paper. Their method relies on dense networks and sufficient covariate variation conditional on w_i . In contrast, my approach leverages exogenous variation from randomized treatment and addresses endogeneity using an IV strategy based on SSIV, without attempting to identify $\lambda(w_i)$.

Corollary 2.1 simplifies the statement of Theorem 2.1 under Assumption 3, and provides the causal interpretation of the coefficients, which is analogous to the Baron-Kenny formulas for mediation (Baron and Kenny, 1986).

Corollary 2.1. *Under Assumption 3, then*

- (1) $\text{DE}(1) = \text{DE}(0) = \text{DE} = \beta_1$;
- (2) $\text{IE}(1) = \text{IE}(0) = \text{IE} = \beta_2 \cdot \{\mathbb{E}(M_i|T_i = 1) - \mathbb{E}(M_i|T_i = 0)\}$;
- (3) $\text{ToE} = \beta_1 + \beta_2 \cdot \{E(M_i|T_i = 1) - \mathbb{E}(M_i|T_i = 0)\}$;
- (4) $\text{SE}(t) = \beta_2 \cdot \{\mathbb{E}(M_i|T_i = t, T_{-i} = 1_{n-1}) - E(M_i|T_i = t, T_{-i} = 0_{n-1})\}$.

Under Assumption 3, where the treatment T_i does not interact with the mediator M_i , $\text{DE}(t)$ and $\text{IE}(t)$ do not depend on the value of t . Furthermore, Corollary 2.1 shows that β_1 captures the direct effect of the treatment, while the indirect and spillover effects depend on β_2 , scaled by the magnitude of changes in the mediator in response to changes in T_i and T_{-i} . When the post-intervention network A^{post} is uncorrelated with the treatment, the mediator does not respond to changes in T_i , resulting in a zero indirect effect and making the total effect equivalent to the direct effect. However, the spillover effect would still be present. Additionally, with M_i in (2.8), $\mathbb{E}(M_i|T_i = t, T_{-i} = 1_{n-1}) - \mathbb{E}(M_i|T_i = t, T_{-i} = 0_{n-1}) = 1$, indicating that β_2 captures the spillover effect $\text{SE}(t)$ defined in (2.6). Corollary 2.1 justifies the emphasis on the estimation and inference of β_1 and β_2 in this paper.

As a result, even in a randomized setting, if the causal effects of interest involve the network, such as the $\text{IE}(t)$ in (2.5) and the $\text{SE}(t)$ in (2.6), endogeneity becomes a concern, potentially biasing the estimation of β_2 . Given the correlation between T_i and M_i , this bias in estimating β_2 can further affect the estimation of β_1 .

3 IV estimation

In Section 3.1, I discuss the properties of M_i in (2.8) and the conditions that give rise to endogeneity. I then consider IV estimation to address the endogeneity when

$\lambda(w_i) \neq 0$. Section 3.2 analyzes the asymptotic properties the IV estimators using SSIV, showing their inconsistency in denser networks. To overcome this limitation, Section 3.3 proposes a modification to SSIV that restores consistency in such settings.

3.1 Discussion of endogeneity issue

The properties of the IV estimator hinge on the features of M_i . As a fractional measure, M_i may exhibit limited variation and potential multicollinearity. Moreover, its cross-unit dependence can affect the convergence rate. To better understand the properties of M_i , I decompose it as $M_i = \xi_i + r_i^*$, where

$$\xi_i = \frac{\mathbb{E}(A_{ij}^{\text{post}} T_j \mid T_i, w_i)}{\mathbb{E}(A_{ij}^{\text{post}} \mid T_i, w_i)} = \mathbb{P}(T_j = 1 \mid A_{ij}^{\text{post}} = 1, T_i, w_i), \quad (3.1)$$

and r_i^* captures deviations from this expectation. Intuitively, ξ_i reflects the conditional probability that a neighbor of i is treated, given i 's treatment status and characteristics. This decomposition separates M_i into an individual-specific component ξ_i , and a remainder r_i^* that depends on all others' treatment and characteristics. The analysis considers two cases depending on whether ξ_i is constant:

- (a) ξ_i is i.i.d. across i with constant variance;
- (b) $\xi_i = \pi$, implying zero variance.

The term ξ_i reflects whether network changes are induced by treatment. Case (b) arises when A_{ij}^{post} is mean independent of T_j given T_i and w_i , i.e., $\mathbb{E}(T_j \mid A_{ij}^{\text{post}}, T_i, w_i) = \pi$. This includes scenarios where the network is static ($A^{\text{pre}} = A^{\text{post}}$), evolves exogenously, or depends on T_j only in a mean-independent way. Case (b) has two important implications. First, in Case (b), there is no endogeneity even in the presence of unobserved confounders:

$$\mathbb{E}[M_i u_i] = \mathbb{E} \left[\frac{\sum_{j=1}^n A_{ij}^{\text{post}} T_j}{\sum_{j=1}^n A_{ij}^{\text{post}}} u_i \right] = \pi \mathbb{E}[u_i] = 0.$$

The second equality follows from the conditional mean independence of A_{ij}^{post} and T_j , and from Assumption 1, which ensures A_{ik}^{post} is independent of T_j for $k \neq j$. This orthogonality, which arises from the row normalization of the mediator, has been

exploited in the construction of the normalized SSIV (Adão et al., 2019; Goldsmith-Pinkham et al., 2020; Borusyak et al., 2022). Second, whether ξ_i is degenerate determines the variation in the regressor M_i , influencing the consistency conditions and convergence rate of the IV estimator. Treatment-induced network changes introduce additional variation in the mediator, potentially improving convergence. Comparing Cases (a) and (b) highlights how the network’s responsiveness to treatment shapes inference.

Remark 3.1. *Empirical studies often assume static networks and/or rely on pre-intervention data to avoid endogeneity. However, replacing A^{post} with A^{pre} in regression introduces bias, compromising identification of direct, indirect, and spillover effects. Define $M_i^{pre} = \sum_j A_{ij}^{pre} T_j / \sum_j A_{ij}^{pre}$ and regress Y_i on $X_i^{pre} = (1, T_i, M_i^{pre})$ with population coefficients β^{pre} . Then,*

$$\beta_1^{pre} = \beta_1 + \beta_2 \frac{\text{Cov}(T_i, M_i)}{\text{Var}(T_i)} = \text{ToE} \text{ and } \beta_2^{pre} = \beta_2 \frac{\text{Cov}(M_i^{pre}, M_i)}{\text{Var}(M_i^{pre})}.$$

Hence, β_1^{pre} recovers the total effect, while β_2^{pre} provides an attenuated estimate of the indirect effect. Pre-intervention networks suffices for total effect estimation, but cannot disentangle direct and indirect effects or recover spillovers.

3.2 Asymptotic properties of SSIV estimators

Endogeneity may arise when $\lambda(w_i) \neq 0$. To identify β_2 , one needs instruments that are uncorrelated with u_i and induce sufficient variation in M_i . I address this using the SSIV approach. Borusyak and Hull (2023) propose a general method for constructing SSIVs by combining exogenous shocks with non-exogenous exposure, adjusting for expected treatment. Applying their method, I construct the linear SSIV for M_i : $Z_i^{\text{SSIV}} = \sum_{j=1}^n A_{ij}^{\text{pre}} (T_j - \pi)$, which uses residualized treatment $T_j - \pi$ as the shock and the pre-intervention network as the share.

I assess the validity of SSIV, and thus the identification of β , by verifying two conditions: exogeneity and relevance. Exogeneity holds by construction, due to random assignment and the centering of treatment around its assignment probability.

$$\mathbb{E}[Z_i^{\text{SSIV}} u_i] = \mathbb{E} \left[\left(\sum_{j=1}^n A_{ij}^{\text{pre}} (T_j - \pi) \right) u_i \right] = \sum_{j=1}^n \mathbb{E}[A_{ij}^{\text{pre}} u_i] \cdot \mathbb{E}[T_j - \pi] = 0. \quad (3.2)$$

The key challenge is the relevance of SSIV. [Borusyak and Hull \(2023\)](#), Assumption 3) assumes weak mutual dependence of the instrument, which in my setting implies that the network must be sufficiently sparse. I estimate β using the instrument vector $Z_i = (1, T_i, Z_i^{\text{SSIV}})$, and denote the resulting coefficient vector by $\hat{\beta}^{\text{IV}}$, obtained from IV estimation using the stacked matrix Z formed from the rows Z_i . I analyze the relevance condition in detail and characterize the sparsity regimes under which SSIV yields consistent estimators, as formalized in [Theorem 3.1](#). In line with [Borusyak and Hull \(2023\)](#), I show that consistency requires both pre- and post-treatment networks to be sufficiently sparse.

Theorem 3.1. *Under Assumptions 1 and 3,*

(a) *if $\mathbb{V}(\xi_i) > 0$ with $\max\{q_n^{\text{pre}}, q_n^{\text{post}}\} \prec n^{-1/2}$, then*

$$\hat{\beta}^{\text{IV}} - \beta = O_{\mathbb{P}}\left(\sqrt{n} \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}\right);$$

(b) *if $\mathbb{V}(\xi_i) = 0$ with $\max\{q_n^{\text{pre}}, q_n^{\text{post}}\} \prec \sqrt{n}q_n^{\text{post}}$, then*

$$\hat{\beta}_1^{\text{IV}} - \beta_1 = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}} \max\left\{\frac{\max\{q_n^{\text{pre}}, q_n^{\text{post}}\}}{\sqrt{q_n^{\text{post}}}}, 1\right\}\right).$$

Moreover, with $\max\{q_n^{\text{pre}}, q_n^{\text{post}}\} \prec n^{-1/2}$, then

$$\hat{\beta}_0^{\text{IV}} - \beta_0 = O_{\mathbb{P}}\left(\sqrt{n} \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}\right) \text{ and } \hat{\beta}_2^{\text{IV}} - \beta_2 = O_{\mathbb{P}}\left(\sqrt{n} \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}\right).$$

[Theorem 3.1](#) characterizes the regimes under which $\hat{\beta}^{\text{IV}}$ is consistent in Cases (a) and (b). The consistency conditions and convergence rates for $\hat{\beta}_0^{\text{IV}}$ and $\hat{\beta}_2^{\text{IV}}$ are the same across both cases, depending on q_n^{pre} and q_n^{post} , and may be slower than the standard \sqrt{n} rate—scaled by the degree in the denser network, $n \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}$. In contrast, $\hat{\beta}_1^{\text{IV}}$ converges faster in Case (b) than in Case (a), and is consistent under weaker conditions. In this setting, the estimation of β_1 is less affected by cross-unit dependence in M_i , and converges more quickly than the estimator for β_2 .

[Theorem 3.1](#) shows that the consistency of SSIV estimators fails when networks are relatively dense, i.e., $\max\{q_n^{\text{pre}}, q_n^{\text{post}}\} \succcurlyeq n^{-1/2}$. The intuition is that SSIV relies on individual-level variation in the number of treated friends, driven by random treatment assignment. As networks become denser, this variation diminishes, weakening

identification.³

Corollary 3.1 simplifies the results of Theorem 3.1 under $q_n^{\text{pre}} \asymp q_n^{\text{post}}$.

Corollary 3.1. *Suppose $q_n^{\text{pre}} \asymp q_n^{\text{post}}$. Under Assumptions 1 and 3,*

- (a) *if $\mathbb{V}(\xi_i) > 0$ with $q_n^{\text{post}} \prec n^{-1/2}$, then $\hat{\beta}^{\text{IV}} - \beta = O_{\mathbb{P}}(\sqrt{n}q_n^{\text{post}})$;*
(b) *if $\mathbb{V}(\xi_i) = 0$ it holds that $\hat{\beta}_1^{\text{IV}} - \beta_1 = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$. Moreover, with $q_n^{\text{post}} \prec n^{-1/2}$, then $\hat{\beta}_0^{\text{IV}} - \beta_0 = O_{\mathbb{P}}(\sqrt{n}q_n^{\text{post}})$ and $\hat{\beta}_2^{\text{IV}} - \beta_2 = O_{\mathbb{P}}(\sqrt{n}q_n^{\text{post}})$.*

I now establish the asymptotic normality of the IV estimator $\hat{\beta}^{\text{IV}}$, focusing on regimes where it is consistent. Let $V_{\text{num}}^{\text{IV}} = \mathbb{V}(\sum_{i=1}^n Z_i u_i)$ denote the variance of the numerator in the centered estimator $\hat{\beta}^{\text{IV}} - \beta$. It can be shown that

$$V_{\text{num}}^{\text{IV}} = \begin{pmatrix} \sum_{i=1}^n \mathbb{E}(u_i^2) & \pi \sum_{i=1}^n \mathbb{E}(u_i^2) & 0 \\ \pi \sum_{i=1}^n \mathbb{E}(u_i^2) & \pi \sum_{i=1}^n \mathbb{E}(u_i^2) & \pi(1-\pi) \sum_{i=1}^n \sum_{j=1}^n \mathbb{E}(A_{ij}^{\text{pre}} u_i u_j) \\ 0 & \pi(1-\pi) \sum_{i=1}^n \sum_{j=1}^n \mathbb{E}(A_{ij}^{\text{pre}} u_i u_j) & \pi(1-\pi) \sum_{i=1}^n \mathbb{E}\left(\left(\sum_{j=1}^n A_{ij}^{\text{pre}} u_j\right)^2\right) \end{pmatrix}. \quad (3.3)$$

The (2, 3), (3, 2) and (3, 3) elements of $V_{\text{num}}^{\text{IV}}$ in (3.3) account for the dependence of Z_i^{SSIV} across i , even though u_i is i.i.d.. Let \hat{u}_i^{IV} denote the residual from the IV fits with Z_i . Define $\hat{V}_{\text{num}}^{\text{IV}}$ as the plug-in estimator of $V_{\text{num}}^{\text{IV}}$:

$$\hat{V}_{\text{num}}^{\text{IV}} = \begin{pmatrix} \sum_{i=1}^n (\hat{u}_i^{\text{IV}})^2 & \pi \sum_{i=1}^n (\hat{u}_i^{\text{IV}})^2 & 0 \\ \pi \sum_{i=1}^n (\hat{u}_i^{\text{IV}})^2 & \pi \sum_{i=1}^n (\hat{u}_i^{\text{IV}})^2 & \pi(1-\pi) \sum_{i=1}^n \sum_{j=1}^n A_{ij}^{\text{pre}} \hat{u}_i^{\text{IV}} \hat{u}_j^{\text{IV}} \\ 0 & \pi(1-\pi) \sum_{i=1}^n \sum_{j=1}^n A_{ij}^{\text{pre}} \hat{u}_i^{\text{IV}} \hat{u}_j^{\text{IV}} & \pi(1-\pi) \sum_{i=1}^n \left(\sum_{j=1}^n A_{ij}^{\text{pre}} \hat{u}_j^{\text{IV}}\right)^2 \end{pmatrix}.$$

The asymptotic variance of $\hat{\beta}^{\text{IV}}$ is then estimated by $\hat{V}^{\text{IV}} = (Z^\top X)^{-1} \hat{V}_{\text{num}}^{\text{IV}} (X^\top Z)^{-1}$. Theorem 3.2 establishes the asymptotic normality of the IV estimators $\hat{\beta}^{\text{IV}}$, along with a consistent variance estimator.

³A common alternative in the peer effects literature is the “peer-of-peer” IV (Bramoullé et al., 2009; De Giorgi et al., 2020). When peers of peers are not direct peers, their characteristics affect outcomes only indirectly, yielding valid instruments. Identification similarly requires sparse, non-overlapping peer networks, which aligns with the conditions for SSIV validity.

Theorem 3.2. *Under Assumptions 1 and 3, and with $\max\{q_n^{\text{pre}}, q_n^{\text{post}}\} \prec n^{-1/2}$, then $(\hat{V}^{\text{IV}})^{-1/2} (\hat{\beta}^{\text{IV}} - \beta) \xrightarrow{d} \mathcal{N}(0, I_3)$.*

Theorem 3.2 highlights two key insights. First, the standard heteroskedasticity-consistent IV variance estimator assumes i.i.d. data and overlooks the cross-unit dependence introduced by the shift-share instrument Z_i^{SSIV} , leading to misleading inference. In contrast, the proposed estimator $\hat{V}_{\text{num}}^{\text{IV}}$ explicitly accounts for this dependence, yielding more accurate standard errors. This aligns with findings from Adão et al. (2019), who show that conventional cluster-robust errors in shift-share designs can understate uncertainty due to residual correlation across units with similar exposure shares. Second, the asymptotic normality result exhibits a self-normalization property that removes the need to know the sparsity rates q_n^{pre} and q_n^{post} . Although the convergence rate of $\hat{\beta}^{\text{IV}}$ may depend on q_n^{pre} and q_n^{post} , this dependence is offset by the scaling of the variance estimator, ensuring that standard t -tests remain valid. This robustness is particularly valuable given that q_n^{post} is typically unidentifiable in practice (Bickel et al., 2011). Similar self-normalization behavior arises in network centrality regression (Cai, 2022) and cluster-robust inference (Hansen and Lee, 2019).

3.3 Modification of SSIV

I begin by formally examining the failure of the SSIV relevance condition, which motivates the proposed modification. To simplify the argument, consider the special case where $q_n^{\text{pre}} = q_n^{\text{post}} = q_n$, and recall the decomposition $M_i = \xi_i + r_i^*$. The first-stage relevance term can be decomposed as:

$$\frac{1}{n} \sum_{i=1}^n M_i Z_i^{\text{SSIV}} = \frac{1}{n} \sum_{i=1}^n \xi_i Z_i^{\text{SSIV}} + \frac{1}{n} \sum_{i=1}^n r_i^* Z_i^{\text{SSIV}}.$$

Since ξ_i is a function of T_i and w_i , it is uncorrelated with Z_i^{SSIV} , making the first term mean-zero and purely noise, with variance $O(nq_n^2)$ that increases with network density due to stronger cross-unit dependence. Here’s a smoother and more compact version: In contrast, r_i^* incorporates information from other units and can be shifted by Z_i^{SSIV} , yielding a stable signal with nonzero mean $O(1)$ and small variance $O(1/n)$. As networks become denser, the growing noise from ξ_i overwhelms this signal, weakening the first stage and resulting in inconsistency.

To address this, the goal is to remove the noise without sacrificing the identifying signal. Ideally, one would project out ξ_i from Z_i^{SSIV} and use the residual as the new instrument, exploiting their orthogonality. Although ξ_i is unknown, it depends on the latent w_i , whose structure can be recovered via spectral analysis of the pre-intervention network; removing components of Z_i associated with w_i thus achieves a similar effect. Under Assumption 4, the graphon is approximately low-rank, so the matrix $q_n^{\text{pre}} G_n^{\text{pre}}$, with entries $G_{n,ij}^{\text{pre}} = g^{\text{pre}}(w_i, w_j)$ and eigen-decomposition $\sum_{k=1}^n \lambda_k^* \psi_k^* \psi_k^{*\top}$ (with $\lambda_1^* \geq \lambda_2^* \geq \dots \geq \lambda_n^*$), is well-approximated by its top r components. The oracle-modified SSIV is the residual from projecting Z_i^{SSIV} onto $\{\psi_k^*(w_i)\}_{k=1}^r$. In practice, this is approximated using the top r eigenvectors $\hat{\psi}_k$ of A^{pre} . The procedure is as follows:

- (1) Perform eigenvalue decomposition of $A^{\text{pre}} = \sum_{k=1}^n \hat{\lambda}_k \hat{\psi}_k \hat{\psi}_k^\top$, and extract the top r eigenvectors $\{\hat{\psi}_k\}_{k=1}^r$;
- (2) Project Z_i^{SSIV} onto the span of $\{\hat{\psi}_k\}_{k=1}^r$ and use the residual as the modified SSIV.

The modified SSIV, denoted Z_i^{DE} (for “denoised”), is given by $Z_i^{\text{DE}} = Z_i^{\text{SSIV}} - \sum_{k=1}^r \hat{\gamma}_k \hat{\psi}_k(w_i)$, where the projection coefficients are $\hat{\gamma}_k = \sum_{i=1}^n \hat{\psi}_k(w_i) Z_i^{\text{SSIV}}$, exploiting the orthogonality of eigenvectors. The resulting modified instrument Z_i^{DE} is orthogonal to the space spanned by $\{\hat{\psi}_k(w_i)\}_{k=1}^r$, effectively removing variation linked to w_i while retaining identifying variation from treatment exposure to others. This projection reduces noise in the relevance term and restores consistency.

Remark 3.2. *There is a notable connection between SSIV and the indirect effect estimator in Li and Wager (2022), who analyze treatment effect asymptotics under network interference with graphon-generated networks. Their framework assumes potential outcome for individual i is $Y_i(t_i, t_{-i}) = f_i(t_i, \sum_{j=1}^n A_{ij} t_j / \sum_{j=1}^n A_{ij})$, where $f_i \in \mathcal{F}$ allows arbitrary dependence on the latent variable w_i . In contrast to my setting, they assume the network remains unchanged after the intervention. They propose an unbiased estimator for the spillover effect:*

$$\hat{\tau}_{\text{IND}}^U = \frac{1}{n} \sum_{i=1}^n Y_i \left(\frac{\sum_{j \neq i} A_{ij} T_j}{\pi} - \frac{\sum_{j \neq i} A_{ij} (1 - T_j)}{1 - \pi} \right) = \frac{1}{\pi(1 - \pi)} \frac{1}{n} \sum_{i=1}^n Y_i \left(\sum_{j \neq i} A_{ij} (T_j - \pi) \right).$$

This estimator is constructed as the difference between the Horvitz–Thompson estimators for the total and direct effects, implicitly utilizing a shift-share IV structure

to address endogeneity.

In the network literature, especially in inference based on eigenvectors (Cai, 2022; Li and Wager, 2022), it is common to assume that the graphon admits a finite-rank decomposition: $g(w_i, w_j) = \sum_{k=1}^r \lambda_k \psi_k(w_i) \psi_k(w_j)$, where the eigenfunctions $\{\psi_k\}$ are orthonormal, i.e., $\mathbb{E}[\psi_k^2(w_i)] = 1$ and $\mathbb{E}[\psi_k(w_i) \psi_l(w_i)] = 0$ for $k \neq l$. This structure arises in several canonical models, including the stochastic block model (Holland et al., 1983) and random dot product graphs (Young and Scheinerman, 2007). I assume that the pre-intervention graphon g^{pre} admits a low-rank approximation based on its leading eigenvalues, as formalized in Assumption 4(b). This encompasses a broader class of models, including latent space models (Hoff et al., 2002), which belongs to the broader class of inhomogeneous Erdős–Rényi models and includes the homophily and beta models as special cases.

Assumption 4. *There exists a constant r such that $r < n$ such that*

$$(a) \min_{k \in \{1, \dots, r-1\}} (\lambda_k - \lambda_{k+1}) \asymp n q_n^{\text{pre}}; \quad (b) \left\| \sum_{k=r+1}^n \lambda_k^* \psi_k^* \psi_k^{*\top} \right\|_{\text{op}} = O_{\mathbb{P}}(q_n^{\text{pre}}).$$

Assumption 4(a) requires the minimum eigen-gap, i.e., the spacing between an eigenvalue and the rest of the spectrum, to be sufficiently large. Several papers propose eigenvector estimators robust to small eigen-gaps (Cheng et al., 2021; Li et al., 2022). Although these may extend my methods, such extensions lie beyond this paper’s scope. Assumption 4(b) is analogous to sparsity assumptions in high-dimensional statistics, where the signal is concentrated in a small number of components, and the remainder contributes only weakly.

To estimate β , I use an IV regression with the modified SSIV vector $\tilde{Z}_i^{\text{DE}} = (1, T_i, Z_i^{\text{DE}})$. Stacking these row vectors yields the instrument matrix \tilde{Z}^{DE} . Let $\hat{\beta}^{\text{DE}}$ denote the coefficient vector obtained from this IV regression.

Theorem 3.3. *Suppose $q_n^{\text{pre}} \succ \frac{\log(n)}{\log(\log(n))}/n$. Under Assumptions 1, 3 and 4,*

(a) *if $\mathbb{V}(\xi_i) > 0$ with $\max\{q_n^{\text{pre}}, q_n^{\text{post}}\} \prec \sqrt{q_n^{\text{pre}}}$, then*

$$\hat{\beta}^{\text{DE}} - \beta = O_{\mathbb{P}} \left(\frac{\max\{q_n^{\text{pre}}, q_n^{\text{post}}\}}{\sqrt{q_n^{\text{pre}}}} \right);$$

(b) *if $\mathbb{V}(\xi_i) = 0$, it holds that $\hat{\beta}_1^{\text{DE}} - \beta_1 = O_{\mathbb{P}} \left(\frac{1}{\sqrt{n}} \right)$. Moreover, with $\max\{q_n^{\text{pre}}, q_n^{\text{post}}\} \prec$*

$\sqrt{q_n^{pre}}$, then

$$\hat{\beta}_0^{DE} - \beta_0 = O_{\mathbb{P}}\left(\frac{\max\{q_n^{pre}, q_n^{post}\}}{\sqrt{q_n^{pre}}}\right) \text{ and } \hat{\beta}_2^{DE} - \beta_2 = O_{\mathbb{P}}\left(\frac{\max\{q_n^{pre}, q_n^{post}\}}{\sqrt{q_n^{pre}}}\right).$$

The modified SSIV requires accurate eigenvector estimation, which can be challenging in very sparse networks. Nevertheless, this condition only requires $\frac{\log(n)}{\log \log(n)}/n$ sparsity, which is substantially weaker than the $n^{-1/2}$ threshold where standard SSIV fails. As a result, the modified SSIV shares the consistency region of the original SSIV and further extends into denser regimes where the latter becomes inconsistent.

Remark 3.3. To clarify the condition $\max\{q_n^{pre}, q_n^{post}\} \prec \sqrt{q_n^{pre}}$, consider two cases:

- (1) If $q_n^{pre} \prec q_n^{post}$, then the condition holds when $\sqrt{q_n^{pre}} \succ q_n^{post}$, meaning the pre-intervention network must not be too sparse relative to the post.
- (2) If $q_n^{pre} \succcurlyeq q_n^{post}$, then the condition reduces to $q_n^{pre} \prec 1$, ruling out the case of a dense pre-intervention network.

The modified SSIV approach is not applicable when both networks are dense. In such settings, alternative strategies—such as the matching method proposed by [Auerbach \(2022\)](#) or the control function approach of [Johnsson and Moon \(2021\)](#)—may be more appropriate, provided suitable regularity conditions hold.

Corollary 3.2 simplifies the results of Theorem 3.3 under $q_n^{post} \succcurlyeq q_n^{pre}$.

