Dynamic Reward Design[*]

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Abstract

This paper studies a dynamic screening model in which a principal hires an agent with limited liability. The agent's private cost of working is an i.i.d. draw from a continuous distribution. His working status is publicly observable. The limited liability constraint requires that payments remain nonnegative at all times. In this setting, despite costs being i.i.d. and the payoffs being additively separable across periods, the optimal mechanism does not treat each period independently. Instead, it features backloading payments and requires the agent to work in consecutive periods. Specifically, I characterize conditions under which the optimal mechanism either grants the agent flexibility to start working in any period or restricts the starting period to the first. In either case, once the agent begins working, he is incentivized to work consecutively until the end.

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1

1   Introduction

This paper studies dynamic screening problems in which a principal hires an agent with limited liability to work over multiple periods. The agent's private cost of working is stochastic and independently and identically distributed (i.i.d.) across time. This variation of cost over time is common in reality. For example, Uber drivers' private cost of working may be relatively high when they have family obligations or other duties, or when they simply feel inclined to rest at home; conversely, it may be lower when they are free of other duties and feel energetic and eager to drive. On the other hand, the principal always prefers the agent to work more—just as the Uber platform benefits from drivers taking more rides, regardless of drivers' private working costs.

Additionally, workers are often subject to a limited liability constraint, which implies that the payments from the principal to the agent must be nonnegative in all periods. In other words, the principal cannot require the agent to pay. This constraint is common to many real-world settings. The rationale behind it is twofold. First, it is the convention. Compensation schemes are often designed with this constraint in mind. Second, agents, such as drivers, are often liquidity-constrained and unable to afford upfront payments.This paper investigates what the optimal reward scheme is in a setting with these two key components: i.i.d. private costs and limited liability.

One possible approach is to treat each period independently since the costs are i.i.d. and the principal's payoff is additively separable across time. However, we show that the principal can achieve a better outcome by designing a dynamic mechanism that links the agent's performance across periods, using compensation from past work to incentivize future effort. Another possibility is to discipline the agent by exploiting the law of large numbers as the number of periods approaches infinity, as in the quota mechanism proposed by Jackson and Sonnenschein (2007). However, this method is not applicable in our setting, in which the number of periods is finite and can be as small as two periods. Arguably, the assumption that the number of periods is finite is a good description of many real-world situations.

Then what is the optimal mechanism in this setting? We show that the optimal contract involves backloaded payments and requires the agent to work in consecutive periods. Proposition 1 demonstrates that the optimal mechanism backloads all payments to the end. This strengthens the principal's ability to incentivize the agent, enabling the principal to induce effort dynamics that would not be sustainable under other compensation

2

schemes. More importantly, Theorem 1 establishes that the optimal mechanism prescribes that the agent to work consecutively once he begins. Specifically, the optimal contract takes one of two forms: a consecutive-working menu or an always-working mechanism. The consecutive-working menu allows the agent to choose the period in which to start working, after which he is incentivized to work in all subsequent periods. In other words, the agent uses this flexibility to delay the start of working if his costs are high in the early periods.

In contrast, the always-working mechanism requires that the agent start working in the first period and continue in all subsequent periods. Thus, the optimal mechanism either grants the agent flexibility on when to start working or restricts the starting period to the first. Theorem 2 identifies the conditions under which each mechanism is optimal, based on the principal's valuation of the agent's work relative to monetary transfer. When such a valuation is large, the always-working mechanism is optimal, whereas when it is small, the consecutive-working menu is optimal.

To the best of our knowledge, this is the first paper to investigate the optimality of contracts with a consecutive work structure, even though this structure is commonly observed in practice. For instance, ride share platforms like Uber and Lyft frequently offer bonuses to drivers for completing a specified number of consecutive rides, with the bonus backloaded and paid after the final trip. According to Uber's policy: "Complete the required number of trips without rejecting, unfulfilling a trip, or going offline between trips. Your extra earnings are added to the last trip receipt."[1] More generally, the principal-agent model this paper analyzes can also be applied to scenarios in which the principal seeks to incentivize the agent to take actions other than working, such as consuming products or services. The reward structure this paper identifies is common in such settings as well. For example, Starbucks offers a reward program that grants customers "stars" based on the number of consecutive days they purchase coffee, with stars awarded at the end of the streak. Similarly, the gaming industry offers a reward scheme of this kind. For instance, Microsoft's Xbox rewards players for logging in and completing tasks over consecutive weeks, with the rewards given at the end of the streak.

To understand the intuition behind the optimal mechanism, it is helpful to introduce a key concept in dynamic mechanism design: a deposit can be a powerful tool for the princi-

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[1]See this website for the exact term and condition.

pal to discipline the agent. Consider the following example. Suppose the principal charges the agent a $100 deposit on day 0 and promises to return it on day 2 if the agent follows the work instructions on day 1. Assume the agent's cost of working on day 1 is a draw from the uniform distribution over $[0, 10]$, and the work instruction requires the agent to work for sure on day 1 in exchange for $5 compensation. A risk-neutral agent would agree to this contract, as the expected payment equals the expected cost. However, if the agent's realized cost on day 1 turns out to be high (say, 10), he will still work because he fears losing the deposit. In this way, the principal can achieve the first-best outcome—requiring the agent to work for sure on day 1 while paying the agent only the expected cost. In contrast, when a deposit is infeasible due to limited liability, the agent would work only if his cost is below 5, leading to the principal's losing surplus and allowing the agent to earn an information rent.

With deposits, the principal can achieve the first-best outcome in every period. This result crucially hinges on the agent having "deep pockets" and is not viable with limited liability. Nevertheless, we show that the principal can still discipline the agent using a mechanism exploiting a similar logic. Specifically, the principal can backload all payments to the end, which play the role of the deposit. These promised yet postponed payments give the principal leverage over the agent: if the agent quits, he forfeits all these payments. Each deferred payment can be repeatedly utilized in every subsequent period to discipline the agent and extract surplus. Over time, as these postponed payments accumulate, the principal's ability to regulate the agent and extract more surplus grows, making the agent increasingly reluctant to quit. This is the key intuition behind why the optimal mechanism has a consecutive-working format.

The specific logic behind the consecutive-working feature of the optimal mechanism is as follows: under a mild distributional condition, for any given history path, the optimal promised payment for initiating work is large enough to incentivize the agent to work until the end, while allowing the principal to extract all surplus in periods after the agent has started working. More specifically, in any period where the agent has not yet started, the optimal mechanism employs a threshold rule: the agent with a current type below a specified cutoff is asked to start working and continue until the end. The principal then compensates the agent with the cutoff cost plus an information rent for the current period, and the average cost for each future period. Under the distributional condition, when the agent's type is at or below the cutoff, he is not only willing to participate in the current pe-

riod but also happy to follow the rule and work to the end for any future cost realizations.[2] Proposition 8 in the Appendix characterizes the necessary and sufficient distributional condition. Assumption 1 in the main text offers a simpler sufficient condition: the expected value of the cost must exceed the difference between the upper and lower bounds of the cost distribution. A further sufficient condition is that the support of the cost distribution is not too wide [3] Moreover, consecutive working not only allows the principal to extract full surplus in all subsequent periods for any given history path, but it also does not interfere with other history paths and, therefore, does not affect the principal's expected payoff from them (Proposition 4). Thus, it is optimal for the principal to assign consecutive working across all history paths once the agent has started.

In summary, the optimal mechanism gives the agent flexibility in deciding when to start working; however, once he begins, he is incentivized to work consecutively until the end. Notably, the optimal contract distorts the timing of when the agent starts working but not the decisions he makes after starting. Importantly, the optimal contract may induce the agent to delay starting for a long period, despite the costs being i.i.d.. We characterize the precise dynamics in section 4; they differ significantly from those in the absence of limited liability, where the agent is induced to start working immediately.

In section 6, we explore an extension in which the principal is able to offer stochastic mechanisms. We demonstrate in Proposition 6 that for every optimal deterministic mechanism, there exists a stochastic improvement. The reason for this is that the principal can optimally allocate the promised payments—determining her future leverage over the agent—across different realizations of the recommended action, thereby facilitating better regulation in various scenarios. That said, focusing on deterministic mechanisms remains meaningful, as stochastic mechanisms may have credibility issues and can be difficult to implement in practice. Nonetheless, the study of stochastic mechanisms is useful because it shows that the main forces under deterministic mechanisms remain true and can be exploited more flexibly and effectively under stochastic contracts.

This paper makes two main contributions. First, to the best of our knowledge, it is the first work in the dynamic mechanism design literature that provides an explanation for the

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[2]While the agent may incur a negative ex-post payoff and regret at the end, it is in his best interest to participate ex-ante and in every interim period.

[3]One might think the principal could benefit by allowing the agent to shirk in periods with very high costs after he has started, as a way to incentivize an earlier start. However, under the conditions we identify, the optimal contract doesn't need to do so.

widely used practice of offering schemes that reward agents based on the number of consecutive periods they work (or consume). Second, the analysis shows how to solve a novel dynamic mechanism design problem with limited liability constraints and a continuum of types. We believe this approach could prove useful in other problems with limited liability.

The rest of the paper is organized as follows. The next section discusses the related literature. Section 3 introduces the model, while Section 4 sets up and simplifies the dynamic optimization problem. Section 5 solves the optimization problem and presents the main results. Finally, Section 6 explores possible extensions, and Section 7 concludes the paper. All proofs are in the appendix.

2 Related Literature

This paper contributes to the large literature on revenue-maximizing dynamic mechanism design (screening). See, for example, Baron and Besanko (1984), Besanko (1985), Courty and Li (2000), Battaglini (2005), Eső and Szentes (2007), Boleslavsky and Said (2012), Pavan et al. (2014), Bergemann and Välimäki (2019). Similar to these studies, this paper seeks to find the mechanmism that maximizes the principal's expected payoff given the agent's time-evolving private information.

This paper differs from the existing literature in two key aspects. First, we consider a setting in which the agent has limited liability so that the payments from the principal must be nonnegative at all times. While this constraint is common in labor contracts, it has not been studied in the dynamic mechanism design literature. Second, most of the existing work assumes imperfectly correlated private information and focuses on how the persistence of private types influences the information rents the principal must pay to incentivize the agent to reveal his type (as captured by the impulse function in Pavan et al. (2014)) and the corresponding optimal mechanisms. In contrast, the independence of costs in our setting renders these questions trivial without the limited liability constraint. Specifically, the agent's current private information does not influence future information rents, which allows the optimal contract to achieve the first-best outcome in every period. Instead, this paper investigates how the principal should optimally incentivize the agent when payments must be nonnegative in all periods.

More closely related to our work is the literature on dynamic screening whose models

include the constraint that the agent's stage payoff must remain nonnegative at all times, including Krishna et al. (2013), Krasikov and Lamba (2021), and Krähmer and Strausz (2024). This nonnegativity constraint on the stage payoff is stricter than our nonnegative payment constraint, since it requires the payment minus the cost to be nonnegative at all times. In other words, their lower bound on payments is endogenous and strictly higher than the one in this paper (which is zero). Thus, our paper occupies a middle ground between these two strands of literature. In those papers, assuming nonnegative flow payoffs is reasonable, since the agent's task is to produce goods, and a cash-strapped agent may require payments to cover production input costs. In contrast, our assumption of nonnegative payments more accurately describes the labor market in which the agent can bear the cost of working in each period and the principal can defer compensations but cannot demand a deposit from the agent.

The central problem in the dynamic screening literature is how to incentivize the agent to reveal his evolving private information over time, with the main focus traditionally placed on the interim incentive compatibility (IC) constraints. . In contrast, significantly less attention has been given to the interim individual rationality (IR) constraints, which must be satisfied in all periods. This is because much of the literature assumes that the agent has deep pockets and can afford to pay a substantial upfront deposit,[4] which would make it unprofitable for the agent to quit and forfeit the deposit mid-game. Under this assumption, the interim IR constraints are trivially satisfied, and the principal needs to enforce only the ex-ante IR constraint. In contrast, in our setting, the limited liability constraint prevents the principal from requiring a deposit. As a result, the interim IR constraints play an important role, and addressing these constraints for any given history introduces the main technical difficulty. Efficient dynamic mechanisms, such as the dynamic pivot mechanism in Bergemann and Välimäki (2010), also account for interim IR constraints, but not for the limited liability constraints that are central to our analysis.

From this perspective, our paper is also related to the literature on dynamic screening with ex-post IR constraints. For instance, Krähmer and Strausz (2015) considers the case in which the consumer has withdrawal rights requiring that the seller ensure that the buyer's ex-post payoff be nonnegative. It finds that the static contract is optimal under certain conditions. Bergemann et al. (2020) further establishes necessary and sufficient conditions for the optimality of static contracts for a single buyer with two types. Additionally, Mirrokni

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[4]Also referred to as "posting a bond" in the literature.

et al. (2020) and Ashlagi et al. (2023) impose ex-post IR constraints in dynamic auction and multiple-product selling contexts, respectively. In our setting, the interim IR constraints play an important role, but the ex-post IR constraints may not be satisfied, since they are not imposed as a requirement. It is reasonable for the ex-post IR constraint to be violated in our context, since the agent in the labor market cannot return or withdraw the work they have already completed, unlike the situation in goods markets.

Battaglini (2005) investigates the optimal contract in a setting of repeated bilateral trade, where the buyer's type follows a Markov process with two types. He finds that as soon as the buyer's type becomes high, the allocation becomes efficient. While this finding may appear similar to our results, they are fundamentally different. When types are independent and the agent possesses no private information at the outset, the result in Battaglini (2005) indicates that the optimal contract achieves efficiency in all periods (see also Baron and Besanko (1984) and Pavan et al. (2014)). In contrast, with limited liability, the optimal contract may induce the agent to wait a very long time before he starts working, which is inefficient. Furthermore, once the agent starts working, he continues until the end, irrespective of the time at which he started and of his type when starting.[5]

This paper is also related to the literature on financial contracting with limited liability. See, for example, Clementi and Hopenhayn (2006), DeMarzo and Sannikov (2006), and Biais et al. (2007). The key difference is that these papers study dynamic moral hazard problems while we study dynamic adverse selection problems. Board (2011) examines the dynamic optimal contract for the holdup problem in a context in which the agent has no private information. The optimal contract in this setting shares a similarity with that in ours: the agent earns rent only in the first trade (or work), after which the allocation becomes efficient.[6]

Backloading payments or rents is a common feature of the optimal contracts in this paper as well as in the literature on dynamic screening with nonnegative stage payoffs, dynamic moral hazard, and dynamic holdup. This practice arises from the assumptions that the agent is risk-neutral and operates under limited liability.

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[5]Our analysis also allows for a continuum of types, whereas Battaglini (2005) assumes only two types. Characterizing where the IR constraints bind is significantly more demanding with a continuum of types.

[6]Since the agent does not possess any private information, the rent in Board (2011) is not an information rent; rather, it is compensation provided to prevent the agent from holding up the principal.

Additionally, this paper relates to Halac et al. (2016), which studies dynamic contracts in experimentation. Its setting involves both adverse selection and moral hazard, leading to different results. Specifically, its optimal contract is a menu of stopping times, while ours is a menu of starting times. The related literature on revenue management is discussed in Gershkov and Moldovanu (2014). A key distinction between that work and this paper is that the former focuses on the optimal allocation of fixed inventory to a dynamic population (i.e., randomly arriving customers), whereas our setting involves a single fixed agent and no inventory.

Finally, this paper connects to the literature on linking mechanisms, such as in Jackson and Sonnenschein (2007), Frankel (2014, 2016), and Ball and Kattwinkel (2024). In these studies, the optimal mechanism is a quota mechanism that links the agent's actions across periods: the agent has the flexibility to choose any action in each period, but the total number of actions taken over all periods must satisfy specified quotas. In contrast, our optimal mechanism features consecutive linking, where the agent has the flexibility to choose the starting period, after which he is incentivized to work continuously to the end. All these papers as well as ours demonstrate the advantages of dynamic mechanisms, even in i.i.d. environments.[7]

3 Model

A principal (she) incentivizes an agent (he) to work. The principal benefits from the agent's work time while bearing the cost of paying him to work. We impose a limited liability constraint, which requires that the payment from the principal must be nonnegative at all times. The agent's payoff is the total payment he receives minus the total cost of working. The agent's cost of working in each period is his private information, while his action of working or shirking is publicly observable. The principal's goal is to design a mechanism to maximize her expected payoff, given this adverse selection problem with dynamically disclosed private information and the limited liability constraint.

Let N be the total number of periods. Denote $\{1, \cdots, N\}$ as $\mathcal{N}$. The agent's cost in each period, $\tilde{\theta}_t$, is an independent and identically distributed (i.i.d.) random variable

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[7]While these studies require either a large or infinite number of periods to apply the law of large numbers or achieve stationarity, our results highlight the benefits for any finite number of periods.

with continuous probability density function f and cumulative distribution function F.[8] We assume that $f(\theta) > 0$ for all $\theta \in \Theta \equiv [\underline{\theta}, \bar{\theta}]$. We use a tilde to differentiate between random variables and their realizations; for example, θ_t represents the realization of the random variable $\tilde{\theta}_t$. At the beginning of each period, $\tilde{\theta}_t$ is realized and it is the agent's private information. Let $h_t \equiv (\theta_1, \theta_2, \cdots, \theta_t) \in \Theta^t$ denote the history of realized costs up to period t.[9] Let $\boldsymbol{\theta} \equiv (\tilde{\theta}_1, \tilde{\theta}_2, \cdots, \tilde{\theta}_N)$ denote the random vector of the agent's complete type profile. We use $E_{\boldsymbol{\theta}}(\cdot)$ for the expectation over $\boldsymbol{\theta}$ and $E_{\boldsymbol{\theta}}^t(\cdot)$ for the expectation over the future costs $(\tilde{\theta}_{t+1}, \tilde{\theta}_{t+2}, \cdots, \tilde{\theta}_N)$ after h_t is realized.

The principal commits to a mechanism at the beginning of the game. In each period, the agent first observes his cost θ_t and then reports a message to the mechanism, which specifies an action recommendation and a corresponding payment for that period. The agent then takes a publically observable action. By the Revelation Principle, we restrict our attention to direct mechanisms where the agent is truthful and obedient on the equilibrium path.[10] Formally, a direct mechanism is a sequence of action and payment functions $\{x_t, p_t\}_{t=1}^N$, where $x_t : \Theta^t \to \{0, 1\}$ is the action rule in period t that maps h_t to the binary action space, with 1 and 0 representing work and shirk, respectively.[11] The payment rule $p_t : \Theta^t \to \mathbb{R}_+$ maps h_t to a nonnegative payment transferred to the agent in period t.[12] In the truthful and obedient equilibrium of a direct mechanism, the agent's expected payoff U is defined as

$$U = E_{\boldsymbol{\theta}} \left[\sum_{t=1}^N \left(p_t - \tilde{\theta}_t \cdot x_t \right) \right].$$

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[8]We assume i.i.d. costs for technical simplicity. Meanwhile, the result under this assumption carries some meaningful information. If the agent's costs were persistent, consecutive working benefits the principal would be more obvious, as a low cost today implies a higher chance of a low cost tomorrow. Nonetheless, we show that even with independent costs, the optimal mechanism still features consecutive working.

[9]h_0 denotes $\emptyset$.

[10]In equilibrium, the agent truthfully reports his type and follows the action recommendation. For obedience, we assume that the principal will end the game if the agent doesn't follow the recommendation.

[11]In section 6, we consider the case where the action space is $[0, 1]$, and x_t represents the probability of working.