Corollary 3.2. Suppose $q_n^{post} \succcurlyeq q_n^{pre} \succ \frac{\log(n)}{\log \log(n)}/n$. Under Assumptions 1, 3 and 4,

- (a) if $\mathbb{V}(\xi_i) > 0$ with $\sqrt{q_n^{pre}} \succ q_n^{post}$, then $\hat{\beta}^{DE} - \beta = O_{\mathbb{P}}\left(q_n^{post}/\sqrt{q_n^{pre}}\right)$;
- (b) if $\mathbb{V}(\xi_i) = 0$, it holds that $\hat{\beta}_1^{DE} - \beta_1 = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$. Moreover, with $\sqrt{q_n^{pre}} \succ q_n^{post}$, then $\hat{\beta}_0^{DE} - \beta_0 = O_{\mathbb{P}}\left(q_n^{post}/\sqrt{q_n^{pre}}\right)$ and $\hat{\beta}_2^{DE} - \beta_2 = O_{\mathbb{P}}\left(q_n^{post}/\sqrt{q_n^{pre}}\right)$.

Corollary 3.2, in line with Corollary 3.1, shows that under Case (a), $\hat{\beta}$ converges at a rate slower than \sqrt{n} , but faster than the unmodified SSIV estimator, which converges at rate $1/(\sqrt{n}q_n^{post})$. In Case (b), when the post-intervention network A^{post} is conditionally mean independent of treatment, $\hat{\beta}_1^{DE}$ is less affected by dependence across units and achieves the usual \sqrt{n} rate. In contrast, $\hat{\beta}_0^{DE}$ and $\hat{\beta}_2^{DE}$ may still converge more slowly.

Remark 3.4. *A key takeaway is that the relevance condition analysis offers broader insights into SSIV applications. When the endogenous regressor admits a decomposition $M_i = \xi_i + r_i^*$, with ξ_i depending only on unit i 's own information, ξ_i contributes noise that may weaken instrument relevance. The proposed denoising procedure offers a simple, general strategy to address this issue and potentially restore consistency.*

I now establish the asymptotic normality of $\hat{\beta}^{\text{DE}}$. Define $\mu_k^u = \sum_{i=1}^n u_i \psi_k^*(w_i)$ and $\eta_i^u = u_i - \sum_{k=1}^r \mu_k^u \psi_k^*(w_i)$, where η_i^u captures the residual variation in u_i orthogonal to the top r eigenvectors. Define $V_{\text{num}}^{\text{DE}} = \mathbb{V} \left(\sum_{i=1}^n \tilde{Z}_i^{\text{DE}} u_i \right)$:

$$V_{\text{num}}^{\text{DE}} = \begin{pmatrix} \sum_{i=1}^n \mathbb{E}(u_i^2) & \pi \sum_{i=1}^n \mathbb{E}(u_i^2) & 0 \\ \pi \sum_{i=1}^n \mathbb{E}(u_i^2) & \pi \sum_{i=1}^n \mathbb{E}(u_i^2) & 0 \\ 0 & 0 & \pi(1-\pi) \sum_{i=1}^n \sum_{j=1}^n \mathbb{E}(A_{ij}^{\text{pre}}(\eta_i^u)^2) \end{pmatrix}. \quad (3.4)$$

Comparing $V_{\text{num}}^{\text{DE}}$ in (3.4) with $V_{\text{num}}^{\text{IV}}$ in (3.3), note that the off-diagonal terms (3, 2) and (2, 3) vanish in (3.4) but are nonzero in (3.3). Moreover, the (3, 3) entry in (3.4) accounts only for the diagonal component of the variance, whereas in (3.3), it includes covariance terms across units. This proposed modification effectively reduces noise by mitigating dependence induced by the latent variable w_i , thereby improving the stability of the modified estimator.

Let \hat{u}_i^{DE} denote the residual from the IV regression using the modified instrument \tilde{Z}_i^{DE} . Define $\hat{\mu}_k^u = \sum_{i=1}^n \hat{u}_i^{\text{DE}} \hat{\psi}_{ki}$ and $\hat{\eta}_i^u = \hat{u}_i^{\text{DE}} - \sum_{k=1}^r \hat{\mu}_k^u \hat{\psi}_{ki}$. The plug-in estimator of the numerator variance component is given by:

$$\hat{V}_{\text{num}}^{\text{DE}} = \begin{pmatrix} \sum_{i=1}^n (\hat{u}_i^{\text{DE}})^2 & \pi \sum_{i=1}^n (\hat{u}_i^{\text{DE}})^2 & 0 \\ \pi \sum_{i=1}^n (\hat{u}_i^{\text{DE}})^2 & \pi \sum_{i=1}^n (\hat{u}_i^{\text{DE}})^2 & 0 \\ 0 & 0 & \pi(1-\pi) \sum_{i=1}^n \sum_{j=1}^n A_{ij}^{\text{pre}}(\hat{\eta}_i^u)^2 \end{pmatrix}.$$

The variance estimator for $\hat{\beta}^{\text{DE}}$ is then constructed as $\hat{V}^{\text{DE}} = ((\tilde{Z}^{\text{DE}})^\top X)^{-1} \hat{V}_{\text{num}}^{\text{DE}} (X^\top \tilde{Z}^{\text{DE}})^{-1}$. Theorem 3.4 establishes the asymptotic normality of the IV estimator $\hat{\beta}^{\text{DE}}$ based on the modified SSIV, and confirms the consistency of the associated variance estimator.

Theorem 3.4. Suppose $q_n^{\text{pre}} \succ \frac{\log(n)}{\log(\log(n))}/n$. Under Assumptions 1, 3 and 4, and with $\max\{q_n^{\text{pre}}, q_n^{\text{post}}\} \prec \sqrt{q_n^{\text{pre}}}$, then $(\hat{V}^{\text{DE}})^{-1/2} (\hat{\beta}^{\text{DE}} - \beta) \xrightarrow{d} \mathcal{N}(0, I_3)$.

Theorem 3.4 justifies inference based on the usual t -test for $\hat{\beta}^{\text{DE}}$. Although a direct comparison of the asymptotic variances of $\hat{\beta}^{\text{IV}}$ and $\hat{\beta}^{\text{DE}}$ is not analytically tractable, simulation results in Section 4 show that, in regimes where both estimators are consistent, the modified SSIV does not increase variance.

4 Monte Carlo Simulation

In this section, I present simulation evidence to support the theoretical results. Treatment assignments are drawn independently as $T_i \stackrel{\text{i.i.d.}}{\sim} \text{Bern}(0.5)$. I consider the following two designs for generating the network structure:

Design 1: Homophily Model

- $g^{\text{pre}}(w_i, w_j) = 1 - (\Phi(w_i) - \Phi(w_j))^2$.
- $g^{\text{post}}(w_i, w_j, T_i, T_j) = 1 - (\Phi(w_i)(1 - T_i) - \Phi(w_j)(1 - T_j))^2$.

Design 2: Homophily Model

- $g^{\text{pre}}(w_i, w_j) = 1 - (\Phi(w_i) - \Phi(w_j))^2$.
- $g^{\text{post}}(w_i, w_j, T_i, T_j) = 1 - (\Phi(w_i(1 - T_i)) - \Phi(w_j(1 - T_j)))^2$.

I use Φ to denote the CDF of $\mathcal{N}(0, 1)$. The matrices A^{pre} and A^{post} are generated according to (2.1) and (2.2), with $w_i \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1)$, $\eta_{ij} \stackrel{\text{i.i.d.}}{\sim} U[0, 1]$ for $i < j$, and $\eta_{ij} = \eta_{ji}$. I set $q_n^{\text{pre}} = q_n^{\text{post}} = q_n$. In both designs, the resulting graphon has $r = 2$. Although both are homophily models and share the same pre-intervention network, they differ in how treatment alters the post-intervention network, which in turn affects the mediator M_i and the performance of the IV estimators. Table 1 reports the within-sample mean and standard error of M_i across sample sizes and sparsity levels for both designs. With $q_n = n^{-1}$, standard errors are stable across designs. For $q_n = n^{-1/2}$, they decline with sample size, more sharply under Design 2. At $q_n = 1$, Design 1 shows little change in standard errors, while Design 2 exhibits a 50% drop from $n = 200$ to $n = 800$, indicating reduced variation in M_i and potential near-collinearity. Design

Table 1: M_i in different designs

Design	statistics	q_n					
		$n = 200$			$n = 800$		
		-1	-1/2	1	-1	-1/2	1
1	mean	0.282	0.507	0.507	0.284	0.508	0.508
	std	0.420	0.199	0.134	0.422	0.170	0.135
2	mean	0.312	0.523	0.523	0.313	0.523	0.523
	std	0.425	0.141	0.012	0.427	0.099	0.006

1 corresponds to Case (a) with non-degenerate ξ_i , and Design 2 to Case (b) with constant ξ_i .

The outcome is generated as follows:

$$Y_i = \beta_0 + \beta_1 T_i + \beta_2 \frac{\sum_{j=1}^n A_{ij}^{\text{post}} T_j}{\sum_{j=1}^n A_{ij}^{\text{post}}} + (w_i + \varepsilon_i)/2$$

with $(\beta_0, \beta_1, \beta_2) = (1, 1, 0.5)$, $\varepsilon_i \stackrel{\text{i.i.d.}}{\sim} U[-1, 1]$ and $\varepsilon_i \perp\!\!\!\perp w_i$. The resulting error term is $u_i = (w_i + \varepsilon_i)/2$.

Tables 2 and 3 present the results of IV estimation using SSIV and modified SSIV with $q_n = \frac{\log(n)}{\log(\log(n))}/n$ and $q_n = n^{-1/5}$, respectively. Each table reports results for $n = 200$ (top) and $n = 800$ (bottom), with 5,000 replications. The left panel reports the mean, standard deviation, estimated standard error, and 95% CI coverage of the IV estimator using SSIV, labeled as “ $\hat{\beta}^{\text{IV}}$ ”, “ $\text{std}(\hat{\beta}^{\text{IV}})$ ”, “s.e.”, and “coverage”. The right panel shows the corresponding statistics for the modified SSIV estimator, denoted “ $\hat{\beta}^{\text{DE}}$.” Results for β_0 are omitted for brevity.

I begin with Table 2, where the network is relatively sparse. In this regime, Theorem 3.1 guarantees consistency of the SSIV estimators $\hat{\beta}^{\text{IV}}$, which is supported by the simulation: estimates concentrate around the true values. As sample size increases from 200 to 800, the standard deviation of $\hat{\beta}_1^{\text{IV}}$ roughly halves, consistent with \sqrt{n} convergence, while $\hat{\beta}_2^{\text{IV}}$ converges more slowly. Consistent with Theorem 3.2, the 95% CI coverage is close to nominal using our variance estimator. Table 3 reports IV estimation results for $q_n = n^{-1/5}$, a regime denser than the threshold for SSIV validity. As predicted by Theorem 3.1, $\hat{\beta}^{\text{IV}}$ is inconsistent under Design 1: estimates vary widely and standard deviations do not decline with sample size. Under Design 2, $\hat{\beta}_1^{\text{IV}}$ remains consistent at rate \sqrt{n} , while $\hat{\beta}_2^{\text{IV}}$ does not. Applying the modified SSIV reduces noise and improves accuracy: $\hat{\beta}_1^{\text{DE}}$ is \sqrt{n} -consistent, and $\hat{\beta}_2^{\text{DE}}$ converges at rate $1/\sqrt{q_n}$, with both estimators achieving near-nominal 95% CI coverage.

Table 2: Simulation results of IV estimation with $q_n = \frac{\log(n)}{\log(\log(n))}/n$

Design	β	SSIV				modified SSIV			
		$\hat{\beta}^{IV}$	$\text{std}(\hat{\beta}^{IV})$	s.e.	coverage	$\hat{\beta}^{DE}$	$\text{std}(\hat{\beta}^{DE})$	s.e.	coverage
$n = 200$									
1	$\beta_1 = 1$	0.999	0.083	0.087	0.954	1.000	0.083	0.082	0.940
	$\beta_2 = 0.5$	0.505	0.150	0.157	0.952	0.505	0.168	0.210	0.983
2	$\beta_1 = 1$	1.000	0.082	0.084	0.949	1.000	0.082	0.084	0.950
	$\beta_2 = 0.5$	0.502	0.137	0.142	0.955	0.502	0.138	0.145	0.956
$n = 800$									
1	$\beta_1 = 1$	1.000	0.040	0.042	0.957	1.000	0.040	0.042	0.956
	$\beta_2 = 0.5$	0.500	0.076	0.076	0.948	0.501	0.078	0.081	0.958
2	$\beta_1 = 1$	1.000	0.041	0.041	0.950	1.000	0.041	0.041	0.950
	$\beta_2 = 0.5$	0.501	0.068	0.069	0.957	0.501	0.068	0.069	0.953

Table 3: Simulation results of IV estimation with $q_n = n^{-1/5}$

Design	β	SSIV				modified SSIV			
		$\hat{\beta}^{IV}$	$\text{std}(\hat{\beta}^{IV})$	s.e.	coverage	$\hat{\beta}^{DE}$	$\text{std}(\hat{\beta}^{DE})$	s.e.	coverage
$n = 200$									
1	$\beta_1 = 1$	0.946	0.330	0.369	0.972	0.997	0.144	0.167	0.952
	$\beta_2 = 0.5$	0.944	2.257	2.041	0.959	0.537	0.750	0.846	0.978
2	$\beta_1 = 1$	1.000	0.091	0.086	0.950	1.000	0.083	0.084	0.950
	$\beta_2 = 0.5$	0.526	1.562	1.773	0.976	0.507	0.669	0.777	0.973
$n = 800$									
1	$\beta_1 = 1$	0.879	1.380	0.453	0.973	0.998	0.092	0.103	0.953
	$\beta_2 = 0.5$	1.278	8.703	2.507	0.957	0.509	0.463	0.521	0.971
2	$\beta_1 = 1$	1.000	0.047	0.042	0.948	1.000	0.041	0.041	0.948
	$\beta_2 = 0.5$	0.514	1.921	1.991	0.973	0.489	0.435	0.489	0.969

5 Empirical Application

In this section, I revisit [Prina \(2015\)](#), which conducted an RCT to assess the impact of offering access to formal savings on households' financial situations in 19 villages in Nepal with 915 households. I now provide a detailed description of the empirical setting of the RCT. Half of the female household heads were randomly offered the savings accounts, while the other half were not. The financial links were measured by asking survey questions such as "who did you exchange loans or gifts with?" The network experienced considerable reshuffling, despite the total number of links remaining nearly constant (328 at baseline, 329 at endline). Of these, only 73 links persisted from baseline, while 255 links were broken after the intervention, and 256 new links were formed by the endline.

Table 4 reports OLS and IV estimates (using both SSIV and modified SSIV)

across various household expenditure categories, with standard errors in parentheses. Outcomes are measured in Nepalese rupees over the 30 days prior to the endline survey and include spending on festivals, fish, and education-related items (textbooks and supplies). Education-related outcomes are restricted to households with school-age children (ages 6–16). Results for β_0 are omitted for brevity.

The divergence between OLS and SSIV estimates—whether in sign or significance—highlights bias from endogeneity due to unobserved confounders. The SSIV results reveal that treatment influences outcomes through distinct channels. For fish consumption, the direct effect of access to savings accounts is not significant, but the indirect effect—mediated through treated peers—is significantly positive. Spillover effects from having all friends treated increase fish expenditure by Rs. 252.91 compared to having no friends treated. For total education expenditures, textbooks, and school supplies, the direct effects of access to savings accounts are positive and significant, while the indirect effects via the fraction of treated friends are not. Access to free savings accounts increases spending by Rs. 574.43 (total education), Rs. 223.26 (textbooks), and Rs. 104.47 (supplies), holding peer treatment constant. These results demonstrate that our method successfully isolates direct effects from network-mediated effects, clarifying the channels through which the intervention operates. When $\hat{\beta}_2^{IV}$ is not significant, $\hat{\beta}_1^{OLS}$ and $\hat{\beta}_1^{IV}$ closely align, reflecting the mitigating effect of random assignment on omitted variable bias. Moreover, the IV estimates from SSIV and modified SSIV are highly similar, despite the sparsity of the pre-intervention network, suggesting that the eigenvector-based modification does not introduce substantial noise.

For comparison, Table 4 includes results from Prina (2015, Tables 5 and 6). Their analysis estimates the average treatment effect using a regression controlling for treatment assignment, baseline outcomes, and covariates, interpreting the treatment indicator as the intention-to-treat (ITT) due to non-compliance. Align with the OLS results in the top panel, for outcomes where $\hat{\beta}_2^{IV}$ is insignificant, the ITT and $\hat{\beta}_1^{IV}$ estimates align closely—indicating that random assignment mitigates bias from unobserved confounders in estimating the direct effect.

Table 4: Estimation result based on data from [Prina \(2015\)](#)

Method	$\hat{\beta}$	Festivals & Ceremonies	Fish	Total Exp. in Education	Exp. on Textbooks	Exp. on Sch. Supplies
OLS	$\hat{\beta}_1^{\text{OLS}}$	267.112* (141.412)	48.505 (39.683)	562.276** (281.922)	246.481** (105.114)	103.767** (45.880)
	$\hat{\beta}_2^{\text{OLS}}$	-157.612 (119.425)	-55.696 (35.796)	87.445 (336.824)	-91.316 (111.511)	22.194 (56.184)
SSIV	$\hat{\beta}_1^{\text{IV}}$	232.207 (152.369)	33.276 (42.002)	574.427* (299.153)	223.261* (117.319)	104.469** (48.935)
	$\hat{\beta}_2^{\text{IV}}$	549.756 (961.761)	252.913* (150.221)	-109.182 (1299.093)	284.441 (666.867)	10.823 (219.498)
Denoised SSIV	$\hat{\beta}_1^{\text{DE}}$	272.332 (155.618)	27.030 (42.708)	576.798* (295.529)	211.983* (114.828)	100.048** (48.519)
	$\hat{\beta}_2^{\text{DE}}$	-263.414 (1314.334)	379.510* (206.481)	-147.559 (1395.153)	466.935 (644.052)	82.376 (219.500)
Prina (2015)	ITT	248.070** (123.670)	45.756 (39.055)	667.560** (320.630)	213.740* (120.220)	115.950** (49.980)

Note: Dependent variables are expressed in Nepalese rupees (the exchange rate was roughly Rs. 70 to USD 1 during the study period). The significance level of the estimators is indicated with stars in the usual manner: “****” means significant at $\alpha = 1\%$, “***” means significant at $\alpha = 5\%$, and “*” means significant at $\alpha = 10\%$. [Prina \(2015\)](#) did not report estimates for “Fish,” so this entry is left blank.

6 Conclusion

This paper studies the identification and inference of treatment effects in RCTs with network interference, focusing on two key challenges: (1) unobserved confounders affecting both network formation and outcomes, and (2) treatment-induced changes to the network structure. The framework distinguishes between direct effects and indirect effects mediated through network changes. To address endogeneity, I propose IV estimation using SSIV for sparse networks and introduce a denoised modification for denser networks. Consistent variance estimators are provided for all estimators to ensure valid inference. More broadly, the paper highlights how network density may weaken the SSIV relevance condition, and the proposed modification offers insights for other network-mediated settings. Simulation studies support the theoretical results, and the methods are applied to the RCT in [Prina \(2015\)](#) as an empirical illustration.

This paper opens several avenues for future research. A natural extension would be to incorporate endogenous and correlated peer effects, along with covariates, into the linear model. An expanding body of research highlights the importance of mea-

suring and accounting for network changes when evaluating the welfare impacts of policies. While the literature extensively examines optimal treatment assignment under interference, the challenge of designing effective policies that account for endogenous network evolution remains unresolved. Furthermore, my methods rely on detailed panel network data, but relaxing this requirement to single-period or aggregated network data would be desirable. I leave these questions for future research.

Supplementary Material

Notation Let N_i denote the degree of unit i in network A , i.e., $N_i = \sum_{j=1}^n A_{ij}$. Define $g_0(i, j) = g^{\text{pre}}(w_i, w_j)$, $g_1(i, j) = g^{\text{post}}(w_i, w_j, T_i, T_j)$, and $g_0(k) = \mathbb{E}(g_0(j, k)|w_k)$. Let $X_n = \Omega_{\mathbb{P}}(b_n)$, if for any $\delta > 0$, there exists $M, N > 0$, s.t. $\mathbb{P}(|X_n| \leq Mb_n) \leq \delta$ for any $n > N$. Let $\text{diag}(a_1, \dots, a_k)$ be the $k \times k$ diagonal matrix with diagonal entries a_1, \dots, a_k . Denote by $\|\cdot\|_{\text{F}}$ the Frobenius norm. Define $X_{i0} = 1, X_{i1} = T_i, X_{i2} = M_i$.

A Proof of results in Section 2.2

Proof of Theorem 2.1. The proof follows the standard mediation analysis framework. We focus on the discrete case; the continuous and mixed cases follow analogously. It suffices to show that

$$\mathbb{E}[Y_i(t_i, M_i(t'_i, T_{-i})) | w_i = w] = \mathbb{E}[\mathbb{E}(Y_i | M_i, T_i = t_i, w_i = w) | T_i = t'_i, w_i = w]. \quad (\text{A.1})$$

To establish this, fix $T_{-i} = t_{-i}$ and $w_i = w$ and consider:

$$\begin{aligned} & \mathbb{E}[Y_i(t_i, M_i(t'_i, T_{-i})) | T_{-i} = t_{-i}, w_i = w] \\ &= \sum_{m_i \in \mathcal{M}} \mathbb{E}(Y_i(t_i, m_i) | M_i(t'_i, t_{-i}) = m_i, T_{-i} = t_{-i}, w_i = w) \mathbb{P}(M_i(t'_i, t_{-i}) = m_i | T_{-i} = t_{-i}, w_i = w) \\ &= \sum_{m_i \in \mathcal{M}} \mathbb{E}(Y_i(t_i, m_i) | T_{-i} = t_{-i}, w_i = w) \mathbb{P}(M_i(t'_i, t_{-i}) = m_i | T_{-i} = t_{-i}, w_i = w) \\ &= \sum_{m_i \in \mathcal{M}} \mathbb{E}(Y_i(t_i, m_i) | w_i = w) \mathbb{P}(M_i = m_i | T_i = t'_i, T_{-i} = t_{-i}, w_i = w) \\ &= \sum_{m_i \in \mathcal{M}} \mathbb{E}(Y_i | T_i = t_i, M_i = m_i, w_i = w) \mathbb{P}(M_i = m_i | T_i = t'_i, T_{-i} = t_{-i}, w_i = w) \end{aligned}$$

Taking expectations over $T_{-i}|w_i = w$ yields the desired conditional equality in (A.1). Finally, marginalizing over the distribution of w completes the proof. \square

Proof of Corollary 2.1. The result follows from Theorem 2.1 under Assumption 3. \square

B Proof of results in Section 3.2

B.1 Taylor expansion of M_i

To simplify the notations, define

$$R_{ij} = \frac{A_{ij}^{\text{post}} T_j}{\mathbb{E}(A_{ij}^{\text{post}} T_j | T_i, w_i)} - \frac{A_{ij}^{\text{post}}}{\mathbb{E}(A_{ij}^{\text{post}} | T_i, w_i)}, \quad (\text{B.1})$$

$$U_{ij} = A_{ij}^{\text{post}} T_j - \mathbb{E}(A_{ij}^{\text{post}} T_j | T_i, w_i), \quad (\text{B.2})$$

$$W_{ij} = A_{ij}^{\text{post}} - \mathbb{E}(A_{ij}^{\text{post}} | T_i, w_i), \quad (\text{B.3})$$

which satisfies $\mathbb{E}(R_{ij} | T_i, w_i) = \mathbb{E}(U_{ij} | T_i, w_i) = \mathbb{E}(W_{ij} | T_i, w_i) = 0$ by construction. Define $x_i = \frac{1}{n-1} \sum_{j \neq i} A_{ij}^{\text{post}} T_j$ and $y_i = \frac{1}{n-1} \sum_{j \neq i} A_{ij}^{\text{post}}$. To facilitate the proof, we expand M_i at the conditional mean of (x_i, y_i) : $\theta_i = (\mathbb{E}(A_{ij}^{\text{post}} T_j | T_i, w_i), \mathbb{E}(A_{ij}^{\text{post}} | T_i, w_i))$, we have $M_i = \xi_i + r_{0,i} + r_{1,i}$ where ξ_i is defined in (3.1) and

$$r_{0,i} = \xi_i \frac{1}{n-1} \sum_{k \neq i} R_{ik},$$

$$r_{1,i} = \frac{1}{(n-1)^2} \sum_{k \neq i} \left(-\frac{U_{ik} W_{ik}}{\theta_i^*(y)^2} + \frac{2\theta_i^*(x)}{\theta_i^*(y)^3} W_{ik}^2 \right) + \frac{1}{(n-1)^2} \sum_{(k,l): k \neq l} \left(-\frac{U_{ik} W_{il}}{\theta_i^*(y)^2} + \frac{2\theta_i^*(x)}{\theta_i^*(y)^3} W_{ik} W_{il} \right).$$

for some $\theta_i^* = (\theta_i^*(x), \theta_i^*(y))$ between θ_i and (x_i, y_i) . Note that ξ_i is either a function of T_i and w_i , making it i.i.d. across units and independent of n , or it equals π when A^{post} is conditionally independent of T . Define $\mu_{r_{1,i}} = \mathbb{E}(r_{1,i} | T_i, w_i)$. By direct algebra, we can show that

$$\mu_{r_{1,i}} = \frac{1}{n-1} \mathbb{E} \left(-\frac{U_{ik} W_{ik}}{\theta_i^*(y)^2} + \frac{2\theta_i^*(x)}{\theta_i^*(y)^3} W_{ik}^2 \mid T_i, w_i \right) = O_{\mathbb{P}} \left(\frac{1}{nq_n^{\text{post}}} \right). \quad (\text{B.4})$$

Lemma B.1. *Define a_i and b_i as any functions of T_i and w_i with constant variance. Under Assumption 1, we have*

$$\mathbb{E} \left[(r_{0,i} + r_{1,i} - \mu_{r_{1,i}})^2 \right] = O \left(\frac{1}{nq_n^{\text{post}}} \right), \quad (\text{B.5})$$

$$\text{Cov}(a_i(r_{0,i} + r_{1,i} - \mu_{r_{1,i}}), b_j(r_{0,j} + r_{1,j} - \mu_{r_{1,j}})) = O\left(\frac{1}{n}\right) \text{ for any } i \neq j. \quad (\text{B.6})$$

Proof. See proof in Section D.1 of the Online Appendix. \square

Lemma B.2. *Let $a_i \in \{1, T_i\}$. Under Assumption 1, then*

$$\frac{1}{n} \sum_{i=1}^n a_i(r_{0,i} + r_{1,i} - \mu_{r_{1,i}}) = O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{post}}}\right), \quad (\text{B.7})$$

$$\frac{1}{n} \sum_{i=1}^n a_i M_i = \frac{1}{n} \sum_{i=1}^n a_i(\xi_i + \mu_{r_{1,i}}) + O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{post}}}\right). \quad (\text{B.8})$$

Proof. (B.7) follows directly from Lemma B.1. (B.8) is an immediate result of (B.7) and the decomposition of M_i . \square

B.2 Auxiliary Lemmas

Lemma B.3 is a key lemma that established the performance of the SSIV estimator.