[12]Both the action rule and the payment rule do not depend on the agent's historical actions for the following reasons. First, the obedience constraints ensure that the agent will follow the prescribed action in each period. Second, the past actions do not affect the agent's future type realizations. Thus, even if we formulate the mechanism as a function of the agent's historical actions, the past actions should be functions of the historical types themselves.

The principal's expected payoff V is defined as

$$V = E_{\boldsymbol{\theta}} \left[\sum_{t=1}^{N} \left(\alpha \cdot x_t - p_t \right) \right].$$

We assume $\alpha \geq \bar{\theta}$, making work always efficient.[13] The principal's objective is to maximize her expected payoff over feasible mechanisms that ensure the agent participates, truthfully reports his costs and follows the action recommendations in each period, without making any payments to the principal.

4 Dynamic Optimization

4.1 Set up

In our setting, a mechanism $\{x_t, p_t\}_{t=1}^{N}$ is implementable if it satisfies periodic incentive compatibility (IC-t), individual rationality (IR-t), obedience and limited liability (LL-t) constraints. Since we assume that the principal will end the game if the agent doesn't follow the recommendation, obedience is implied by the participation constraint. Let $u_t(h_{t-1}, \theta_t, \hat{\theta}_t)$ denote the agent's expected payoff calculated in period t, assuming truthful future reports and given the past reports h_{t-1}, current cost realization θ_t and current report $\hat{\theta}_t$, i.e.,

$$u_t(h_{t-1}, \theta_t, \hat{\theta}_t) \equiv p_t(h_{t-1}, \hat{\theta}_t) - \theta_t \cdot x_t(h_{t-1}, \hat{\theta}_t) +$$
$$E_{\boldsymbol{\theta}}^t \left[\sum_{i=t+1}^{N} p_i(h_{t-1}, \hat{\theta}_t, \{\tilde{\theta}_k\}_{k=t+1}^{i}) - \tilde{\theta}_i \cdot x_i(h_{t-1}, \hat{\theta}_t, \{\tilde{\theta}_k\}_{k=t+1}^{i}) \right].$$

To simpily notation, we use $u_t(h_{t-1}, \theta_t)$ to denote $u_t(h_{t-1}, \theta_t, \theta_t)$. Definition 1 formalizes the implementability conditions.

Definition 1. *A mechanism $\{x_t, p_t\}_{t=1}^{N}$ is implementable if the following conditions hold:*

$$\theta_t \in \arg\max_{\hat{\theta}_t \in \Theta} u_t(h_{t-1}, \theta_t, \hat{\theta}_t) \tag{IC-t}$$

$$u_t(h_{t-1}, \theta_t) \geq 0 \tag{IR-t}$$

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[13]This is similar to the assumption in the classic monopoly seller model, where the seller's cost is assumed to be lower than the buyer's valuation, making trade always efficient.

$$p_t(h_{t-1}, \theta_t) \geq 0 \qquad\qquad \text{(LL-t)}$$

for all t, $\theta_t \in \Theta$ and $h_{t-1} \in \Theta^{t-1}$.

The dynamic mechanism design literature typically requires the agent to be on-path truthful, meaning it is optimal for the agent to report truthfully given truthful past reports. In our i.i.d. type setting, on-path truthfulness implies off-path truthfulness, indicating that it remains optimal for the agent to be truthful even if he has lied in the past. As discussed in Pavan et al. (2014), this is because the agent's expected payoff in any period t depends only on his current true type and past reports, but not on his past true types. Consequently, we can impose a stronger IC constraint that requires the agent to be truthful across all histories.

In the literature without limited liability, the principal can require the agent to post a sufficiently large bond in the initial period, making it unprofitable for the agent to quit and forfeit the bond mid-game. Consequently, the mechanism designer only needs to ensure the IR constraint is satisfied in period 1[14]. However, in our setting, limited liability prevents the principal from charging the agent a deposit. Therefore, the principal must ensure that the agent will not quit in any period by making the periodic IR constraints (IR-t) hold for all t.

The principal's optimization problem can be written as finding implementable action and payment rules $\{x_t, p_t\}_{t=1}^N$ that maximize the principal's payoff V subject to the IC-t, IR-t, and LL-t constraints for all t. Formally, the principal's problem can be written as the following:

$$\max_{\{x_t, p_t\}_{t=1}^N} E_{\boldsymbol{\theta}} \left[\sum_{t=1}^N \alpha \cdot x_t - p_t \right]$$

$$\text{s.t.} \quad \text{IC-t, IR-t, LL-t} \quad \forall\, t.$$

Proposition 1. *It is without loss of generality to pay the agent only at the very end, i.e. $p_t = 0$ for all $t < N$.*

Proposition 1 states that paying the agent only at the very end is without loss of generality. This is because, for any implementable mechanism that pays the agent in interim periods, the principal can backload all payments to the last period while maintaining the mechanism's implementability. This can be demonstrated as follows: First, backloading does not change the agent's incentive to be truthful in each period. The total current and

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[14]Or in period 0 before the realization of any private information, depending on the timing of the contract.

expected future payments for each report remain the same; they are merely postponed. Although the agent's expected payoff in each period increases due to accumulated delayed payments from the past, this does not impact the agent's current incentive as it depends only on past reports. Therefore, the expected payoff difference between truth-telling and lying remains the same. Second, backloading the payments makes the agent's participation constraints easier to satisfy. When the agent chooses to participate or quit the mechanism in each period, while quitting always yields a zero payoff, the expected payoff from participating increases due to the payments that would have been made in previous periods being deferred to the end period.

Given that the agent's payoff is quasi-linear, it is a standard result in the dynamic mechanism design literature that backloading payments is without loss of generality. However, the presence of limtied liability constraint makes it not just a convenient setup but a necessary feature of the optimal mechanism. As discussed in Section 5, a key aspect of our optimal mechanism is that it progressively strengthens the principal's ability to discipline the agent as time passes and the accrued payment increases. This accumulated payment provides the principal with leverage: if the agent quits, they forfeit all promised payments. Consequently, delaying payments allows the principal to extract surplus from the agent more effectively. This necessity of backloading payments is also observed in other dynamic screening models with nonnegative stage payoff constraints, such as Krishna et al. (2013), Ashlagi et al. (2023), and Krähmer and Strausz (2024), as well as in dynamic moral hazard models with limited liability constraints, for example, Clementi and Hopenhayn (2006), DeMarzo and Sannikov (2006), and Biais et al. (2007).

Proposition 1 allows us to simplify the principal's optimization problem by focusing on pay-at-the-end mechanisms, denoted as $(\{x_t\}_{t=1}^N, p_N)$, where $p_N : \Theta^N \to \mathbb{R}_+$. Then, the expression for $u_t(h_{t-1}, \theta_t)$ becomes:

$$u_t(h_{t-1}, \theta_t) = -\theta_t \cdot x_t(h_{t-1}, \theta_t) + E_{\boldsymbol{\theta}}^t \left[p_N(h_{t-1}, \theta_t, \{\tilde{\theta}_k\}_{k=t+1}^N) - \sum_{i=t+1}^N \tilde{\theta}_i \cdot x_i(h_{t-1}, \theta_t, \{\tilde{\theta}_k\}_{k=t+1}^i) \right].$$

Note that, in the pay-at-the-end mechanism, the limited liability constraints for $t < N$ are automatically satisfied by construction. Additionally, the IR constraint in the last period

(IR-N) implies the limited liability constraint for $t = N$. Specifically,

$$p_N(h_{N-1}, \theta_N) - \theta_N \cdot x_N(h_{N-1}, \theta_N) \geq 0 \tag{IR-N}$$

$$\Rightarrow p_N(h_{N-1}, \theta_N) \geq \theta_N \cdot x_N(h_{N-1}, \theta_N) \geq 0. \tag{LL-N}$$

Therefore, we can omit all the limited liability constraints, simplifying the principal's optimization problem to the following:

$$\max_{\{x_t\}_{t=1}^N, p_N} E_{\boldsymbol{\theta}} \left[\sum_{t=1}^N \alpha \cdot x_t - p_N \right]$$

$$\text{s.t.} \quad \text{IC-t, IR-t} \quad \forall\, t.$$

4.2 Optimization over thresholds

Next, we will demonstrate that the principal's problem is equivalent to optimizing over threshold action rules. First, we will use Lemma 1 to show that incentive-compatible action rules must be threshold rules. Then, we will apply Proposition 2 to illustrate that, as in the static screening model, the payment rule p_N can be determined by the action rules. These two results allow us to reformulate the principal's problem as an optimization over threshold rules, thereby reducing the dimensionality of the optimization problem from functional spaces to real spaces.

Lemma 1. *A mechanism is incentive-compatible if and only if, for all t and h_{t-1},*

(i) $x_t(h_{t-1}, \theta_t)$ *is non-increasing in θ_t.*

(ii) $u_t(h_{t-1}, \theta_t) = u_t(h_{t-1}, \bar{\theta}) + \int_{\theta_t}^{\bar{\theta}} x_t(h_{t-1}, \theta_t) d\theta_t.$

Corollary 1. $x_t(h_{t-1}, \theta_t)$ *is a threshold rule:* $x_t(h_{t-1}, \theta_t) = 1$ *when* $\theta_t \leq c_t(h_{t-1})$, *and* $x_t(h_{t-1}, \theta_t) = 0$ *when* $\theta_t > c_t(h_{t-1})$.

Lemma 1 characterizes the necessary and sufficient conditions for a mechanism to be incentive-compatible. When t and h_{t-1} are fixed, $x_t(h_{t-1}, \theta_1)$ can be viewed as a function of θ_t only. In this scenario, the conditions for a mechanism to be incentive-compatible are

the same as in the static screening model. The first condition requires that the action rule x_t is non-increasing in the period t cost θ_t. This is intuitive, as the agent's payoff satisfies the single-crossing property. Combining this with the fact that the action space is discrete, i.e. $\{0, 1\}$, we derive Corollary 1, which states that the action rules must be threshold rules. We use $c_t(h_{t-1})$ to denote the threshold in period t given history h_{t-1}. The second condition states that the agent's period t expected payoff u_t is pinned down by the action rule x_t and the expected utility of the highest type $\bar{\theta}$, i.e. $u_t(h_{t-1}, \bar{\theta})$. This is a standard result derived from the envelope theorem in Milgrom and Segal (2002).

Lemma 2. *(Payoff Equivalence) For any incentive compatible action rules $\{x_t\}_{t=1}^{N}$, u_t is a function of $\{x_t\}_{t=1}^{N}$ and $u_1(\bar{\theta})$. Denote $\hat{u}_t(h_t) = \int_{\theta_1}^{\bar{\theta}} x_1(\theta_1) d\theta_1 + \sum_{i=1}^{t-1} x_i(h_i)\theta_i + \sum_{i=2}^{t} \left(\int_{\theta_i}^{\bar{\theta}} x_i(h_{i-1}, \theta_i) d\theta_i - \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_i) x_i(h_{i-1}, \theta_i) d\theta_i \right)$. Then the following hold for any t and h_t:*

$$u_t(h_t) = u_1(\bar{\theta}) + \hat{u}_t(h_t).$$

Lemma 2 shows that the agent's expected utility in each period t, u_t, is pinned down by the action rules up to that period, $\{x_i\}_{i=1}^{t}$, and the expected utility of the highest type in the first period, $u_1(\bar{\theta})$. We decompose the agent's expected utility into two components: $u_t = \hat{u}_t + u_1(\bar{\theta})$, where $\hat{u}_t$ is a function of the action rules. In Lemma 1, we show that u_t is determined by x_t and $u_t(h_{t-1}, \bar{\theta})$, where $u_t(h_{t-1}, \bar{\theta})$ may vary for each t and h_{t-1}. However, in Lemma 2, we use a single, unique $u_1(\bar{\theta})$. This is because each $u_t(h_{t-1}, \bar{\theta})$ is also a function of the action rules and $u_1(\bar{\theta})$. To illustrate this, consider the case when $t = 2$ and the realized cost in the first period is θ_1. Then we have,

$$u_2(\theta_1, \theta_2) = u_2(\theta_1, \bar{\theta}) + \int_{\theta_2}^{\bar{\theta}} x_2(\theta_1, \theta_2) d\theta_2.$$

Meanwhile, from the agent's perspective in period 1, after the realization of θ_1 and before the realization of θ_2, his expected utility $u_1(\theta_1)$ equals the expectation of the expected

15

payoff in period 2 minus the cost of working in period 1, i.e.

$$
\begin{aligned}
u_1(\theta_1) &= E_{\tilde{\theta}_2}\left[u_2(\theta_1, \tilde{\theta}_2)\right] - \theta_1 \cdot x_1(\theta_1) \\
&= \int_{\underline{\theta}}^{\bar{\theta}} \left(u_2(\theta_1, \bar{\theta}) + \int_{\tilde{\theta}_2}^{\bar{\theta}} x_2(\theta_1, \theta_2) d\theta_2 \right) dF(\theta_2) - \theta_1 \cdot x_1(\theta_1) \\
&= u_2(\theta_1, \bar{\theta}) + \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_2) x_2(\theta_1, \theta_2) d\theta_2 - \theta_1 \cdot x_1(\theta_1).
\end{aligned}
$$

Combining this with the fact that $u_1(\theta_1) = u_1(\bar{\theta}) + \int_{\theta_1}^{\bar{\theta}} x_1(\theta_1) d\theta_1$, we can express $u_2(\theta_1, \bar{\theta})$ as a function of the action rules and $u_1(\bar{\theta})$. Similarly, $u_3(h_2, \bar{\theta})$ is a function of the action rules and $u_2(h_1, \bar{\theta})$, which itself is a function of the action rules and $u_1(\bar{\theta})$. This pattern continues, allowing us to express $u_t(h_{t-1}, \bar{\theta})$ as a function of the action rules and $u_1(\bar{\theta})$ for all t and h_{t-1}.

Lemma 3. *For an incentive-compatible mechanism to be implementable, $u_1(\bar{\theta})$ must satisfy:*

$$
u_1(\bar{\theta}) + \hat{u}_t(h_t) \geq 0
$$

for all t and h_t. In the optimal mechanism, the following holds:

$$
u_1^*(\bar{\theta}) = \inf\{u_1(\bar{\theta}) : u_1(\bar{\theta}) + \hat{u}_t(h_{t-1}, \bar{\theta}) \geq 0, \forall\ h_{t-1} \in \Theta^{t-1}, t \leq N-1\}.
$$

Lemma 3 characterizes the optimal choice of $u_1(\bar{\theta})$ as the smallest value that ensures the mechanism is implementable. Unlike in the static screening model or dynamic screening model without limited liability (see Pavan et al. (2014)), where the principal can set $u(\bar{\theta})$ to the lowest possible value of 0, in our setting, the principal must choose $u_1(\bar{\theta})$ to guarantee the periodic IR constraints hold everywhere. Lemma 2 shows that $u_t = u_1(\bar{\theta}) + \hat{u}_t$. Therefore, the smallest $u_1(\bar{\theta})$ that ensures the mechanism is implementable is the infimum of $u_1(\bar{\theta})$ such that $u_1(\bar{\theta}) + \hat{u}_t(h_t) \geq 0$ for all t and h_t. Any value smaller than this would violate some periodic IR constraint, rendering the mechanism not implementable. Conversely, any value larger than this would be suboptimal for the principal, as it would reduce the mechanism's profitability.

Proposition 2. *(Revenue Equivalence) For any incentive-compatible action rules $\{x_t\}_{t=1}^N$, the payment rule p_N that makes the mechanism implementable and maximizes the principal's expected payoff is pinned down by the action rules as follows:*

$$p(h_N) = \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_1)x_1(\theta_1)d\theta_1 + \sum_{i=1}^N \left(x_i(h_i)\theta_i + \int_{\theta_i}^{\bar{\theta}} x_i(h_{i-1},\theta_i)d\theta_i - \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_i)x_i(h_{i-1},\theta_i)d\theta_i \right) + u_1^*(\bar{\theta}).$$

Where $u_1^*(\bar{\theta})$ is also determined by the action rules: $u_1^*(\bar{\theta}) = \inf\{u_1(\bar{\theta}) : u_1(\bar{\theta}) + \hat{u}_t(h_{t-1},\bar{\theta}) \geq 0, \forall\, h_{t-1} \in \Theta^{t-1}, t \leq N-1\}.$

Proposition 2 provides the formula by which the payment rule p_N is determined by the action rules. In Lemma 3, we give the optimal choice of $u_1(\bar{\theta})$, based on which the agent's expected utility in the last period can be expressed as:

$$u_N(h_N) = \hat{u}_N(h_N) + u_1^*(\bar{\theta}).$$

Combining this with the fact that u_N also equals the final payment p_N minus the cost of working in the last period, i.e. $u_N(h_N) = p_N(h_N) - \theta_N \cdot x_N(h_N)$, we can formulate p_N as a function of the action rules and $u_1^*(\bar{\theta})$, which itself is determined by the action rules.

The logic behind the lemmas and the proposition in this section is as follows. Lemma 2 and 3 serve as intermediate steps to establish Proposition 2. Specifically, we use Lemma 2 to show that the agent's expected payoff in each period is pinned down by the action rules and by the expected payoff of the highest type in the first period, $u_1(\bar{\theta})$. Lemma 3 then characterizes the optimal choice of $u_1(\bar{\theta})$, ensuring the mechanism is implementable and maximizes the principal's expected payoff. Finally, Proposition 2 provides the formula by which the payment rule p_N is determined by the action rules and the optimally chosen $u_1^*(\bar{\theta})$, which also depends on the action rules.

So far, we have shown that the principal's optimization problem can be formulated as an optimization over action rules which are threshold rules. This simplifies the original problem from an optimization over functional spaces to an optimization over real spaces. However, the challenge remains that there are uncountably many histories given the continuous type space. In the next section, we will address this by showing that the agent's working status history is a sufficient statistic for the cost history, thereby reducing the dimensionality of the principal's optimization problem.

4.3 Reduce dimensionality

To make the principal's optimization problem more tractable, Proposition 3 demonstrates that the agent's working status history serves as a sufficient statistic for the cost history in determining the future thresholds. This reduction in dimensionality shifts the optimization problem from addressing uncountably many thresholds to optimizing over finitely many thresholds. Let w_t denote the working status history up to period t, i.e. $w_t \equiv (x_1, x_2, \cdots, x_t) \in \{0, 1\}^t$. We will show that the threshold c_t depends on h_{t-1} only through w_{t-1}.

Proposition 3. *For any implementable mechanism, the working status history w_t is a sufficient statistic for the cost history h_t in determining the future thresholds. This means that the thresholds $c_t(h_{t-1})$ can be written as $c_t(w_{t-1})$.*

The intuition behind this proposition is as follows. First, the principal's payoff from each period is additively separable. Second, since costs are i.i.d., historical costs do not provide any information about future costs. Because of these two reasons, one might think that the future threshold selection is independent of past costs. However, there is one (and only one) link between them: the accumulated promised yet postponed payment that the principal owes the agent for working in the past.[15] It affects the future threshold selection as it provides the principal with negotiation power to discipline the agent. Since the accumulated due payments depend only on the working status history given the cutoff rules, it suffices to condition the future thresholds on the working status history only.[16]

Technically speaking, future thresholds interact with past cost history through $u_1^*(\bar{\theta})$, which is the smallest $u_1(\bar{\theta})$ that makes all the periodic IR constraints hold. Specifically, past cost history determines the accumulated promised payment, which in turn determines the range of future thresholds that satisfy the IR constraints. The future thresholds after the same working status history appear symmetrically in the objective function, implying that their optimal values should be the same.

Another angle to consider this result is: given i.i.d. costs, the only reason the principal would want to differentiate future thresholds given the same working status history is to

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[15]This is the part of the final payment that depends on past cost realizations and past working status.