Lemma B.3. *Define ϕ_i as an i.i.d. random variable with constant variance. Define the conditional expectation $Q_j^\phi = \mathbb{E}\left[\frac{A_{ij}^{pre} \phi_i}{q_n^{pre}} | w_j\right]$. Under Assumptions 1 and 3,*

$$\frac{1}{n^2 q_n^{pre}} \sum_{i=1}^n Z_i^{SSIV} \phi_i = \frac{1}{n} \sum_{i=1}^n (T_i - \pi) Q_i^\phi + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n} \sqrt{n q_n^{pre}}}\right). \quad (\text{B.9})$$

Proof. See proof in Section D.1 of the Online Appendix. \square

Lemma B.4. *Under Assumption 1, we have*

$$\frac{1}{n^2 q_n^{pre}} \sum_{i=1}^n Z_i^{SSIV} M_i = \frac{\mathbb{E}[Z_i^{SSIV}(r_{0,i} + r_{1,i} - \mu_{r_{1,i}})]}{n q_n^{pre}} + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right). \quad (\text{B.10})$$

Furthermore, we can show that

$$\frac{1}{n} \sum_{i=1}^n Z_i^{SSIV} M_i - \overline{Z^{SSIV}} \cdot \overline{M} = \begin{cases} O_{\mathbb{P}}\left(\frac{q_n^{pre}}{\max\{q_n^{pre}, q_n^{post}\}}\right) + O_{\mathbb{P}}\left(\frac{q_n^{pre}}{\sqrt{n} q_n^{post}}\right) & \text{if } \mathbb{V}(\xi_i) = 0; \\ O_{\mathbb{P}}\left(\frac{q_n^{pre}}{\max\{q_n^{pre}, q_n^{post}\}}\right) + O_{\mathbb{P}}(\sqrt{n} q_n^{pre}) & \text{if } \mathbb{V}(\xi_i) > 0. \end{cases} \quad (\text{B.11})$$

Proof. See proof in Section D.1 of the Online Appendix. \square

B.3 Consistency

Define $D_n^{\text{IV}} = \text{diag}(1, 1, 1/(nq_n^{\text{pre}}))$, the diagonal matrix that scales the third coordinate. The closed-form of $(D_n^{\text{IV}} Z^\top X)^{-1}$ is

$$(D_n^{\text{IV}} Z^\top X)^{-1} = \frac{1}{\det(D_n^{\text{IV}} Z^\top X)} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \quad (\text{B.12})$$

where

$$\begin{aligned} b_{11} &= \frac{1}{nq_n^{\text{pre}}} (\bar{T} \cdot \overline{MZ^{\text{SSIV}}} - \overline{TM} \cdot \overline{TZ^{\text{SSIV}}}), \quad b_{12} = \frac{1}{nq_n^{\text{pre}}} (-\bar{T} \cdot \overline{MZ^{\text{SSIV}}} + \overline{M} \cdot \overline{TZ^{\text{SSIV}}}), \\ b_{13} &= \overline{T} \cdot \overline{(1-T)M}, \quad b_{21} = \frac{1}{nq_n^{\text{pre}}} (-\bar{T} \cdot \overline{MZ^{\text{SSIV}}} + \overline{Z^{\text{SSIV}}} \cdot \overline{TM}), \\ b_{22} &= \frac{1}{nq_n^{\text{pre}}} (\overline{MZ^{\text{SSIV}}} - \overline{M} \cdot \overline{Z^{\text{SSIV}}}), \quad b_{23} = \overline{T} \cdot \overline{M} - \overline{TM}, \\ b_{31} &= \frac{1}{nq_n^{\text{pre}}} (\bar{T} \cdot \overline{TZ^{\text{SSIV}}} - \overline{T} \cdot \overline{Z^{\text{SSIV}}}), \quad b_{32} = \frac{1}{nq_n^{\text{pre}}} (-\overline{TZ^{\text{SSIV}}} + \overline{T} \cdot \overline{Z^{\text{SSIV}}}), \quad b_{33} = \overline{T}(1 - \overline{T}), \end{aligned}$$

and $\det(D_n^{\text{IV}} Z^\top X) = b_{33}b_{22} - b_{32}b_{23}$. Define $b_{22}^* = \frac{\mathbb{E}[Z_i^{\text{SSIV}}(r_{0,i} + r_{1,i})]}{nq_n^{\text{pre}}} \asymp \frac{1}{n \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}}$. Below, we show the probability limit of $(D_n^{\text{IV}} Z^\top X)^{-1}$.

Theorem B.1. *Under Assumption 1, then*

Case (a)

$$(D_n^{\text{IV}} Z^\top X)^{-1} = \frac{\begin{pmatrix} \pi b_{22}^* & -\pi b_{22}^* & a_{13}^* \\ -\pi b_{22}^* & b_{22}^* & -\text{Cov}(T_i, \xi_i + \mu_{r_1, i}) \\ 0 & 0 & \pi(1 - \pi) \end{pmatrix} + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)}{\pi(1 - \pi)b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)};$$

Case (b)

$$(D_n^{\text{IV}} Z^\top X)^{-1} = \frac{\begin{pmatrix} \pi b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) & -\pi b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) & a_{13}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) \\ -\pi b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) & b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{nnq_n^{\text{post}}}}\right) & O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{\text{post}}}}\right) \\ O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) & O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}\sqrt{nq_n^{\text{pre}}}}\right) & \pi(1 - \pi) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) \end{pmatrix}}{\pi(1 - \pi)b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{nnq_n^{\text{post}}}}\right)}.$$

Proof. By Lemmas B.2 and B.4, we have

$$b_{11} = \pi b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right), b_{12} = -\pi b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right), b_{13} = b_{13}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right), b_{21} = -\pi b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right).$$

By Lemma B.3, we have $b_{31} = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$ and $b_{32} = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}\sqrt{nq_n^{\text{pre}}}}\right)$. With i.i.d. data, we have $b_{33} = \pi(1 - \pi) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$. For other terms, we consider Cases (a) and (b).

Case (a) Lemmas B.2 and B.4 imply that

$$b_{22} = b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right), \quad b_{23} = -\text{Cov}(T_i, \xi_i + \mu_{r_1, i}) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right),$$

and hence $\det(D_n^{\text{IV}} Z^{\top} X) = \pi(1 - \pi)b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$.

Case (b) Similarly, Lemmas B.2 and B.4 imply that

$$b_{22} = b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{nnq_n^{\text{post}}}}\right), \quad b_{23} = O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{\text{post}}}}\right),$$

and hence $\det(D_n^{\text{IV}} Z^{\top} X) = \pi(1 - \pi)b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{nnq_n^{\text{post}}}}\right)$. □

Proof of Theorem 3.1. By Lemma B.3, we have

$$\frac{1}{n^2 q_n^{\text{pre}}} \sum_{i=1}^n Z_i^{\text{SSIV}} = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) \quad \text{and} \quad \frac{1}{n^2 q_n^{\text{pre}}} \sum_{i=1}^n Z_i^{\text{SSIV}} u_i = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right). \quad (\text{B.13})$$

(B.13) then implies that $\frac{1}{n^2 q_n^{\text{pre}}} \sum_{i=1}^n (Z_i^{\text{SSIV}} - \bar{Z}^{\text{SSIV}}) u_i = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$. Under i.i.d. data, $\frac{1}{n} \sum_{i=1}^n u_i = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$ and $\frac{1}{n} \sum_{i=1}^n (T_i - \bar{T}) u_i = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$.

Case (a) By Theorem B.1 and the closed form of $\hat{\beta}^{\text{IV}} - \beta$, we conclude:

$$\hat{\beta}_1^{\text{IV}} - \beta_1 = \frac{b_{22}^* \cdot O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) - O_{\mathbb{P}}(1) \cdot O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) + O_{\mathbb{P}}\left(\frac{1}{n}\right)}{\pi(1-\pi)b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)},$$

$$\hat{\beta}_2^{\text{IV}} - \beta_2 = \frac{\pi(1-\pi) \cdot O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) + O_{\mathbb{P}}\left(\frac{1}{n}\right)}{\pi(1-\pi)b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)}.$$

Hence, $\hat{\beta}^{\text{IV}}$ is consistent when $\max\{q_n^{\text{pre}}, q_n^{\text{post}}\} \prec \frac{1}{\sqrt{n}}$, with $\hat{\beta}^{\text{IV}} - \beta = O_{\mathbb{P}}(\sqrt{n} \max\{q_n^{\text{pre}}, q_n^{\text{post}}\})$.

Case (b) By Theorem B.1, for $\hat{\beta}_1^{\text{IV}} - \beta_1$, we have

$$\hat{\beta}_1^{\text{IV}} - \beta_1 = \frac{b_{22}^* \cdot O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) + O_{\mathbb{P}}\left(\frac{1}{n\sqrt{n}q_n^{\text{post}}}\right)}{\pi(1-\pi)b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{nn}q_n^{\text{post}}}\right)} = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}} \cdot \max\left\{\frac{\max\{q_n^{\text{pre}}, q_n^{\text{post}}\}}{\sqrt{q_n^{\text{post}}}}, 1\right\}\right),$$

where the second equality follows from $b_{22}^* \asymp \frac{1}{n \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}}$. Thus, $\hat{\beta}_1^{\text{IV}}$ is consistent if $\max\{q_n^{\text{pre}}, q_n^{\text{post}}\} \prec \sqrt{n}q_n^{\text{post}}$. For $\hat{\beta}_2^{\text{IV}} - \beta_2$, we have

$$\hat{\beta}_2^{\text{IV}} - \beta_2 = \frac{\pi(1-\pi) \cdot O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) + O_{\mathbb{P}}\left(\frac{1}{n}\right)}{\pi(1-\pi)b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{nn}q_n^{\text{post}}}\right)} = O_{\mathbb{P}}\left(\sqrt{n} \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}\right).$$

Therefore, $\hat{\beta}_2^{\text{IV}}$ is consistent if $\max\{q_n^{\text{pre}}, q_n^{\text{post}}\} \prec 1/\sqrt{n}$. The same condition implies consistency of $\hat{\beta}_0^{\text{IV}}$, with $\hat{\beta}_0^{\text{IV}} - \beta_0 = O_{\mathbb{P}}\left(\sqrt{n} \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}\right)$. \square

B.4 Asymptotic normality

Theorem B.2 below establishes the CLT with weak dependence, which makes use of the proof of Ross (2011, Theorem 3.6) and is the unconditional version of Leung (2020, Theorem 4). Let H_1, \dots, H_n be real-valued random variables. Define the indicator of dependency as $D_{ij} = 1\{A_{ij}^{\diamond} + \max_k A_{ik}^{\diamond} A_{kj}^{\diamond} + 1\{i = j\} > 0\}$ where $\diamond \in \{\text{pre}, \text{post}\}$. In words, two units are dependent if they are directly connected, share at least one common friend, or refer to the same index. An $n \times n$ binary symmetric matrix D with (i, j) th element being D_{ij} is a dependency graph on H if for any two disjoint subsets $I_1, I_2 \subseteq \{1, \dots, n\}$, we have $\{H_i : i \in I_1\} \perp\!\!\!\perp \{H_i : i \in I_2\}$ conditional on the event that $D_{ij} = 0$ for all $i \in I_1$ and $j \in I_2$. By construction, $D_{ii} = 1$ for all i . Define $\mathcal{S}_i = \{j : D_{ij} = 1\}$ and $S_i = |\mathcal{S}_i| = \sum_{j=1}^n D_{ij}$. Define $\sigma_n^2 = \mathbb{V}(\sum_{i=1}^n H_i)$. Let $(D^3)_{ij}$ denote the (i, j) th entry of the third matrix power of D .

Theorem B.2 (CLT for local dependence). *Under the following assumptions, we have $\sigma_n^{-1} \sum_{i=1}^n H_i \xrightarrow{d} \mathcal{N}(0, 1)$ as $n \rightarrow \infty$: (a) $\sigma_n^2 = O(n)$; (b) $\max_i \mathbb{E}(|H_i|^4) = O(1)$; (c) $n^{-1} \sum_{i=1}^n S_i^2 = o_{\mathbb{P}}(\sqrt{n})$, $n^{-1} \sum_{i=1}^n S_i^3 = o_{\mathbb{P}}(n)$, and $n^{-1} \sum_{(i,j)} (D^3)_{ij} = o_{\mathbb{P}}(n)$.*

Proof. Convergence in the Wasserstein distance $\Delta(U, V)$ implies convergence in the Kolmogorov metric (see Ross (2011, Proposition 1.2)), which in turn implies weak convergence. Thus, the theorem holds if $\Delta\left(\frac{1}{\sigma_n} \sum_{i=1}^n H_i, \mathcal{N}(0, 1)\right) \xrightarrow{p} 0$. The distance

is bounded above by

$$\begin{aligned} \Delta \left(\frac{1}{\sigma_n} \sum_{i=1}^n H_i, \mathcal{N}(0, 1) \right) &\leq \frac{1}{\sigma_n^3} \sum_{i=1}^n \mathbb{E} \left[\left| H_i \left(\sum_{j \in \mathcal{S}_i} H_j \right)^2 \right| \right] + \frac{\sqrt{2}}{\sqrt{\pi} \sigma_n^2} \sqrt{\text{Var} \left(\sum_{i=1}^n \sum_{j \in \mathcal{S}_i} H_i H_j \right)} \\ &\leq \max_i \mathbb{E} [|H_i|^3] \frac{n}{\sigma_n^3} \frac{1}{n} \sum_{i=1}^n S_i^2 + \frac{1}{\sqrt{n}} \frac{n}{\sigma_n^2} \sqrt{\frac{4}{\pi} \max_i \mathbb{E} [H_i^4]} \sqrt{\frac{4}{n} \sum_{i=1}^n S_i^3 + \frac{3}{n} \sum_{i=1}^n \sum_{j=1}^n (G^3)_{ij}}, \end{aligned}$$

which is $o_{\mathbb{P}}(1)$ under Assumptions (a)–(c). \square

We verify Assumption (c) in Theorem B.2 when $nq_n^\diamond \asymp 1$ for $\diamond \in \{\text{pre}, \text{post}\}$.

Lemma B.5. *Suppose $nq_n^\diamond \asymp 1$ for $\diamond \in \{\text{pre}, \text{post}\}$. Then*

$$(i) \frac{1}{n} \sum_{i=1}^n S_i^2 = o_{\mathbb{P}}(\sqrt{n}); \quad (ii) \frac{1}{n} \sum_{i=1}^n S_i^3 = o_{\mathbb{P}}(n); \quad (iii) \frac{1}{n} \sum_{(i,j)} (D^3)_{ij} = o_{\mathbb{P}}(n).$$

Proof. See proof in Section D.1 of the Online Appendix. \square

Proof of Theorem 3.2. We complete the proof in three steps.

Step 1: Asymptotic normality. We establish the asymptotic normality of the numerator $D_n^{\text{IV}} \cdot \frac{1}{n} \sum_{i=1}^n Z_i u_i$, considering two cases.

Case 1: $nq_n^{\text{pre}} \succ 1$. By (B.13), we have

$$\frac{1}{n^2 q_n^{\text{pre}}} \sum_{i=1}^n Z_i^{\text{SSIV}} u_i = \frac{1}{n} \sum_{i=1}^n (T_i - \pi) \mathbb{E} \left[\frac{A_{ij}^{\text{pre}} u_j}{q_n^{\text{pre}}} | w_i \right] + O_{\mathbb{P}} \left(\frac{1}{\sqrt{n} \sqrt{nq_n^{\text{pre}}}} \right),$$

which is the average of i.i.d. terms plus a negligible remainder $o_{\mathbb{P}}(1/\sqrt{n})$. Therefore, asymptotic normality of $D_n^{\text{IV}} \cdot \frac{1}{n} \sum_{i=1}^n Z_i u_i$ follows from a standard Cramér–Wold argument and the Lindeberg–Lévy CLT.

Case 2: $nq_n^{\text{pre}} \asymp 1$. It suffices to verify the assumptions in Theorem B.2. Assumption (a) holds since $D_n^{\text{IV}} V_{\text{num}}^{\text{IV}} D_n^{\text{IV}} = O(n)$. Assumption (b) is satisfied by boundedness of X_i . Assumption (c) follows from Lemma B.5, applied to A^{pre} under the regime $nq_n^{\text{pre}} \asymp 1$. By Theorem B.2 and the Cramér–Wold device, we conclude that $(V_{\text{num}}^{\text{IV}})^{-1/2} \sum_{i=1}^n Z_i u_i \xrightarrow{d} \mathcal{N}(0, I_3)$.

Step 2: consistent variance estimator. We show that $D_n^{\text{IV}} (\hat{V}_{\text{num}}^{\text{IV}} - V_{\text{num}}^{\text{IV}}) D_n^{\text{IV}} = o_{\mathbb{P}}(1)$. We verify the consistency of each entry separately.

(1): Consistency of the (1, 1)th element of $\hat{V}_{\text{num}}^{\text{IV}}$. We can show $\frac{1}{n} \sum_{i=1}^n (\hat{u}_i^{\text{IV}})^2 = \mathbb{E}(u_i^2) + o_{\mathbb{P}}(1)$ holds by $\hat{\beta}^{\text{IV}} - \beta = o_{\mathbb{P}}(1)$, and bounded support of T_i and M_i .

(2): Consistency of (3,3)th element of $\hat{V}_{\text{num}}^{\text{IV}}$. Expanding the square:

$$\frac{1}{n} \sum_{j=1}^n \left(\sum_{i \neq j} A_{ij}^{\text{pre}} \hat{u}_i^{\text{IV}} \right)^2 = \frac{1}{n} \left[\begin{aligned} & \sum_{j=1}^n \left(\sum_{i \neq j} A_{ij}^{\text{pre}} u_i \right)^2 + \sum_{j=1}^n \left(\sum_{i \neq j} A_{ij}^{\text{pre}} (\hat{\beta}^{\text{IV}} - \beta)^\top X_i \right)^2 \\ & - 2 \sum_{j=1}^n \left(\sum_{i \neq j} A_{ij}^{\text{pre}} u_i \sum_{i \neq j} A_{ij}^{\text{pre}} (\hat{\beta}^{\text{IV}} - \beta)^\top X_i \right) \end{aligned} \right]. \quad (\text{B.14})$$

For the first term of (B.14), the expectation is $\mathbb{E} \left[\frac{1}{n} \sum_{j=1}^n \left(\sum_{i \neq j} A_{ij}^{\text{pre}} u_i \right)^2 \right] = O((nq_n^{\text{pre}})^2)$.

The variance involves at most six-way dependence from adjacency terms and is $\text{Var} \left(\frac{1}{n} \sum_{j=1}^n \left(\sum_{i \neq j} A_{ij}^{\text{pre}} u_i \right)^2 \right) = O(n^3(q_n^{\text{pre}})^4)$. Hence, by Chebyshev's inequality:

$$\frac{1}{n} \sum_{j=1}^n \left(\sum_{i \neq j} \frac{A_{ij}^{\text{pre}} u_i}{nq_n^{\text{pre}}} \right)^2 = \frac{1}{n} \mathbb{E} \left[\sum_{j=1}^n \left(\sum_{i \neq j} \frac{A_{ij}^{\text{pre}} u_i}{nq_n^{\text{pre}}} \right)^2 \right] + O_{\mathbb{P}} \left(\frac{1}{\sqrt{n}} \right). \quad (\text{B.15})$$

For the second term of (B.14), we apply the triangle inequality:

$$\frac{1}{n} \sum_{j=1}^n \left(\sum_{i \neq j} A_{ij}^{\text{pre}} (\hat{\beta}^{\text{IV}} - \beta)^\top X_i \right)^2 \leq \sum_{k=0}^2 (\hat{\beta}_k^{\text{IV}} - \beta_k)^2 \frac{1}{n} \sum_{j=1}^n \left(\sum_{i \neq j} A_{ij}^{\text{pre}} X_{ik} \right)^2 = o_{\mathbb{P}}((nq_n^{\text{pre}})^2) \quad (\text{B.16})$$

by $\hat{\beta}^{\text{IV}} - \beta = o_{\mathbb{P}}(1)$ and $\frac{1}{n} \sum_{j=1}^n \left(\sum_{i \neq j} A_{ij}^{\text{pre}} X_{ik} \right)^2 = O_{\mathbb{P}}((nq_n^{\text{pre}})^2)$ for $k = 0, 1, 2$ by boundedness of X_{ik} . For the third term of (B.14):

$$\begin{aligned} & \frac{1}{n} \sum_{j=1}^n \left(\sum_{i \neq j} A_{ij}^{\text{pre}} u_i \right) \left(\sum_{i \neq j} A_{ij}^{\text{pre}} (\hat{\beta}^{\text{IV}} - \beta)^\top X_i \right) \\ & \leq \sum_{k=0}^2 \left| \hat{\beta}_k^{\text{IV}} - \beta_k \right| \frac{1}{n} \sum_{j=1}^n \left[\left(\sum_{i \neq j} A_{ij}^{\text{pre}} u_i \right)^2 + \left(\sum_{i \neq j} A_{ij}^{\text{pre}} X_{ik} \right)^2 \right] = o_{\mathbb{P}}((nq_n^{\text{pre}})^2), \quad (\text{B.17}) \end{aligned}$$

where the final equality follows from $\hat{\beta}^{\text{IV}} - \beta = o_{\mathbb{P}}(1)$, and $\frac{1}{n} \sum_{j=1}^n \left(\sum_{i \neq j} A_{ij}^{\text{pre}} u_i \right)^2 = O_{\mathbb{P}}((nq_n^{\text{pre}})^2)$. Combining (B.15), (B.16), and (B.17), we conclude:

$$\frac{1}{n(nq_n^{\text{pre}})^2} \sum_{j=1}^n \left(\sum_{i \neq j} A_{ij}^{\text{pre}} \hat{u}_i^{\text{IV}} \right)^2 = \frac{1}{n(nq_n^{\text{pre}})^2} \sum_{j=1}^n \mathbb{E} \left[\left(\sum_{i \neq j} A_{ij}^{\text{pre}} u_i \right)^2 \right] + o_{\mathbb{P}}(1).$$

(3): Consistency of (2, 3) element of $\hat{V}_{\text{num}}^{\text{IV}}$. By expanding \hat{u}_i^{IV} :

$$\frac{1}{n} \sum_{i=1}^n \sum_{j \neq i} A_{ij}^{\text{pre}} \hat{u}_i^{\text{IV}} \hat{u}_j^{\text{IV}} = \frac{1}{n} \left[\begin{aligned} & \sum_{i=1}^n \sum_{j \neq i} A_{ij}^{\text{pre}} u_i u_j - 2 \sum_{i=1}^n \sum_{j \neq i} A_{ij}^{\text{pre}} u_i (\hat{\beta}^{\text{IV}} - \beta)^\top X_j \\ & + \sum_{i=1}^n \sum_{j \neq i} A_{ij}^{\text{pre}} (\hat{\beta}^{\text{IV}} - \beta)^\top X_i X_j^\top (\hat{\beta}^{\text{IV}} - \beta) \end{aligned} \right]. \quad (\text{B.18})$$

By analogous argument to the (2, 3) element of $\hat{V}_{\text{num}}^{\text{IV}}$, so we can show that

$$\frac{1}{n^2 q_n^{\text{pre}}} \sum_{i=1}^n \sum_{j \neq i} A_{ij}^{\text{pre}} \hat{u}_i^{\text{IV}} \hat{u}_j^{\text{IV}} = \frac{1}{n^2 q_n^{\text{pre}}} \sum_{i=1}^n \sum_{j \neq i} \mathbb{E} [A_{ij}^{\text{pre}} u_i u_j] + o_{\mathbb{P}}(1).$$

Step 3: Combine the results. By Steps 1–2 and CMT, $(\hat{V}_{\text{num}}^{\text{IV}})^{-1/2} \sum_{i=1}^n Z_i u_i \xrightarrow{d} \mathcal{N}(0, I_3)$. Applying Theorem B.1 and CMT again then completes the proof. \square

C Proof of results in Section 3.3

C.1 Useful lemmas

Define $q_n^{\text{pre}} \tilde{G}_n^{\text{pre}} = \sum_{k=1}^r \lambda_k^* \psi_k^* \psi_k^{*\top}$, which is formed from the leading r terms of $q_n^{\text{pre}} G_n^{\text{pre}}$.

Lemma C.1. Suppose $q_n \succ \frac{\log n}{\log \log n} / n$. Under Assumptions 1 and 4, we have

$$(1) \|A^{\text{pre}} - q_n^{\text{pre}} G_n^{\text{pre}}\|_{\text{op}} = O_{\mathbb{P}} \left(\sqrt{n q_n^{\text{pre}}} \left(\frac{\log n}{\log \log n} \right)^{1/4} \right);$$

$$(2) \|A^{\text{pre}} - q_n^{\text{pre}} \tilde{G}_n^{\text{pre}}\|_{\text{op}} = O_{\mathbb{P}} \left(\sqrt{n q_n^{\text{pre}}} \left(\frac{\log n}{\log \log n} \right)^{1/4} \right).$$

Proof. See proof in Section D.2 of the Online Appendix. \square

Lemma C.2. Suppose $q_n^{\text{pre}} \succ \frac{\log n}{\log \log n} / n$. Let ψ_k^* denote the vector of $\psi_k^*(w_i)$. There exists an $r \times r$ orthogonal matrix \hat{R} , where r is defined in Assumption 4, such that if we write $\hat{\Psi}^R = \hat{\Psi} \hat{R}$, and let $\hat{\psi}_k^R$ be the k -th column of $\hat{\Psi}^R$ for $k \leq r$, then under Assumptions 1 and 4, for $l = 1, \dots, n$, we have

$$(\psi_l^*)^\top \left(\hat{\psi}_k^R - \psi_k^* \right) = O_{\mathbb{P}} \left(\sqrt{\frac{\log n}{\log \log n}} / (n q_n^{\text{pre}}) \right). \quad (\text{C.1})$$

Proof. See proof in Section D.2 of the Online Appendix. \square

Let ϕ_i be any i.i.d. random variable with constant variance. Define the projection coefficient $\mu_{n,k}^\phi = \sum_{i=1}^n \phi_i \psi_k^*(w_i)$ and the projection residual from projecting ϕ_i onto the span of $\{\psi_k^*(w_i)\}_{k=1}^r$ as $\eta_i^\phi = \phi_i - \sum_{k=1}^r \mu_{n,k}^\phi \psi_k^*(w_i)$.