[16]If the action rules were not threshold rules, such as if the action space were continuous, x_t would still be a sufficient summary statistic of θ_t, although it might not simplify the problem as much if x_t took continuous values.

provide incentives for truth-telling in the past. However, this will not happen as the same promised payment for working adequately addresses the information rent needed for truth-telling. Thus, the principal does not need to use different continuation payoffs to provide incentives for truth-telling in the past.

Proposition 3 reduces the number of thresholds to a finite number. Specifically, it now equals the number of possible working status histories, which is 2^t for w_t. Therefore, we only need to optimize over $2^0 + 2^1 + \cdots + 2^{N-1} = 2^N - 1$ thresholds. For instance, if there are three periods, then we only need to optimally choose seven thresholds: $c_1, c_2(1), c_2(0), c_3(1,1), c_3(1,0), c_3(0,1), c_3(0,0)$ over the space Θ^7. Let $\mathbf{1}_n$ denote the row vector of n ones. The principal's optimization problem can then be rewritten as follows:

$$\max_{\{c_t(w_{t-1})\}_{t=1}^N} E_{\boldsymbol{\theta}} \left[\alpha \cdot w_N \mathbf{1}_N^\mathsf{T} - p_N(w_N) \right].$$

Alternatively, we can treat w_N as a random variable taking values from $\{0,1\}^N$ with the following probability mass function:

$$Pr(w_N) = \prod_{t=1}^N \left[x_t \cdot F(c_t(w_{t-1})) + (1 - x_t)\big(1 - F(c_t(w_{t-1}))\big) \right],$$

Then, the principal's optimization problem can be expressed as:

$$\max_{\{c_t(w_{t-1})\}_{t=1}^N} \sum_{w_N \in \{0,1\}^N} Pr(w_N) \cdot (\alpha \cdot w_N \mathbf{1}_N^\mathsf{T} - p_N(w_N))$$

$$= \max_{\{c_t(w_{t-1})\}_{t=1}^N} E_{w_N} \left[\alpha \cdot w_N \mathbf{1}_N^\mathsf{T} - p_N(w_N) \right].$$

According to Proposition 2, under threshold rules, the payment rule $p_N(w^N)$ becomes:

$$p_N(w^N) = \sum_{t=1}^N x_t \cdot c_t(w_{t-1}) - \sum_{t=2}^N \int_{\underline{\theta}}^{c_t(w_{t-1})} F(\theta_t) d\theta_t + u_1^*(\bar{\theta}). \tag{1}$$

$u_1^*(\bar{\theta})$ is the smallest value that satisfies all the IR constraints. Proposition 3 also reduces the number of IR constraints to a finite number, transforming $u_1^*(\bar{\theta})$ from an infimum function

to a minimum function:

$$u_1^*(\bar{\theta}) = - \inf_{h_{t-1} \in \Theta^{t-1}, t \in \mathcal{N}} \hat{u}_t(h_{t-1}, \bar{\theta})$$

$$= - \min_{w_{t-1} \in \{0,1\}^{t-1}, t \in \mathcal{N}} \left(\sum_{i=1}^{t-1} x_i \cdot c_i(w_{i-1}) - \sum_{i=2}^{t} \int_{\underline{\theta}}^{c_i(w_{i-1})} F(\theta_i) d\theta_i \right).$$

Therefore, the principal's optimization problem becomes:

$$\max_{\{c_t(w_{t-1})\}_{t=1}^N} E_{w_N} \left[\alpha \cdot w_N \mathbf{1}_N^\mathsf{T} - \sum_{t=1}^N x_t \cdot c_t(w_{t-1}) + \sum_{t=2}^N \int_{\underline{\theta}}^{c_t(w_{t-1})} F(\theta_t) d\theta_t + u_1^*(\bar{\theta}) \right]. \quad (2)$$

The incentive compatibility constraints are automatically satisfied by the threshold rules and the formulation of the payment rule $p(w^N)$ as stated in Proposition 2. In addition, the periodic participation constraints are satisfied by the selection of $u_1^*(\bar{\theta})$. Therefore, we can now omit these constraints from the optimization problem.

Proposition 3 reduces the dimensionality of the thresholds and transforms $u_1^*(\bar{\theta})$ from an infimum function to a minimum function. However, the optimization problem remains challenging due to the inclusion of the minimum function in the objective, turning it into a nested optimization problem. We will address this problem in the following section.

4.4 Simplifying the optimization problem

To tackle the nested optimization problem in (2), we will first solve the inner minization problem to find $u_1^*(\bar{\theta})$. Specifically, we show that one of the IR constraints must be binding and can therefore determine the value of $u_1^*(\bar{\theta})$. Additionally, we add constraints to ensure that other IR constraints are also satisfied. Consequently, we reformulate the nested optimization problem as a standard optimization problem with constraints.

Lemma 4 identifies a property of the optimal mechanism that facilitates the setup of Proposition 4 and serves as the foundation for the main result in section 5. Specifically, Lemma 4 states that the optimal mechanism either prescribes the agent to work in every period or sets the thresholds for starting to work, given that the agent has not worked before, as interior points. In other words, in the second case, the optimal mechanism does not let the agent work or shirk for sure if he has not engaged in work yet. Let $\mathbf{0}_n$ denote the row vector of n zeros. Then, $c_1, c_2(0), c_3(\mathbf{0}_2), \cdots, c_N(\mathbf{0}_{N-1})$ represent the thresholds

for working in periods 1 through N, given that the agent has consistently shirked. For simplicity, we denote $c_t(\mathbf{0}_{t-1})$ as $c_t(\mathbf{0})$. Let c_t^* represent the thresholds under the optimal mechanism. This property is formally stated as follows:

Lemma 4. *For $\alpha \geq \bar{\theta}$, there are only two possibilities for the optimal rules: either the optimal action rule is to assign the agent to work in every period, or the optimal thresholds satisfy $c_t^*(\mathbf{0}) \in (\underline{\theta}, \bar{\theta})$ for all $t \in \mathcal{N}$.*

Lemma 4 presents two key pieces of information. Firstly, it states that if the principal intends for the agent to work for all cost realizations in some period t, provided the agent has not worked previously, she should choose the first period. This is because the principal's expected payoff from setting $c_t(\mathbf{0})$ to $\bar{\theta}$ decreases over time. Once the agent starts working for all cost realizations, the principal can achieve a "relative" first best in all subsequent periods by asking him to work for sure and paying the average cost. [17] Thus, asking the agent to start working for all cost realizations in an earlier period provides the principal with more periods to extract surplus from the agent. For the same reason, if the principal asks the agent to work for sure in the first period, she should also ask him to keep working in every future period.

Secondly, Lemma 4 also indicates that $c_t^*(\mathbf{0})$ are strictly above $\underline{\theta}$ for all t. This is primarily because working is efficient when the cost is very small, say close to $\underline{\theta}$, given that $\alpha > \underline{\theta}$. If any threshold equals $\underline{\theta}$, implying that the agent always shirks in that period, then the principal can be strictly better off by inducing the agent to work at least a little in that period, regardless of whether this changes $u_1^*(\bar{\theta})$.

Proposition 4. *When $c_t^*(\mathbf{0}) \in (\underline{\theta}, \bar{\theta})$ for all $t \in \mathcal{N}$, the principal's optimization problem is equivalent to the following constrained optimization problem:*

$$\max_{\{c_t(w_{t-1})\}_{t=1}^N} E_{w_N} \left[\alpha \cdot \sum_{t=1}^N x_t - \sum_{t=1}^N x_t \cdot c_t(w_{t-1}) + \sum_{t=2}^N \int_{\underline{\theta}}^{c_t(w_{t-1})} F(\theta_t) d\theta_t - \sum_{t=2}^N \int_{\underline{\theta}}^{c_t(\mathbf{0})} F(\theta_t) d\theta_t \right]$$

$$s.t. \ \sum_{i=1}^{t-1} x_i \cdot c_i(w_{i-1}) - \sum_{i=2}^t \int_{\underline{\theta}}^{c_i(w_{i-1})} F(\theta_i) d\theta_i \geq -\sum_{t=2}^N \int_{\underline{\theta}}^{c_t(\mathbf{0})} F(\theta_t) d\theta_t, \ \forall \, t \in \mathcal{N}, w_t \in \{0,1\}^t.$$

$$(3)$$

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[17]The term "relative" is used here because the principal achieves the first best relative to the previous mechanism. Specifically. the principal still has to pay the agent an additional $u_1^*(\bar{\theta})$ amount, but conditional on this payment remaining the same, she can be seen as achieving the first best in every future period.

In Proposition 4, we discuss the case where the principal does not choose to ask the agent to work in every period. In this scenario, we demonstrate that the IR constraint for the work history of shirking in all N periods, i.e., $w_N = \mathbf{0}_N$, must be binding. Otherwise, we show that the principal could be strictly better off by increasing the threshold $c_N(\mathbf{0}_{N-1})$ while maintaining all the IR constraints. Thus, the following relationship must always hold:

$$u_1^*(\bar{\theta}) + \hat{u}_N(\mathbf{0}_N) = 0. \qquad \text{(IR constraint for } w_N = \mathbf{0}_N)$$

This implies that $u_1^*(\bar{\theta}) = -\hat{u}_N(\mathbf{0}_N) = \sum_{t=2}^{N} \int_{\underline{\theta}}^{c_t(\mathbf{0})} F(\theta_t)d\theta_t$. [18] This relationship allows us to replace the minimum function in the objective function with $-\sum_{t=2}^{N} \int_{\underline{\theta}}^{c_t(\mathbf{0})} F(\theta_t)d\theta_t$. Although we still need to add constraints to ensure that all other terms in the minimum function are no smaller than this, i.e. the IR constraints for other work histories should also hold.

5 Main Results

5.1 Properties of the optimal mechanism

We have identified in Lemma 4 that the optimal mechanism either assigns the agent to work for sure in every period or sets $c_t^*(\mathbf{0})$ as interior points. To determine the optimal mechanism when $c_t^*(\mathbf{0})$ are all interior, we solve the relaxed optimization problem in Theorem 1 by disregarding the constraints in (3). We then verify that the solution satisfies all constraints under appropriate Assumption.

First, the relaxed solution features consecutive working: once the agent starts working, the relaxed optimal action rule prescribes him to work for sure in every future period, i.e., $c_t(w_{t-1}) = \bar{\theta}$ for all $w_{t-1} \neq \mathbf{0}_{t-1}$ and $w_{t-1} \neq \emptyset$. This is because the choice of these thresholds does not affect the principal's expected payoff from other work histories, as these thresholds do not appear in $u_1^*(\bar{\theta})$. Since work is always efficient when $\alpha > \bar{\theta}$, the relaxed optimal choices of these thresholds should be $\bar{\theta}$.

Second, for the relaxed solution to be feasible, we need to ensure that the IR constraints for all work histories are satisfied. We can separate the IR constraints into two parts: those

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[18]Lemma 4 ensures that when the principal does not assign the agent to work in every period, $c_t(\mathbf{0})$ exists for all t as the working status history $\mathbf{0}_t$ is on-path for all t.

for consistent shirking histories, where $w_t = \mathbf{0}_t$ for all $t \in \mathcal{N}$, and those for other work histories. In the first category, the work history that generates the lowest agent's expected payoff is $w_N = \mathbf{0}_N$.[19] The IR constraint for this work history is automatically satisfied by construction, as we set $u_1^*(\bar{\theta})$ such that the IR constraint for work history $\mathbf{0}_N$ binds. In the second category, the IR constraints that are relatively harder to satisfy are those for work histories corresponding to the second period that the agent works, i.e., $(\mathbf{0}_t, \mathbf{1}_2)$. This is because these are the first periods in which the agent works for all cost realizations. If the agent is willing to participate even with cost $\bar{\theta}$ here, then he will be willing to work for sure in any future periods, as the final payment is fixed while the cost of working in past periods is considered sunk costs. In other words, if the IR constraint for work history $(\mathbf{0}_t, \mathbf{1}_2)$ holds, then the IR constraints for work histories $(\mathbf{0}_t, \mathbf{1}_m)$ will hold for all $m > 2$.[20]

Let us illustrate the relationship of the IR constraints using Figure 1. It displays all possible work histories as a tree structure, with each branch representing the agent's assigned action in each period given the work history and current cost realization. Note that we only need to ensure the IR constraint holds for the highest point on each branch, as the agent's expected payoff is minimized when the current cost equals $\bar{\theta}$, given any fixed work history.[21] The first category of consistent shirking histories comprises the branches above the blue box; they are satisfied as the IR constraint for work history $\mathbf{0}_N$ binds. All the remaining branches belong to the second category, among which the IR constraints for histories in the blue box are the most challenging to satisfy. This is because the IR constraint for each branch in the blue box implies the IR constraints for all the branches to the right of it. Therefore, the feasibility of the relaxed solution hinges on whether the IR constraints for the work histories in the blue box hold.

The IR constraints for work histories in the blue box, i.e., $(\mathbf{0}_t, \mathbf{1}_2)$, are the following:

$$c_{t+1}(\mathbf{0}) + \sum_{i=t+2}^{N} \int_{\underline{\theta}}^{c_i(\mathbf{0})} F(\theta_i) d\theta_i + E(\theta) - \bar{\theta} \geq 0.$$

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[19] One can verify that $u_t(\mathbf{0}_t)$ decreases in t. The intuition is that the agent has fewer periods to start working and obtain information rents as time passes.

[20] The IR constraint for work history $(\mathbf{0}_t, \mathbf{1}_1)$ is implied by the IR constraint for work history $\mathbf{0}_{t+1}$, as the agent needs to receive information rents to start working compared to shirking in period $t + 1$. Thus, these IR constraints hold given that the IR constraints in the first category are satisfied.

[21] For this reason, when we refer to the IR constraint for a certain branch, we mean the IR constraint for the highest point on this branch, i.e., when the last cost realization is $\bar{\theta}$.
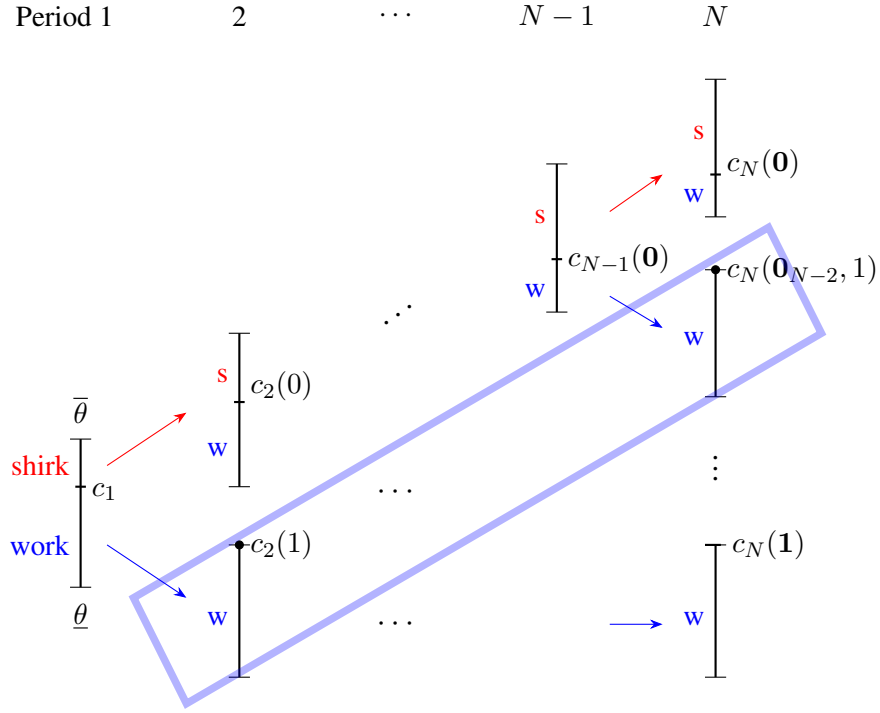
Figure 1: Work histories and relevant IR constraints

For all $0 \le t \le N - 2$. Therefore, to ensure all these IR constraints hold, we only need to guarantee that

$$\min_t \left(c_{t+1}(\mathbf{0}) + \sum_{i=t+2}^{N} \int_{\underline{\theta}}^{c_i(\mathbf{0})} F(\theta_i) d\theta_i \right) + E(\theta) - \bar{\theta} \ge 0.$$

When all $c_t(\mathbf{0})$ are interior points of the support, we have:

$$\min_t \left(c_{t+1}(\mathbf{0}) + \sum_{i=t+2}^{N} \int_{\underline{\theta}}^{c_i(\mathbf{0})} F(\theta_i) d\theta_i \right) \ge \min_t c_{t+1}(\mathbf{0}) > \underline{\theta}.$$

Assumption 1. $\underline{\theta} + E(\theta) \ge \bar{\theta}$.

Thus, a straightforward and simple sufficient condition for the feasibility of the relaxed solution is provided by Assumption 1. Note that this condition is not necessary for all IR constraints to hold. So, the conclusion in the following theorem (Theorem 1) may still hold even if Assumption 1 is violated. Finding the necessary and sufficient condition requires solving a system of equations that might not have analytical solutions. However, for any

24

given distribution, one can always solve this system numerically to determine the necessary and sufficient condition. In the appendix, we provide the necessary and sufficient condition for the feasibility of the relaxed solution in Proposition 8.

Assumption 1 is mild and satisfied by many distributions. For example, a stronger version of Assumption 1 is that $\underline{\theta} \geq \frac{\bar{\theta}}{2}$, which implies that the costs are not excessively heterogeneous—the lower bound of the support is at least half of the upper bound. This condition merely requires a relatively narrow range of cost variation, simplifying the verification process.

When the solution to the relaxed optimization problem is feasible and $c_1 \neq \bar{\theta}$, we refer to this optimal mechanism as a consecutive-working menu. Essentially, this mechanism allows for the possibility, with positive probability, that the agent may start working in any period $t \in \mathcal{N}$. Once the agent begins working, he is prescribed to continue working for all cost realizations in every subsequent period until the end. Definition 2 formalizes the action rules and the corresponding payment rule for a consecutive-working menu.

Definition 2. *A consecutive-working menu is a mechanism whose action rules have the following properties:*

(i) $c_t(\mathbf{0}) \in (\underline{\theta}, \bar{\theta})$ *for all $t \in \mathcal{N}$ (possible to start working in any period).*

(ii) $c_t(w_{t-1}) = \bar{\theta}$ *for all $w_{t-1} \notin \{\mathbf{0}_{t-1}, \emptyset\}$ (consecutive working).*

The final payment rule for work history $(\mathbf{0}_t, \mathbf{1}_{N-t})$ is:

$$p_N = c_{t+1}(\mathbf{0}) + (N - t - 1)E(\theta) + \sum_{i=t+2}^{N} \int_{\underline{\theta}}^{c_i(\mathbf{0})} F(\theta_i)d\theta_i$$

for all $t < N - 1$, $p_N = c_N(\mathbf{0})$ for $t = N - 1$, and $p_N = 0$ for $t = N$.

If the solution to the relaxed optimization problem results in $c_1 = \bar{\theta}$, then the optimal mechanism degenerates to an always-working mechanism.[22] In this case, the agent is required to work for sure in every period from the beginning to the end. Definition 3 formalizes the action rules and the corresponding payment rule for an always-working mechanism.

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[22] In this scenario, $c_t(\mathbf{0})$ do not exist as they are off-path. Here, $u_1^*(\bar{\theta})$ can equal to its lowest possible value, 0. It is then straightforward to verify that the solution to the relaxed problem is feasible.