Lemma C.3. *Suppose $q_n^{\text{pre}} \succ \frac{\log n}{\log \log n}/n$. Under Assumptions 1 and 4, we have*

$$\frac{1}{n} \sum_{i=1}^n (Z_i^{\text{SSIV}} + \hat{\delta}_i) \phi_i = \frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}} \eta_i^\phi + o_{\mathbb{P}} \left(\sqrt{q_n^{\text{pre}}} \right). \quad (\text{C.2})$$

Proof. See proof in Section D.2 of the Online Appendix. \square

Define the projection coefficient $\mu_{n,k}^r = \sum_{i=1}^n (r_{0,i} + r_{1,i} - \mu_{r_1,i}) \psi_k^*(w_i)$ and the projection residual $\eta_i^r = (r_{0,i} + r_{1,i} - \mu_{r_1,i}) - \sum_{k=1}^r \mu_{n,k}^r \psi_k^*(w_i)$.

Lemma C.4. *Suppose $q_n^{\text{pre}} \succ \frac{\log n}{\log \log n}/n$. Under Assumptions 1 and 4, then*

$$\frac{1}{n} \sum_{i=1}^n (Z_i^{\text{SSIV}} + \hat{\delta}_i) M_i = \frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}} \eta_i^r + O_{\mathbb{P}} \left(\sqrt{q_n^{\text{pre}}} \right). \quad (\text{C.3})$$

Proof. See proof in Section D.2 of the Online Appendix. \square

C.2 Consistency

Define $D_n^{\text{DE}} = \text{diag}(1, 1, 1/\sqrt{nq_n^{\text{pre}}})$. We first show the probability limit of $(D_n^{\text{DE}}(\tilde{Z}^{\text{DE}})^\top X)^{-1}$, which has the closed-form expression:

$$\left(D_n^{\text{DE}}(\tilde{Z}^{\text{DE}})^\top X \right)^{-1} = \frac{1}{\det(D_n^{\text{DE}}(\tilde{Z}^{\text{DE}})^\top X)} \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \quad (\text{C.4})$$

where

$$\begin{aligned} c_{11} &= \frac{1}{\sqrt{nq_n^{\text{pre}}}} (\bar{T} \cdot \overline{MZ^{\text{DE}}} - \overline{TM} \cdot \overline{TZ^{\text{DE}}}), & c_{12} &= \frac{1}{\sqrt{nq_n^{\text{pre}}}} (-\bar{T} \cdot \overline{MZ^{\text{DE}}} + \overline{M} \cdot \overline{TZ^{\text{DE}}}), & c_{13} &= b_{13}, \\ c_{21} &= \frac{1}{\sqrt{nq_n^{\text{pre}}}} (-\bar{T} \cdot \overline{MZ^{\text{DE}}} + \overline{Z^{\text{DE}}} \cdot \overline{TM}), & c_{22} &= \frac{1}{\sqrt{nq_n^{\text{pre}}}} (\overline{MZ^{\text{DE}}} - \overline{M} \cdot \overline{Z^{\text{DE}}}), & c_{23} &= b_{23}, \\ c_{31} &= \frac{1}{\sqrt{nq_n^{\text{pre}}}} (\bar{T} \cdot \overline{TZ^{\text{DE}}} - \overline{T} \cdot \overline{Z^{\text{DE}}}), & c_{32} &= \frac{1}{\sqrt{nq_n^{\text{pre}}}} (-\overline{TZ^{\text{DE}}} + \overline{T} \cdot \overline{Z^{\text{DE}}}), & c_{33} &= b_{33}, \end{aligned}$$

and $\det(D_n^{\text{DE}}(\tilde{Z}^{\text{DE}})^\top X) = c_{33}c_{22} - c_{32}c_{23}$. Define $c_{22}^* = \frac{\mathbb{E}[Z_i^{\text{SSIV}}(r_{0,i} + r_{1,i})]}{\sqrt{nq_n^{\text{pre}}}} \asymp \frac{\min\{q_n^{\text{pre}}, q_n^{\text{post}}\}}{q_n^{\text{post}} \sqrt{nq_n^{\text{pre}}}}$.

Theorem C.1. Suppose $q_n^{\text{pre}} \asymp \frac{\log(n)}{\log(\log(n))}/n$. Under Assumptions 1 and 4, then
Case (a):

$$\left(D_n^{\text{DE}}(\tilde{Z}^{\text{DE}})^\top X\right)^{-1} = \frac{\begin{pmatrix} \pi c_{22}^* & -\pi c_{22}^* & a_{13}^* \\ -\pi c_{22}^* & c_{22}^* & -\text{Cov}(T_i, \xi_i + \mu_{r_1, i}) \\ 0 & 0 & \pi(1-\pi) \end{pmatrix} + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)}{\pi(1-\pi)c_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)}.$$

Case (b):

$$\left(D_n^{\text{DE}}(\tilde{Z}^{\text{DE}})^\top X\right)^{-1} = \frac{\begin{pmatrix} \pi c_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) & -\pi c_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) & a_{13}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) \\ -\pi c_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) & c_{22}^* + O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{\text{post}}}}\right) & O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{\text{post}}}}\right) \\ O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) & O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) & \pi(1-\pi) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) \end{pmatrix}}{\pi(1-\pi)c_{22}^* + O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{\text{post}}}}\right)}.$$

Proof. The proof is analogous to Theorem B.1 with Lemmas C.3 and C.4. \square

Proof of Theorem 3.3. We show the consistency by Theorem C.1. By Lemma C.3, we can show that

$$\frac{1}{n\sqrt{nq_n^{\text{pre}}}} \sum_{i=1}^n (Z_i^{\text{SSIV}} + \hat{\delta}_i) = \frac{1}{n\sqrt{nq_n^{\text{pre}}}} \sum_{i=1}^n Z_i^{\text{SSIV}} \eta_i^1 + O_{\mathbb{P}}\left(\frac{\sqrt{q_n^{\text{pre}}}}{\sqrt{n}}\right) = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$$

and thus

$$\frac{1}{n\sqrt{nq_n^{\text{pre}}}} \sum_{i=1}^n \left((Z_i^{\text{SSIV}} + \hat{\delta}_i) - \frac{1}{n} \sum_{i=1}^n (Z_i^{\text{SSIV}} + \hat{\delta}_i) \right) u_i = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right).$$

Case (a): By the closed-form of $(\hat{\beta}^{\text{DE}} - \beta)$ and combining Theorem C.1:

$$\begin{aligned} \hat{\beta}_1^{\text{DE}} - \beta_1 &= \frac{\left(c_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)\right) O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)}{\pi(1-\pi)c_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)} = \frac{O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)}{\pi(1-\pi)c_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)}, \\ \hat{\beta}_2^{\text{DE}} - \beta_2 &= \frac{\left(\pi(1-\pi) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)\right) O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)}{\pi(1-\pi)c_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)} = \frac{O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)}{\pi(1-\pi)c_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)}. \end{aligned}$$

Therefore, $\hat{\beta}^{\text{DE}}$ is consistent when $\max\{q_n^{\text{pre}}, q_n^{\text{post}}\} \prec \sqrt{q_n^{\text{pre}}}$ with $\hat{\beta}^{\text{DE}} - \beta = O_{\mathbb{P}}\left(\frac{\max\{q_n^{\text{pre}}, q_n^{\text{post}}\}}{\sqrt{q_n^{\text{pre}}}}\right)$.

Case (b): By combining the results from Theorem C.1, we have

$$\hat{\beta}_1^{\text{DE}} - \beta_1 = \frac{\left(c_{22}^* + O\left(\frac{1}{n\sqrt{q_n^{\text{post}}}}\right)\right) O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) + O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{\text{post}}}}\right) O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)}{\pi(1-\pi)c_{22}^* + O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{\text{post}}}}\right)}.$$

The consistency of $\hat{\beta}_1^{\text{DE}}$ always holds with $\hat{\beta}_1^{\text{DE}} - \beta_1 = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$. For $\hat{\beta}_2^{\text{DE}}$, we have

$$\hat{\beta}_2^{\text{DE}} - \beta_2 = \frac{\left(\pi(1-\pi) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)\right) O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) + O_{\mathbb{P}}\left(\frac{1}{n}\right)}{\pi(1-\pi)c_{22}^* + O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{\text{post}}}}\right)}.$$

Therefore, $\hat{\beta}_2^{\text{DE}} - \beta_2 = O_{\mathbb{P}}\left(\frac{\max\{q_n^{\text{pre}}, q_n^{\text{post}}\}}{\sqrt{q_n^{\text{pre}}}}\right)$ provided $\max\{q_n^{\text{pre}}, q_n^{\text{post}}\} \prec \sqrt{q_n^{\text{pre}}}$. Under the same condition, it follows that $\hat{\beta}_0^{\text{DE}} - \beta_0 = O_{\mathbb{P}}\left(\frac{\max\{q_n^{\text{pre}}, q_n^{\text{post}}\}}{\sqrt{q_n^{\text{pre}}}}\right)$. \square

C.3 Asymptotic normality

Proof of Theorem 3.4. We complete the proof in three steps.

Step 1: asymptotic normality. By Lemma C.3, we have

$$\frac{1}{n\sqrt{nq_n^{\text{pre}}}} \sum_{i=1}^n (Z_i^{\text{SSIV}} + \hat{\delta}_i) u_i = \frac{1}{n\sqrt{nq_n^{\text{pre}}}} \sum_{i=1}^n \sum_{j \neq i} A_{ij}^{\text{pre}} (T_j - \pi) \eta_i^u + o_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right). \quad (\text{C.5})$$

By Theorem B.2 with each index corresponding to a pair, we have

$$\begin{aligned} \Delta \left(\frac{1}{\sigma_n} \sum_{(i,j)} H_{ij}, \mathcal{N}(0, 1) \right) &\leq \frac{1}{\sigma_n^3} \sum_{(i,j)} \mathbb{E} \left[\left\| H_{ij} \left(\sum_{(k,l) \in \mathcal{S}(i,j)} H_{kl} \right)^2 \right\| \right] + \frac{\sqrt{2}}{\sqrt{\pi} \sigma_n^2} \sqrt{\text{Var} \left(\sum_{(i,j)} H_{ij} \sum_{(k,l) \in \mathcal{S}(i,j)} H_{kl} \right)} \\ &\leq \frac{C}{\sigma_n^3} \sum_{(i,j)} \mathbb{E} \left[\left(A_{ij}^{\text{pre}} \left(\sum_{(k,l) \in \mathcal{S}(i,j)} H_{kl} \right) \right)^2 \right] + \frac{\sqrt{2}}{\sqrt{\pi} \sigma_n^2} \sqrt{\text{Var} \left(\sum_{(i,j)} H_{ij} \sum_{(k,l) \in \mathcal{S}(i,j)} H_{kl} \right)}. \end{aligned} \quad (\text{C.6})$$

Define $H_{ij} = A_{ij}^{\text{pre}} (T_j - \pi) \eta_i^u$ and $\sigma_n^2 = \text{Var} \left(\sum_{(i,j)} H_{ij} \right)$. It suffices to show the

asymptotic normality of $\sigma_n^{-1} \sum_{(i,j)} H_{ij}$ by showing

$$\frac{1}{\sigma_n^3} \sum_{i,j} \mathbb{E} \left[\left\| H_{ij} \left(\sum_{(k,l) \in \mathcal{S}(i,j)} H_{kl} \right)^2 \right\| \right] = O\left(\sqrt{q_n^{\text{pre}}}\right), \quad (\text{C.7})$$

$$\frac{\sqrt{2}}{\sqrt{\pi} \sigma_n^2} \sqrt{\text{Var} \left(\sum_{(i,j)} H_{ij} \sum_{(k,l) \in \mathcal{S}(i,j)} H_{kl} \right)} = O\left(\sqrt{q_n^{\text{pre}}}\right). \quad (\text{C.8})$$

Then by (C.6) with (C.7) and (C.8), we show $\sigma_n^{-1} \sum_{(i,j)} A_{ij}^{\text{pre}}(T_j - \pi) \eta_i^u \xrightarrow{d} \mathcal{N}(0, 1)$. Together with Cramér–Wold Theorem and (C.5), we can show that $(V_{\text{num}}^{\text{DE}})^{-1/2} \sum_{i=1}^n \tilde{Z}_i^{\text{DE}} u_i \xrightarrow{d} \mathcal{N}(0, I_3)$. We defer the details of (C.7) and (C.8) to Section D.2 of the Online Appendix.

Step 2: consistent variance estimator. By definition and Lemma C.2, we have

$$\hat{\mu}_k^R - \mu_k = \sum_{i=1}^n (\hat{u}_i^{\text{DE}} - u_i) (\hat{\psi}_{ki}^R - \psi_k^*(w_i)) + \sum_{i=1}^n (\hat{u}_i^{\text{DE}} - u_i) \psi_k^*(w_i) + \sum_{i=1}^n u_i (\hat{\psi}_{ki}^R - \psi_k^*(w_i)) = o_{\mathbb{P}}(\sqrt{n}),$$

where each term is controlled as follows:

- For the first term,

$$\sum_{i=1}^n (\hat{u}_i^{\text{DE}} - u_i) (\hat{\psi}_{ki}^R - \psi_k^*(w_i)) = (\hat{\beta}^{\text{DE}} - \beta)^\top \sum_{i=1}^n X_i (\hat{\psi}_{ki}^R - \psi_k^*(w_i)) = O_{\mathbb{P}} \left(\frac{\sqrt{\log n / \log \log n}}{\sqrt{n q_n^{\text{pre}}}} \right) = o_{\mathbb{P}}(\sqrt{n}).$$

- For the second term, $\sum_{i=1}^n (\hat{u}_i^{\text{DE}} - u_i) \psi_k^*(w_i) = (\hat{\beta}^{\text{DE}} - \beta)^\top \sum_{i=1}^n X_i \psi_k^*(w_i) = o_{\mathbb{P}}(\sqrt{n})$.

- For the third term, by Cauchy–Schwarz,

$$\sum_{i=1}^n u_i (\hat{\psi}_{ki}^R - \psi_k^*(w_i)) \leq \sqrt{\sum_{i=1}^n u_i^2} \cdot \left\| \hat{\psi}_k^R - \psi_k^* \right\| = O_{\mathbb{P}} \left(\frac{1}{\sqrt{q_n^{\text{pre}}}} \left(\frac{\log n}{\log \log n} \right)^{1/4} \right) = o_{\mathbb{P}}(\sqrt{n}).$$

Now we show the consistency of the variance estimators one by one.

(1): Consistency of the (1, 1) element of $\hat{V}_{\text{num}}^{\text{DE}}$. The proof is analogous to that of Theorem 3.2 and follows from $\hat{\beta}^{\text{DE}} - \beta = o_{\mathbb{P}}(1)$.

(2): Consistency of the (3, 3) element of $\hat{V}_{\text{num}}^{\text{DE}}$. By definition, $\hat{\eta}_i^u = \hat{u}_i^{\text{DE}} - \sum_{k=1}^r \hat{\mu}_k \hat{\psi}_{ki}$ = $\eta_i^u + (\hat{u}_i^{\text{DE}} - u_i) - \Delta_i$, where $\eta_i^u = u_i - \sum_{k=1}^r \mu_k \psi_k^*(w_i)$ and $\Delta_i = \sum_{k=1}^r (\hat{\mu}_k^R \hat{\psi}_{ki}^R - \mu_k \psi_k^*(w_i))$.

We decompose the variance estimator as:

$$\begin{aligned}
\frac{1}{n^2 q_n^{\text{pre}}} \sum_{(i,j)} A_{ij}^{\text{pre}} (\hat{\eta}_i^u)^2 &= \underbrace{\frac{1}{n^2 q_n^{\text{pre}}} \sum_{(i,j)} A_{ij}^{\text{pre}} (\eta_i^u)^2}_{S_1} + \underbrace{\frac{2}{n^2 q_n^{\text{pre}}} \sum_{(i,j)} A_{ij}^{\text{pre}} (\hat{u}_i^{\text{DE}} - u_i)^2}_{S_2} \\
&+ \underbrace{\frac{2}{n^2 q_n^{\text{pre}}} \sum_{(i,j)} A_{ij}^{\text{pre}} \Delta_i^2}_{S_3} + \underbrace{\frac{2}{n^2 q_n^{\text{pre}}} \sum_{(i,j)} A_{ij}^{\text{pre}} \eta_i^u (\hat{u}_i^{\text{DE}} - u_i - \Delta_i)}_{S_4}. \quad (\text{C.9})
\end{aligned}$$

S_1 : The expectation is $\mathbb{E} \left[\frac{1}{n^2 q_n^{\text{pre}}} \sum_{i,j=1}^n A_{ij}^{\text{pre}} (\eta_i^u)^2 \right] = O(1)$. The variance is

$$\begin{aligned}
&\text{Var} \left(\frac{1}{n^2 q_n^{\text{pre}}} \sum_{i=1}^n (\eta_i^u)^2 \sum_{j=1}^n A_{ij}^{\text{pre}} \right) \\
&= \frac{1}{n^4 (q_n^{\text{pre}})^2} \left[\sum_{i=1}^n \sum_{j=1}^n \mathbb{V}((\eta_i^u)^2 A_{ij}^{\text{pre}}) + \sum_{i=1}^n \sum_{j_1 \neq j_2} \text{Cov}((\eta_i^u)^2 A_{ij_1}^{\text{pre}}, (\eta_i^u)^2 A_{ij_2}^{\text{pre}}) \right. \\
&\quad \left. + \sum_{(i,k)} \sum_{j=1}^n \text{Cov}((\eta_i^u)^2 A_{ij}^{\text{pre}}, (\eta_k^u)^2 A_{kj}^{\text{pre}}) + \sum_{(i,k)} \sum_{(j_1, j_2)} \text{Cov}((\eta_i^u)^2 A_{ij_1}^{\text{pre}}, (\eta_k^u)^2 A_{kj_2}^{\text{pre}}) \right] = O\left(\frac{1}{n}\right).
\end{aligned}$$

Therefore, $\frac{1}{n^2 q_n^{\text{pre}}} \sum_{(i,j)} A_{ij}^{\text{pre}} (\eta_i^u)^2 = \mathbb{E} \left[\frac{1}{n^2 q_n^{\text{pre}}} \sum_{(i,j)} A_{ij}^{\text{pre}} (\eta_i^u)^2 \right] + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$.

S_2 : Since $\hat{\beta}^{\text{DE}} - \beta = o_{\mathbb{P}}(1)$, we have

$$\frac{1}{n^2 q_n^{\text{pre}}} \sum_{(i,j)} A_{ij}^{\text{pre}} (\hat{u}_i^{\text{DE}} - u_i)^2 \leq \frac{3}{n^2 q_n^{\text{pre}}} \sum_{k=0}^2 (\hat{\beta}_k^{\text{DE}} - \beta_k)^2 \sum_{(i,j)} A_{ij}^{\text{pre}} X_{ik}^2 = o_{\mathbb{P}}(1).$$

S_3 : By definition,

$$\frac{1}{n^2 q_n^{\text{pre}}} \sum_{i=1}^n \sum_{j \neq i} A_{ij}^{\text{pre}} \Delta_i^2 = \frac{1}{n^2 q_n^{\text{pre}}} \left[\sum_{k=1}^r \sum_{i=1}^n (\hat{\mu}_k^R \hat{\psi}_{ki}^R - \mu_k \psi_k^*(w_i))^2 \sum_j A_{ij}^{\text{pre}} \right. \\
\left. + \sum_{k \neq l} \sum_{i=1}^n (\hat{\mu}_k^R \hat{\psi}_{ki}^R - \mu_k \psi_k^*(w_i)) (\hat{\mu}_l^R \hat{\psi}_{li}^R - \mu_l \psi_l^*(w_i)) \sum_{j=1}^n A_{ij}^{\text{pre}} \right].$$

Using

$$\hat{\mu}_k^R \hat{\psi}_{ki}^R - \mu_k \psi_k^*(w_i) = (\hat{\mu}_k^R - \mu_k) (\hat{\psi}_{ki}^R - \psi_k^*(w_i)) + (\hat{\mu}_k^R - \mu_k) \psi_k^*(w_i) + \mu_k (\hat{\psi}_{ki}^R - \psi_k^*(w_i)), \quad (\text{C.10})$$

we bound the first term:

$$\begin{aligned} & \frac{1}{n^2 q_n^{\text{pre}}} \sum_{k=1}^r \sum_{i=1}^n (\hat{\mu}_k^R \hat{\psi}_{ki}^R - \mu_k \psi_k^*(w_i))^2 \sum_j A_{ij}^{\text{pre}} \\ & \leq \frac{3}{n^2 q_n^{\text{pre}}} \sum_{k=1}^r \left[\begin{aligned} & (\hat{\mu}_k^R - \mu_k)^2 \sum_i (\hat{\psi}_{ki}^R - \psi_k^*(w_i))^2 \sum_j A_{ij}^{\text{pre}} \\ & + (\hat{\mu}_k^R - \mu_k)^2 \sum_i \psi_k^*(w_i)^2 \sum_j A_{ij}^{\text{pre}} + \mu_k^2 \sum_i (\hat{\psi}_{ki}^R - \psi_k^*(w_i))^2 \sum_j A_{ij}^{\text{pre}} \end{aligned} \right] = o_{\mathbb{P}}(1), \end{aligned}$$

where the bound uses:

$$\begin{aligned} - \sum_i \psi_k^*(w_i)^2 \sum_j A_{ij}^{\text{pre}} & \leq C n q_n^{\text{pre}}, \\ - \sum_i (\hat{\psi}_{ki}^R - \psi_k^*(w_i))^2 \sum_j A_{ij}^{\text{pre}} & \leq \max_i \sum_j A_{ij}^{\text{pre}} \sum_i (\hat{\psi}_{ki}^R - \psi_k^*(w_i))^2 = O_{\mathbb{P}}\left(\sqrt{\frac{\log n}{\log \log n}}\right). \end{aligned}$$

The cross-product term admits a similar bound, and thus $S_3 = o_{\mathbb{P}}(1)$.

S_4 : By definition,

$$\frac{1}{n^2 q_n^{\text{pre}}} \sum_{(i,j)} A_{ij}^{\text{pre}} \eta_i^u (\hat{u}_i^{\text{DE}} - u_i - \Delta_i) = \frac{1}{n^2 q_n^{\text{pre}}} \sum_{(i,j)} A_{ij}^{\text{pre}} \eta_i^u (\hat{u}_i^{\text{DE}} - u_i) - \frac{1}{n^2 q_n^{\text{pre}}} \sum_{(i,j)} A_{ij}^{\text{pre}} \eta_i^u \Delta_i.$$

For the first term, since $\hat{u}_i^{\text{DE}} - u_i = (\hat{\beta}^{\text{DE}} - \beta)^\top X_i$, we have

$$\frac{1}{n^2 q_n^{\text{pre}}} \sum_{(i,j)} A_{ij}^{\text{pre}} \eta_i^u (\hat{u}_i^{\text{DE}} - u_i) = \sum_{k=0}^2 (\hat{\beta}_k^{\text{DE}} - \beta_k) \left(\frac{1}{n^2 q_n^{\text{pre}}} \sum_{(i,j)} A_{ij}^{\text{pre}} \eta_i^u X_{ik} \right).$$

Each summand is $O_{\mathbb{P}}(1)$ and multiplied by $o_{\mathbb{P}}(1)$, so the full expression is $o_{\mathbb{P}}(1)$.

This follows from bounding: $\mathbb{E} \left[\left(\sum_{(i,j)} A_{ij}^{\text{pre}} \eta_i^u M_i \right)^2 \right] \leq C_1 n^2 q_n^{\text{pre}} + C_2 n^3 (q_n^{\text{pre}})^2 + C_3 n^4 (q_n^{\text{pre}})^2$, implying $\sum_{(i,j)} A_{ij}^{\text{pre}} \eta_i^u M_i = O_{\mathbb{P}}(n^2 q_n^{\text{pre}})$. For the second term:

$$\frac{1}{n^2 q_n^{\text{pre}}} \sum_{(i,j)} A_{ij}^{\text{pre}} \eta_i^u \Delta_i = \sum_{k=1}^r \frac{1}{n^2 q_n^{\text{pre}}} \sum_{(i,j)} A_{ij}^{\text{pre}} \eta_i^u (\hat{\mu}_k^R \hat{\psi}_{ki}^R - \mu_k \psi_k^*(w_i)).$$

Recall (C.10). Each of the three resulting terms is $o_{\mathbb{P}}(1)$, by:

$$\begin{aligned} - \sum_{(i,j)} A_{ij}^{\text{pre}} \eta_i^u (\hat{\psi}_{ki}^R - \psi_k^*(w_i)) & \leq \|\eta^u\|_2 \|A^{\text{pre}}(\hat{\psi}_k^R - \psi_k^*)\|_2 = O_{\mathbb{P}}(\sqrt{n} \sqrt{\frac{\log n}{\log \log n}}) = O_{\mathbb{P}}(\sqrt{n q_n^{\text{pre}}}). \\ - \mathbb{E} \left[\left(\sum_{(i,j)} A_{ij}^{\text{pre}} \eta_i^u \psi_k^*(w_i) \right)^2 \right] & \leq C n^3 (q_n^{\text{pre}})^2. \end{aligned}$$

Hence, $\frac{1}{n^2 q_n^{\text{pre}}} \sum_{(i,j)} A_{ij}^{\text{pre}} \eta_i^u \Delta_i = o_{\mathbb{P}}(1)$. Combining both terms, we conclude:

$$\frac{1}{n^2 q_n^{\text{pre}}} \sum_{(i,j)} A_{ij}^{\text{pre}} \eta_i^u (\hat{u}_i^{\text{DE}} - u_i - \Delta_i) = o_{\mathbb{P}}(1).$$

Therefore,

$$\frac{1}{n^2 q_n^{\text{pre}}} \sum_{(i,j)} A_{ij}^{\text{pre}} (\hat{\eta}_i^u)^2 = \mathbb{E} \left[\frac{1}{n^2 q_n^{\text{pre}}} \sum_{(i,j)} A_{ij}^{\text{pre}} (\eta_i^u)^2 \right] + o_{\mathbb{P}}(1).$$

Step 3: Combine the results. By Steps 1–2 and CMT, $(\hat{V}_{\text{num}}^{\text{DE}})^{-1/2} ((\tilde{Z}^{\text{DE}})^{\top} u) \xrightarrow{d} \mathcal{N}(0, I_3)$. Applying Theorem C.1 and CMT again then completes the proof. \square

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Online Appendix to “Endogenous Interference in Randomized Experiments”

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This document has two sections. Section **D** contains proof of lemmas omitted from the paper. Section **E** presents results of IV estimation with normalized SSIV.