Definition 3. *An always-working mechanism prescribes the agent to work in each period t for all possible realizations of θ_t and for all $t \in \mathcal{N}$. The final payment is $p_N = \bar{\theta} + (N - 1) \cdot E(\theta)$*

Theorem 1. *Under Assumption 1, the optimal mechanism is either a consecutive-working menu or an always-working mechanism.*

Theorem 1 summarizes the properties of the optimal mechanism. The always-working mechanism can be viewed as a special case of the consecutive-working menu, where the agent can only start working in the first period. The theorem shows that this is the only degenerate case of the consecutive-working menu that can be optimal. In other words, the optimal mechanism either allows the agent to start working in any of the N periods or restricts the starting period to the first period only.

Both possibilities of the optimal mechanism share the feature that once the agent begins working, he must continue to work consecutively until the end. The intuition behind this is that, under Assumption 1, the principal can achieve a "relative" first best by assigning the agent to work for sure and paying him only the average cost in each subsequent period once he starts working.[23] This is feasible because the payment promised to the agent for starting work, along with the corresponding $u_1^*(\bar{\theta})$, is large enough to incentivize the agent to work for sure in the period following the one he starts working. This amount is backloaded to the end, serving to discipline the agent to follow the prescribed action rules; failure to do so would result in the loss of this amount. This leverage can be utilized repeatedly in every subsequent period to extract all of the agent's surplus in these periods. In fact, the promised payment grows over time as the agent continues to work, enhancing the principal's ability to regulate the agent and increasing the agent's willingness to work in later periods.

From a technical perspective, if we ignore $u_1^*(\theta)$, then the part of the payment dependent on $c_t(w_{t-1})$ can be interpreted as the payment promised for working in period t given the work history w_{t-1}. This payment equals the expected cost of working in that period: under a threshold c_t, the agent's expected cost of working in period t is $F(c_t)c_t - \int_{\underline{\theta}}^{c_t} f(\theta)d\theta$, which exactly matches the part of payment dependent on c_t besides $u_1^*(\theta)$. Therefore, if we can set c_t to $\bar{\theta}$ without affecting $u_1^*(\theta)$, it can be interpreted as achieving the first best in that period, given that $\alpha > \bar{\theta}$. Assumption 1 guarantees that assigning the agent to work for

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[23]The term "relative" highlights that the principal must still pay the agent an additional $u_1^*(\bar{\theta})$.

sure in the period following the one he starts working won't affect $u_1^*(\theta)$, so assigning him to work for sure in subsequent periods won't affect $u_1^*(\theta)$ either. Thus, the principal can be seen as achieving the first best in these periods.

5.2 Characterizing the optimal mechanism

We have identified two possible forms of the optimal mechanism, yet their existence and the conditions determining which one is optimal remain unknown. In this section, we first establish the existence of the optimal mechanism and the properties of the principal's expected payoff under the optimal mechanism in Proposition 5. We then demonstrate in Theorem 2 that, under Assumption 2, there exist a threshold for α. Above this threshold, the optimal mechanism is the always-working mechanism, while below it, the optimal mechanism is a consecutive-working menu.

Proposition 5. *The optimal mechanism exists. V^* is increasing and continuous with respect to α, and it is differentiable almost everywhere.*

In Proposition 5, we first demonstrate the existence of the optimal mechanism. Let $V^*(\alpha)$ denote the principal's expected payoff under the optimal mechanism. We then establish that V^* is increasing and continuous with respect to α. This property facilitates the analysis in Theorem 2.

Assumption 2. $f(\theta)$ *is continuous,* $\frac{F(\theta)}{f(\theta)} + \theta$ *is increasing, and* $f(\bar{\theta}) \leq \frac{1}{(N-1)[\bar{\theta} - E(\theta)]}$.

Theorem 2. *Under Assumption 2, there exists $\hat{\alpha} > \bar{\theta}$ such that: the optimal mechanism is a consecutive-working menu when $\bar{\theta} \leq \alpha < \hat{\alpha}$, and the optimal mechanism is an always-working mechanism when $\alpha \geq \hat{\alpha}$.*

Let us illustrate the form of the optimal mechanism using Figure 2. Consider a restricted optimal mechanism that maximizes the principal's expected payoff among all mechanisms that do not require the agent to work for sure in every period (i.e., at least one on-path threshold c_i does not equal $\bar{\theta}$). We refer to this as the optimal non-always-working mechanism. The optimal mechanism is either the optimal non-always-working mechanism or the always-working mechanism. Let V^{naw}, V^{aw}, V^{cm} denote the principal's expected

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payoff under the optimal non-always-working mechanism, the always-working mechanism, and the optimal consecutive-working menu, respectively. Then, we have $V^* = \max\{V^{naw}, V^{aw}\}$.
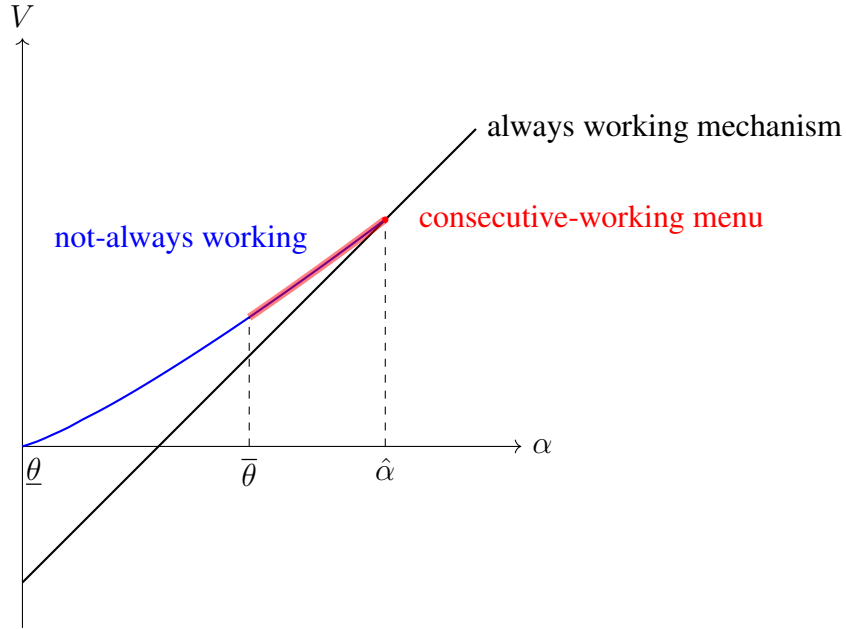

Figure 2: Comparison of the principal's expected payoff

When $\alpha < \bar{\theta}$, it is intuitive that assigning the agent to work for all cost realizations wouldn't be the optimal, so it must be that $V^* = V^{naw}$. When $\alpha \geq \bar{\theta}$, which is the focus of this paper, the optimal non-always-working mechanism, if it exists, must be the optimal consecutive-working menu according to Theorem 1, i.e., $V^{naw} = V^{cm}$. Lemma 5 shows that V^{cm} is strictly greater than V^{aw} when $\alpha = \bar{\theta}$. By continuity, there must exist a region to the right of $\bar{\theta}$ where the optimal mechanism is still the consecutive-working menu. Finally, we demonstrate that the optimal mechanism will eventually switch to the always-working mechanism when α is sufficiently large and will not switch back to the consecutive-working menu. This is primarily because the slope of V^{cm}, whenever it exits, is strictly less than the slope of V^{aw}, as shown in Lemma 9. Hence, we reach the main result of this paper, formalized in Theorem 2.

The following lemma provides an intermediate step in the proof of Theorem 2. It establishes the existence of a region to the right of $\bar{\theta}$ in which the optimal mechanism is characterized by the consecutive-working menu. However, if $V^{cm} = V^{aw}$ at $\alpha = \bar{\theta}$, the con-

clusion of Theorem 2 remains valid, but the region where the consecutive-working menu is optimal reduces to an empty set.

Lemma 5. *Under Assumption 2, V^{cm} exists and is strictly greater than V^{aw} when $\alpha = \bar{\theta}$.*

The main reason behind the result in Lemma 5 is as follows: when $\alpha = \bar{\theta}$, $\frac{\partial V}{\partial c_1}$ can be expressed as $M - \left(\frac{F(c_1)}{f(c_1)} + c_1 \right)$, where M denotes the sum of all terms that do not depend on c_1 and can therefore be treated as a constant. We show that M is greater than $\underline{\theta}$. When $c_1 = \underline{\theta}$, $\frac{F(c_1)}{f(c_1)} + c_1$ equals $\underline{\theta}$, and thus is below M. Assumption 2 states that $\frac{F(c_1)}{f(c_1)} + c_1$ is increasing in c_1 and guarantees that $\frac{F(c_1)}{f(c_1)} + c_1$ exceeds M when $c_1 = \bar{\theta}$. Consequently, the optimal c_1 must be an interior point of the support. Combined with the properties of the optimal mechanism established in Theorem 1, we conclude that the optimal mechanism is a consecutive-working menu when $\alpha = \bar{\theta}$.
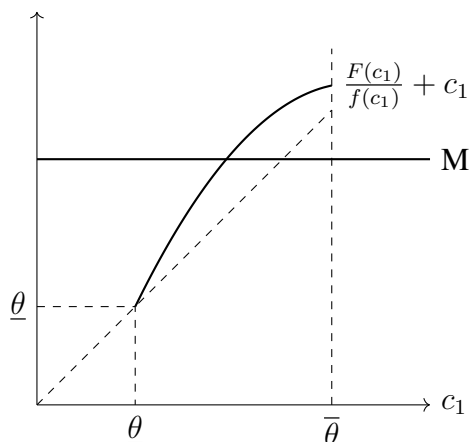

Figure 3: Relationship between M and $\frac{F(c_1)}{f(c_1)} + c_1$

Assumption 2 primarily guarantees that the optimal c_1 is an interior point of the support when $\alpha = \bar{\theta}$ by ensuring a single crossing property of $\frac{\partial V}{\partial c_1}$, as depicted in Figure 3. The second requirement of Assumption 2 can be interpreted as ensuring that the virtual cost $\theta + \frac{F(\theta)}{f(\theta)}$ is increasing. The virtual cost is the actual cost plus the information rent that the principal must provide to lower types for inducing type θ to work.[24] The assumption

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[24] This requirement is generally weaker than the increasing hazard rate condition, particularly for distributions with decreasing density.

$f(\bar{\theta}) \leq \frac{1}{(N-1)[\bar{\theta}-E(\theta)]}$ is a technical condition ensuring $\frac{\partial V}{\partial c_1}|_{c_1=\bar{\theta}} < 0$.[25] It can be interpreted as the virtual cost of working for type $\bar{\theta}$ is not too small.[26] Note that this assumption is a sufficient condition, and the main result does not rely solely on it. The necessary and sufficient condition is detailed in Lemma 8 in the appendix.[27]

6   Extensions

In this section, we extend the model to account for stochastic action rules. First, we introduce new notations and implementability conditions to accommodate randomness in the action rules. We then demonstrate in Proposition 6 that stochastic mechanisms can outperform the optimal deterministic mechanism when it is a consecutive-working menu. Next, we explore the optimal implementation of stochastic rules in Proposition 7 and demonstrate that intuition from deterministic mechanisms still holds in the stochastic setting. Finally, we discuss the disadvantages of stochastic mechanisms and why it is still meaningful to focus on deterministic mechanisms.

When the action rule is stochastic, the recommended action in each period becomes a random variable, denoted as $\tilde{x}_t$. Let $q_t(h_t, w_{t-1})$ represent the probability of recommending the agent to work in period t given the cost history h_t and the work history w_{t-1}. Then we have:

$$\tilde{x}_t(h_t, w_{t-1}) = \begin{cases} 1, & \text{with probability } q_t(h_t, w_{t-1}) \\ 0, & \text{with probability } 1 - q_t(h_t, w_{t-1}) \end{cases}.$$

The realization of $\tilde{x}_t$ is denoted as x_t. Additionally, the payment rule should depend on these action realizations, i.e., the work history, and is expressed as $p_N(h_N, w_{N-1})$.

With stochastic rules, we need to ensure the agent's participation at two stages each period t: first, the agent must be willing to participate for every realization of θ_t (referred to as the periodic ex-ante IR constraint); second, the agent's continued participation must be assured after both realizations of $\tilde{x}_t$ (referred to as the periodic ex-post IR constraint).[28]

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[25]This condition is satisfied by many distributions. For instance, for a symmetric truncated normal distribution with $\sigma = 1$ and $b = \frac{\bar{\theta}-\underline{\theta}}{2}$, the condition holds when $e^{b^2/2}/b \geq (N-1)/\sqrt{2\pi}$.

[26]At least as large as $N \cdot \bar{\theta} - (N-1) \cdot E(\theta)$

[27]Similar to Assumption 1, the necessary and sufficient condition is complex and involves solving a system of equations that might not have analytical solutions. However, for any given distribution, one can solve this system numerically to determine the necessary and sufficient condition.

[28]In typical usage, "ex-ante" and "ex-post" refer to the beginning and end of a game. Here, "periodic

It's straightforward that the periodic ex-post IR constraints imply the periodic ex-ante IR constraints. Therefore, we can focus only on the periodic ex-post IR constraints. Similar to the deterministic case, it is without loss of generality to concentrate payments at the end of the period.[29] Let $u_t(ht-1, w_{t-1}; \theta_t)$ and $u_t(h_{t-1}, w_{t-1}; \theta_t, x_t)$ represent the agent's on-path expected payoffs in period t before and after the realization of $\tilde{x}t$, respectively, given the history $ht-1$ and w_{t-1}.[30] The periodic ex-post IR constraints are then given by:

$$-\theta_t + u_t(h_{t-1}, w_{t-1}; \theta_t, x_t = 1) \geq 0,$$

$$u_t(h_{t-1}, w_{t-1}; \theta_t, x_t = 0) \geq 0.$$

For simplicity, the terms "IR constraints" or "periodic IR constraints" will sometimes be used to refer specifically to the periodic ex-post IR constraints when the context is clear. For truth-telling, similar to the deterministic case, we only need to ensure it once in each period t after the realization of θ_t. Using the same proof method as in Lemma 1, we can establish the following lemma.[31]

Lemma 6. *A stochastic mechanism is incentive compatible if and only if for all t, h_{t-1} and w_{t-1}:*

1. *$q_t(h_{t-1}, w_{t-1}; \theta_t)$ is non-increasing in θ_t.*

2. *$u_t(h_{t-1}, w_{t-1}; \theta_t) = u_t(h_{t-1}, w_{t-1}; \bar{\theta}) + \int_{\theta_t}^{\bar{\theta}} q_t(h_{t-1}, w_{t-1}; \theta_t)d\theta_t.$*

When the deterministic optimal mechanism is a consecutive-working menu, the principal assigns the agent to work more frequently in the first period compared to the second-best in the $N = 1$ case, e.g., $c_1^* > x_{SB}$. The principal makes the agent overwork in the first period because she has a higher continuation payoff once the agent starts working. This is because she can achieve the "relative" first-best outcome in every subsequent period if the

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agent works in the first period.[32] Consequently, to increase the chance of a brighter future, the principal bears the cost of paying extra information rent in the first period.

Next, we show that the principal can achieve the same continuation payoff by paying less information rent in the first period when stochastic mechanisms are allowed. For instance, consider reducing the probability of working in the first period for types between x_{SB} and c_1^* by a small amount ϵ, as depicted in Figure 4. This reduction clearly decreases the extra information rents paid in the first period. Meanwhile, it does not affect the principal's continuation payoff, as she can still achieve efficiency in subsequent periods for $\theta_1 \in [x_{SB}, c_1^*]$.

To see this, consider setting the agent's expected payoff to zero if the action realization is to work and transferring all the information rent that the agent deserves to the case when the action realization is to shirk. Specifically, when $x_1 = 1$, the principal promises to pay the agent $\theta_1 + (N-1)E(\theta)$ and prescribes him to work for sure in every subsequent period. This is feasible because, under Assumption 1, the promised but postponed payment for working in the first period, i.e., θ_1, is large enough to incentivize the agent to work in the second period and therefore all future periods. When $x_1 = 0$, the principal promises to pay the agent the information rent he deserves for working in the first period plus the average cost of working in the future, i.e.,

$$\frac{u_1^*(\bar{\theta}) + \int_{\theta_1}^{\bar{\theta}} q_1(\theta_1)d\theta_1}{\epsilon} + (N-1)E(\theta),$$

and prescribes him to work for sure until the end. If ϵ is small enough, then the promised payment from the first period could be large enough to leverage the agent and achieve efficiency thereafter. Thus, the stochastic mechanism improves the principal's expected payoff by reducing the information rent paid in the first period while maintaining the same continuation payoff. This is formalized in Proposition 6.

Proposition 6. *Under Assumption 1, when the optimal deterministic mechanism is a consecutive-working menu, there exist stochastic mechanisms that outperform the optimal deterministic mechanism.[33]*

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[32]The term "relative" is used because the principal still has to pay the additional $u_1^*(\bar{\theta})$ amount. Conditional on this payment being fixed, the principal can be seen as achieving the first best in every subsequent period. When the context is clear, I will omit this word for simplicity.

[33]This proposition still holds if we relax Assumption 1 to the necessary and sufficient condition in Propo-
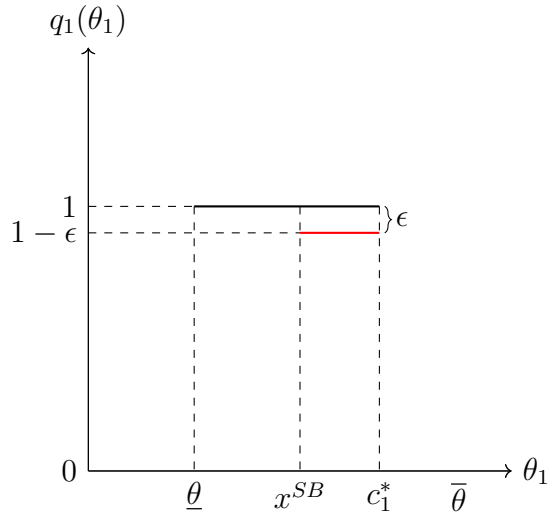
Figure 4: Stochastic improvement for action rule in the first period

The intuition behind Proposition 6 is as follows: in the deterministic optimal mechanism, the principal incurs the cost of paying extra information rent in the first period to increase the likelihood of better outcomes in the future. However, this extra information rent can be costly. The principal opts for this strategy partly due to the discrete nature of the action space, i.e., $\{0,1\}$. When faced with a choice between work or shirk for relatively high types, the principal chooses work. Ideally, the principal would prefer these types to work partially, and a stochastic mechanism allows for this flexibility. As explained above, by transferring all the information rent the agent deserves for working in the first period to the case when the action realization is to shirk, the principal gains enough leverage to discipline the agent and extract all his surplus in future periods for both action realizations. This approach is somewhat similar to making the action space continuous, giving the principal the ability to assign the relatively high types to work less and therefore pay less information rent in the first period.

A key feature of the stochastic improvement is that the principal can strengthen her future incentive power in the case when the agent shirks by transferring all the information rents to this scenario, without affecting her continuation payoff when the agent works. Usually, the principal has stronger future incentive power once the agent works because she can accrue promised payments and use them to leverage the agent in subsequent periods.

sition 8, as long as $c_1^* + u_1^*(\bar{\theta}) \neq \min_t \left(c_{t+1}^*(\mathbf{0}) + \sum_{i=t+2}^N \int_{\underline{\theta}}^{c_i^*(\mathbf{0})} F(\theta_i) d\theta_i \right)$. Here c_t^* and $u_1^*(\bar{\theta})$ are from the optimal deterministic mechanism.