D Proof of Auxillary Lemmas

D.1 Proof of Lemmas in Appendix **B**

*Proof of Lemma **B.1**.* We first show **(B.5)**. By C_r inequality, we have

$$\mathbb{E} [(r_{0,i} + r_{1,i} - \mu_{r_{1,i}})^2] \leq 2^3 [\mathbb{E}(r_{0,i}^2) + \mathbb{E}(r_{1,i}^2) + \mathbb{E}(\mu_{r_{1,i}}^2)]. \quad (\text{D.1})$$

Bound for $\mathbb{E}[r_{0,i}^2]$: By $\mathbb{E}[R_{ij}|T_i, w_i] = 0$:

$$\mathbb{E} [r_{0,i}^2|T_i, w_i] = \frac{\xi_i^2}{(n-1)^2} \mathbb{E} \left[\left(\sum_{j \neq i} R_{ij} \right)^2 |T_i, w_i \right] = \frac{\xi_i^2}{(n-1)^2} \sum_{j \neq i} \mathbb{E} [R_{ij}^2|T_i, w_i].$$

Recall the definition of R_{ij} in **(B.1)** and moment bounds on the treatment and adjacency terms, we obtain $\mathbb{E}[r_{0,i}^2] \leq \frac{C}{nq_n^{\text{post}}}$.

Bound for $\mathbb{E}[r_{1,i}^2]$: Define $O_{ij} = U_{ij}W_{ij}$ and $V_{ij} = W_{ij}^2$. Expanding $r_{1,i}^2$ and applying the AM–GM inequality:

$$r_{1,i}^2 \leq \frac{2^3}{(n-1)^4} \left[\left(\frac{1}{\theta_i^*(y)^2} \sum_{j \neq i} O_{ij} \right)^2 + \left(\frac{2\theta_i^*(x)}{\theta_i^*(y)^3} \sum_{j \neq i} V_{ij} \right)^2 + \left(\frac{1}{\theta_i^*(y)^2} \sum_{(j,l)} U_{ij}W_{il} \right)^2 + \left(\frac{2\theta_i^*(x)}{\theta_i^*(y)^3} \sum_{(j,l)} W_{ij}W_{il} \right)^2 \right].$$

For the first term, we expand $\mathbb{E} \left[\left(\sum_{j \neq i} O_{ij} \right)^2 \mid T_i, w_i \right] = \mathbb{E} \left[\left(\sum_{j \neq i} O_{ij}^2 + \sum_{j_1 \neq j_2} O_{ij_1} O_{ij_2} \right) \mid T_i, w_i \right]$.

Each moment is bounded by $C \cdot q_n^{\text{post}}$, so: $\mathbb{E} \left[\left(\sum_{j \neq i} O_{ij} \right)^2 \right] \leq C(nq_n^{\text{post}})^2$.

For the third term, we expand: $\mathbb{E} \left[\left(\sum_{j \neq l} U_{ij} W_{il} \right)^2 \mid T_i, w_i \right] = \mathbb{E} \left[\left(\sum_{j \neq l} U_{ij}^2 W_{il}^2 \right) \mid T_i, w_i \right]$

by the definition of U_{ij} and W_{ij} . It yields: $\mathbb{E} \left[\left(\sum_{j \neq l} U_{ij} W_{il} \right)^2 \right] \leq C(nq_n^{\text{post}})^2$. The second term and the final term are bounded analogously. Hence, $\mathbb{E}[r_{1,i}^2] \leq \frac{C}{(nq_n^{\text{post}})^2}$.

Bound for $\mathbb{E}[\mu_{r_1,i}^2]$: From the definition of $\mu_{r_1,i}$, we have $\mathbb{E}[\mu_{r_1,i}^2] = O\left(\frac{1}{(nq_n^{\text{post}})^2}\right)$. Combining the terms in (D.1), we complete showing (B.5).

We complete the proof of (B.6) by showing the following terms. To simplify presentation, we define $E_{ij} = -\frac{U_{ij}W_{ij}}{\theta_i^*(y)^2} + \frac{2\theta_i^*(x)}{\theta_i^*(y)^3}W_{ij}^2$ and $E_{ijk} = -\frac{U_{ij}W_{ik}}{\theta_i^*(y)^2} + \frac{2\theta_i^*(x)}{\theta_i^*(y)^3}W_{ij}W_{ik}$.

(a) The first term is nonzero when R_{ik} and R_{jk} share a common second index k :

$$\text{Cov}(a_i r_{0,i}, b_j r_{0,j}) = \frac{1}{(n-1)^2} \sum_{k \neq i,j} \text{Cov}(a_i \xi_i R_{ik}, b_j \xi_j R_{jk}) = O\left(\frac{1}{n}\right).$$

(b) The second term is nonzero when the two sides share a common index:

$$\text{Cov}(a_i r_{0,i}, b_j r_{1,j}) = \frac{1}{(n-1)^3} \sum_{k \neq i,j} \left[\text{Cov}(a_i \xi_i R_{ik}, b_j E_{jk}) + \text{Cov}(a_i \xi_i R_{ik}, b_j E_{jki}) \right] = O\left(\frac{1}{n^2 q_n^{\text{post}}}\right).$$

(c) The last term is nonzero when the two sides share a common index:

$$\begin{aligned} \text{Cov}(a_i r_{1,i}, b_j r_{1,j}) &= \frac{1}{(n-1)^4} \left[\begin{aligned} &\sum_{k \neq i,j} \text{Cov}(a_i E_{ik}, b_j E_{jk}) + \sum_{k \neq i,j} \text{Cov}(a_i E_{ik}, b_j E_{jki}) \\ &+ \sum_{k \neq i,j} \text{Cov}(a_i E_{ikj}, b_j E_{jk}) + \sum_{(k,l):k \neq l} \text{Cov}(a_i E_{ikl}, b_j E_{jkl}) \end{aligned} \right] \\ &= O\left(\frac{1}{n^3 (q_n^{\text{post}})^2}\right) + O\left(\frac{1}{n^2}\right). \end{aligned}$$

We omit the proof for the remaining terms involving $\mu_{r_1,i}$, which can be shown by (B.4). Combining results from parts (a)–(c) above, we conclude (B.6). \square

Proof of Lemma B.3. By index reordering,

$$\frac{1}{n^2 q_n^{\text{pre}}} \sum_{i=1}^n Z_i^{\text{SSIV}} \phi_i = \frac{1}{n^2 q_n^{\text{pre}}} \sum_{i=1}^n \sum_{j \neq i} A_{ij}^{\text{pre}} (T_j - \pi) \phi_i = \frac{1}{n} \sum_{j=1}^n (T_j - \pi) \frac{1}{n q_n^{\text{pre}}} \sum_{i \neq j} A_{ij}^{\text{pre}} \phi_i.$$

Given w_j , the terms $\frac{A_{ij}^{\text{pre}} \phi_i}{q_n^{\text{pre}}}$ are i.i.d.. Then we have

$$\mathbb{E} \left[\left(\sum_{i \neq j} \frac{A_{ij}^{\text{pre}} \phi_i}{(n-1)q_n^{\text{pre}}} - Q_j^\phi \right)^2 \right] = \frac{1}{n-1} \mathbb{E} \left[\left(\frac{A_{ij}^{\text{pre}} \phi_i}{q_n^{\text{pre}}} - Q_j^\phi \right)^2 \right] \leq \frac{1}{(n-1)q_n^{\text{pre}}} \mathbb{E} \left[\frac{A_{ij}^{\text{pre}} \phi_i^2}{q_n^{\text{pre}}} \right] \leq \frac{C}{(n-1)q_n^{\text{pre}}}.$$

We complete the proof by showing that

$$\mathbb{E} \left[\left(\frac{1}{n} \sum_{j=1}^n (T_j - \pi) \left(\sum_{i \neq j} \frac{A_{ij}^{\text{pre}} \phi_i}{nq_n^{\text{pre}}} - Q_j^\phi \right) \right)^2 \right] = \frac{\pi(1-\pi)}{n^2} \sum_{j=1}^n \mathbb{E} \left[\left(\sum_{i \neq j} \frac{A_{ij}^{\text{pre}} \phi_i}{nq_n^{\text{pre}}} - Q_j^\phi \right)^2 \right] \leq \frac{C}{n^2 q_n^{\text{pre}}}.$$

□

Proof of Lemma B.4. To show (B.10), we decompose it into

$$\frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}} M_i = \frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}} (\xi_i + \mu_{r_1, i}) + \frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}} (r_{0, i} + r_{1, i} - \mu_{r_1, i}).$$

For the first term, we can show by Lemma B.3 and (B.4) that

$$\frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}} \xi_i = O_{\mathbb{P}}(\sqrt{n} q_n^{\text{pre}}) \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}} \mu_{r_1, i} = O_{\mathbb{P}}\left(\frac{q_n^{\text{pre}}}{\sqrt{n} q_n^{\text{post}}}\right). \quad (\text{D.2})$$

For $\frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}} r_{0, i}$, the expectation satisfies

$$\mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}} r_{0, i} \right] = \frac{1}{n(n-1)} \sum_{i=1}^n \mathbb{E} \left[\xi_i \sum_{k \neq i} A_{ik}^{\text{pre}} (T_k - \pi) R_{ik} \right] \leq C \cdot \frac{q_n^{\text{pre}}}{\max\{q_n^{\text{pre}}, q_n^{\text{post}}\}}.$$

The variance is $\text{Var} \left(\frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}} r_{0, i} \right) = \frac{1}{n^2} \sum_{i=1}^n \text{Var} (Z_i^{\text{SSIV}} r_{0, i}) + \frac{1}{n^2} \sum_{i \neq j} \text{Cov} (Z_i^{\text{SSIV}} r_{0, i}, Z_j^{\text{SSIV}} r_{0, j})$.

For the diagonal terms,

$$\begin{aligned} \text{Var} (Z_i^{\text{SSIV}} r_{0, i}) &\leq \mathbb{E} [(Z_i^{\text{SSIV}})^2 r_{0, i}^2] = \frac{\xi_i^2}{(n-1)^2} \mathbb{E} \left[\left(\sum_{j \neq i} A_{ij}^{\text{pre}} (T_j - \pi) \right)^2 \left(\sum_{j \neq i} R_{ij} \right)^2 \right] \\ &= \frac{\xi_i^2}{(n-1)^2} \mathbb{E} \left[\sum_{(j, k)} A_{ij}^{\text{pre}} A_{ik}^{\text{pre}} (T_j - \pi)(T_k - \pi) \sum_{(\ell, m)} R_{i\ell} R_{im} \right] \leq C \cdot \frac{q_n^{\text{pre}}}{q_n^{\text{post}}}. \end{aligned} \quad (\text{D.3})$$

For the off-diagonal terms,

$$\begin{aligned} \text{Cov} (Z_i^{\text{SSIV}} r_{0,i}, Z_j^{\text{SSIV}} r_{0,j}) &= \frac{1}{(n-1)^2} \text{Cov} \left(\xi_i \sum_{k \neq i} A_{ik}^{\text{pre}} (T_k - \pi) \sum_{k \neq i} R_{ik}, \xi_j \sum_{k \neq j} A_{jk}^{\text{pre}} (T_k - \pi) \sum_{k \neq j} R_{jk} \right) \\ &= O \left(\frac{(q_n^{\text{pre}})^2}{n \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}^2} \right). \end{aligned} \quad (\text{D.4})$$

Combining both terms: $\text{Var} \left(\frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}} r_{0,i} \right) = O \left(\frac{q_n^{\text{pre}}}{n q_n^{\text{post}}} \right)$. Therefore,

$$\frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}} r_{0,i} = \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}} r_{0,i} \right] + O_{\mathbb{P}} \left(\frac{\sqrt{q_n^{\text{pre}}}}{\sqrt{n q_n^{\text{post}}}} \right). \quad (\text{D.5})$$

For $\frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}} r_{1,i}$, the expectation satisfies

$$\mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}} r_{1,i} \right] = \frac{1}{n(n-1)^2} \sum_{i=1}^n \mathbb{E} \left[\sum_{k \neq i} A_{ik}^{\text{pre}} (T_k - \pi) \left(-\frac{U_{ik} W_{ik}}{\theta_i^*(y)^2} + \frac{2\theta_i^*(x)}{\theta_i^*(y)^3} W_{ik}^2 \right) \right] \leq C \cdot \frac{\min\{q_n^{\text{pre}}, q_n^{\text{post}}\}}{n(q_n^{\text{post}})^2}.$$

The variance is $\text{Var} \left(\frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}} r_{1,i} \right) = \frac{1}{n^2} \sum_{i=1}^n \text{Var} (Z_i^{\text{SSIV}} r_{1,i}) + \frac{1}{n^2} \sum_{i \neq j} \text{Cov} (Z_i^{\text{SSIV}} r_{1,i}, Z_j^{\text{SSIV}} r_{1,j})$.

For the diagonal terms, similar to the argument of (D.3), $\text{Var} (Z_i^{\text{SSIV}} r_{1,i}) = O \left(\frac{q_n^{\text{pre}}}{n(q_n^{\text{post}})^2} \right)$.

For the off-diagonal terms, similar to the argument of (D.4), $\text{Cov} (Z_i^{\text{SSIV}} r_{1,i}, Z_j^{\text{SSIV}} r_{1,j}) = O \left(\frac{(q_n^{\text{pre}})^2}{n(q_n^{\text{post}})^2} \right)$. Therefore, $\text{Var} \left(\frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}} r_{1,i} \right) = O \left(\frac{q_n^{\text{pre}}}{n^2 (q_n^{\text{post}})^2} \right)$. Hence,

$$\frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}} r_{1,i} = \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}} r_{1,i} \right] + O_{\mathbb{P}} \left(\frac{\sqrt{q_n^{\text{pre}}}}{n q_n^{\text{post}}} \right) + O_{\mathbb{P}} \left(\frac{q_n^{\text{pre}}}{\sqrt{n} q_n^{\text{post}}} \right). \quad (\text{D.6})$$

Combining (D.5) and (D.6), we obtain

$$\frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}} (r_{0,i} + r_{1,i}) = \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}} (r_{0,i} + r_{1,i}) \right] + O_{\mathbb{P}} \left(\frac{\sqrt{q_n^{\text{pre}}}}{\sqrt{n} q_n^{\text{post}}} \right) + O_{\mathbb{P}} \left(\frac{q_n^{\text{pre}}}{\sqrt{n} q_n^{\text{post}}} \right). \quad (\text{D.7})$$

Therefore, (D.2) and (D.7) together imply (B.10).

To show (B.11), we can rewrite the expression as

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}} M_i - \overline{Z^{\text{SSIV}}} \cdot \overline{M} &= \frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}} (\xi_i + \mu_{r_{1,i}}) - \overline{Z^{\text{SSIV}}} \frac{1}{n} \sum_{i=1}^n (\xi_i + \mu_{r_{1,i}}) \\ &\quad + \frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}} (r_{0,i} + r_{1,i} - \mu_{r_{1,i}}) - \overline{Z^{\text{SSIV}}} \frac{1}{n} \sum_{i=1}^n (r_{0,i} + r_{1,i} - \mu_{r_{1,i}}). \end{aligned}$$

By the argument in the proof of Lemma B.4 (see (D.7) for details), we show that

$$\frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}}(r_{0,i} + r_{1,i} - \mu_{r_{1,i}}) = \mathbb{E}[Z_i^{\text{SSIV}}(r_{0,i} + r_{1,i} - \mu_{r_{1,i}})] + O_{\mathbb{P}}\left(\frac{q_n^{\text{pre}}}{\sqrt{nq_n^{\text{post}}}}\right)$$

where $\mathbb{E}[Z_i^{\text{SSIV}}(r_{0,i} + r_{1,i} - \mu_{r_{1,i}})] \asymp \frac{q_n^{\text{pre}}}{\max\{q_n^{\text{pre}}, q_n^{\text{post}}\}}$. By Lemmas B.2 and B.3, we have

$$\overline{Z}^{\text{SSIV}} \frac{1}{n} \sum_{i=1}^n (r_{0,i} + r_{1,i} - \mu_{r_{1,i}}) = O_{\mathbb{P}}\left(\frac{nq_n^{\text{pre}}}{\sqrt{n}}\right) O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{\text{post}}}}\right) = O_{\mathbb{P}}\left(\frac{q_n^{\text{pre}}}{\sqrt{nq_n^{\text{post}}}}\right).$$

For the sum of the first two terms, we consider Cases (a) and (b), respectively.

Case (a) By Lemma B.3, we have

$$\frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}}(\xi_i + \mu_{r_{1,i}}) - \overline{Z}^{\text{SSIV}} \frac{1}{n} \sum_{i=1}^n (\xi_i + \mu_{r_{1,i}}) = O_{\mathbb{P}}(\sqrt{n}q_n^{\text{pre}}).$$

Case (b) By Lemma B.3 and (B.4), we have

$$\frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}}(\xi_i + \mu_{r_{1,i}}) - \overline{Z}^{\text{SSIV}} \frac{1}{n} \sum_{i=1}^n (\xi_i + \mu_{r_{1,i}}) = O_{\mathbb{P}}\left(\frac{q_n^{\text{pre}}}{\sqrt{nq_n^{\text{post}}}}\right).$$

By combining these results, we complete the proof. \square

Proof of Lemma B.5. We omit the proof for (i) since it is analogous to that of (ii). For (ii), we bound it in L_1 norm: $\mathbb{E}\left(\frac{1}{n} \sum_{i=1}^n S_i^3\right) \leq C \cdot \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n (O_i^\diamond + H_i^\diamond)^3\right]$, where $O_i^\diamond = \sum_j A_{ij}^\diamond$ is the degree of node i , and $H_i^\diamond = \sum_{(j,k)} A_{ik}^\diamond A_{kj}^\diamond$ counts 2-hop paths through neighbors. By the inequality $(a+b)^3 \leq 4(a^3 + b^3)$, we have:

$$\mathbb{E}\left(\frac{1}{n} \sum_{i=1}^n S_i^3\right) \leq 4 \cdot \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n (O_i^\diamond)^3\right] + 4 \cdot \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n (H_i^\diamond)^3\right]$$

where for the first term, we have $\mathbb{E}[(O_i^\diamond)^3] = \mathbb{E}[(\sum_j A_{ij}^\diamond)^3] \leq C$ since $A_{ij}^\diamond \in \{0, 1\}$ and entries are sparse, and for the second term, $\mathbb{E}[(H_i^\diamond)^3] = \mathbb{E}[(\sum_{(j,k)} A_{ik}^\diamond A_{kj}^\diamond)^3] \leq C$ by expanding and bounding cross-products. Thus, $\mathbb{E}\left(\frac{1}{n} \sum_{i=1}^n S_i^3\right) \leq C$.

For (iii), by definition: $(D^3)_{ij} = \sum_{k=1}^n (D^2)_{ik} D_{kj} = \sum_{(k,l)} D_{il} D_{lk} D_{kj}$. We show that

$$\sum_{i=1}^n \sum_{j \neq i} (D^3)_{ij} \leq \sum_{i=1}^n \sum_{j \neq i} \sum_{(k,l)} \left(A_{il}^\diamond + \max_h A_{ih}^\diamond A_{hl}^\diamond\right) \left(A_{lk}^\diamond + \max_h A_{lh}^\diamond A_{hk}^\diamond\right) \left(A_{kj}^\diamond + \max_h A_{kh}^\diamond A_{hj}^\diamond\right)$$

$$\leq \sum_{i=1}^n \sum_{j \neq i} \sum_{(k,l)} \left(A_{il}^\diamond + \sum_h A_{ih}^\diamond A_{hl}^\diamond \right) \left(A_{lk}^\diamond + \sum_h A_{lh}^\diamond A_{hk}^\diamond \right) \left(A_{kj}^\diamond + \sum_h A_{kh}^\diamond A_{hj}^\diamond \right) \leq C(nq_n^\diamond)^6 \leq C,$$

where the first inequality follows from $D_{il} \leq A_{il}^\diamond + \max_h A_{ih}^\diamond A_{hl}^\diamond$, the second inequality follows from $\max_h X_h \leq \sum_h X_h$ for nonnegative X_h and the third inequality follows from the fact that each term contains six adjacency indicators. The other cases are handled similarly, so we omit them. By combining these results, we can show that $\frac{1}{n} \sum_{(i,j)} (D^3)_{ij} = o_{\mathbb{P}}(\sqrt{n})$. \square

D.2 Proof of Lemmas in Appendix C

Proof of Lemma C.1. We first show (1). Suppose $q_n^{\text{pre}} \succ \log(n)/n$. By Theorem 5.2 in [Lei and Rinaldo \(2015\)](#) with $d = nq_n^{\text{pre}}$, there exists some constant C such that $\|A^{\text{pre}} - q_n^{\text{pre}} G_n^{\text{pre}}\|_{\text{op}} \leq C\sqrt{nq_n^{\text{pre}}}$ with probability approaching to 1.

Suppose instead $\frac{\log n}{\log \log n}/n \prec q_n^{\text{pre}} \prec \log(n)/n$. By Corollary 3.3 in [Benaych-Georges et al. \(2020\)](#) and by setting their $\varepsilon^2 = \sqrt{\frac{\log n}{\log \log n}}/(nq_n^{\text{pre}})$, we have that with probability approaching to 1, $\|A^{\text{pre}} - q_n^{\text{pre}} G_n^{\text{pre}}\|_{\text{op}} \leq k\sqrt{nq_n^{\text{pre}}} \left(\frac{\log n}{\log \log n}\right)^{1/4}$, where k is a universal constant. For (2), by the triangle inequality, we have

$$\left\| A^{\text{pre}} - q_n^{\text{pre}} \tilde{G}_n^{\text{pre}} \right\|_{\text{op}} \leq \|A^{\text{pre}} - q_n^{\text{pre}} G_n^{\text{pre}}\|_{\text{op}} + \left\| \sum_{k=r+1}^n \lambda_k^* \psi_k^{*\top} \psi_k^* \right\|_{\text{op}} = O_{\mathbb{P}} \left(\sqrt{nq_n^{\text{pre}}} \left(\frac{\log n}{\log \log n}\right)^{1/4} \right).$$

\square

Lemma D.1. *Suppose $q_n^{\text{pre}} \succ \frac{\log n}{\log \log n}/n$. Under Assumptions 1 and 4, for $k \in \{1, \dots, r\}$, then*

$$(1) \left| \hat{\lambda}_k - \lambda_k^* \right| = O_{\mathbb{P}} \left(\sqrt{nq_n^{\text{pre}}} \left(\frac{\log n}{\log \log n}\right)^{1/4} \right); \quad (2) \hat{\lambda}_k = \Omega_{\mathbb{P}}(nq_n^{\text{pre}}).$$

Proof. This is analogous to [Li and Wager \(2022, Lemma 27\)](#). For (1),

$$\left| \hat{\lambda}_k - \lambda_k^* \right| \leq \|A^{\text{pre}} - q_n^{\text{pre}} G_n^{\text{pre}}\|_{\text{op}} = O_{\mathbb{P}} \left(\sqrt{nq_n^{\text{pre}}} \left(\frac{\log n}{\log \log n}\right)^{1/4} \right)$$

by Weyl's Inequality and Lemma C.1. (2) holds by (1) and Assumption 4(a). \square

Lemma D.2 (Theorem 2 in [Yu et al. \(2015\)](#)). *Let Σ and $\hat{\Sigma} \in \mathbb{R}^{p \times p}$ be symmetric, with eigenvalues $\lambda_1 \geq \dots \geq \lambda_p$ and $\hat{\lambda}_1 \geq \dots \geq \hat{\lambda}_p$, respectively. Fix $1 \leq t \leq s \leq p$ and*

assume that $\min(\lambda_{t-1} - \lambda_t, \lambda_s - \lambda_{s+1}) > 0$ where we define $\lambda_0 = \infty$ and $\lambda_{p+1} = -\infty$. Let $d = s - t + 1$, and let $V = (v_t, v_{t+1}, \dots, v_s) \in \mathbb{R}^{p \times d}$ and $\hat{V} = (\hat{v}_t, \hat{v}_{t+1}, \dots, \hat{v}_s) \in \mathbb{R}^{p \times d}$ have orthonormal columns satisfying $\Sigma v_j = \lambda_j v_j$ and $\hat{\Sigma} \hat{v}_j = \hat{\lambda}_j \hat{v}_j$ for $j = t, t+1, \dots, s$. Then there exists an orthogonal matrix $\hat{O} \in \mathbb{R}^{d \times d}$ such that

$$\|\hat{V}\hat{O} - V\|_{\text{F}} \leq \frac{2^{3/2} \min\left(d^{1/2} \|\hat{\Sigma} - \Sigma\|_{\text{op}}, \|\hat{\Sigma} - \Sigma\|_{\text{F}}\right)}{\min(\lambda_{t-1} - \lambda_t, \lambda_s - \lambda_{s+1})}. \quad (\text{D.8})$$

Proof of Lemma C.2. We largely follow the proof of Li and Wager (2022, Lemma 8). By applying (D.8) to $\hat{\Psi}$ and Ψ^* , and together with Lemma C.1 and Lemma D.2, we get that there exists an $r \times r$ orthogonal matrix \hat{R} such that

$$\left\| \hat{\Psi}\hat{R} - \Psi^* \right\|_{\text{F}} = O_{\mathbb{P}} \left(\frac{1}{\sqrt{nq_n^{\text{pre}}}} \left(\frac{\log n}{\log \log n} \right)^{1/4} \right). \quad (\text{D.9})$$

Define $\hat{\Psi}^R = \hat{\Psi}\hat{R}$. Let $\hat{\psi}_k^R$ and ψ_k^* denote the k -th columns of $\hat{\Psi}^R$ and Ψ^* , respectively. Then we have

$$\left\| \hat{\psi}_k^R - \psi_k^* \right\| = O_{\mathbb{P}} \left(\frac{1}{\sqrt{nq_n^{\text{pre}}}} \left(\frac{\log n}{\log \log n} \right)^{1/4} \right). \quad (\text{D.10})$$

For $l \leq r$, we will show that $\left\| (\Psi^*)^{\top} (\hat{\Psi}^R - \Psi^*) \right\|_{\text{F}}$ is small. Let $\hat{V}^{\top} V = O_1 D O_2^{\top}$ be the singular value decomposition of $\hat{V}^{\top} V$, then \hat{O} is constructed by taking $\hat{O} = O_1 O_2^{\top}$. The construction of \hat{O} ensures that $(\hat{V}\hat{O})^{\top} V$ is symmetric, i.e., $(\hat{V}\hat{O})^{\top} V = O_2 O_1^{\top} O_1 D O_2^{\top} = O_2 D O_2^{\top}$. This implies that $(\hat{\Psi}^R)^{\top} \Psi^*$ is symmetric. By definition,

$$\|\hat{V}\hat{O} - V\|_{\text{F}}^2 = \text{tr} \left(\hat{O}^{\top} \hat{V}^{\top} \hat{V} \hat{O} - (\hat{V}\hat{O})^{\top} V - V^{\top} \hat{V} \hat{O} + V^{\top} V \right) = 2 \text{tr}(I_r - D).$$

If we write $(\hat{\Psi}^R)^{\top} \Psi^* = O_{\Psi} D_{\Psi} O_{\Psi}^{\top}$, then by (D.9), we have

$$\left\| \hat{\Psi}\hat{R} - \Psi^* \right\|_{\text{F}}^2 = \text{tr}(I_r - D_{\Psi}) = O_{\mathbb{P}} \left(\frac{1}{nq_n^{\text{pre}}} \sqrt{\frac{\log n}{\log \log n}} \right).$$

Note that $(\Psi^*)^{\top} (\hat{\Psi}^R - \Psi^*) = (\Psi^*)^{\top} \hat{\Psi}^R - I_r = O_{\Psi} D_{\Psi} O_{\Psi}^{\top} - I_r$. Therefore,

$$\left\| (\Psi^*)^{\top} (\hat{\Psi}^R - \Psi^*) \right\|_{\text{F}}^2 = \text{tr} \left((O_{\Psi} D_{\Psi} O_{\Psi}^{\top} - I_r)^2 \right) = \text{tr} \left((D_{\Psi} - I_r)^2 \right) = O_{\mathbb{P}} \left(\left(\frac{1}{nq_n^{\text{pre}}} \sqrt{\frac{\log n}{\log \log n}} \right)^2 \right).$$

It implies that (C.1) holds for any $k, l \in \{1, \dots, r\}$.

For $l > r$, we have $(\psi_l^*)^\top (\hat{\Psi}^R - \Psi^*) = (\psi_l^*)^\top \hat{\Psi}^R$ by orthogonality. Note that

$$\left\| (\psi_l^*)^\top \hat{\Psi}_r^R \right\| = \left\| (\psi_l^*)^\top \hat{\Psi}_r \hat{R} \right\| = \left\| (\psi_l^*)^\top \hat{\Psi}_r \hat{\Lambda}_r \hat{R} \hat{R}^\top \hat{\Lambda}_r^{-1} \hat{R} \right\|$$

where $\hat{\Lambda}_r$ is the $r \times r$ diagonal matrix with $\hat{\lambda}_1, \dots, \hat{\lambda}_r$ on its diagonal. We complete the proof by bounding $\|(\psi_l^*)^\top \hat{\Psi}_r \hat{\Lambda}_r \hat{R}\|$. Note that

$$(\psi_l^*)^\top \hat{\Psi}_r \hat{\Lambda}_r \hat{R} = (\psi_l^*)^\top A^{\text{pre}} \hat{\Psi}_r^R = (\psi_l^*)^\top (A^{\text{pre}} - q_n^{\text{pre}} \tilde{G}_n^{\text{pre}}) (\hat{\Psi}_r^R - \Psi_r^*) + (\psi_l^*)^\top (A^{\text{pre}} - q_n^{\text{pre}} \tilde{G}_n^{\text{pre}}) \Psi_r^*.$$

The first term can be easily bounded by

$$\begin{aligned} \left\| (\psi_l^*)^\top (A^{\text{pre}} - q_n^{\text{pre}} \tilde{G}_n^{\text{pre}}) (\hat{\Psi}_r^R - \Psi_r^*) \right\| &\leq \|\psi_l^*\| \left\| A^{\text{pre}} - q_n^{\text{pre}} \tilde{G}_n^{\text{pre}} \right\|_{\text{op}} \left\| \hat{\Psi}_r^R - \Psi_r^* \right\| \\ &= O_{\mathbb{P}} \left(\sqrt{\frac{\log n}{\log \log n}} \right), \end{aligned} \quad (\text{D.11})$$

where the last inequality follows from Lemma C.1 and (D.9). For the second term, we rewrite it in the summation form:

$$\begin{aligned} (\psi_l^*)^\top (A^{\text{pre}} - q_n^{\text{pre}} \tilde{G}_n^{\text{pre}}) \psi_k^* &= (\psi_l^*)^\top (A^{\text{pre}} - q_n^{\text{pre}} G_n^{\text{pre}}) \psi_k^* + (\psi_l^*)^\top \left(\sum_{h=r+1}^n \lambda_h^* \psi_h^{*\top} \psi_h^* \right) \psi_k^* \\ &= \sum_{(i,j)} \psi_{li}^* \psi_{kj}^* (A_{ij}^{\text{pre}} - q_n^{\text{pre}} g^{\text{pre}}(w_i, w_j)). \end{aligned}$$

As A_{ij}^{pre} are independent given w , we have:

$$\begin{aligned} \mathbb{E} \left[\left(\sum_{i \neq j} \psi_{li}^* \psi_{kj}^* (A_{ij}^{\text{pre}} - q_n^{\text{pre}} g_0(i, j)) \right)^2 \right] &= \sum_{i \neq j} \mathbb{E} \left[(\psi_{li}^*)^2 (\psi_{kj}^*)^2 \mathbb{E} \left[(A_{ij}^{\text{pre}} - q_n^{\text{pre}} g_0(i, j))^2 \mid w \right] \right] \\ &\leq \sum_{i \neq j} \mathbb{E} \left[(\psi_{li}^*)^2 (\psi_{kj}^*)^2 q_n^{\text{pre}} g_0(i, j) \right] \leq q_n^{\text{pre}} \mathbb{E} \left[\sum_{i \neq j} (\psi_{li}^*)^2 (\psi_{kj}^*)^2 \right] \leq C q_n^{\text{pre}}. \end{aligned} \quad (\text{D.12})$$

Combining (D.11) and (D.12), we can show that $\|(\psi_l^*)^\top \hat{\Psi}_r \hat{\Lambda}_r \hat{R}\| = O_{\mathbb{P}} \left(\sqrt{\frac{\log n}{\log \log n}} \right)$.

Lemma D.1 shows that $\hat{\lambda}_k = \Omega_{\mathbb{P}}(n q_n^{\text{pre}})$ for $k \leq r$. Thus

$$\left\| (\psi_l^*)^\top \hat{\Psi}_r^R \right\| = \left\| (\psi_l^*)^\top \hat{\Psi}_r \hat{R} \right\| = \left\| (\psi_l^*)^\top \hat{\Psi}_r \hat{\Lambda}_r \hat{R} \hat{R}^\top \hat{\Lambda}_r^{-1} \hat{R} \right\| = O_{\mathbb{P}} \left(\sqrt{\frac{\log n}{\log \log n}} / (n q_n^{\text{pre}}) \right).$$

It implies that (C.1) holds for $k \in \{1, \dots, r\}$ and $l > r$, and we complete the proof.

□

Lemma D.3. Suppose $q_n^{\text{pre}} \succ \frac{\log n}{\log \log n}/n$. Define $\hat{\psi}_k^R$ as in Lemma C.2. Under Assumptions 1 and 4, then $\left\| A^{\text{pre}} \left(\hat{\psi}_k^R - \psi_k^* \right) \right\| = O_{\mathbb{P}} \left(\sqrt{\frac{\log n}{\log \log n}} \right)$.

Proof of Lemma D.3. By decomposing the target expression:

$$\begin{aligned} \left\| A^{\text{pre}} \left(\hat{\psi}_k^R - \psi_k^* \right) \right\| &\leq \left\| q_n^{\text{pre}} \tilde{G}_n^{\text{pre}} \left(\hat{\psi}_k^R - \psi_k^* \right) \right\| + \left\| \left(A^{\text{pre}} - q_n^{\text{pre}} \tilde{G}_n^{\text{pre}} \right) \left(\hat{\psi}_k^R - \psi_k^* \right) \right\| \\ &\leq \left\| \sum_{l=1}^r \lambda_l^* \psi_l^* \psi_l^{*\top} \left(\hat{\psi}_k^R - \psi_k^* \right) \right\| + \left\| A^{\text{pre}} - q_n^{\text{pre}} \tilde{G}_n^{\text{pre}} \right\|_{\text{op}} \left\| \hat{\psi}_k^R - \psi_k^* \right\| = O_{\mathbb{P}} \left(\sqrt{\frac{\log n}{\log \log n}} \right) \end{aligned}$$

by (D.10) and Lemma C.1 and Lemma C.2. □

Lemma D.4. Suppose $q_n^{\text{pre}} \succ \frac{\log n}{\log \log n}/n$. Define \hat{R} and $\hat{\Psi}^R$ as in Lemma C.2, and let $\hat{\gamma}^R = \hat{R}^\top \hat{\gamma}$. Under Assumptions 1 and 4, it holds that $\sum_{k=1}^r \hat{\gamma}_k \hat{\psi}_{ki}^R = \sum_{k=1}^r \hat{\gamma}_k^R \hat{\psi}_{ki}^R$ and $\hat{\gamma}_k^R = O_{\mathbb{P}}(nq_n^{\text{pre}})$.

Proof of Lemma D.4. We follow the proof of Li and Wager (2022, Lemma 29). The first result holds by construction. Multiplying both sides by \hat{R} , we obtain

$$\hat{\gamma}_k^R = \sum_{i=1}^n \hat{\psi}_{ki}^R Z_i^{\text{SSIV}} = \sum_{i=1}^n \psi_k^*(w_i) Z_i^{\text{SSIV}} + \sum_{i=1}^n \left(\hat{\psi}_{ki}^R - \psi_k^*(w_i) \right) Z_i^{\text{SSIV}} =: S_1 + S_2.$$

We analyze two terms separately. Rewrite $S_1 = \sum_{i,j} \psi_k^*(w_i) A_{ij}^{\text{pre}} (T_j - \pi)$. We bound S_1 in L_2 norm:

$$\mathbb{E}[S_1^2] = \pi(1 - \pi) \sum_{j=1}^n \left(\sum_{i=1}^n \mathbb{E}[\psi_k^*(w_i)^2 A_{ij}^{\text{pre}}] + \sum_{(i_1, i_2)} \mathbb{E}[\psi_k^*(w_{i_1}) \psi_k^*(w_{i_2}) A_{i_1 j}^{\text{pre}} A_{i_2 j}^{\text{pre}}] \right) \leq C(nq_n^{\text{pre}})^2.$$

Hence,

$$\sum_{i=1}^n \psi_k^*(w_i) Z_i^{\text{SSIV}} = O_{\mathbb{P}}(nq_n^{\text{pre}}). \quad (\text{D.13})$$

For $S_2 = \sum_{j=1}^n (T_j - \pi) \sum_{i \neq j} A_{ij}^{\text{pre}} (\hat{\psi}_{ki}^R - \psi_k^*(w_i))$, we bound it in L_2 norm: $\mathbb{E}[S_2^2] = \pi(1 - \pi) \cdot \mathbb{E} \left[\left\| A^{\text{pre}} (\hat{\psi}_k^R - \psi_k^*) \right\|^2 \right] = O \left(\frac{\log n}{\log \log n} \right)$ by Lemma D.3. Therefore,

$$\sum_{i=1}^n \left(\hat{\psi}_{ki}^R - \psi_k^*(w_i) \right) Z_i^{\text{SSIV}} = O_{\mathbb{P}} \left(\sqrt{\frac{\log n}{\log \log n}} \right). \quad (\text{D.14})$$

Combining (D.13) and (D.14) gives $\hat{\gamma}_k^R = O_{\mathbb{P}}(nq_n^{\text{pre}})$ as claimed. \square

By construction, the residual is orthogonal to each basis function: $\sum_{i=1}^n \eta_i^\phi \psi_k^*(w_i) = 0$. Furthermore, we have $\mu_{n,l}^\phi = O_{\mathbb{P}}(\sqrt{n})$, which follows from the bound on its second moment: $\mathbb{E}\left((\mu_{n,l}^\phi)^2\right) = \sum_{i=1}^n \mathbb{E}\left(\phi_i^2 (\psi_{li}^*)^2\right) + \sum_{(i,j)} \mathbb{E}\left(\phi_i \psi_{li}^* \phi_j \psi_{lj}^*\right) = O(n)$.

Proof of Lemma C.3. We can rewrite the expression as:

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n (Z_i^{\text{SSIV}} + \hat{\delta}_i) \phi_i &= \frac{1}{n} \sum_{i=1}^n \left(Z_i^{\text{SSIV}} + \sum_{k=1}^r \hat{\gamma}_k \hat{\psi}_{ki} \right) \left(\sum_{l=1}^r \mu_{n,l}^\phi \psi_l^*(w_i) + \eta_i^\phi \right) \\ &= \frac{1}{n} \sum_{i=1}^n \left(Z_i^{\text{SSIV}} + \sum_{k=1}^r \hat{\gamma}_k^R \hat{\psi}_{ki}^R \right) \eta_i^\phi + \frac{1}{n} \sum_{l=1}^r \mu_{n,l}^\phi \sum_{i=1}^n \left(Z_i^{\text{SSIV}} + \sum_{k=1}^r \hat{\gamma}_k^R \hat{\psi}_{ki}^R \right) \left(\psi_l^*(w_i) - \hat{\psi}_{li}^R \right) \\ &=: S_1 + S_2. \end{aligned}$$

For the first part of S_1 , define $S_{11} = \frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}} \eta_i^\phi$. Since T_j is mean-zero after centering and η_i^ϕ is mean-zero and independent of T_j , the expectation of S_{11} is zero. Its variance is given by

$$\text{Var}(S_{11}) = \frac{1}{n^2} \left[\begin{aligned} &\sum_{(i,j)} \text{Var}\left(A_{ij}^{\text{pre}}(T_j - \pi)\eta_i^\phi\right) + \sum_{(i,j)} \text{Cov}\left(A_{ij}^{\text{pre}}(T_j - \pi)\eta_i^\phi, A_{ji}^{\text{pre}}(T_i - \pi)\eta_j^\phi\right) \\ &+ \sum_{\substack{(i,j,k) \\ \text{all distinct}}} \text{Cov}\left(A_{ij}^{\text{pre}}(T_j - \pi)\eta_i^\phi, A_{ik}^{\text{pre}}(T_k - \pi)\eta_k^\phi\right) \\ &+ \sum_{\substack{(i,j,k) \\ \text{all distinct}}} \text{Cov}\left(A_{ij}^{\text{pre}}(T_j - \pi)\eta_i^\phi, A_{kj}^{\text{pre}}(T_j - \pi)\eta_k^\phi\right) \\ &+ \sum_{\substack{(i,j,k,l) \\ \text{all distinct}}} \text{Cov}\left(A_{ij}^{\text{pre}}(T_j - \pi)\eta_i^\phi, A_{kl}^{\text{pre}}(T_l - \pi)\eta_k^\phi\right) \end{aligned} \right].$$

The last three terms vanish due to the construction of η_i^ϕ , which is orthogonal to all ψ_k^* by definition, and the independence across units. For example, consider the term:

$$\frac{1}{n^2} \sum_{(i,j,k): \text{all distinct}} \text{Cov}\left(A_{ij}^{\text{pre}}(T_j - \pi)\eta_i^\phi, A_{kj}^{\text{pre}}(T_j - \pi)\eta_k^\phi\right).$$

This simplifies to

$$\pi(1-\pi) \cdot \frac{1}{n^2} \sum_{j=1}^n (q_n^{\text{pre}})^2 \cdot \sum_{(l_1, l_2)} \lambda_{l_1}^\phi \lambda_{l_2}^\phi \mathbb{E} \left[\psi_{l_1}^*(w_j) \psi_{l_2}^*(w_j) \left(\sum_i \psi_{l_1}^*(w_i) \eta_i^\phi \right) \left(\sum_k \psi_{l_2}^*(w_k) \eta_k^\phi \right) \right],$$

which is zero by orthogonality: $\sum_i \eta_i^\phi \psi_l^*(w_i) = 0$ for all l . Thus, only the first term

remains:

$$\text{Var}(S_{11}) = \frac{1}{n^2} \sum_{(i,j)} \text{Var} \left(A_{ij}^{\text{pre}} (T_j - \pi) \eta_i^\phi \right) = O(q_n^{\text{pre}}), \quad (\text{D.15})$$

since A_{ij}^{pre} is Bernoulli with mean $q_n^{\text{pre}} g_0(w_i, w_j)$ and η_i^ϕ has bounded variance. Therefore, $S_{11} = O_{\mathbb{P}}(\sqrt{q_n^{\text{pre}}})$.

For the second part of S_1 , define $S_{12} = \frac{1}{n} \sum_{i=1}^n \left(\sum_{k=1}^r \hat{\gamma}_k^R \hat{\psi}_{ki}^R \right) \eta_i^\phi$. We expand η_i^ϕ in the basis $\{\psi_l^*(w_i)\}_{l=1}^n$, i.e., $\eta_i^\phi = \sum_{l=1}^n \alpha_l \psi_l^*(w_i)$ with $\sum_{l=1}^n \alpha_l^2 = n$. Because $\sum_{i=1}^n \eta_i^\phi \psi_k^*(w_i) = 0$ for $k \leq r$, we know that $\alpha_k = 0$ for $k \leq r$. Thus, $\eta_i^\phi = \sum_{l=r+1}^n \alpha_l \psi_l^*(w_i)$. We bound $\sum_{i=1}^n \eta_i^\phi (\hat{\psi}_{ki}^R - \psi_k^*(w_i)) = \sum_{l=r+1}^n \alpha_l \psi_l^*(\hat{\psi}_k^R - \psi_k^*)$:

$$\begin{aligned} & \mathbb{E} \left[\left(\sum_{i=1}^n \eta_i^\phi (\hat{\psi}_{ki}^R - \psi_k^*(w_i)) \right)^2 \right] = \mathbb{E} \left[\left(\sum_{l=r+1}^n \alpha_l \psi_l^*(\hat{\psi}_k^R - \psi_k^*) \right)^2 \right] \\ &= \mathbb{E} \left[\sum_{l=r+1}^n \alpha_l^2 \left(\psi_l^*(\hat{\psi}_k^R - \psi_k^*) \right)^2 \right] + \mathbb{E} \left[\sum_{(l_1, l_2) > r} \alpha_{l_1} \alpha_{l_2} \left(\psi_{l_1}^*(\hat{\psi}_k^R - \psi_k^*) \right) \left(\psi_{l_2}^*(\hat{\psi}_k^R - \psi_k^*) \right) \right] \\ &= O \left(n \left(\sqrt{\frac{\log n}{\log \log n}} / (n q_n^{\text{pre}}) \right)^2 \right). \end{aligned}$$

by Lemma C.2, and thus $\sum_{i=1}^n \eta_i^\phi (\hat{\psi}_{ki}^R - \psi_k^*(w_i)) = O_{\mathbb{P}} \left(\frac{1}{\sqrt{n q_n^{\text{pre}}}} \sqrt{\frac{\log n}{\log \log n}} \right)$. Since $\hat{\gamma}_k^R = O_{\mathbb{P}}(n q_n^{\text{pre}})$ (by Lemma D.4), we conclude:

$$S_{12} = \sum_{k=1}^r \frac{\hat{\gamma}_k^R}{n} \sum_{i=1}^n \eta_i^\phi (\hat{\psi}_{ki}^R - \psi_k^*(w_i)) = O_{\mathbb{P}} \left(\sqrt{\frac{\log n}{\log \log n}} / n \right).$$

For S_2 , we have

$$S_2 = \frac{1}{n} \sum_{l=1}^r \mu_{n,l}^\phi \sum_{i=1}^n \left(\psi_l^*(w_i) - \hat{\psi}_{li}^R \right) \left(Z_i^{\text{SSIV}} + \sum_{k=1}^r \hat{\gamma}_k^R \hat{\psi}_{ki}^R \right) =: S_{21} + S_{22}.$$

For S_{21} , we have

$$S_{21} = \frac{1}{n} \sum_{l=1}^r \mu_{n,l}^\phi \sum_{i=1}^n \left(\psi_l^*(w_i) - \hat{\psi}_{li}^R \right) Z_i^{\text{SSIV}} = \frac{1}{n} \sum_{l=1}^r \mu_{n,l}^\phi \sum_{j=1}^n (T_j - \pi) \sum_{i=1}^n A_{ij}^{\text{pre}} \left(\psi_l^*(w_i) - \hat{\psi}_{li}^R \right).$$

By $\mu_{n,l}^\phi = O_{\mathbb{P}}(\sqrt{n})$ and $\sum_{j=1}^n (T_j - \pi) \sum_{i=1}^n A_{ij}^{\text{pre}} \left(\psi_l^*(w_i) - \hat{\psi}_{li}^R \right) = O_{\mathbb{P}} \left(\sqrt{\frac{\log n}{\log \log n}} \right)$, we

conclude: $S_{21} = O_{\mathbb{P}}\left(\sqrt{\frac{\log n}{\log \log n}/n}\right)$. For S_{22} , we have:

$$S_{22} = \frac{1}{n} \sum_{k=1}^r \sum_{l=1}^r \mu_{n,l}^{\phi} \hat{\gamma}_k^R \sum_{i=1}^n (\psi_l^*(w_i) - \hat{\psi}_{li}^R) \hat{\psi}_{ki}^R.$$

For each pair (k, l) ,

$$\begin{aligned} \left| \sum_{i=1}^n (\psi_l^*(w_i) - \hat{\psi}_{li}^R) \hat{\psi}_{ki}^R \right| &= \left| (\psi_l^* - \hat{\psi}_l^R)^{\top} \hat{\psi}_k^R \right| = \left| (\psi_l^* - \hat{\psi}_l^R)^{\top} (\hat{\psi}_k^R - \psi_k^*) + (\psi_l^* - \hat{\psi}_l^R)^{\top} \psi_k^* \right| \\ &\leq \left\| \psi_l^* - \hat{\psi}_l^R \right\| \left\| \hat{\psi}_k^R - \psi_k^* \right\| + \left| (\psi_l^* - \hat{\psi}_l^R)^{\top} \psi_k^* \right| = O_{\mathbb{P}}\left(\sqrt{\frac{\log n}{\log \log n}/(nq_n^{\text{pre}})}\right) \end{aligned}$$

by Lemma C.2 and (D.10). Together with $\mu_{n,l}^{\phi} = O_{\mathbb{P}}(\sqrt{n})$ and $\hat{\gamma}_k^R = O_{\mathbb{P}}(nq_n^{\text{pre}})$, we obtain $S_{22} = O_{\mathbb{P}}\left(\sqrt{\frac{\log n}{\log \log n}/n}\right)$. Therefore, S_{11} is the leading term of order $O_{\mathbb{P}}(\sqrt{q_n^{\text{pre}}})$, and we can show (C.2). \square

Proof of Lemma C.4. By the decomposition of M_i , we have

$$\frac{1}{n} \sum_{i=1}^n (Z_i^{\text{SSIV}} + \hat{\delta}_i) M_i = \frac{1}{n} \sum_{i=1}^n (Z_i^{\text{SSIV}} + \hat{\delta}_i) (\xi_i + \mu_{r_1,i}) + \frac{1}{n} \sum_{i=1}^n (Z_i^{\text{SSIV}} + \hat{\delta}_i) (r_{0,i} + r_{1,i} - \mu_{r_1,i}).$$

First, by Lemma C.3, we can show

$$\frac{1}{n} \sum_{i=1}^n (Z_i^{\text{SSIV}} + \hat{\delta}_i) (\xi_i + \mu_{r_1,i}) = \frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}} \eta_i^{(\xi_i + \mu_{r_1,i})} + O_{\mathbb{P}}(q_n^{\text{pre}}) = O_{\mathbb{P}}\left(\sqrt{q_n^{\text{pre}}}\right).$$

Second, we have

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n (Z_i^{\text{SSIV}} + \hat{\delta}_i) (r_{0,i} + r_{1,i} - \mu_{r_1,i}) &= \frac{1}{n} \sum_{i=1}^n \left(Z_i^{\text{SSIV}} + \sum_{k=1}^r \hat{\gamma}_k \hat{\psi}_{ki} \right) \left(\sum_{l=1}^r \mu_{n,l}^r \psi_l^*(w_i) + \eta_i^r \right) \\ &= \frac{1}{n} \sum_{i=1}^n \left(Z_i^{\text{SSIV}} + \sum_{k=1}^r \hat{\gamma}_k \hat{\psi}_{ki} \right) \eta_i^r + \frac{1}{n} \sum_{l=1}^r \mu_{n,l}^r \sum_{i=1}^n \left(Z_i^{\text{SSIV}} + \sum_{k=1}^r \hat{\gamma}_k \hat{\psi}_{ki} \right) (\psi_l^*(w_i) - \hat{\psi}_{li}^R) \\ &=: S_1 + S_2. \end{aligned}$$

For S_1 , we have $S_1 = S_{11} + S_{12}$. For S_{11} , we have

$$\mathbb{E}(S_{11}) = \mathbb{E} \left(\frac{1}{n} \sum_{i=1}^n \sum_{j \neq i} A_{ij}^{\text{pre}} (T_j - \pi) \eta_i^r \right) \asymp \frac{\min\{q_n^{\text{pre}}, q_n^{\text{post}}\}}{q_n^{\text{post}}}$$

and

$$\begin{aligned} \text{Var}(S_{11}) &= \frac{1}{n^2} \left[\begin{aligned} &\sum_{(i,j)} \text{Var} (A_{ij}^{\text{pre}} (T_j - \pi) \eta_i^r) \\ &+ \sum_{(i,j)} \text{Cov} (A_{ij}^{\text{pre}} (T_j - \pi) \eta_i^r, A_{ji}^{\text{pre}} (T_i - \pi) \eta_j^r) + \sum_{(i,j,k)} \text{Cov} (A_{ij}^{\text{pre}} (T_j - \pi) \eta_i^r, A_{ik}^{\text{pre}} (T_k - p) \eta_i^r) \\ &+ \sum_{(i,j,k)} \text{Cov} (A_{ij}^{\text{pre}} (T_j - \pi) \eta_i^r, A_{kj}^{\text{pre}} (T_j - \pi) \eta_k^r) + \sum_{(i,j,k,l)} \text{Cov} (A_{ij}^{\text{pre}} (T_j - \pi) \eta_i^r, A_{kl}^{\text{pre}} (T_l - p) \eta_k^r) \end{aligned} \right] \\ &= \frac{1}{n^2} \sum_{(i,j)} \text{Var} (A_{ij}^{\text{pre}} (T_j - \pi) \eta_i^r) = O \left(\frac{q_n^{\text{pre}}}{n q_n^{\text{post}}} \right). \end{aligned}$$

Therefore,

$$S_{11} = O_{\mathbb{P}} \left(\frac{q_n^{\text{pre}}}{\max\{q_n^{\text{pre}}, q_n^{\text{post}}\}} \right) + O_{\mathbb{P}} \left(\sqrt{\frac{q_n^{\text{pre}}}{n q_n^{\text{post}}}} \right).$$

For S_{12} , we can show that

$$\frac{1}{n} \sum_{i=1}^n \left(\sum_{k=1}^r \hat{\gamma}_k \hat{\psi}_{ki} \right) \eta_i^r = \sum_{k=1}^r \frac{\hat{\gamma}_k}{n} \left(\sum_{i=1}^n \eta_i^r (\hat{\psi}_{ki} - \psi_k^*(w_i)) \right) = O_{\mathbb{P}} \left(\frac{1}{n \sqrt{q_n^{\text{post}}}} \sqrt{\frac{\log n}{\log \log n / n}} \right)$$

by Lemma C.2 and Lemma D.4. Therefore,

$$S_1 = \frac{1}{n} \sum_{i=1}^n \left(Z_i^{\text{SSIV}} + \sum_{k=1}^r \hat{\gamma}_k \hat{\psi}_{ki} \right) \eta_i^r = O_{\mathbb{P}} \left(\frac{q_n^{\text{pre}}}{\max\{q_n^{\text{pre}}, q_n^{\text{post}}\}} \right) + O_{\mathbb{P}} \left(\sqrt{\frac{q_n^{\text{pre}}}{n q_n^{\text{post}}}} \right).$$

For S_2 , we apply the same proof in Lemma C.3. By definition and Lemma B.2,

$$\mu_{n,k}^r = \sum_{i=1}^n (r_{0,i} + r_{1,i} - \mu_{r_1,i}) \psi_k^*(w_i) = O_{\mathbb{P}} \left(\frac{1}{\sqrt{n q_n^{\text{post}}}} \right),$$

and this implies that $S_2 = O_{\mathbb{P}} \left(\frac{1}{n \sqrt{q_n^{\text{post}}}} \sqrt{\frac{\log n}{\log \log n / n}} \right)$. To conclude,

$$\frac{1}{n} \sum_{i=1}^n (Z_i^{\text{SSIV}} + \hat{\delta}_i) (r_{0,i} + r_{1,i} - \mu_{r_1,i}) = \mathbb{E} \left(\frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}} \eta_i^r \right) + O_{\mathbb{P}} \left(\frac{\sqrt{q_n^{\text{pre}}}}{\sqrt{n q_n^{\text{post}}}} \right).$$

□

Below we provide the proof of (C.7) and (C.8).