33

Conversely, the principal has less incentive power when the agent shirks since she does not have to promise any payment for shirking.[34] Stochastic mechanisms provide the principal with the flexibility to relocate the promised payment across both action realizations, thus improving her overall continuation payoff. Proposition 7 formalizes this idea by showing that, under Assumption 1, the optimal way to implement a stochastic action rule is to transfer all the information rent to the case when the agent shirks.[35]

Proposition 7. *Under Assumption 1, for any stochastic action rule $\tilde{x}_t(h_{t-1}, w_{t-1}; \theta_t)$, the optimal way to implement it is to let:*

1. $u_t(h_{t-1}, w_{t-1}; \theta_t, x_t = 1) = 0$, *and*

2. $u_t(h_{t-1}, w_{t-1}; \theta_t, x_t = 0) = u_t(h_{t-1}, w_{t-1}; \bar{\theta}) + \int_{\theta_t}^{\bar{\theta}} q_t(h_{t-1}, w_{t-1}; \theta_t) d\theta_t$.

Despite the existence of stochastic improvements, focusing on deterministic mechanisms remains reasonable for the following two reasons. First, the optimal stochastic mechanism is challenging to characterize since the principal's problem can no longer be simplified to optimizing over real spaces, e.g. thresholds. Instead, the optimization must occur over functional spaces, specifically the Cartesian product of uncountably many functional spaces, which significantly increases the complexity and intractability of the problem. Second, and more importantly, stochastic mechanisms may suffer from credibility issues if the agent does not fully trust the principal. When the principal implements an outcome attached with a positive probability, it is difficult to verify whether the randomization is conducted as promised. In other words, the agent might be unable to detect deviations in the action recommendation, which could lead to a lack of trust. If the agent believes that the principal has limited commitment, this mistrust could deter the agent from participating truthfully, making stochastic mechanisms hard to implement in practice.

That being said, our discussion on stochastic mechanisms remains meaningful as they largely preserve the intuition established for the deterministic case. They align with the main idea we want to convey about leveraging an agent with private information to work in a dynamic setting—through promised yet postponed payments. Meanwhile, such exploration enriches our comprehension of how this tool can be used more flexibly and powerfully under randomness.

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[34]According to the formula for the price in (1), the accumulated promised payment for shirking in any period t is actually negative: $-\int_{\underline{\theta}}^{c_t} F(\theta) d\theta$.

[35]Assumption 1 is a sufficient condition that ensures transferring all information rents to the case when the agent shirks will not harm the principal's continuation payoff when the action realization is to work.

7 Conclusion

From traditional labor contracts to reward schemes on modern internet platforms, many mechanisms require work or consumption over consecutive periods. This paper identifies a simple and canonical economic setting—dynamic screening with i.i.d. stochastic costs and limited liability constraints—in which the optimal mechanism naturally features this consecutiveness (along with backloaded payments).

The key intuition behind the optimality of consecutive working is as follows: first, the principal backloads all payments to the end, using them as leverage to discipline the agent—if the agent quits early, he forfeits all accumulated payments. Under the condition we identify, for any given history path, the optimal promised payment for initiating work is large enough to incentivize the agent to work until the end, while allowing the principal to extract all surplus in periods where the agent has already started working. Moreover, we show that this won't affect the principal's expected payoff from other history paths. Since work is efficient and the principal can extract the full surplus, she optimally assigns the agent to continue working to the end once he has started.

Compared to the optimal contract without limited liability, where the agent is induced to work immediately and efficiency is achieved in every period, the optimal mechanism under limited liability distorts the starting time—potentially inducing the agent to delay work for a long time—but does not distort actions after work has begun. Once the agent starts, he is incentivized to work continuously to the end, achieving efficiency in every subsequent period. This results from a trade-off between the cost of inducing the agent to start working early and the benefit of extracting surplus over more periods.

Moreover, we demonstrate that stochastic mechanisms can sometimes outperform deterministic ones, as the principal can optimally allocate promised payments—thus adjusting her future leverage over the agent—across different realizations of the recommended action. This flexibility enables better regulation in various scenarios. The main intuition established for deterministic mechanisms remains valid in the stochastic setting, though it can be applied more flexibly and effectively under randomness. Fully characterizing the optimal stochastic mechanism is left for future research.

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Appendix A Proof of results

A.1 Notations in the proofs

Let $w_{i:j}$ denote the working status history from period i to period j, i.e., $w_{i:j} = (w_i, w_{i+1}, \cdots, w_j)$. Let $T(w_t)$ denote the agent's on-path expected total future working periods after the working status history w_t:

$$T(w_t) = E_{w_N} \left[\sum_{i=t+1}^{N} \mathbb{1}\{\theta_i \leq c_i(w_{i-1})\} \,\middle|\, w_t \right].$$

We use $P(w_t)$ to denote the expected final payment minus the accumulated payments up until (and including) period t, given the working status history w_t:

$$P(w_t) = E_{w_N} \left[\sum_{i=t+1}^{N} x_i \cdot c_i(w_{i-1}) - \int_{\underline{\theta}}^{c_i(w_{i-1})} F(\theta_i) d\theta_i \,\middle|\, w_t \right] + u_1^*(\bar{\theta}).$$

Let $\hat{V}(w_t)$ denote the following:

$$\alpha \cdot T(w_t) - P(w_t).$$

In addition, we use $T^*(w_t)$, $P^*(w_t)$ and $\hat{V}^*(w_t)$ to denote the same quantities under the optimal mechanism. Finally, we use V^{aw} and V^{cm} to denote the principal's expected payoff under the always-working mechanism and the consecutive-working menu, respectively.

A.2 Proof of Proposition 1

Proof. For any implementable mechanism $\{x_t, p_t\}_{t=1}^{N}$, we can construct an alternative mechanism $\{x_t, \bar{p}_t\}_{t=1}^{N}$ that has the same action rules but pays the agent only at the end. $\bar{p}_t$ is defined as follows: $\bar{p}_t = 0$ for all $t < N$ and $\bar{p}_N = \sum_{t=1}^{N} p_t$. Then the IC-t constraint in the alternative mechanism is:

$$\theta_t \in \arg\max_{\hat{\theta}_t \in \Theta} -\theta_t \cdot x_t(h_{t-1}, \hat{\theta}_t) + E_{\boldsymbol{\theta}}^t \left[\bar{p}_N(h_{t-1}, \hat{\theta}_t, \{\tilde{\theta}_k\}_{k=t+1}^{N}) - \sum_{i=t+1}^{N} \tilde{\theta}_i \cdot x_i(h_{t-1}, \hat{\theta}_t, \{\tilde{\theta}_k\}_{k=t+1}^{i}) \right]$$

$$(4)$$

39

By the definition of $\bar{p}_N$, $E^t_{\boldsymbol{\theta}}\left[\bar{p}_N(h_{t-1}, \hat{\theta}_t, \{\tilde{\theta}_k\}^N_{k=t+1})\right]$ can be rewritten as:

$$E^t_{\boldsymbol{\theta}}\left[\sum_{i=1}^{t-1} p_i(h_{t-1}) + p_t(h_{t-1}, \hat{\theta}_t) + \sum_{i=t+1}^{N} p_i(h_{t-1}, \hat{\theta}_t, \{\tilde{\theta}_k\}^i_{k=t+1})\right]$$

$$= \sum_{i=1}^{t-1} p_i(h_{t-1}) + p_t(h_{t-1}, \hat{\theta}_t) + E^t_{\boldsymbol{\theta}}\left[\sum_{i=t+1}^{N} p_i(h_{t-1}, \hat{\theta}_t, \{\tilde{\theta}_k\}^i_{k=t+1})\right]$$

Therefore, the agent's on-path expected payoff calculated in period t under the alternative mechanism is:

$$p_t(h_{t-1}, \theta_t) - \theta_t \cdot x_t(h_{t-1}, \hat{\theta}_t) + E^t_{\boldsymbol{\theta}}\left[\sum_{i=t+1}^{N} p_i(h_{t-1}, \hat{\theta}_t, \{\tilde{\theta}_k\}^i_{k=t+1}) - \tilde{\theta}_i \cdot x_i(h_{t-1}, \hat{\theta}_t, \{\tilde{\theta}_k\}^i_{k=t+1})\right] + \sum_{i=1}^{t-1} p_i(h_{t-1})$$

Therefore, the IC-t and IR-t constraints in the alternative mechanism are:

$$\theta_t \in \arg\max_{\hat{\theta}_t \in \Theta} \ u_t(h_{t-1}, \hat{\theta}_t) + \sum_{i=1}^{t-1} p_i(h_{t-1}) \tag{IC-t}$$

$$u_t(h_{t-1}, \theta_t) + \sum_{i=1}^{t-1} p_i(h_{t-1}) \geq 0 \tag{IR-t}$$

$\sum_{i=1}^{t-1} p_i(h_{t-1})$ is a constant term that only depends on past reports and therefore doesn't affect the agent's current incentive. Thus, the IC-t constraint in the alternative mechanism is the same as in the original mechanism. However, this constant term makes the IR-t constraint in the alternative mechanism easier to satisfy, as $\sum_{i=1}^{t-1} p_i(h_{t-1})$ is non-negative due to the limited liability constraints. Lastly, the limited liability constraints in the alternative mechanism are satisfied by construction. Thus, the alternative mechanism is also implementable. Furthermore, the principal's expected payoff remains the same because the action rules x_t are the same and the total expected payment is the same

$$E_{\boldsymbol{\theta}}[\bar{p}_N] = E_{\boldsymbol{\theta}}\left[\sum_{t=1}^{N} p_t\right]$$

Therefore, for any implementable mechanism, there exists a pay-at-the-end mechanism that implements the same action rules and provides the same expected payoff for the principal. Thus, it is sufficient to consider mechanisms that only pay the agent at the end. $\qquad \square$

A.3 Proof of Lemma 1

Proof. For the necessary part, to show that IC implies (i), for any t and h_t, consider two types with $\theta_t > \theta_t'$. The IC-t constraint for θ_t implies:

$$- \theta_t \cdot x_t(h_{t-1}, \theta_t) + E_{\boldsymbol{\theta}}^t \left[p_N(h_{t-1}, \theta_t, \{\tilde{\theta}_k\}_{k=t+1}^N) - \sum_{i=t+1}^N \tilde{\theta}_i \cdot x_i(h_{t-1}, \theta_t, \{\tilde{\theta}_k\}_{k=t+1}^i) \right]$$

$$\geq - \theta_t \cdot x_t(h_{t-1}, \theta_t') + E_{\boldsymbol{\theta}}^t \left[p_N(h_{t-1}, \theta_t', \{\tilde{\theta}_k\}_{k=t+1}^N) - \sum_{i=t+1}^N \tilde{\theta}_i \cdot x_i(h_{t-1}, \theta_t', \{\tilde{\theta}_k\}_{k=t+1}^i) \right]$$

$$(5)$$

The IC-t constraint for θ_t' implies:

$$- \theta_t' \cdot x_t(h_{t-1}, \theta_t') + E_{\boldsymbol{\theta}}^t \left[p_N(h_{t-1}, \theta_t', \{\tilde{\theta}_k\}_{k=t+1}^N) - \sum_{i=t+1}^N \tilde{\theta}_i \cdot x_i(h_{t-1}, \theta_t', \{\tilde{\theta}_k\}_{k=t+1}^i) \right]$$

$$\geq - \theta_t' \cdot x_t(h_{t-1}, \theta_t) + E_{\boldsymbol{\theta}}^t \left[p_N(h_{t-1}, \theta_t, \{\tilde{\theta}_k\}_{k=t+1}^N) - \sum_{i=t+1}^N \tilde{\theta}_i \cdot x_i(h_{t-1}, \theta_t, \{\tilde{\theta}_k\}_{k=t+1}^i) \right]$$

$$(6)$$

Adding up inequalities (5) and (6) gives:

$$-\theta_t \cdot x_t(h_{t-1}, \theta_t) - \theta_t' \cdot x_t(h_{t-1}, \theta_t') \geq -\theta_t \cdot x_t(h_{t-1}, \theta_t') - \theta_t' \cdot x_t(h_{t-1}, \theta_t)$$

$$\Leftrightarrow (\theta_t - \theta_t') \cdot [x_t(h_{t-1}, \theta_t) - x_t(h_{t-1}, \theta_t')] \leq 0$$

$\theta_t > \theta_t'$ implies that $x_t(h_{t-1}, \theta_t) \leq x_t(h_{t-1}, \theta_t')$. Thus, incentive compatibility implies that $x_t(h_{t-1}, \theta_t)$ is non-increasing in θ_t, for any t and h_{t-1}.

Next, to show IC implies (ii). Note that $u_t(h_{t-1}, \theta_t)$ equals the following:

$$u_t(h_{t-1}, \theta_t) = -\theta_t \cdot x_t(h_{t-1}, \theta_t) + E_{\boldsymbol{\theta}}^t \left[p_N(h_{t-1}, \theta_t, \{\tilde{\theta}_k\}_{k=t+1}^N) - \sum_{i=t+1}^N \tilde{\theta}_i \cdot x_i(h_{t-1}, \theta_t, \{\tilde{\theta}_k\}_{k=t+1}^i) \right]$$

The envelope theorem in Milgrom and Segal (2002) implies that,

$$\frac{\partial u_t(h_{t-1}, \theta_t)}{\partial \theta_t} = -x_t(h_{t-1}, \theta_t)$$

41

Hence

$$u_t(h_{t-1}, \theta_t) = u_t(h_{t-1}, \bar{\theta}) + \int_{\theta_t}^{\bar{\theta}} x_t(h_{t-1}, \theta_t) d\theta_t$$

For the sufficient part, we need to show that, for all t and h_{t-1}, no type θ_t wants to deviate to a different type θ_t' if conditions (i) and (ii) hold. This is equivalent to showing that

$$u_t(h_{t-1}, \theta_t) \geq -\theta_t \cdot x_t(h_{t-1}, \theta_t') + E_{\boldsymbol{\theta}}^t \left[p_N(h_{t-1}, \theta_t', \{\tilde{\theta}_k\}_{k=t+1}^N) - \sum_{i=t+1}^N \tilde{\theta}_i \cdot x_i(h_{t-1}, \theta_t', \{\tilde{\theta}_k\}_{k=t+1}^i) \right]$$

$$= -\theta_t \cdot x_t(h_{t-1}, \theta_t') + E_{\boldsymbol{\theta}}^t \left[p_N(h_{t-1}, \theta_t', \{\tilde{\theta}_k\}_{k=t+1}^N) - \sum_{i=t+1}^N \tilde{\theta}_i \cdot x_i(h_{t-1}, \theta_t', \{\tilde{\theta}_k\}_{k=t+1}^i) \right]$$

$$- \theta_t' \cdot x_t(h_{t-1}, \theta_t') + \theta_t' \cdot x_t(h_{t-1}, \theta_t')$$

$$= -\theta_t \cdot x_t(h_{t-1}, \theta_t') + u_t(h_{t-1}, \theta_t') + \theta_t' \cdot x_t(h_{t-1}, \theta_t')$$

$$= u_t(h_{t-1}, \theta_t') + x_t(h_{t-1}, \theta_t') \cdot (\theta_t' - \theta_t)$$

It's without loss of generality to assume that $\theta_t > \theta_t'$. Then, we want to show the following hold:

$$u_t(h_{t-1}, \theta_t) - u_t(h_{t-1}, \theta_t') \geq x_t(h_{t-1}, \theta_t') \cdot (\theta_t' - \theta_t)$$

$$\Leftrightarrow -\int_{\theta_t'}^{\theta_t} x_t(h_{t-1}, \theta_t) d\theta_t \geq -\int_{\theta_t'}^{\theta_t} x_t(h_{t-1}, \theta_t') d\theta_t \qquad \text{(by (ii))}$$

$$\Leftrightarrow \int_{\theta_t'}^{\theta_t} x_t(h_{t-1}, \theta_t') d\theta_t \geq \int_{\theta_t'}^{\theta_t} x_t(h_{t-1}, \theta_t) d\theta_t$$

Condition (i) implies that the integrand of the left-hand side is pointwise greater than the integrand of the right-hand side. Therefore, the inequality holds, completing the proof of sufficiency. $\qquad \square$

A.4 Proof of Lemma 2

Proof. Let's prove by induction. First, we want to show that if $u_{t-1}(h_{t-1}) = u_1(\bar{\theta}) + \hat{u}_{t-1}(h_{t-1})$, then $u_t(h_t) = u_1(\bar{\theta}) + \hat{u}_t(h_t)$. According to Lemma 1, we have

$$u_t(h_t) = u_t(h_{t-1}, \theta_t)$$
$$= u_t(h_{t-1}, \bar{\theta}) + \int_{\theta_t}^{\bar{\theta}} x_t(h_{t-1}, \theta_t) d\theta_t \tag{7}$$

By the definition of u_t, we also have

$$u_{t-1}(h_{t-1}) = \int_{\underline{\theta}}^{\bar{\theta}} u_t(h_{t-1}, \tilde{\theta}_t) dF(\theta_t) - x_{t-1}(h_{t-1}) \cdot \theta_{t-1}$$
$$= u_t(h_{t-1}, \bar{\theta}) + \int_{\underline{\theta}}^{\bar{\theta}} \int_{\tilde{\theta}_t}^{\bar{\theta}} x_t(h_{t-1}, \theta_t) d\theta_t dF(\theta_t) - x_{t-1}(h_{t-1}) \cdot \theta_{t-1}$$
$$= u_t(h_{t-1}, \bar{\theta}) + \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_t) x_t(h_{t-1}, \theta_t) d\theta_t - x_{t-1}(h_{t-1}) \cdot \theta_{t-1}$$

The reason for the first equality is that the agent's expected payoff in period $t - 1$, before the realization of θ_t, equals the expectation of his expected payoff in period t minus the cost of working in period $t - 1$. The second equality comes from substituting u_t with equation (7). The third equality comes from integration by parts. Furthermore, since we assume that $u_{t-1}(h_{t-1}) = u_1(\bar{\theta}) + \hat{u}_{t-1}(h_{t-1})$, the following holds

$$u_t(h_{t-1}, \bar{\theta}) + \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_t) x_t(h_{t-1}, \theta_t) d\theta_t - x_{t-1}(h_{t-1}) \cdot \theta_{t-1} = u_1(\bar{\theta}) + \hat{u}_{t-1}(h_{t-1})$$

$$\Leftrightarrow u_t(h_{t-1}, \bar{\theta}) = u_1(\bar{\theta}) + \hat{u}_{t-1}(h_{t-1}) - \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_t) x_t(h_{t-1}, \theta_t) d\theta_t + x_{t-1}(h_{t-1}) \cdot \theta_{t-1}$$

Plugging this into equation (7) and combining it with the definition of $\hat{u}$, we get

$$u_t(h_t) = u_t(h_{t-1}, \bar{\theta}) + \int_{\theta_t}^{\bar{\theta}} x_t(h_{t-1}, \theta_t) d\theta_t$$
$$= u_1(\bar{\theta}) + \hat{u}_{t-1}(h_{t-1}) - \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_t) x_t(h_{t-1}, \theta_t) d\theta_t + x_{t-1}(h_{t-1}) \theta_{t-1} + \int_{\theta_t}^{\bar{\theta}} x_t(h_{t-1}, \theta_t) d\theta_t$$
$$= u_1(\bar{\theta}) + \hat{u}_t(h_t)$$

Lastly, let's show that $u_1(\theta_1) = u_1(\bar{\theta}) + \hat{u}_1(\theta_1)$. According to Lemma 1, we have

$$u_1(\theta_1) = u_1(\bar{\theta}) + \int_{\theta_1}^{\bar{\theta}} x_1(\theta_1)d\theta_1$$

$$= u_1(\bar{\theta}) + \hat{u}_1(\theta_1)$$

Thus, by induction, we have $u_t(h_t) = u_1(\bar{\theta}) + \hat{u}_t(h_t)$ for any t and h_t. □

A.5 Proof of Lemma 3

Proof. For an incentive compatible mechanism to be implementable, the periodic IR constraints must hold for all t and h_t. According to Lemma 2, $u_t = u_1(\bar{\theta}) + \hat{u}_t(h_t)$. Thus, the periodic IR constraints are equivalent to: $u_1(\bar{\theta}) + \hat{u}_t(h_t) \geq 0$ for all t and h_t.