Proof of Theorem 3.4 (Step 1). By definition of $\mathcal{S}_{(i,j)}$, we have

$$\sum_{(k,l) \in \mathcal{S}_{(i,j)}} H_{kl} = H_{ij} + H_{ji} + \sum_{k \neq i,j} H_{ik} + \sum_{k \neq i,j} H_{ki} + \sum_{k \neq i,j} H_{jk} + \sum_{k \neq i,j} H_{kj}. \quad (\text{D.16})$$

Then we bound the first term of (C.6) by considering the terms in (D.16).

(1): $H_{ij} + H_{ji}$. By definition,

$$\mathbb{E} [A_{ij}^{\text{pre}} (H_{ij} + H_{ji})^2] = \mathbb{E} [A_{ij}^{\text{pre}} ((T_j - \pi)\eta_i^u + (T_i - \pi)\eta_j^u)^2] \leq Cq_n^{\text{pre}}.$$

(2): $\sum_{k \neq i,j} H_{ik}$. We bound it by independence between $(T_{k_1} - \pi)$ and $(T_{k_2} - \pi)$:

$$\begin{aligned} & \mathbb{E} \left[A_{ij}^{\text{pre}} \left(\sum_{k \neq i,j} H_{ik} \right)^2 \right] = \mathbb{E} \left[A_{ij}^{\text{pre}} \left(\sum_{k \neq i,j} H_{ik}^2 + \sum_{(k_1, k_2)} H_{ik_1} H_{ik_2} \right) \right] \\ &= \mathbb{E} \left[A_{ij}^{\text{pre}} \left(\sum_{k \neq i,j} A_{ik}^{\text{pre}} (T_k - \pi)^2 (\eta_i^u)^2 + \sum_{(k_1, k_2)} A_{ik_1}^{\text{pre}} (T_{k_1} - \pi) \eta_i^u A_{ik_2}^{\text{pre}} (T_{k_2} - \pi) \eta_i^u \right) \right] \\ &= \mathbb{E} \left[A_{ij}^{\text{pre}} \sum_{k \neq i,j} A_{ik}^{\text{pre}} (T_k - \pi)^2 (\eta_i^u)^2 \right] \leq Cn(q_n^{\text{pre}})^2 \end{aligned}$$

(3): $\sum_{k \neq i,j} H_{ki}$. It can be bounded above by

$$\begin{aligned} \mathbb{E} \left[A_{ij}^{\text{pre}} \left(\sum_{k \neq i,j} H_{ki} \right)^2 \right] &= \mathbb{E} \left[A_{ij}^{\text{pre}} \left(\sum_{k \neq i,j} A_{ki}^{\text{pre}} (T_i - \pi)^2 (\eta_k^u)^2 + (T_i - \pi)^2 \sum_{(k_1, k_2)} A_{k_1 i}^{\text{pre}} \eta_{k_1}^u A_{k_2 i}^{\text{pre}} \eta_{k_2}^u \right) \right] \\ &= \mathbb{E} \left[A_{ij}^{\text{pre}} (T_i - \pi)^2 \sum_{k \neq i,j} A_{ki}^{\text{pre}} (\eta_k^u)^2 \right] \leq Cn(q_n^{\text{pre}})^2, \end{aligned}$$

where the last equality follows from

$$\begin{aligned} & \mathbb{E} \left[A_{ij}^{\text{pre}} (T_i - \pi)^2 \sum_{(k_1, k_2)} A_{k_1 i}^{\text{pre}} \eta_{k_1}^u A_{k_2 i}^{\text{pre}} \eta_{k_2}^u \right] \\ &= \pi(1 - \pi)(q_n^{\text{pre}})^3 \mathbb{E} \left[f_0(i, j) \sum_{k_1} \sum_{l_1=1}^n \lambda_{l_1}^* \psi_{l_1}^*(w_{k_1}) \psi_{l_1}^*(w_i) \eta_{k_1}^u \sum_{k_2} \sum_{l_2=1}^n \lambda_{l_2}^* \psi_{l_2}^*(w_{k_2}) \psi_{l_2}^*(w_i) \eta_{k_2}^u \right] = 0. \end{aligned}$$

By the variance calculation in (D.15), we have $\sigma_n^2 = O(n^2 q_n^{\text{pre}})$. By combining these three cases, we show (C.7). To show (C.8), we decompose it into several pieces in terms of (D.16), and consider these terms one by one.

(1): $H_{ij} + H_{ji}$. By definition,

$$\begin{aligned} \text{Var} \left(\sum_{(i,j)} H_{ij} (H_{ij} + H_{ji}) \right) &= \text{Var} \left(\sum_{(i,j)} A_{ij}^{\text{pre}} (T_j - \pi) \eta_i^u ((T_j - \pi) \eta_i^u + (T_i - \pi) \eta_j^u) \right) \\ &= \left[\begin{aligned} &\sum_{(i,j)} \text{Var} (A_{ij}^{\text{pre}} (T_j - \pi) \eta_i^u ((T_j - \pi) \eta_i^u + (T_i - \pi) \eta_j^u)) \\ &+ \sum_{(i,j,k)} \text{Cov} (A_{ij}^{\text{pre}} (T_j - \pi) \eta_i^u ((T_j - \pi) \eta_i^u + (T_i - \pi) \eta_j^u), A_{ik}^{\text{pre}} (T_k - \pi) \eta_i^u ((T_k - \pi) \eta_i^u + (T_i - \pi) \eta_k^u)) \end{aligned} \right] \\ &\leq C n^3 (q_n^{\text{pre}})^2. \end{aligned}$$

(2): $\sum_{k \neq i,j} H_{ik}$. We bound the variance:

$$\begin{aligned} \text{Var} \left(\sum_{(i,j)} H_{ij} \sum_{k \neq i,j} H_{ik} \right) &\leq \text{Var} \left(\sum_{(i,j,k)} A_{ij}^{\text{pre}} A_{ik}^{\text{pre}} (T_j - \pi) (T_k - \pi) (\eta_i^u)^2 \right) \\ &= \left[\begin{aligned} &\sum_{(i,j,k)} \text{Var} (A_{ij}^{\text{pre}} A_{ik}^{\text{pre}} (T_j - \pi) (T_k - \pi) (\eta_i^u)^2) \\ &+ \sum_{(i_1, i_2, j, k)} \mathbb{E} [A_{i_1 j}^{\text{pre}} A_{i_1 k}^{\text{pre}} A_{i_2 j}^{\text{pre}} A_{i_2 k}^{\text{pre}} (T_j - \pi)^2 (T_k - \pi)^2 (\eta_{i_1}^u)^2 (\eta_{i_2}^u)^2] \end{aligned} \right] \leq C_1 n^3 (q_n^{\text{pre}})^2 + C_2 n^4 (q_n^{\text{pre}})^4. \end{aligned}$$

(3): $\sum_{k \neq i,j} H_{ki}$. We bound the variance:

$$\begin{aligned} \text{Var} \left(\sum_{(i,j)} H_{ij} \sum_{k \neq i,j} H_{ki} \right) &= \text{Var} \left(\sum_{(i,j,k)} A_{ij}^{\text{pre}} A_{ki}^{\text{pre}} (T_j - \pi) (T_i - \pi) \eta_i^u \eta_k^u \right) \\ &= \left[\begin{aligned} &\sum_{(i,j,k)} \text{Var} (A_{ij}^{\text{pre}} A_{ki}^{\text{pre}} (T_j - \pi) (T_i - \pi) \eta_i^u \eta_k^u) \\ &+ \sum_{(i,j,k_1,k_2)} \mathbb{E} [A_{ij}^{\text{pre}} A_{k_1 i}^{\text{pre}} A_{k_2 i}^{\text{pre}} (T_j - \pi)^2 (T_i - \pi)^2 (\eta_i^u)^2 \eta_{k_1}^u \eta_{k_2}^u] \end{aligned} \right] \leq C_1 n^3 (q_n^{\text{pre}})^2. \end{aligned}$$

(4): $\sum_{k \neq i,j} H_{kj}$. Recall that $\sum_{i=1}^n \eta_i \psi_k^*(w_i) = 0$.

$$\begin{aligned} \text{Var} \left(\sum_{(i,j)} H_{ij} \sum_{k \neq i,j} H_{kj} \right) &\leq \mathbb{E} \left[\left(\sum_{(i,j,k)} A_{ij}^{\text{pre}} (T_j - \pi)^2 \eta_i^u A_{kj}^{\text{pre}} \eta_k^u \right)^2 \right] \\ &= \sum_{\substack{(i_1, j_1, k_1) \\ (i_2, j_2, k_2)}} \mathbb{E} [(T_{j_1} - \pi)^2 (T_{j_2} - \pi)^2 A_{i_1 j_1}^{\text{pre}} \eta_{i_1}^u A_{k_1 j_1}^{\text{pre}} \eta_{k_1}^u A_{i_2 j_2}^{\text{pre}} \eta_{i_2}^u A_{k_2 j_2}^{\text{pre}} \eta_{k_2}^u] \end{aligned}$$

$$\begin{aligned}
&= \pi^2(1 - \pi)^2 \left[\sum_{(i,j_1,j_2,k)} \mathbb{E} [A_{ij_1}^{\text{pre}} A_{kj_1}^{\text{pre}} A_{kj_2}^{\text{pre}} (\eta_i^u)^2 (\eta_k^u)^2] + \sum_{\substack{(i,j_1,j_2) \\ \text{all distinct}}} \mathbb{E} [A_{ij_1}^{\text{pre}} A_{ij_2}^{\text{pre}} (\eta_i^u)^4] \right] \\
&\leq C_1 n^4 (q_n^{\text{pre}})^3 + C_2 n^3 (q_n^{\text{pre}})^2.
\end{aligned}$$

By combining these results, we show (C.8). \square

E Normalized SSIV

Borusyak et al. (2022) suggest the following normalized SSIV for M_i :

$$Z_i^{\text{alt}} = \sum_{j=1}^n A_{ij}^{\text{pre}} T_j / \sum_{j=1}^n A_{ij}^{\text{pre}}. \quad (\text{E.1})$$

The exclusion restriction holds by the randomness of the shocks and the normalization of the shares: $\mathbb{E} [Z_i^{\text{alt}} u_i] = \pi \cdot \mathbb{E} \left[\frac{\sum_{j=1}^n A_{ij}^{\text{pre}}}{\sum_{j=1}^n A_{ij}^{\text{pre}}} u_i \right] = 0$. Borusyak et al. (2022) state that the relevance condition holds when individual units are mostly exposed to only a small number of shocks. In this section, we quantify the regime where the normalized SSIV in (E.1) yields consistent estimators.

E.1 Consistency

For estimation, we consider the IV estimation with the IV vector: $\tilde{Z}_i^{\text{alt}} = (1, T_i, Z_i^{\text{alt}})$. Let $\hat{\beta}^{\text{alt}}$ denote the vector of coefficients obtained from the above IV regression. Define \tilde{Z}^{alt} as the $n \times 3$ matrix obtained by stacking \tilde{Z}_i^{alt} .

Theorem E.1. *Under Assumptions 1 and 3,*

- (a) *if $\text{Var}(\xi_i) > 0$ with $\max\{q_n^{\text{pre}}, q_n^{\text{post}}\} \prec 1/\sqrt{n}$, then $\hat{\beta}^{\text{alt}} - \beta = O_{\mathbb{P}}(\sqrt{n} \max\{q_n^{\text{pre}}, q_n^{\text{post}}\})$;*
- (b) *if $\text{Var}(\xi_i) = 0$ with $\max\{q_n^{\text{pre}}, q_n^{\text{post}}\} \prec \sqrt{n} q_n^{\text{post}}$, then $\hat{\beta}_1^{\text{alt}} - \beta_1 = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}} \max\left\{\frac{\max\{q_n^{\text{pre}}, q_n^{\text{post}}\}}{\sqrt{q_n^{\text{post}}}}, 1\right\}\right)$.*
- Moreover, with $\max\{q_n^{\text{pre}}, q_n^{\text{post}}\} \prec 1/\sqrt{n}$, then $\hat{\beta}_0^{\text{alt}} - \beta_0 = O_{\mathbb{P}}(\sqrt{n} \max\{q_n^{\text{pre}}, q_n^{\text{post}}\})$ and $\hat{\beta}_2^{\text{alt}} - \beta_2 = O_{\mathbb{P}}(\sqrt{n} \max\{q_n^{\text{pre}}, q_n^{\text{post}}\})$.*

Theorem E.1 shows that the consistency regime for the normalized SSIV in (E.1) coincides with that of the original SSIV Z_i^{SSIV} in Theorem 3.1. This is unsurprising, since both instruments rely on the same core component, $\sum_{j=1}^n A_{ij}^{\text{post}} T_j$, which is endogenous because it correlates with the error term u_i . Normalizing or centering this

sum by its expectation effectively removes that endogeneity. Corollary S1 simplifies Theorem E.1 to the special case where $q_n^{\text{pre}} \asymp q_n^{\text{post}}$.

Corollary S1. *Suppose $q_n^{\text{pre}} \asymp q_n^{\text{post}}$. Under Assumptions 1 and 3,*

(a) *if $\text{Var}(\xi_i) > 0$ with $q_n^{\text{post}} \prec \frac{1}{\sqrt{n}}$, then $\hat{\beta}^{\text{alt}} - \beta = O_{\mathbb{P}}(\sqrt{n}q_n^{\text{post}})$;*

(b) *if $\text{Var}(\xi_i) = 0$ with $q_n^{\text{post}} \prec \frac{1}{\sqrt{n}}$, then $\hat{\beta}_0^{\text{alt}} - \beta_0 = O_{\mathbb{P}}(\sqrt{n}q_n^{\text{post}})$ and $\hat{\beta}_2^{\text{alt}} - \beta_2 = O_{\mathbb{P}}(\sqrt{n}q_n^{\text{post}})$. Moreover, it always holds that $\hat{\beta}_1^{\text{alt}} - \beta_1 = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$.*

E.2 Proof of Theorem E.1

We prove Theorem E.1 by first showing Lemma E.1 and Lemma E.2.

Lemma E.1. *Define ϕ_i as an i.i.d. random variable with constant variance. Define the conditional expectation $Q_j^\phi = \mathbb{E}\left[\frac{A_{ij}^{\text{pre}}\phi_i}{q_n^{\text{pre}}g_0(i)}|w_j\right]$. Under Assumptions 1 and 3,*

$$\frac{1}{n} \sum_{i=1}^n (Z_i^{\text{alt}} - \pi)\phi_i = \frac{1}{n} \sum_{i=1}^n (T_i - \pi)Q_i^\phi + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}\sqrt{n}q_n^{\text{pre}}}\right). \quad (\text{E.2})$$

Proof of Lemma E.1. We make use of the proof of Li and Wager (2022, Theorem 4). By reordering the index, we have

$$\frac{1}{n} \sum_{i=1}^n (Z_i^{\text{alt}} - \pi)\phi_i = \frac{1}{n} \sum_{i=1}^n \frac{\sum_{j=1}^n A_{ij}^{\text{pre}}(T_j - \pi)}{\sum_{j=1}^n A_{ij}^{\text{pre}}} \phi_i = \frac{1}{n} \sum_{j=1}^n (T_j - \pi) \sum_{i \neq j} \frac{A_{ij}^{\text{pre}} \phi_i}{N_i}$$

where

$$\sum_{i \neq j} \frac{A_{ij}^{\text{pre}} \phi_i}{N_i} = \sum_{i \neq j} \frac{A_{ij}^{\text{pre}} \phi_i}{(n-1)q_n^{\text{pre}}g_0(i)} - \sum_{i \neq j} \frac{A_{ij}^{\text{pre}} \phi_i (N_i - (n-1)q_n^{\text{pre}}g_0(i))}{(n-1)q_n^{\text{pre}}g_0(i)N_i}. \quad (\text{E.3})$$

First term in (E.3): Given w_j , the terms $\frac{A_{ij}^{\text{pre}} \phi_i}{q_n^{\text{pre}}g_0(i)}$ are i.i.d. across i , so

$$\mathbb{E} \left[\left(\frac{1}{n-1} \sum_{i \neq j} \frac{A_{ij}^{\text{pre}} \phi_i}{q_n^{\text{pre}}g_0(i)} - Q_j^\phi \right)^2 \right] \leq \frac{1}{n-1} \mathbb{E} \left[\frac{A_{ij}^{\text{pre}} \phi_i^2}{(q_n^{\text{pre}}g_0(i))^2} \right] \leq \frac{C}{(n-1)q_n^{\text{pre}}}. \quad (\text{E.4})$$

This implies that $\frac{1}{n-1} \sum_{i \neq j} \frac{A_{ij}^{\text{pre}} \phi_i}{q_n^{\text{pre}}g_0(i)}$ approximates Q_j^ϕ with an error of order $O_{\mathbb{P}}\left(\frac{1}{nq_n^{\text{pre}}}\right)$.

Second term in (E.3): We start by noting that

$$\frac{A_{ij}^{\text{pre}} \phi_i(N_i - (n-1)q_n^{\text{pre}} g_0(i))}{q_n^{\text{pre}} g_0(i) N_i} = \frac{A_{ij}^{\text{pre}} \phi_i((N_i - A_{ij}^{\text{pre}} + 1) - (n-1)q_n^{\text{pre}} g_0(i))}{q_n^{\text{pre}} g_0(i) (N_i - A_{ij}^{\text{pre}} + 1)}. \quad (\text{E.5})$$

Conditional on w , the term $N_i - A_{ij}^{\text{pre}}$ follows $\text{Binomial}(n-1, q_n^{\text{pre}} g_0(i))$. For a binomial random variable $X \sim \text{Binomial}(n, p)$, it holds that $\mathbb{E}\left(\frac{1}{X+1}\right) = \frac{1-(1-p)^{n+1}}{(n+1)p}$. Applying this identity, we compute

$$\begin{aligned} \mathbb{E}\left[\frac{(N_i - A_{ij}^{\text{pre}} + 1) - (n-1)q_n^{\text{pre}} g_0(i)}{(n-1)q_n^{\text{pre}} g_0(i) (N_i - A_{ij}^{\text{pre}} + 1)} \mid w\right] &= \frac{1}{(n-1)q_n^{\text{pre}} g_0(i)} - \mathbb{E}\left[\frac{1}{N_i - A_{ij}^{\text{pre}} + 1} \mid w\right] \\ &= \frac{1}{(n-1)q_n^{\text{pre}} g_0(i)} - \frac{1 - (1 - q_n^{\text{pre}} g_0(i))^n}{nq_n^{\text{pre}} g_0(i)} = \frac{1}{n(n-1)q_n^{\text{pre}} g_0(i)} + \frac{(1 - q_n^{\text{pre}} g_0(i))^n}{nq_n^{\text{pre}} g_0(i)} \\ &\leq \frac{1}{n(n-1)q_n^{\text{pre}} g_0(i)} + \frac{e^{-Cnq_n^{\text{pre}}}}{nq_n^{\text{pre}} g_0(i)} \leq C \cdot \frac{1}{(nq_n^{\text{pre}})^2}. \end{aligned} \quad (\text{E.6})$$

By the Cauchy–Schwarz inequality and [Li and Wager \(2022, Lemma 15\)](#), we have

$$\begin{aligned} &\mathbb{E}\left[\left(\frac{(N_i - A_{ij}^{\text{pre}} + 1) - (n-1)q_n^{\text{pre}} g_0(i)}{(n-1)q_n^{\text{pre}} g_0(i) (N_i - A_{ij}^{\text{pre}} + 1)}\right)^2 \mid w\right] \\ &\leq \frac{1}{(n-1)^2 (q_n^{\text{pre}} g_0(i))^2} \sqrt{\mathbb{E}\left(\left((N_i - A_{ij}^{\text{pre}} + 1) - (n-1)q_n^{\text{pre}} g_0(i)\right)^4\right)} \sqrt{\mathbb{E}\left(\frac{1}{(N_i - A_{ij}^{\text{pre}} + 1)^4}\right)} \\ &\leq C \frac{1}{((n-1)q_n^{\text{pre}})^2} \frac{nq_n^{\text{pre}}}{(nq_n^{\text{pre}})^2} \leq C \cdot \frac{1}{(nq_n^{\text{pre}})^3}. \end{aligned} \quad (\text{E.7})$$

Let $B_{ij} = \frac{A_{ij}^{\text{pre}} \phi_i(N_i - (n-1)q_n^{\text{pre}} g_0(i))}{(n-1)q_n^{\text{pre}} g_0(i) N_i}$. We bound the second moment as $\mathbb{E}\left[\left(\sum_{i \neq j} B_{ij}\right)^2\right] = \sum_{i \neq j} \mathbb{E}[B_{ij}^2] + \sum_{i \neq k} \mathbb{E}[B_{ij} B_{kj}]$.

For the diagonal term: By using (E.7),

$$\mathbb{E}[B_{ij}^2] = \mathbb{E}\left[A_{ij}^{\text{pre}} \phi_i^2 \left(\frac{(N_i - A_{ij}^{\text{pre}} + 1) - (n-1)q_n^{\text{pre}} g_0(i)}{(n-1)q_n^{\text{pre}} g_0(i) (N_i - A_{ij}^{\text{pre}} + 1)}\right)^2\right] \leq \frac{C}{n(nq_n^{\text{pre}})^2}.$$

For the off-diagonal term: Conditional on w , A_{ij}^{pre} , A_{jk}^{pre} , $\frac{N_j - (n-1)q_n^{\text{pre}} g_0(w_j)}{N_j}$ and $\frac{N_k - (n-1)q_n^{\text{pre}} g_0(w_k)}{N_k}$ are all independent. By using (E.6):

$$\begin{aligned} \mathbb{E}[B_{ij} B_{ik}] &= \mathbb{E}\left[\left(q_n^{\text{pre}}\right)^2 g_0(i, j) g_0(i, k) \phi_j \phi_k \mathbb{E}\left[\frac{N_j - (n-1)q_n^{\text{pre}} g_0(w_j)}{(n-1)q_n^{\text{pre}} g_0(w_j) N_j} \mid w\right] \mathbb{E}\left[\frac{N_k - (n-1)q_n^{\text{pre}} g_0(w_k)}{(n-1)q_n^{\text{pre}} g_0(w_k) N_k} \mid w\right]\right] \\ &\leq C \cdot \frac{1}{n^2 (nq_n^{\text{pre}})^2}. \end{aligned}$$

Combining the diagonal and off-diagonal terms, we conclude:

$$\mathbb{E} \left[\left(\sum_{i \neq j} \frac{A_{ij}^{\text{pre}} \phi_i (N_i - (n-1)q_n^{\text{pre}} g_0(i))}{(n-1)q_n^{\text{pre}} g_0(i) N_i} \right)^2 \right] \leq \frac{C}{(nq_n^{\text{pre}})^2}. \quad (\text{E.8})$$

Combining (E.4) and (E.8), we get $\mathbb{E} \left[\left(\sum_{i \neq j} \frac{A_{ij}^{\text{pre}}}{N_i} \phi_i - Q_j^\phi \right)^2 \right] \leq \frac{C}{nq_n^{\text{pre}}}$, which yields:

$$\frac{1}{n} \sum_{j=1}^n (T_j - \pi) \left(\sum_{i \neq j} \frac{A_{ij}^{\text{pre}} \phi_i}{N_i} - Q_j^\phi \right) = O_{\mathbb{P}} \left(\frac{1}{\sqrt{n} \sqrt{nq_n^{\text{pre}}}} \right)$$

and thus completes the proof. \square

Define

$$S_1^0 = \frac{1}{n} \sum_{j=1}^n (T_j - \pi) \sum_{i \neq j} \frac{A_{ij}^{\text{pre}} r_{0,i}}{(n-1)q_n^{\text{pre}} g_0(i)}. \quad (\text{E.9})$$

Lemma E.2. *Under Assumptions 1 and 3, we have*

$$\frac{1}{n} \sum_{i=1}^n (Z_i^{\text{alt}} - \pi) M_i = \mathbb{E}(S_1^0) + O_{\mathbb{P}} \left(\frac{1}{n\sqrt{q_n^{\text{post}}}} \right) + O_{\mathbb{P}} \left(\frac{1}{\sqrt{n}} \right). \quad (\text{E.10})$$

Proof of Lemma E.2. By the decomposition of M_i , we have

$$\frac{1}{n} \sum_{i=1}^n (Z_i^{\text{alt}} - \pi) M_i = \frac{1}{n} \sum_{i=1}^n (Z_i^{\text{alt}} - \pi) \xi_i + \frac{1}{n} \sum_{i=1}^n (Z_i^{\text{alt}} - \pi) r_{0,i} + \frac{1}{n} \sum_{i=1}^n (Z_i^{\text{alt}} - \pi) r_{1,i}.$$

First term: By Lemma E.1, we have

$$\frac{1}{n} \sum_{i=1}^n (Z_i^{\text{alt}} - \pi) \xi_i = \frac{1}{n} \sum_{i=1}^n (T_i - \pi) \mathbb{E} \left[\frac{A_{ij}^{\text{pre}} \xi_j}{q_n^{\text{pre}} g_0(j)} | w_i \right] + O_{\mathbb{P}} \left(\frac{1}{n\sqrt{q_n^{\text{pre}}}} \right) = O_{\mathbb{P}} \left(\frac{1}{\sqrt{n}} \right). \quad (\text{E.11})$$

Second term: Rewrite $\frac{1}{n} \sum_{i=1}^n (Z_i^{\text{alt}} - \pi) r_{0,i} = S_1^0 - S_2^0$ where S_1^0 defined in (E.9) and

$$S_2^0 = \frac{1}{n} \sum_{j=1}^n (T_j - \pi) \sum_{i \neq j} \frac{A_{ij}^{\text{pre}} r_{0,i} (N_i - (n-1)q_n^{\text{pre}} g_0(i))}{(n-1)q_n^{\text{pre}} g_0(i) N_i}. \quad (\text{E.12})$$

For S_1^0 , the expectation is

$$\mathbb{E}[S_1^0] = \frac{1}{n} \sum_{j=1}^n \mathbb{E} \left[(T_j - \pi) \sum_{i \neq j} \frac{A_{ij}^{\text{pre}} \xi_i R_{ij}}{(n-1)^2 q_n^{\text{pre}} g_0(i)} \right] \asymp \frac{1}{n \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}}.$$

The variance is

$$\mathbb{V}[S_1^0] = \frac{1}{(n-1)^2 (n q_n^{\text{pre}})^2} \left[\begin{aligned} & \sum_{j=1}^n \sum_{i \neq j} \text{Var} \left(\frac{(T_j - \pi) A_{ij}^{\text{pre}} r_{0,i}}{g_0(i)} \right) + \sum_{j=1}^n \sum_{(i_1, i_2)} \text{Cov} \left(\frac{(T_j - \pi) A_{i_1 j}^{\text{pre}} r_{0, i_1}}{g_0(i_1)}, \frac{(T_j - \pi) A_{i_2 j}^{\text{pre}} r_{0, i_2}}{g_0(i_2)} \right) \\ & + \sum_{(j_1, j_2)} \sum_{i \neq j_1, j_2} \text{Cov} \left(\frac{(T_{j_1} - \pi) A_{i j_1}^{\text{pre}} r_{0, i}}{g_0(i)}, \frac{(T_{j_2} - \pi) A_{i j_2}^{\text{pre}} r_{0, i}}{g_0(i)} \right) \\ & + \sum_{(j_1, j_2)} \sum_{(i_1, i_2)} \text{Cov} \left(\frac{(T_{j_1} - \pi) A_{i_1 j_1}^{\text{pre}} r_{0, i_1}}{g_0(i_1)}, \frac{(T_{j_2} - \pi) A_{i_2 j_2}^{\text{pre}} r_{0, i_2}}{g_0(i_2)} \right) \end{aligned} \right].$$

Bounding each component, we find:

- Variance term is $O(n^2)$ terms of order $O\left(\frac{q_n^{\text{pre}}}{n q_n^{\text{post}}}\right)$;
- Within- j and across- i covariance is $O(n^3)$ terms of order $O\left(\frac{(q_n^{\text{pre}})^2}{n}\right)$;
- Across- j and within- i covariances is $O(n^3)$ terms of order $O\left(\frac{(q_n^{\text{pre}})^2}{n^2 \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}^2}\right)$;
- Across- j and i covariances is $O(n^4)$ terms of order $O\left(\frac{(q_n^{\text{pre}})^2}{n^2 \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}^2}\right)$ and $O\left(\frac{(q_n^{\text{pre}})^2}{n^2 q_n^{\text{post}}}\right)$

Combining these bounds, we conclude:

$$S_1^0 = \mathbb{E}[S_1^0] + O_{\mathbb{P}} \left(\frac{1}{n \sqrt{q_n^{\text{post}}}} \right). \quad (\text{E.13})$$

Recall (E.5). Define $B_{ij} = \frac{A_{ij}^{\text{pre}} r_{0,i} (N_i - (n-1) q_n^{\text{pre}} g_0(i))}{(n-1) q_n^{\text{pre}} g_0(i) N_i}$, so that $S_2^0 = \frac{1}{n} \sum_{j=1}^n (T_j - \pi) \sum_{i \neq j} B_{ij}$.

We bound the second moment of S_2^0 as follows:

$$\mathbb{E} [(S_2^0)^2] = \frac{1}{n^2} \left[\begin{aligned} & \sum_{j=1}^n \mathbb{E} \left[(T_j - \pi)^2 \left(\sum_{i \neq j} B_{ij}^2 + \sum_{(i_1, i_2)} B_{i_1 j} B_{i_2 j} \right) \right] \\ & + \sum_{(j, k)} \mathbb{E} \left[(T_j - \pi)(T_k - \pi) \left(\sum_{i \neq j, k} B_{ij} B_{ik} + \sum_{(i_1, i_2)} B_{i_1 j} B_{i_2 k} \right) \right] \end{aligned} \right].$$

We analyze the components of the moment bound for S_2^0 in turn. Conditional on w , A_{ij}^{pre} and $\frac{N_i - (n-1) q_n^{\text{pre}} g_0(i)}{N_i}$ are independent.

For $\mathbb{E} [(T_j - \pi)^2 B_{ij}^2]$:

$$\mathbb{E} [(T_j - \pi)^2 B_{ij}^2] = \mathbb{E} \left[\mathbb{E} [(T_j - \pi)^2 A_{ij}^{\text{pre}} r_{0,i}^2 | w] \mathbb{E} \left[\left(\frac{(N_i - A_{ij}^{\text{pre}} + 1) - (n-1) q_n^{\text{pre}} g_0(i)}{(n-1) q_n^{\text{pre}} g_0(i) (N_i - A_{ij}^{\text{pre}} + 1)} \right)^2 | w \right] \right]$$

$$\leq \frac{C}{(nq_n^{\text{pre}})^3} \cdot \mathbb{E} \left[\frac{(T_j - \pi)^2}{(n-1)^2} A_{ij}^{\text{pre}} \xi_i^2 \sum_{k \neq i} R_{ik}^2 \right] = O \left(\frac{1}{n^4 (q_n^{\text{pre}})^2 q_n^{\text{post}}} \right).$$

For $\mathbb{E}[(T_j - \pi)^2 B_{i_1 j} B_{i_2 j}]$: Since $A_{i_1 j}^{\text{pre}}$, $A_{i_2 j}^{\text{pre}}$, and the $\frac{N_{i_1} - (n-1)q_n^{\text{pre}} g_0(i_1)}{(n-1)q_n^{\text{pre}} g_0(i_1) N_{i_1}}$ and $\frac{N_{i_2} - (n-1)q_n^{\text{pre}} g_0(i_2)}{(n-1)q_n^{\text{pre}} g_0(i_2) N_{i_2}}$ are independent conditional on w , we have

$$\begin{aligned} & \mathbb{E}[(T_j - \pi)^2 B_{i_1 j} B_{i_2 j}] \\ &= \mathbb{E} \left[\mathbb{E}[(T_j - \pi)^2 A_{i_1 j}^{\text{pre}} A_{i_2 j}^{\text{pre}} r_{0, i_1} r_{0, i_2} | w] \mathbb{E} \left[\frac{N_{i_1} - (n-1)q_n^{\text{pre}} g_0(i_1)}{(n-1)q_n^{\text{pre}} g_0(i_1) N_{i_1}} | w \right] \mathbb{E} \left[\frac{N_{i_2} - (n-1)q_n^{\text{pre}} g_0(i_2)}{(n-1)q_n^{\text{pre}} g_0(i_2) N_{i_2}} | w \right] \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[\frac{(T_j - \pi)^2}{(n-1)^2} \xi_{i_1} \xi_{i_2} A_{i_1 j}^{\text{pre}} A_{i_2 j}^{\text{pre}} R_{i_1 j} R_{i_2 j} | w \right] \mathbb{E} \left[\frac{N_{i_1} - (n-1)q_n^{\text{pre}} g_0(i_1)}{(n-1)q_n^{\text{pre}} g_0(i_1) N_{i_1}} | w \right] \mathbb{E} \left[\frac{N_{i_2} - (n-1)q_n^{\text{pre}} g_0(i_2)}{(n-1)q_n^{\text{pre}} g_0(i_2) N_{i_2}} | w \right] \right] \\ &\leq C \cdot \frac{\min\{q_n^{\text{pre}}, q_n^{\text{post}}\}^2}{(nq_n^{\text{post}})^2} \cdot \frac{1}{(nq_n^{\text{pre}})^4} = O \left(\frac{1}{n^6 (q_n^{\text{pre}})^2 \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}^2} \right). \end{aligned}$$

For $\mathbb{E}((T_j - \pi)(T_k - \pi)B_{ij}B_{ik})$: By (E.7),

$$\begin{aligned} & \mathbb{E}((T_j - \pi)(T_k - \pi)B_{ij}B_{ik}) \\ &= \mathbb{E} \left[\mathbb{E}[(T_j - \pi)(T_k - \pi)A_{ij}^{\text{pre}} A_{ik}^{\text{pre}} r_{0, i}^2 | w] \mathbb{E} \left[\left(\frac{N_i - (n-1)q_n^{\text{pre}} g_0(i)}{(n-1)q_n^{\text{pre}} g_0(i) N_i} \right)^2 | w \right] \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[\frac{(T_j - \pi)(T_k - \pi)}{(n-1)^2} \xi_i^2 A_{ij}^{\text{pre}} A_{ik}^{\text{pre}} R_{ij} R_{ik} | w \right] \mathbb{E} \left[\left(\frac{N_i - (n-1)q_n^{\text{pre}} g_0(i)}{(n-1)q_n^{\text{pre}} g_0(i) N_i} \right)^2 | w \right] \right] = O \left(\frac{1}{n^5 q_n^{\text{pre}} \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}^2} \right) \end{aligned}$$

For $\mathbb{E}((T_j - \pi)(T_k - \pi)B_{i_1 j}B_{i_2 k})$: By (E.6),

$$\begin{aligned} & \mathbb{E}((T_j - \pi)(T_k - \pi)B_{i_1 j}B_{i_2 k}) \\ &= \mathbb{E} \left[\mathbb{E}[(T_j - \pi)(T_k - \pi)A_{i_1 j}^{\text{pre}} A_{i_2 k}^{\text{pre}} r_{0, i_1} r_{0, i_2} | w] \mathbb{E} \left[\frac{N_{i_1} - (n-1)q_n^{\text{pre}} g_0(i_1)}{(n-1)q_n^{\text{pre}} g_0(i_1) N_{i_1}} | w \right] \mathbb{E} \left[\frac{N_{i_2} - (n-1)q_n^{\text{pre}} g_0(i_2)}{(n-1)q_n^{\text{pre}} g_0(i_2) N_{i_2}} | w \right] \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[\frac{(T_j - \pi)(T_k - \pi)}{(n-1)^2} \xi_{i_1} \xi_{i_2} A_{i_1 j}^{\text{pre}} A_{i_2 k}^{\text{pre}} R_{i_1 j} R_{i_2 k} | T, w \right] \mathbb{E} \left[\frac{N_{i_1} - (n-1)q_n^{\text{pre}} g_0(i_1)}{(n-1)q_n^{\text{pre}} g_0(i_1) N_{i_1}} | w \right] \mathbb{E} \left[\frac{N_{i_2} - (n-1)q_n^{\text{pre}} g_0(i_2)}{(n-1)q_n^{\text{pre}} g_0(i_2) N_{i_2}} | w \right] \right] \\ &\leq C \frac{1}{(nq_n^{\text{pre}})^4} \frac{\min\{q_n^{\text{pre}}, q_n^{\text{post}}\}^2}{n^2 (q_n^{\text{post}})^2} = O \left(\frac{1}{n^6 (q_n^{\text{pre}})^2 \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}^2} \right). \end{aligned}$$

By combining these results, we have

$$\mathbb{E}[(S_2^0)^2] = O \left(\frac{1}{n^4 (q_n^{\text{pre}})^2 q_n^{\text{post}}} \right) + O \left(\frac{1}{n^4 (q_n^{\text{pre}})^2 \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}^2} \right)$$

and thus

$$S_2^0 = O_{\mathbb{P}} \left(\frac{1}{n^2 q_n^{\text{pre}} \sqrt{q_n^{\text{post}}}} \right) + O_{\mathbb{P}} \left(\frac{1}{n^2 q_n^{\text{pre}} \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}} \right). \quad (\text{E.14})$$

Therefore, by combining (E.13) and (E.14), we have

$$\frac{1}{n} \sum_{i=1}^n (Z_i^{\text{alt}} - \pi) r_{0,i} = \mathbb{E}[S_1^0] + O_{\mathbb{P}} \left(\frac{1}{n\sqrt{q_n^{\text{post}}}} \right) + O_{\mathbb{P}} \left(\frac{1}{n^2 q_n^{\text{pre}} \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}} \right). \quad (\text{E.15})$$

Third term: Rewrite $\frac{1}{n} \sum_{i=1}^n (Z_i^{\text{alt}} - \pi) r_{1,i} = S_1^1 - S_2^1$ where

$$S_1^1 = \frac{1}{n} \sum_{j=1}^n (T_j - \pi) \sum_{i \neq j} \frac{A_{ij}^{\text{pre}} r_{1,i}}{(n-1)q_n^{\text{pre}} g_0(i)} \quad \text{and} \quad S_2^1 = \frac{1}{n} \sum_{j=1}^n (T_j - \pi) \sum_{i \neq j} \frac{A_{ij}^{\text{pre}} r_{1,i} (N_i - (n-1)q_n^{\text{pre}} g_0(i))}{(n-1)q_n^{\text{pre}} g_0(i) N_i}.$$

Again, for S_1^1 , the expectation is

$$\mathbb{E}[S_1^1] = \frac{1}{n(n-1)^3 q_n^{\text{pre}}} \sum_{j=1}^n \mathbb{E} \left[(T_j - \pi) \sum_{i \neq j} \frac{A_{ij}^{\text{pre}}}{g_0(i)} \left(-\frac{U_{ij} W_{ij}}{\theta_i^*(y)^2} + \frac{2\theta_i^*(x)}{\theta_i^*(y)^3} W_{ij}^2 \right) \right] \asymp \frac{1}{n^2 q_n^{\text{post}} \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}},$$

and the variance is

$$\begin{aligned} \text{Var}[S_1^1] &= \frac{1}{(n-1)^2 (nq_n^{\text{pre}})^2} \left[\sum_{j=1}^n \sum_{i \neq j} \text{Var} \left(\frac{(T_j - \pi) A_{ij}^{\text{pre}} r_{1,i}}{g_0(i)} \right) + \sum_{j=1}^n \sum_{(i_1, i_2)} \text{Cov} \left(\frac{(T_j - \pi) A_{i_1 j}^{\text{pre}} r_{1,i_1}}{g_0(i_1)}, \frac{(T_j - \pi) A_{i_2 j}^{\text{pre}} r_{1,i_2}}{g_0(i_2)} \right) \right. \\ &\quad + \sum_{(j_1, j_2)} \sum_{i \neq j_1, j_2} \text{Cov} \left(\frac{(T_{j_1} - \pi) A_{i j_1}^{\text{pre}} r_{1,i}}{g_0(i)}, \frac{(T_{j_2} - \pi) A_{i j_2}^{\text{pre}} r_{1,i}}{g_0(i)} \right) \\ &\quad \left. + \sum_{(j_1, j_2)} \sum_{(i_1, i_2)} \text{Cov} \left(\frac{(T_{j_1} - \pi) A_{i_1 j_1}^{\text{pre}} r_{1,i_1}}{g_0(i_1)}, \frac{(T_{j_2} - \pi) A_{i_2 j_2}^{\text{pre}} r_{1,i_2}}{g_0(i_2)} \right) \right] \\ &= \frac{1}{(n-1)^2 (nq_n^{\text{pre}})^2} (S_{11}^1 + S_{12}^1 + S_{13}^1 + S_{14}^1). \end{aligned}$$

Recall $E_{ik} = -\frac{U_{ik} W_{ik}}{\theta_i^*(y)^2} + \frac{2\theta_i^*(x)}{\theta_i^*(y)^3} W_{ik}^2$ and $E_{ikl} = -\frac{U_{ik} W_{il}}{\theta_i^*(y)^2} + \frac{2\theta_i^*(x)}{\theta_i^*(y)^3} W_{ik} W_{il}$. For S_{11}^1 , we bound the variance of the term:

$$\mathbb{V} \left(\frac{(T_j - \pi) A_{ij}^{\text{pre}} r_{1,i}}{g_0(i)} \right) \leq \frac{1}{(n-1)^4} \mathbb{E} \left[\frac{(T_j - \pi)^2}{g_0(i)^2} A_{ij}^{\text{pre}} \left(\sum_{k \neq i} E_{ik} + \sum_{(k,l)} E_{ikl} \right)^2 \right] \asymp \frac{q_n^{\text{pre}}}{(nq_n^{\text{post}})^2}. \quad (\text{E.16})$$

For S_{12}^1 ,

$$\begin{aligned} &\text{Cov} \left(\frac{(T_j - \pi) A_{i_1 j}^{\text{pre}} r_{1,i_1}}{g_0(i_1)}, \frac{(T_j - \pi) A_{i_2 j}^{\text{pre}} r_{1,i_2}}{g_0(i_2)} \right) \\ &= \frac{1}{(n-1)^4} \text{Cov} \left(\frac{(T_j - \pi) A_{i_1 j}^{\text{pre}}}{g_0(i_1)} \left(\sum_{k \neq i_1} E_{i_1 k} + \sum_{(k,l)} E_{i_1 k l} \right), \frac{(T_j - \pi) A_{i_2 j}^{\text{pre}}}{g_0(i_2)} \left(\sum_{k \neq i_2} E_{i_2 k} + \sum_{(k,l)} E_{i_2 k l} \right) \right) \asymp \frac{(q_n^{\text{pre}})^2}{(nq_n^{\text{post}})^2} \end{aligned} \quad (\text{E.17})$$

For S_{13}^1 , we have

$$\begin{aligned} & \text{Cov} \left(\frac{(T_{j_1} - \pi) A_{ij_1}^{\text{pre}} r_{1,i}}{g_0(i)}, \frac{(T_{j_2} - \pi) A_{ij_2}^{\text{pre}} r_{1,i}}{g_0(i)} \right) \\ &= \frac{1}{(n-1)^4} \text{Cov} \left(\frac{(T_{j_1} - \pi) A_{ij_1}^{\text{pre}}}{g_0(i)} \left(\sum_{k \neq i} E_{ik} + \sum_{(k,l)} E_{ikl} \right), \frac{(T_{j_2} - \pi) A_{ij_2}^{\text{pre}}}{g_0(i)} \left(\sum_{k \neq i} E_{ik} + \sum_{(k,l)} E_{ikl} \right) \right) \asymp \frac{(q_n^{\text{pre}})^2}{n^3 q_n^{\text{post}} \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}^2}. \end{aligned} \quad (\text{E.18})$$

For S_{14}^1 , we have

$$\begin{aligned} & \text{Cov} \left(\frac{(T_{j_1} - \pi) A_{i_1 j_1}^{\text{pre}} r_{1,i_1}}{g_0(i_1)}, \frac{(T_{j_2} - \pi) A_{i_2 j_2}^{\text{pre}} r_{1,i_2}}{g_0(i_2)} \right) \\ &= \frac{1}{(n-1)^4} \text{Cov} \left(\frac{(T_{j_1} - \pi) A_{i_1 j_1}^{\text{pre}}}{g_0(i_1)} \left(\sum_{k \neq i_1} E_{i_1 k} + \sum_{(k,l)} E_{i_1 k l} \right), \frac{(T_{j_2} - \pi) A_{i_2 j_2}^{\text{pre}}}{g_0(i_2)} \left(\sum_{k \neq i_2} E_{i_2 k} + \sum_{(k,l)} E_{i_2 k l} \right) \right) \asymp \frac{(q_n^{\text{pre}})^2}{n^3 q_n^{\text{post}} \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}^2}. \end{aligned} \quad (\text{E.19})$$

By combining these results, we have $\text{Var}(S_1^1) = O\left(\frac{1}{n^3 (q_n^{\text{post}})^2}\right)$. In particular, this implies that

$$S_1^1 = O_{\mathbb{P}} \left(\frac{1}{\sqrt{nnq_n^{\text{post}}}} \right) + O_{\mathbb{P}} \left(\frac{1}{n^2 q_n^{\text{post}} \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}} \right). \quad (\text{E.20})$$

Next, for S_2^1 , define $C_{ij} = \frac{A_{ij}^{\text{pre}} r_{1,i} (N_i - (n-1) q_n^{\text{pre}} g_0(i))}{(n-1) q_n^{\text{pre}} g_0(i) N_i}$. Then we bound S_2^1 in L_2 norm:

$$\mathbb{E}[(S_2^1)^2] = \frac{1}{n^2} \left[\begin{aligned} & \sum_{j=1}^n \mathbb{E} \left[(T_j - \pi)^2 \sum_{i \neq j} C_{ij}^2 \right] + \sum_{j=1}^n \mathbb{E} \left[(T_j - \pi)^2 \sum_{(i_1, i_2)} C_{i_1 j} C_{i_2 j} \right] \\ & + \sum_{(j,k)} \sum_{i \neq j} \mathbb{E} [(T_j - \pi)(T_k - \pi) C_{ij} C_{ik}] + \sum_{(j,k)} \sum_{(i_1, i_2)} \mathbb{E} [(T_j - \pi)(T_k - \pi) C_{i_1 j} C_{i_2 k}]. \end{aligned} \right]$$

For the first term, by (E.7) and analogous argument to (E.16), we have

$$\begin{aligned} \mathbb{E}[(T_j - \pi)^2 C_{ij}^2] &= \mathbb{E} \left[\mathbb{E} [(T_j - \pi)^2 A_{ij}^{\text{pre}} r_{1,i}^2 | w] \mathbb{E} \left[\left(\frac{(N_i - (n-1) q_n^{\text{pre}} g_0(i))}{(n-1) q_n^{\text{pre}} N_i} \right)^2 | w \right] \right] \\ &\leq \frac{1}{(n q_n^{\text{pre}})^3} \mathbb{E} [(T_j - \pi)^2 A_{ij}^{\text{pre}} r_{1,i}^2] = O \left(\frac{1}{n^5 (q_n^{\text{post}})^2 (q_n^{\text{pre}})^2} \right). \end{aligned}$$

For the second term, by (E.6) and analogous argument to (E.17), we have

$$\mathbb{E}[(T_j - \pi)^2 C_{i_1 j} C_{i_2 j}] = O \left(\frac{1}{n^6 (q_n^{\text{post}})^2 (q_n^{\text{pre}})^2} \right).$$

For the third term, by (E.7) and analogous argument to (E.18), we have

$$\mathbb{E}[(T_j - \pi)(T_k - \pi) C_{ij} C_{ik}] = O \left(\frac{\min\{q_n^{\text{pre}}, q_n^{\text{post}}\}^2}{n^6 (q_n^{\text{pre}})^3 (q_n^{\text{post}})^3} \right).$$

For the fourth term, by (E.6) and analogous argument to (E.19), we have

$$\mathbb{E}[(T_j - \pi)(T_k - \pi)C_{i_1j}C_{i_2k}] = O\left(\frac{\min\{q_n^{\text{pre}}, q_n^{\text{post}}\}^2}{n^8 q_n^{\text{post}} (q_n^{\text{pre}})^4}\right) + O\left(\frac{\min\{q_n^{\text{pre}}, q_n^{\text{post}}\}}{n^7 (q_n^{\text{post}})^2 (q_n^{\text{pre}})^3}\right).$$

By combining these results, we have $\mathbb{E}[(S_2^1)^2] = O\left(\frac{1}{n^5 (q_n^{\text{post}})^2 (q_n^{\text{pre}})^2}\right)$ and thus

$$S_2^1 = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n} n^2 q_n^{\text{pre}} q_n^{\text{post}}}\right). \quad (\text{E.21})$$

In particular, combining (E.20) and (E.21) implies that

$$\frac{1}{n} \sum_{i=1}^n (Z_i^{\text{alt}} - \pi) r_{1,i} = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n} n q_n^{\text{post}}}\right) + O_{\mathbb{P}}\left(\frac{1}{n^2 q_n^{\text{post}} \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}}\right). \quad (\text{E.22})$$

By combining (E.15) and (E.22), we have

$$\frac{1}{n} \sum_{i=1}^n (Z_i^{\text{alt}} - \pi)(r_{0,i} + r_{1,i}) = \mathbb{E}[S_1^0] + O_{\mathbb{P}}\left(\frac{1}{n \sqrt{q_n^{\text{post}}}}\right). \quad (\text{E.23})$$

Combining (E.11), (E.15) and (E.23) together completes the proof. \square

Proof of Theorem E.1. The proof is analogous to that of Theorem 3.2 but by Lemma E.1 and Lemma E.2. We start with deriving the probability limit of $(\tilde{Z}^{\text{alt}\top} X)^{-1}$:

$$\left(\tilde{Z}^{\text{alt}\top} X\right)^{-1} = \frac{1}{\det(\tilde{Z}^{\text{alt}\top} X)} \begin{pmatrix} * & * & * \\ * & d_{22} & d_{23} \\ * & d_{32} & d_{33} \end{pmatrix}$$

where $d_{22} = \overline{M Z^{\text{alt}}} - \overline{M} \cdot \overline{Z^{\text{alt}}}$, $d_{23} = b_{23}$, $d_{32} = -\overline{T Z^{\text{alt}}} + \overline{T} \cdot \overline{Z^{\text{alt}}}$, $d_{33} = b_{33}$ and $\det(\tilde{Z}^{\text{alt}\top} X) = d_{33}d_{22} - d_{32}d_{23}$. Let $*$ denote the terms that are not relevant. By Lemma E.1, we have $d_{32} = O_{\mathbb{P}}\left(\frac{1}{n \sqrt{q_n^{\text{pre}}}}\right)$ and $\frac{1}{n} \sum_{i=1}^n (Z_i^{\text{alt}} - \overline{Z^{\text{alt}}}) u_i = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$.

We consider the remaining terms under Cases (a) and (b), respectively.

Case (a) By Lemma E.2, we have $d_{22} = \mathbb{E}(S_1^0) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) + o_{\mathbb{P}}\left(\frac{1}{n \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}}\right)$. By Lemma B.2, we have $d_{23} = -\text{Cov}(T_i, \xi_i + \mu_{r_{1,i}}) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) = O_{\mathbb{P}}(1)$. By combining the results above, we have

$$\det(\tilde{Z}^{\text{alt}\top} X) = \pi(1 - \pi)\mathbb{E}(S_1^0) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) + o_{\mathbb{P}}\left(\frac{1}{n \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}}\right).$$

Case (b) By Lemma E.2, we have $d_{22} = \mathbb{E}(S_1^0) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{nnq_n^{\text{post}}}}\right) + o_{\mathbb{P}}\left(\frac{1}{n \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}}\right)$.

By Lemma B.2, we have $d_{23} = O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{\text{post}}}}\right)$. Hence,

$$\det(\tilde{Z}^{\text{alt}\top} X) = \pi(1 - \pi)\mathbb{E}(S_1^0) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{nnq_n^{\text{post}}}}\right) + o_{\mathbb{P}}\left(\frac{1}{n \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}}\right).$$

The convergence rate of $\hat{\beta}^{\text{alt}}$ follows from the above results. □

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