In the optimal mechanism, the principal wants to choose the smallest $u_1(\bar{\theta})$ such that all the periodic IR constraints hold. Otherwise, the principal can always decrease all p_N by a small amount, such that the mechanism remains implementable while yielding a higher expected payoff. Thus, the optimal $u_1^*(\bar{\theta})$ must satisfy the following:

$$u_1^*(\bar{\theta}) = \inf\{u_1(\bar{\theta}) : u_t(h_t) \geq 0 \quad \forall\, t, h_t \in \Theta^t\} \tag{8}$$

For any given h_{t-1}, $\hat{u}_t(h_{t-1}, \theta_t)$ is a decreasing function of θ_t.

$$\hat{u}_t(h_t) = \int_{\theta_t}^{\bar{\theta}} x_t(h_{t-1}, \theta_t)d\theta_t - \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_t)x_t(h_{t-1}, \theta_t)d\theta_t + \int_{\theta_1}^{\bar{\theta}} x_1(\theta_1)d\theta_1$$

$$+ \sum_{i=2}^{t-1} \int_{\theta_i}^{\bar{\theta}} x_i(h_{i-1}, \theta_i)d\theta_i - \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_i)x_i(h_{i-1}, \theta_i)d\theta_i + \sum_{i=1}^{t-1} x_i(h_i)\theta_i$$

Only the first term, $\int_{\theta_t}^{\bar{\theta}} x_t(h_{t-1}, \theta_t)d\theta_t$, is a function of θ_t and it is decreasing in θ_t as $x_t(h_{t-1}, \theta_t) \geq 0$ for all $\theta_t \in \Theta$. Thus, $\hat{u}_t(h_{t-1}, \theta_t)$ archieves its minimum when $\theta_t = \bar{\theta}$. Therefore, we can drop all the inequality constraints in (8) where the last element of the history is not $\bar{\theta}$. This completes the proof. □

44

A.6 Proof of Proposition 2

Proof. According to Lemma 2 and Lemma 3, under the optimal mechanism, we have

$$u_N(h_N) = u_1^*(\bar{\theta}) + \hat{u}_N(h_N)$$

$$= u_1^*(\bar{\theta}) + \int_{\theta_1}^{\bar{\theta}} x_1(\theta_1)d\theta_1 + \sum_{i=1}^{N-1} x_i(h_i)\theta_i + \sum_{i=2}^{N} \left(\int_{\theta_i}^{\bar{\theta}} x_i(h_{i-1}, \theta_i)d\theta_i - \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_i)x_i(h_{i-1}, \theta_i)d\theta_i \right)$$

In addition, by the definition of $u_N(h_N)$, we also know that

$$u_N(h_N) = p(h_N) - x_N(h_N) \cdot \theta_N$$

By combining these two equations, we obtain the desired results. $\qquad\square$

A.7 Proof of Proposition 3

Proof. Under the threshold rules, the principal's optimization problem is formulated as the following:

$$\max_{\{c_t(w_{t-1})\}_{t=1}^N} E_{\boldsymbol{\theta}} \left[\sum_{t=1}^N \alpha \cdot x_t(h_{t-1}, \tilde{\theta}_t) \right] - E_{\boldsymbol{\theta}} \left[p_N(h_N) \right]$$

$$= \max_{\{c_t(w_{t-1})\}_{t=1}^N} E_{\boldsymbol{\theta}} \left[\sum_{t=1}^N \alpha \cdot \mathbb{1}\{\tilde{\theta}_t \leq c_t(h_{t-1})\} \right] - E_{\boldsymbol{\theta}} \left[p_N(h_N) \right]$$

$$= \max_{\{c_t(w_{t-1})\}_{t=1}^N} E_{\boldsymbol{\theta}} \left[\sum_{t=1}^N \alpha \cdot \mathbb{1}\{\tilde{\theta}_t \leq c_t(h_{t-1})\} \right]$$

$$- E_{\boldsymbol{\theta}} \left[\int_{\underline{\theta}}^{\bar{\theta}} F(\theta_1)x_1(\theta_1)d\theta_1 + \sum_{t=1}^N \left(x_t(h_t)\tilde{\theta}_t + \int_{\tilde{\theta}_t}^{\bar{\theta}} x_t(h_{t-1}, \theta_t)d\theta_t - \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_t)x_t(h_{t-1}, \theta_t)d\theta_t \right) + u_1^*(\bar{\theta}) \right]$$

$$= \max_{\{c_t(w_{t-1})\}_{t=1}^N} E_{\boldsymbol{\theta}} \left[\sum_{t=1}^N \alpha \cdot \mathbb{1}\{\tilde{\theta}_t \leq c_t(h_{t-1})\} \right]$$

$$- E_{\boldsymbol{\theta}} \left[\sum_{t=1}^N \mathbb{1}\{\tilde{\theta}_t \leq c_t(h_{t-1})\} \cdot c_t(h_{t-1}) - \sum_{t=2}^N \int_{\underline{\theta}}^{c_t(h_{t-1})} F(\theta_t)d\theta_t + u_1^*(\bar{\theta}) \right]$$

$$= \max_{\{c_t(w_{t-1})\}_{t=1}^N} E_{\boldsymbol{\theta}} \left[\sum_{t=1}^N \mathbb{1}\{\tilde{\theta}_t \leq c_t(h_{t-1})\} \cdot (\alpha - c_t(h_{t-1})) - \sum_{t=2}^N \int_{\underline{\theta}}^{c_t(h_{t-1})} F(\theta_t)d\theta_t \right] + u_1^*(\bar{\theta})$$

The second equality arises from substituting the formula for $p(h_N)$ from Proposition 2. In addition, according to Lemma 2, we have that

$$
\begin{aligned}
u_1^*(\bar{\theta}) &= \inf\{u_1(\bar{\theta}) : u_1(\bar{\theta}) + \hat{u}_t(h_{t-1}, \bar{\theta}) \geq 0, \forall\, h_{t-1} \in \Theta^{t-1}, t \leq N-1\} \\
&= \sup_{h_t \in \Theta^{t-1}, t \in \mathcal{N}} \left\{ -\hat{u}_t(h_{t-1}, \bar{\theta}) \right\} \\
&= - \inf_{h_t \in \Theta^{t-1}, t \in \mathcal{N}} \left\{ \hat{u}_t(h_{t-1}, \bar{\theta}) \right\} \\
&= - \inf_{h_t \in \Theta^{t-1}, t \in \mathcal{N}} \left\{ \int_{\theta_1}^{\bar{\theta}} x_1(\theta_1)d\theta_1 + \sum_{i=1}^{t-1} x_i(h_i)\theta_i + \sum_{i=2}^{t} \int_{\theta_i}^{\bar{\theta}} x_i(h_{i-1}, \theta_i)d\theta_i - \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_i)x_i(h_{i-1}, \theta_i)d\theta_i \right\} \\
&= - \inf_{h_t \in \Theta^{t-1}, t \in \mathcal{N}} \left\{ \sum_{i=1}^{t-1} \mathbb{1}\{\theta_i \leq c_i(h_{i-1})\}c_i(h_{i-1}) - \sum_{i=2}^{t} \int_{\underline{\theta}}^{c_i(h_{i-1})} F(\theta_i)d\theta_i \right\}
\end{aligned}
$$

The third equality uses the definition of $\hat{u}_t$ in Lemma 2. Therefore, the principal's optimization problem can be expressed as the following:

$$
\begin{aligned}
\max_{\{c_t(w_{t-1})\}_{t=1}^{N}} \Bigg\{ E_{\boldsymbol{\theta}} \Bigg[\sum_{t=1}^{N} \mathbb{1}\{\tilde{\theta}_t \leq c_t(h_{t-1})\} \cdot (\alpha - c_t(h_{t-1})) - \sum_{t=2}^{N} \int_{\underline{\theta}}^{c_t(h_{t-1})} F(\theta_t)d\theta_t \Bigg] \\
- \inf_{h_t \in \Theta^{t-1}, t \in \mathcal{N}} \left\{ \sum_{i=1}^{t-1} \mathbb{1}\{\theta_i \leq c_i(h_{i-1})\}c_i(h_{i-1}) - \sum_{i=2}^{t} \int_{\underline{\theta}}^{c_i(h_{i-1})} F(\theta_i)d\theta_i \right\} \Bigg\}
\end{aligned}
$$

Note that the cost in each period t enters the objective function only in the form of $\mathbb{1}\{\theta_t \leq c_t(h_{t-1})\}$, which is the working status in that period. That is why the working status history is a sufficient statistic for the cost history.

Specifically, in the expectation term, the thresholds $c_t(h_{t-1})$ are additively separable from the past cost history, interacting only with the current cost θ_t [36]. However, in the infimum term, the thresholds $c_t(h_{t-1})$ interact with the past cost history, as they are added to terms involving past costs and then compared with other histories. Nevertheless, the past cost history only shows up through the working status history, i.e. θ_i appears solely in the form of $\mathbb{1}\{\theta_i \leq c_i(h_{i-1})\}$ for all $i < t$. Consequently, the thresholds $c_t(h_{t-1})$ for cost histories h_{t-1} that generate the same working status history w_{t-1} appear symmetrically in the objective function. To see this, consider θ_1 and θ_1' that generates the same w_1, then the thresholds $c_t(\theta_1, \theta_2, \cdots, \theta_{t-1})$ and $c_t(\theta_1', \theta_2, \cdots, \theta_{t-1})$ appear symmetrically in the infimum term for all $\{\theta_2, \cdots, \theta_{t-1}\} \in \Theta^{t-2}$ and all $t \in \{2, \cdots, N\}$. Terms involving θ_1

---

[36]It is also in the form of working status, i.e. $\mathbb{1}\{\theta_t \leq c_t(h_{t-1})\}$.

are:

$$w_1 \cdot c_1 + \sum_{t=2}^{m-1} \mathbb{1}\{\theta_t \leq c_t(\theta_1, \theta_2, \cdots, \theta_{t-1})\} c_t(\theta_1, \theta_2, \cdots, \theta_{t-1}) - \sum_{t=2}^{m} \int_{\underline{\theta}}^{c_t(\theta_1, \theta_2, \cdots, \theta_{t-1})} F(\theta_t) d\theta_t$$

And terms involving θ_1' are:

$$w_1 \cdot c_1 + \sum_{t=2}^{m-1} \mathbb{1}\{\theta_t \leq c_t(\theta_1', \theta_2, \cdots, \theta_{t-1})\} c_t(\theta_1', \theta_2, \cdots, \theta_{t-1}) - \sum_{t=2}^{m} \int_{\underline{\theta}}^{c_t(\theta_1', \theta_2, \cdots, \theta_{t-1})} F(\theta_t) d\theta_t$$

where $m \in \{t, \cdots, N\}$. Consequently, the optimal values of these thresholds should be the same. We can therefore rewrite these thresholds as $c_t(w_1, \theta_2, \cdots, \theta_{t-1})$. This pattern continues, leading us to the conclusion that it is sufficient to set up thresholds as $c_t(w_{t-1})$ for all t.

$\square$

A.8 Lemma 7 and the proof

Lemma 7. $c_t^*(w_{t-1}) \neq \underline{\theta}$ for all $t \in \mathcal{N}$ and for all $w_{t-1} \in \{0, 1\}^{t-1}$.

Proof. Let's prove this by contradiction. Suppose there exists some t and w_{t-1} such that $c_t^*(w_{t-1}) = \underline{\theta}$. Then, consider an alternative set of thresholds that equal the original ones except for the following thresholds:

$$\begin{cases} c_t'(w_{t-1}) = \underline{\theta} + \epsilon \\ c_k'(w_{t-1}, 1, w_{t+1:k}) = c_k^*(w_{t-1}, 0, w_{t+1:k}), \quad \forall k \in \{t+1, \cdots, N\}, \ w_{t+1:k} \in \{0, 1\}^{k-t}. \end{cases}$$

Basically, we change the threshold for working in period t given history w_{t-1} from $\underline{\theta}$ to $\underline{\theta} + \epsilon$, and let the future thresholds after working in period t, which were previously off-path and need not be specified, equal the future thresholds after shirking in period t. Payments will be adjusted accordingly following Proposition 2. We then argue that the principal's expected payoff can always strictly increase in the two possible scenarios after the alternation.

Case 1: $u_1^*(\bar{\theta})$ remains the same. Let Δ represent the change in the principal's expected

47

payoff. Then, we have,

$$
\begin{aligned}
\Delta = & Pr(w_{t-1}) \left[\alpha \cdot \left(F(\underline{\theta} + \epsilon) + V^*(w^{i-1}, 0) \right) - F(\underline{\theta} + \epsilon) \left(P^*(w^{i-1}, 0) + \underline{\theta} + \epsilon - \int_{\underline{\theta}}^{\underline{\theta}+\epsilon} F(\theta) d\theta \right) \right. \\
& \left. - \left(1 - F(\underline{\theta} + \epsilon) \right) \left(P^*(w^{i-1}, 0) - \int_{\underline{\theta}}^{\underline{\theta}+\epsilon} F(\theta) d\theta \right) \right] - Pr(w_{t-1}) \left(\alpha \cdot V^*(w^{i-1}, 0) - P^*(w^{i-1}, 0) \right)
\end{aligned}
$$

After the alternation, the principal benefits from the increased probability of the agent working in period t following history w_{t-1}, which rises from zero to $F(\underline{\theta} + \epsilon)$. According to the final payment formula (1), the principal now incurs an additional payment of $\underline{\theta} + \epsilon - \int_{\underline{\theta}}^{\underline{\theta}+\epsilon} F(\theta) d\theta$ when the agent works, while saving $\int_{\underline{\theta}}^{\underline{\theta}+\epsilon} F(\theta) d\theta$ when the agent shirks in period t after history w_{t-1}. After simplification, we have,

$$
\begin{aligned}
\Delta &= Pr(w_{t-1}) \left[\alpha \cdot F(\underline{\theta} + \epsilon) - F(\underline{\theta} + \epsilon)(\underline{\theta} + \epsilon) + \int_{\underline{\theta}}^{\underline{\theta}+\epsilon} F(\theta) d\theta \right] \\
&> Pr(w_{t-1}) \left(\alpha - \underline{\theta} - \epsilon \right) \cdot F(\underline{\theta} + \epsilon) \\
&> 0
\end{aligned}
$$

The last inequality holds because ϵ is small. In other words, for any $\alpha > \underline{\theta}$, we can always find a sufficiently small ϵ such that $\alpha - \underline{\theta} - \epsilon > 0$. Therefore, the principal's expected payoff strictly increases after the alternation.

Case 2: $u_1^*(\bar{\theta})$ changes after the alternation. Let $\delta_{u_1^*(\bar{\theta})}$ represents the change in $u_1^*(\bar{\theta})$ after the alternation. Then, the change in the principal's expected payoff is:

$$
\Delta = Pr(w_{t-1}) \left[\alpha \cdot F(\underline{\theta} + \epsilon) - F(\underline{\theta} + \epsilon)(\underline{\theta} + \epsilon) + \int_{\underline{\theta}}^{\underline{\theta}+\epsilon} F(\theta) d\theta \right] - \delta_{u_1^*(\bar{\theta})}
$$

In addition, we have the following relationship:

$$
\delta_{u_1^*(\bar{\theta})} \leq \int_{\underline{\theta}}^{\underline{\theta}+\epsilon} F(\theta) d\theta
$$

When $u_1^*(\bar{\theta})$ changes, it must be caused by a change in the IR constraint of some work history whose initial t elements are $\{w_{t-1}, 0\}$, previously either slack or binding, now becoming binding. Conversely, The IR constraints for histories that begin with $\{w_{t-1}, 1\}$ are

now all slack, as they each include an additional positive term:

$$\underline{\theta} + \epsilon - \int_{\underline{\theta}}^{\underline{\theta}+\epsilon} F(\theta)d\theta = \underline{\theta} + \epsilon - F(\theta) \cdot \theta|_{\underline{\theta}}^{\underline{\theta}+\epsilon} + \int_{\underline{\theta}}^{\underline{\theta}+\epsilon} \theta \cdot f(\theta)d\theta \quad \text{(Intergration by parts)}$$

$$= \underline{\theta} + \epsilon - F(\underline{\theta}+\epsilon)(\underline{\theta}+\epsilon) + \int_{\underline{\theta}}^{\underline{\theta}+\epsilon} F(\theta)d\theta$$

$$= (\underline{\theta}+\epsilon) \cdot (1 - F(\underline{\theta}+\epsilon)) + \int_{\underline{\theta}}^{\underline{\theta}+\epsilon} F(\theta)d\theta$$

$$> 0$$

For work histories starting with $\{w_{t-1}, 0\}$, their IR constraints now include an additional negative term: $-\int_{\underline{\theta}}^{\underline{\theta}+\epsilon} F(\theta)d\theta$. Consequently, the maximum change in $u_1^*(\bar{\theta})$ occurs when any of the original IR constraints for these work histories used to be binding. In this case, the change in $u_1^*(\bar{\theta})$ amounts to $\int_{\underline{\theta}}^{\underline{\theta}+\epsilon} F(\theta)d\theta$. Therefore, we have,

$$\Delta \geq Pr(w_{t-1}) \left[\alpha \cdot F(\underline{\theta}+\epsilon) - F(\underline{\theta}+\epsilon)(\underline{\theta}+\epsilon) + \int_{\underline{\theta}}^{\underline{\theta}+\epsilon} F(\theta)d\theta \right] - \int_{\underline{\theta}}^{\underline{\theta}+\epsilon} F(\theta)d\theta$$

$$= Pr(w_{t-1}) \left[\alpha \cdot F(\underline{\theta}+\epsilon) - F(\underline{\theta}+\epsilon)(\underline{\theta}+\epsilon) \right] - \left(1 - Pr(w_{t-1})\right) \int_{\underline{\theta}}^{\underline{\theta}+\epsilon} F(\theta)d\theta$$

$$> Pr(w_{t-1})(\alpha - \underline{\theta} - \epsilon)F(\underline{\theta}+\epsilon) - \left(1 - Pr(w_{t-1})\right)F(\underline{\theta}+\epsilon)\epsilon$$

$$= Pr(w_{t-1})F(\underline{\theta}+\epsilon) \left[\alpha - \underline{\theta} - \epsilon - \frac{1 - Pr(w_{t-1})}{Pr(w_{t-1})}\epsilon \right]$$

$$= Pr(w_{t-1})F(\underline{\theta}+\epsilon) \left[\alpha - \underline{\theta} - \frac{1}{Pr(w_{t-1})}\epsilon \right]$$

$$> 0$$

The second inequality holds because $\int_{\underline{\theta}}^{\underline{\theta}+\epsilon} F(\theta)d\theta < \int_{\underline{\theta}}^{\underline{\theta}+\epsilon} F(\underline{\theta}+\epsilon)d\theta$. The last inequality holds because ϵ is small. In other words, for any $\alpha > \underline{\theta}$, we can always find a sufficiently small ϵ such that $\frac{1}{Pr(w_{t-1})}\epsilon + \underline{\theta} < \alpha$ given a fixed $Pr(w_{t-1})$. Therefore, the principal's expected payoff strictly increases after the alternation in this case as well.

Thus far, we have demonstrated that the principal's expected payoff strictly increases in both possible scenarios following the alternation. Therefore, we have reached a contradiction, concluding the proof. $\qquad\square$

A.9   Proof of Lemma 4

Proof. To illustrate this, suppose there exist $m > 1$ such that $m = \arg\min\{t : c_t^*(\mathbf{0}) = \bar{\theta}\}$. It means that m is the smallest period such that the agent is assigned to work for every realization of cost given that he has not worked before in the optimal mechanism, and m is not the first period. Then, we want to show that it is impossible by finding a contradiction.

Note that given any work history, the first best outcome the principal can achieve in each future period is to have the agent work for sure and only pay him the average cost of working [37]. Next, we want to show that if we set $c_j(w_{j-1}) = \bar{\theta}$ for all $m - 1 \leq j \leq N$ and $w_{j-1} = \{\mathbf{0}_{m-2}, \mathbf{1}_{j-m+1}\}$ [38], then the principal can be better off as she can achieve a "relative" first best in every period following the work history $\mathbf{0}_{m-2}$. [39] Recall that the formula for the final payment is the following:

$$p(w_N) = \sum_{t=1}^{N} x_t \cdot c_t(w_{t-1}) - \sum_{t=2}^{N} \int_{\underline{\theta}}^{c_t(w_{t-1})} F(\theta_t)d\theta_t + u_1^*(\bar{\theta})$$

If we can let $c_j(w_{j-1}) = \bar{\theta}$ for all $m - 1 \leq j \leq N$ and $w_{j-1} = \{\mathbf{0}_{m-2}, \mathbf{1}_{j-m+1}\}$ without having to increase $u_1^*(\bar{\theta})$, then the principal archives the first best in every period after the work history $\mathbf{0}_{m-2}$. This is because the payment then becomes:

$$p(w_N) = u_1^*(\bar{\theta}) + \sum_{t=1}^{m-2} x_t \cdot c_t(w_{t-1}) - \sum_{t=2}^{m-2} \int_{\underline{\theta}}^{c_t(w_{t-1})} F(\theta_t)d\theta_t + \sum_{t=m-1}^{N} \left[\bar{\theta} - \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_t)d\theta_t \right]$$

$$= u_1^*(\bar{\theta}) - \sum_{t=2}^{m-2} \int_{\underline{\theta}}^{c_t(w_{t-1})} F(\theta_t)d\theta_t + (N - m + 2) \cdot E(\theta)$$

Here the payment has three components: $u_1^*(\bar{\theta})$, $-\sum_{t=2}^{m-2} \int_{\underline{\theta}}^{c_t(w_{t-1})} F(\theta_t)d\theta_t$ and $(N - m + 2) \cdot E(\theta)$. The summation term only depends on the action rules before and during period $m - 2$. This implies that if $u_1^*(\bar{\theta})$ does not increase after we set $c_j(w_{j-1}) = \bar{\theta}$ for all

---

[37] The first best scenario is to ask the agent to work whenever $\alpha > \theta_t$ and to compensate the agent with the cost θ_t. Since $\alpha > \bar{\theta}$, the first best is to assign the agent to work for sure. In this case, paying the agent θ_t is equivalent to paying the average cost $E(\theta_t)$ for all realization of θ_t from the perspectives of the players' expected payoffs.

[38] $\mathbf{0}_0$ and $\mathbf{1}_0$ represent $\emptyset$.

[39] The term "relative" is used here because the principal archieves the first best relative to the previous mechanism. Specifically. the principal still has to pay the agent an additional $u_1^*(\bar{\theta})$ amount, but conditional on it remaining the same, she can be seen as achieving the first best in every period following the work history $\mathbf{0}_{m-2}$.

$m - 1 \leq j \leq N$ and $w_{j-1} = \{\mathbf{0}_{m-2}, \mathbf{1}_{j-m+1}\}$, then the principal only pays the average cost of working in every subsequent period following the work history $\mathbf{0}_{m-2}$, while asking the agent to work for sure. In this scenario, the principal's expected payoff will definitely increases as she achieves the first best in each of the subsequent periods, and it won't negativity affect the payoff from other work histories since it does not increase $u_1^*(\bar{\theta})$ [40]. Next, we will show that asking the agent to work for sure after the work history $\mathbf{0}_{m-2}$ indeed does not increase $u_1^*(\bar{\theta})$. This is equivalent to showing that the IR constraints are slack for all work histories containing $\mathbf{0}_{m-2}$ given the $u_1^*(\bar{\theta})$ from the original mechanism, since the IR conditions for all other histories remain the same.

The IR constraint from the original mechanism for work history $\{\mathbf{0}_{m-1}, 1\}$ implies:

$$u_1^*(\bar{\theta}) + \hat{u}_m(\mathbf{0}_{m-1}, 1) \geq 0$$

$$\iff \quad u_1^*(\bar{\theta}) - \sum_{t=2}^{m} \int_{\underline{\theta}}^{c_t^*(\mathbf{0})} F(\theta_t) d\theta_t \geq 0$$

$$\iff \quad u_1^*(\bar{\theta}) \geq \sum_{t=2}^{m} \int_{\underline{\theta}}^{c_t^*(\mathbf{0})} F(\theta_t) d\theta_t \tag{9}$$

Next, we will show that the IR constraints are slack for all work histories containing $\mathbf{0}_{m-2}$ given the $u_1^*(\bar{\theta})$ from the original mechanism that satisfies (9). Specifically, IR constraint in period $j \geq m - 1$ after work history $\{\mathbf{0}_{m-2}, \mathbf{1}_{j-m+2}\}$ is:

$$u_1^*(\bar{\theta}) - \sum_{t=2}^{m-2} \int_{\underline{\theta}}^{c_t^*(\mathbf{0})} F(\theta_t) d\theta_t + \sum_{t=m-1}^{j-1} \left(\bar{\theta} - \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_t) d\theta_t \right) - \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_j) d\theta_j$$

$$\geq \int_{\underline{\theta}}^{c_{m-1}^*(\mathbf{0})} F(\theta_{m-1}) d\theta_{m-1} + \int_{\underline{\theta}}^{c_m^*(\mathbf{0})} F(\theta_m) d\theta_m + \sum_{t=m-1}^{j-1} \left(\bar{\theta} - \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_t) d\theta_t \right) - \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_j) d\theta_j$$

$$= \int_{\underline{\theta}}^{c_{m-1}^*(\mathbf{0})} F(\theta_{m-1}) d\theta_{m-1} + (j - m + 1) \cdot (\bar{\theta} - \int_{\underline{\theta}}^{\bar{\theta}} F(\theta) d\theta)$$

$$= \int_{\underline{\theta}}^{c_{m-1}^*(\mathbf{0})} F(\theta_{m-1}) d\theta_{m-1} + (j - m + 1) \cdot E(\theta)$$

$$> 0$$

Where the first inequality is a result of (9) and the first equality comes from the fact that

[40] In the proof of Lemma **??**, we use the first-order condition method to similarly argue that asking the agent to work for certain is optimal when it does not increase $u_1^*(\bar{\theta})$.

$c_m^*(\mathbf{0}) = \bar{\theta}$. The last inequality is strict since $c_{m-1}^*(\mathbf{0}) > \underline{\theta}$ and the distribution has positive density everywhere on the support. So far, we have shown that assigning the agent to work for sure after the work history $\mathbf{0}_{m-2}$ does not increase $u_1^*(\bar{\theta})$. Therefore, the principal's expected payoff will be increased by doing so. This increase is strict since initially $c_{m-1}^*(\mathbf{0}) < \bar{\theta}$, and now we raise it to $\bar{\theta}$, achieving the first best in that period. Therefore, we arrive at a contradiction: m is not the smallest period such that the agent is assigned to work for every realization of cost given that he has not worked before in the optimal mechanism.

Thus, there are only two possibilities for the optimal mechanism: either $m = 1$, or $c_t^*(\mathbf{0}) \in (\underline{\theta}, \bar{\theta})$ for all $t \in \mathcal{N}$. The fact that $c_t^*(\mathbf{0}) > \underline{\theta}$ is established in Lemma 7. When $m = 1$, then the principal can achieve the first best in every period starting from the second period by assigning the agent to always work. This is because the final payment becomes:

$$c_1^* + (N - 1) \cdot E(\theta) + u_1^*(\bar{\theta}) = \bar{\theta} + (N - 1) \cdot E(\theta)$$

The equality holds because $c_1^* = \bar{\theta}$ and $u_1^*(\bar{\theta}) = 0$. The value of $u_1^*(\bar{\theta}) = 0$ is verified by showing that all IR constraints hold even if $u_1^*(\bar{\theta})$ is set to its lowest possible value, which is 0. To illustrate this, consider the IR constraint for the work history $\mathbf{1}_t$ for $t \geq 1$ is:

$$
\begin{aligned}
&u_1^*(\bar{\theta}) + c_1^* + \sum_{i=2}^{t-1} \left(\bar{\theta} - \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_i) d\theta_i \right) - \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_t) d\theta_t \\
&= u_1^*(\bar{\theta}) + \bar{\theta} + (t - 2) \cdot E(\theta) - \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_t) d\theta_t \\
&= u_1^*(\bar{\theta}) + (t - 1) \cdot E(\theta) \\
&\geq u_1^*(\bar{\theta}) \\
&\geq 0
\end{aligned}
$$

Thus, when $m = 1$, the optimal strategy is to ask the agent to always work. This completes the proof. $\qquad \square$

A.10 Proof of Proposition 4

Proof. First of all, we want to show that when $c_i^*(\mathbf{0}) < \bar{\theta}$ for all $1 \leq i \leq N$, the following holds:

$$\min_{x_t \in \{0,1\}^t, t \in \mathcal{N}} \left\{ \sum_{i=1}^{t-1} w_i \cdot c_i^*(w_{i-1}) - \sum_{i=2}^{t} \int_{\underline{\theta}}^{c_i^*(w_{i-1})} F(\theta_i) d\theta_i \right\} = -\sum_{t=2}^{N} \int_{\underline{\theta}}^{c_t^*(\mathbf{0})} F(\theta_t) d\theta_t \tag{10}$$

Suppose this is not true, then we have

$$-u_1^*(\bar{\theta}) < -\sum_{t=2}^{N} \int_{\underline{\theta}}^{c_t^*(\mathbf{0})} F(\theta_t) d\theta_t \tag{11}$$

Next, consider increasing $c_N^*(\mathbf{0})$ by a small amount ϵ such that (11) still holds, while keeping all other thresholds the same. Since $\sum_{t=2}^{N} \int_{\underline{\theta}}^{c_t^*(\mathbf{0})} F(\theta_t) d\theta_t$ is the only term in the minimum function involving $c_N^*(\mathbf{0})$, (11) implies that $u_1^*(\bar{\theta})$ remains unchanged. Payments will be adjusted accordingly following Proposition 2. Denote the probability of the work history w_{N-1} being $\mathbf{0}_{N-1}$ as $Pr(\mathbf{0}_{N-1})$. Then the change in the principal's expected payoff is:

$$Pr(\mathbf{0}_{N-1}) \cdot \left\{ \alpha \cdot (F(c_N^*(\mathbf{0}) + \epsilon) - F(c_N^*(\mathbf{0}))) - \right.$$

$$\left[F(c_N^*(\mathbf{0}) + \epsilon) \cdot \left(c_N^*(\mathbf{0}) + \epsilon - \int_{\underline{\theta}}^{c_N^*(\mathbf{0}) + \epsilon} F(\theta) d\theta \right) + (1 - F(c_N^*(\mathbf{0}) + \epsilon)) \cdot \left(-\int_{\underline{\theta}}^{c_N^*(\mathbf{0}) + \epsilon} F(\theta) d\theta \right) \right] +$$

$$\left. \left[F(c_N^*(\mathbf{0})) \cdot \left(c_N^*(\mathbf{0}) - \int_{\underline{\theta}}^{c_N^*(\mathbf{0})} F(\theta) d\theta \right) + (1 - F(c_N^*(\mathbf{0}))) \cdot \left(-\int_{\underline{\theta}}^{c_N^*(\mathbf{0})} F(\theta) d\theta \right) \right] \right\}$$

$$= Pr(\mathbf{0}_{N-1}) \cdot \left\{ \alpha \cdot (F(c_N^*(\mathbf{0}) + \epsilon) - F(c_N^*(\mathbf{0}))) - \right.$$

$$\left. \left[F(c_N^*(\mathbf{0}) + \epsilon) \cdot \epsilon + (F(c_N^*(\mathbf{0}) + \epsilon) - F(c_N^*(\mathbf{0}))) \cdot c_N^*(\mathbf{0}) - \int_{c_N^*(\mathbf{0})}^{c_N^*(\mathbf{0}) + \epsilon} F(\theta) d\theta \right] \right\}$$

$$= Pr(\mathbf{0}_{N-1}) \cdot \left[(F(c_N^*(\mathbf{0}) + \epsilon) - F(c_N^*(\mathbf{0}))) \cdot (\alpha - c_N^*(\mathbf{0})) - \left(F(c_N^*(\mathbf{0}) + \epsilon)\epsilon - \int_{c_N^*(\mathbf{0})}^{c_N^*(\mathbf{0}) + \epsilon} F(\theta) d\theta \right) \right]$$

$$> Pr(\mathbf{0}_{N-1}) \cdot \left[(F(c_N^*(\mathbf{0}) + \epsilon) - F(c_N^*(\mathbf{0}))) \cdot (\alpha - c_N^*(\mathbf{0})) - (F(c_N^*(\mathbf{0}) + \epsilon) \cdot \epsilon - F(c_N^*(\mathbf{0})) \cdot \epsilon) \right]$$

$$=Pr(\mathbf{0}_{N-1})\left(F(c_N^*(\mathbf{0})+\epsilon)-F(c_N^*(\mathbf{0}))\right)\left(\alpha-c_N^*(\mathbf{0})-\epsilon\right)$$

$$>0$$

The first inequality holds because $\int_{c_N^*(\mathbf{0})}^{c_N^*(\mathbf{0})+\epsilon}F(\theta)d\theta > F(c_N^*(\mathbf{0}))\cdot\epsilon$. The last inequality holds since $c_N^*(\mathbf{0}) < \alpha$, so there exists some small ϵ such that $c_N^*(\mathbf{0})+\epsilon < \alpha$. Thus, the principal's expected payoff strictly increases after we increase $c_N^*(\mathbf{0})$ by a small amount ϵ. This contradicts the fact that the original thresholds are optimal. Therefore, (10) must hold. It implies that the IR constraint for the work history $\mathbf{0}_N$ is always binding, which leads to the following relationship:

$$u_1^*(\bar{\theta}) + \hat{u}_N(\mathbf{0}_N) = 0$$

$$\Leftrightarrow u_1^*(\bar{\theta}) = -\hat{u}_N(\mathbf{0}_N) = \sum_{t=2}^{N}\int_{\underline{\theta}}^{c_t^*(\mathbf{0})}F(\theta_t)d\theta_t$$

In other words, $-\sum_{t=2}^{N}\int_{\underline{\theta}}^{c_t^*(\mathbf{0})}F(\theta_t)d\theta_t$ should be the value of the minimum function. Therefore, in the original optimization problem (2), we can replace the minimum function with $-\sum_{t=2}^{N}\int_{\underline{\theta}}^{c_t^*(\mathbf{0})}F(\theta_t)d\theta_t$ and add constraints to ensure that all other terms in the minimum function are no less than $-\sum_{t=2}^{N}\int_{\underline{\theta}}^{c_t^*(\mathbf{0})}F(\theta_t)d\theta_t$, i.e. ensuring that the other IR constraints also hold. Hence, the principal's optimization problem is equivalent to the constrained optimization problem (3). $\qquad\square$

A.11 Proposition 8 and the proof

Proposition 8. *Let $\{c_t^*(\mathbf{0})\}_{t=1}^{N}$ be the solution to the following system of equations:*

$$\left\{Pr(\mathbf{0}_{t-1})\cdot(\alpha - c_t(\mathbf{0}) + \hat{V}(\mathbf{0}_{t-1},1) - \hat{V}(\mathbf{0}_{t-1},0)) = \frac{F(c_t(\mathbf{0}))}{f(c_t(\mathbf{0}))}\right\}_{t=1}^{N} \qquad (12)$$

where $Pr(\mathbf{0}_{t-1}) = \Pi_{i=1}^{t-1}(1 - F(c_i(\mathbf{0}))), \hat{V}(\mathbf{0}_{t-1}, 1) = (N-t)(\alpha - E(\theta)) - \sum_{i=2}^{N} \int_{\underline{\theta}}^{c_i(\mathbf{0})} F(\theta_i)d\theta_i$

and

$$\hat{V}(\mathbf{0}_{t-1}, 0) = \sum_{i=t+1}^{N} \left[\Pi_{j=t+1}^{i-1}(1 - F(c_j(\mathbf{0}))) \right] \cdot F(c_i(\mathbf{0})) \cdot \left[\alpha(N - i + 1) - c_i(\mathbf{0}) + \sum_{j=t+1}^{i} \int_{\underline{\theta}}^{c_j(\mathbf{0})} F(\theta_j)d\theta_j \right.$$

$$\left. - (N - i)E(\theta) \right] - \sum_{i=2}^{N} \int_{\underline{\theta}}^{c_i(\mathbf{0})} F(\theta_i)d\theta_i.$$

If the solution exists and has that $c_1^* \neq \bar{\theta}$, then the necessary and sufficient condition for the solution of the relaxed optimization problem (3) *to be feasible is that the solution satisfies the following condition:*

$$\min_t \left(c_{t+1}^*(\mathbf{0}) + \sum_{i=t+2}^{N} \int_{\underline{\theta}}^{c_i^*(\mathbf{0})} F(\theta_i)d\theta_i \right) + E(\theta) - \bar{\theta} \geq 0. \tag{13}$$

Proof. The system of equations consists of the first-order conditions for the thresholds $c_t(\mathbf{0})$ for all $t \in \mathcal{N}$ in the relaxed optimization problem (3). When the solution exits and $c_1^* \neq \bar{\theta}$, it implies that $c_t(\mathbf{0})$ are interior points of the support for all $t \in \mathcal{N}$. In this case, the necessary and sufficient condition for the solution to be feasible is that the IR constraint holds for all work histories $(\mathbf{0}_t, \mathbf{1}_2)$ as illustrated in section 5. Among these histories, the one with the smallest agent's expected payoff is:

$$\arg\min_{0 \leq t \leq N-2} \left(c_{t+1}^*(\mathbf{0}) + \sum_{i=t+2}^{N} \int_{\underline{\theta}}^{c_i^*(\mathbf{0})} F(\theta_i)d\theta_i \right).$$

Thus, as long as the IR constraint holds for this work history, which is condition (13), then the relaxed solution is feasible. This completes the proof. $\square$

A.12 Proof of Theorem 1

Proof. When $\alpha \geq \bar{\theta}$, according to Lemma 4, there are only two possibilities for the optimal action rule: either $c_i^*(\mathbf{0}) < \bar{\theta}$ for all $i \in \mathcal{N}$, or the optimal rule is to assign the agent to work in every period. When the optimal rule is to assign the agent to work in every period, the optimal mechanism is an always-working mechanism. Thus, we only need to show that when $c_i^*(\mathbf{0}) < \bar{\theta}$ for all $i \in \mathcal{N}$, the optimal mechanism is a consecutive-working menu.

---

[41]$\mathbf{0}_0$ denotes the emptyset.

According to Proposition 4, the principal's optimization problem is equivalent to the constrained optimization problem (3). Let's first solve the relaxed optimization problem, where we remove the constraints in (3). Then, we will argue that the relaxed optimal solution is indeed feasible by showing that it satisfies all the constraints in the original optimization problem. To start with, for work history $w_{N-1} \neq \mathbf{0}$, the first order condition for $c_N(w_{N-1})$ is:

$$
Pr(w_{N-1}) \frac{\partial \left[\alpha \cdot F(c_N) - (F(c_N)c_N - \int_{\underline{\theta}}^{c_N} F(\theta)d\theta) \right]}{\partial c_N}
$$
$$
= Pr(w_{N-1})[\alpha f(c_N) - f(c_N)c_N - F(c_N) + F(c_N)]
$$
$$
= Pr(w_{N-1})f(c_N)(\alpha - c_N)
$$
$$
\geq 0
$$

The last inequality is because $\alpha \geq \bar{\theta}$. So, the objective value is strictly increasing for $c_N \in (\underline{\theta}, \bar{\theta})$. It implies that $c_N^*(w_{N-1}) = \bar{\theta}$ for all $w_{N-1} \neq \mathbf{0}$. Next, assume that $c_i^*(w_{i-1}) = \bar{\theta}$ for all $w_{i-1} \neq \mathbf{0}$ and for all $i \in \{t+1, t+2, \cdots, N\}$. Then, we want to show that $c_t^*(w_{t-1}) = \bar{\theta}$ for all $w_{t-1} \neq \mathbf{0}$. Fix any w_{t-1}, terms in the objective function that involve $c_t(w_{t-1})$ are:

$$
\alpha \cdot Pr(w_{t-1})\Big(F(c_t) + F(c_t) \cdot V^*(w_{t-1}, 1) + (1 - F(c_t)) \cdot V^*(w_{t-1}, 0) \Big)
$$
$$
-Pr(w_{t-1})\Big[F(c_t)\Big(P^*(w_{t-1}, 1) + c_t - \int_{\underline{\theta}}^{c_t} F(\theta)d\theta \Big) + \big(1 - F(c_t)\big)\Big(P^*(w_{t-1}, 0) - \int_{\underline{\theta}}^{c_t} F(\theta)d\theta \Big) \Big]
$$

$V^*(w_{t-1}, 0)$ and $V^*(w_{t-1}, 1)$ should be the same because we assume $c_i^*(w_{i-1}) = \bar{\theta}$ for all $w_{i-1} \neq \mathbf{0}$ and for all $i \in \{t+1, t+2, \cdots, N\}$. This means all the future thresholds after the histories $(w_{t-1}, 0)$ and $(w_{t-1}, 1)$ are the same and equal to $\bar{\theta}$. In addition, $P^*(w_{t-1}, 0)$ and $P^*(w_{t-1}, 1)$ should equal each other because the work histories in and before period $t - 1$ are the same, and the future thresholds after period t are the same. Therefore, the terms in the objective function that involve c_t are:

$$
\alpha \cdot Pr(w_{t-1})F(c_t) - Pr(w_{t-1})\Big(F(c_t)c_t - \int_{\underline{\theta}}^{c_t} F(\theta)d\theta \Big)
$$
$$
= Pr(w_{t-1})\Big(\alpha F(c_t) - F(c_t)c_t + \int_{\underline{\theta}}^{c_t} F(\theta)d\theta \Big)
$$

The first order condition for c_t is: $Pr(w_{t-1})f(c_t)(\alpha - c_t) \geq 0$. Thus the unconstrained

optimal value for $c_t(w_{t-1})$ is also $\bar{\theta}$ when $w_{t-1} \neq \mathbf{0}$. By induction, we have shown that $c_i^*(w_{i-1}) = \bar{\theta}$ for all $w_{i-1} \neq \mathbf{0}$ and for all $i \geq 2$. Therefore, the optimal thresholds in the relaxed optimization problem are such that $c_i^*(w_{i-1}) = \bar{\theta}$ for all $w_{i-1} \neq \mathbf{0}$ and for all $i \geq 2$, and $c_i^*(\mathbf{0}) \in (\underline{\theta}, \bar{\theta})$ for all $i \in \mathcal{N}$. Next, we need to show that the relaxed optimal thresholds are feasible, i.e. the constraints in (3) all hold:

$$\sum_{i=1}^{t-1} x_i \cdot c_i(w_{i-1}) - \sum_{i=2}^{t} \int_{\underline{\theta}}^{c_i(w_{i-1})} F(\theta_i)d\theta_i \geq -\sum_{i=2}^{N} \int_{\underline{\theta}}^{c_i(\mathbf{0})} F(\theta_i)d\theta_i \quad \forall t \in \mathcal{N}, \forall w_t \in \{0,1\}^t$$

The constraint for any work history $w_{m+n} = (\mathbf{0}_m, \mathbf{1}_n)$ where $n \geq 2$ is equivalent to:

$$\sum_{i=1}^{t-1} x_i \cdot c_i(w_{i-1}) - \sum_{i=2}^{t} \int_{\underline{\theta}}^{c_i(w_{i-1})} F(\theta_i)d\theta_i + \sum_{i=2}^{N} \int_{\underline{\theta}}^{c_i(\mathbf{0})} F(\theta_i)d\theta_i \geq 0$$

$$\Leftrightarrow \quad c_{m+1}^*(\mathbf{0}) + \sum_{i=m+2}^{m+n-1} \bar{\theta} - \sum_{i=2}^{m+1} \int_{\underline{\theta}}^{c_i^*(\mathbf{0})} F(\theta_i)d\theta_i - \sum_{i=m+2}^{m+n} \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_i)d\theta_i + \sum_{i=2}^{N} \int_{\underline{\theta}}^{c_i(\mathbf{0})} F(\theta_i)d\theta_i \geq 0$$

$$\Leftrightarrow \quad c_{m+1}^*(\mathbf{0}) + \sum_{i=m+2}^{m+n-1} \left(\bar{\theta} - \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_i)d\theta_i \right) - \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_{m+n})d\theta_{m+n} + \sum_{i=m+2}^{N} \int_{\underline{\theta}}^{c_i(\mathbf{0})} F(\theta_i)d\theta_i \geq 0$$

$$(14)$$

Note that $\bar{\theta} - \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_i)d\theta_i = E(\theta)$, which comes from integration by parts. Thus, the left hand side of the inequality in (14) can be rewritten as:

$$c_{m+1}^*(\mathbf{0}) - \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_{m+n})d\theta_{m+n} + (n-2) \cdot E(\theta) + \sum_{i=m+2}^{N} \int_{\underline{\theta}}^{c_i(\mathbf{0})} F(\theta_i)d\theta_i$$

$$> c_{m+1}^*(\mathbf{0}) - \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_{m+n})d\theta_{m+n}$$

$$> \underline{\theta} - \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_{m+n})d\theta_{m+n}$$

$$= \underline{\theta} + E(\theta) - \bar{\theta}$$

$$\geq 0$$

The first inequality holds because $n \geq 2$, $N \geq m + n \geq m + 2$ and $c_i(\mathbf{0}) > \underline{\theta}$ for all i. The second inequality holds because $c_{m+1}^*(\mathbf{0}) > \underline{\theta}$ according to Lemma 7. The last inequality holds because of Assumption 1.

When $n \leq 1, m + n \geq 2$, the constraint is the following, which holds as $N \geq m + n$.

$$- \sum_{i=2}^{m+n} \int_{\underline{\theta}}^{c_i(\mathbf{0})} F(\theta_i) d\theta_i + \sum_{i=2}^{N} \int_{\underline{\theta}}^{c_i(\mathbf{0})} F(\theta_i) d\theta_i \geq 0$$

When $m + n = 1$, the constraint trivially holds as it becomes $\sum_{i=2}^{N} \int_{\underline{\theta}}^{c_i(\mathbf{0})} F(\theta_i) d\theta_i \geq 0$. So far, we have shown that the relaxed optimal thresholds are feasible. Therefore, the solution to the relaxed optimization problem is also the solution to the original optimization problem, which has the feature that $c_i^*(w_{i-1}) = \bar{\theta}$ for all $w_{i-1} \neq \mathbf{0}$ and for all $i \geq 2$, and $c_i^*(\mathbf{0}) \in (\underline{\theta}, \bar{\theta})$ for all $i \in \mathcal{N}$. It implies that the optimal mechanism is a consecutive-working menu: once the agent starts to work, he will work in every period until the end. Hence, we have shown that when $\alpha \geq \bar{\theta}$, the optimal mechanism is either a consecutive-working menu or an always-working mechanism. $\qquad \square$

A.13 Proof of Proposition 5

Proof. First, to establish the existence of the optimal mechanism, it suffices to show that a solution to the optimization problem (2). For any fixed α, the objective function is continuous with respect to all the thresholds $c_i(w_{i-1})$. The feasible set for each threshold compact, specifically $[\underline{\theta}, \bar{\theta}]$. Since there are finitely many such thresholds, the Cartesian product of these finitely many compact sets remains compact. By the Weierstrass Extreme Value Theorem, a continuous function on a compact set attains its maximum. Thus, the optimization problem (2) has a solution, ensuring the existence of the optimal mechanism.

Second, observe that the objective function is continuous on the product space $[\underline{\theta}, \bar{\theta}]^{2^N - 1} \times [\underline{\theta}, \infty)$, where $[\underline{\theta}, \infty)$ is the feasible set for α. Define $C : [\underline{\theta}, \infty) \to [\underline{\theta}, \bar{\theta}]^{2^N - 1}$ as the correspondence that maps α to the feasible thresholds. For any value of α, the feasible set for the optimal thresholds is always $[\underline{\theta}, \bar{\theta}]^{2^N - 1}$. Therefore, C is a continuous and compact-valued correspondence. By applying the Maximum Theorem, it follows that the principal's optimal expected payoff V^* is continuous with respect to α.

Next, V^* can be expressed as:

$$V^*(\alpha) = \alpha \cdot E_{\boldsymbol{\theta}} \left[\sum_{t=1}^{N} x_t^*(\alpha) \right] - E_{\boldsymbol{\theta}}[p_N^*(\alpha)]$$

where $\{x_t^*(\alpha)\}_{t=1}^N$ denote the optimal action rules for α, and $p_N^*(\alpha)$ is the corresponding payment rule. Then for any pair $\alpha_1 < \alpha_2$, when $\alpha = \alpha_2$, we can at least use the same action rules $\{x_t^*(\alpha_1)\}_{t=1}^N$, which results in a higher principal's expected payoff as α itself is higher. Therefore, V^* is increasing in α.

Lastly, since V^* is increasing, Lebesgue's theorem guarantees that V^* is differentiable almost everywhere. $\qquad\square$

A.14 Proof of Lemma 5

Proof. Terms involving c_1 in the principal's expected payoff under the consecutive-working menu are given by:

$$\alpha \cdot [F(c_1) \cdot N + (1 - F(c_1)) \cdot T^*(0)] - F(c_1)[c_1 + (N-1)E(\theta) + u_1^*(\bar{\theta})] - (1 - F(c_1)) \cdot P^*(0)$$

Here, N is the expected total working period if the agent works in the first period, while $T^*(0)$ is the expected total working period if the agent shirks in the first period. Similarly, $c_1 + (N-1)E(\theta) + u_1^*(\bar{\theta})$ is the expected total payment when the agent works in the first period, and $P^*(0)$ is the expected total payment when the agent shirks in the first period.[42] The first order condition for c_1 is:

$$\alpha \cdot f(c_1)(N - V^*(0)) - f(c_1)[c_1 + (N-1)E(\theta) + u_1^*(\bar{\theta})] - F(c_1) + f(c_1)P^*(0)$$
$$= f(c_1)\left\{\alpha \cdot N - [(N-1)E(\theta) + u_1^*(\bar{\theta})] - [\alpha \cdot T^*(0) - P^*(0)] - \frac{F(c_1)}{f(c_1)} - c_1\right\} \quad (15)$$

Given that the density $f(\theta) > 0$ on the support, to determine the sign of the first-order condition (FOC), we need only examine the sign of the term in the curly brackets. Note that the first part, $\alpha \cdot N - [(N-1)E(\theta) + u_1^*(\bar{\theta})] - [\alpha \cdot T^*(0) - P^*(0)]$, does not depend on c_1. Denote this term as M:

$$M = \alpha \cdot N - [(N-1)E(\theta) + u_1^*(\bar{\theta})] - [\alpha \cdot T^*(0) - P^*(0)]$$
$$= \underbrace{\left\{\alpha \cdot N - \overbrace{[c_1^* + (N-1)E(\theta) + u_1^*(\bar{\theta})]}^{\text{payment if start working in period 1}}\right\}}_{\text{principal's expected payoff if start working in period 1}} - \underbrace{[\alpha \cdot T^*(0) - P^*(0)]}_{\text{principal's expected payoff if start working after period 1}} + c_1^*$$

---

[42]Notably, if the agent shirks in the first period, the final payment does not include terms involving c_1. Consequently, $P^*(0)$represents the expected total payment in this scenario.

Since the principal's expected payoff can be viewed as the weighted average of her expected payoff if the agent starts working in period 1, with weight $F(c_1^*)$, and her expected payoff if the agent starts working after period 1, with weight $1 - F(c_1^*)$. The fact that $c_1^* \in (\underline{\theta}, \bar{\theta}]$ implies that $F(c_1^*) \in (0, 1]$, which in turn implies that the principal's expected payoff if the agent starts working in period 1 is at least as good as her expected payoff if the agent starts working after period 1. Otherwise, she can be better off by reducing c_1^*, which would violate its optimality. [43] Therefore, we have:

$$M \geq c_1^* > \underline{\theta}$$

Let's rewrite the term in the curly bracket in (15) as

$$M - \left[\frac{F(c_1)}{f(c_1)} + c_1 \right]$$

where M can be considered a constant, and the second part as a function of c_1. Now, let's consider the case when $\alpha = \bar{\theta}$. Then M becomes

$$\begin{aligned} M &= N \cdot \bar{\theta} - (N-1)E(\theta) - u_1^*(\bar{\theta}) - \left[\bar{\theta} \cdot V^*(0) - P^*(0) \right] \\ &< N \cdot \bar{\theta} - (N-1)E(\theta) \end{aligned} \tag{16}$$

The inequality holds because $u_1^*(\bar{\theta}) = \sum_{t=2}^{N} \int_{\underline{\theta}}^{c_t(\mathbf{0})^*} F(\theta) d\theta > 0$ and $\bar{\theta} \cdot T^*(0) - P^*(0) = \hat{V}^*(0) > 0$.

Assumption 2 establishes that the function $\frac{F(c_1)}{f(c_1)} + c_1$ is continuous and increasing. At $c_1 = \underline{\theta}$, this function equals $\underline{\theta}$. At $c_1 = \bar{\theta}$,

$$\begin{aligned} \frac{F(\bar{\theta})}{f(\bar{\theta})} + \bar{\theta} &= \frac{1}{f(\bar{\theta})} + \bar{\theta} \\ &\geq (N-1)[\bar{\theta} - E(\theta)] + \bar{\theta} \\ &= N\bar{\theta} - (N-1)E(\theta) \\ &> M \end{aligned}$$

where the first inequality follows from Assumption 2 and the last inequality follows from (16). Thus, the relationship between M and the function $\frac{F(c_1)}{f(c_1)} + c_1$ can be illustrated by

---

[43]Which also increases her expected payoff if the agent starts working in period 1.

Figure 3.

The function $\frac{F(c_1)}{f(c_1)} + c_1$ is below M when $c_1 = \underline{\theta}$. According to Assumption 2, this function is continuous and increasing, and it exceeds M when $c_1 = \bar{\theta}$. Therefore, $\frac{F(c_1)}{f(c_1)} + c_1$ intersects M exactly once from below. This implies that there is a unique interior point where the derivative of V^* with respect to c_1 is positive below this point and negative above it. Consequently, the optimal threshold c_1^* must be this interior point. Hence, by Theorem 1, the optimal mechanism when $\alpha = \bar{\theta}$ must be a consecutive-working menu. We arrived at the conclusion that V^{cm} exists and is strictly larger than V^{aw} when $\alpha = \bar{\theta}$.

$\qquad\square$

A.15 Lemma 8 and the proof

Lemma 8. *Let $\{c_t^*(\mathbf{0})\}_{t=1}^N$ be the solution to the system of equations (12). If $f(\theta)$ is continuous, $\frac{F(\theta)}{f(\theta)} + \theta$ is increasing, then the necessary and sufficient condition for $V^{cm} > V^{aw}$ at $\alpha = \bar{\theta}$ is that the solution satisfies the following condition:*

$$
\frac{1}{f(\bar{\theta})} > (N-1)(\bar{\theta} - E(\theta)) -
$$

$$
\sum_{t=2}^N \left(\Pi_{i=2}^{t-1}(1 - F(c_i^*(\mathbf{0}))) \right) \cdot \left[F(c_t^*(\mathbf{0}))(\bar{\theta} - c_t^*) + \int_{\underline{\theta}}^{c_t^*(\mathbf{0})} F(\theta)d\theta + F(c_t^*(\mathbf{0}))(N-t)(\bar{\theta} - E(\theta)) \right]
$$

Proof. According to Theorem 1, $V^{cm} > V^{aw}$ at $\alpha = \bar{\theta}$ if and only if c_1^* lies in $(\underline{\theta}, \bar{\theta})$. The condition listed in Lemma 8 is equivalent to M, defined in the proof of Lemma 5, being less than $\frac{F(\bar{\theta})}{f(\bar{\theta})} + \bar{\theta}$. It is further equivalent to the condition that $\frac{\partial V}{\partial c_1} < 0$ at $c_1 = \bar{\theta}$. According to the expression of $\frac{\partial V}{\partial c_1}$ derived in the proof of Lemma 5, given that $\frac{F(\theta)}{f(\theta)} + \theta$ is increasing and that $\frac{\partial V}{\partial c_1}|_{c_1=\underline{\theta}} > 0$, $\frac{\partial V}{\partial c_1}|_{c_1=\bar{\theta}} < 0$ is then the necessary and sufficient condition for the optimal c_1 to be an interior point of the support. This completes the proof. $\qquad\square$

A.16 Lemma 9 and the proof

Lemma 9. *When $V^* \neq V^{aw}$, $\frac{\partial V^*}{\partial \alpha} < N$.*

Proof. Since V^* can be expressed as the following:

$$
V^*(\alpha) = \alpha \cdot E_{\boldsymbol{\theta}} \left[\sum_{t=1}^N x_t^*(\alpha) \right] - E_{\boldsymbol{\theta}}[p_N^*(\alpha)]
$$

According to the envelope theorem, we have:

$$\frac{\partial V^*}{\partial \alpha} = E_{\boldsymbol{\theta}} \left[\sum_{t=1}^{N} x_t^*(\alpha) \right]$$

$E_{\boldsymbol{\theta}} \left[\sum_{t=1}^{N} x_t^*(\alpha) \right]$ denotes the agent's expected total working period under the optimal mechanism. When the optimal mechanism is not the always-working mechanism, the agent's expected total working period is strictly less than N. Therefore, $\frac{\partial V^*}{\partial \alpha} < N$ when $V^* \neq V^{aw}$. □

A.17 Proof of Theorem 2

Proof. To begin with, we examine some properties of the principal's expected payoff. First, Proposition 5 demonstrates that V^* always exists. However, V^{naw} may not exist. This is due to the fact that excluding the always-working mechanism renders the set of feasible thresholds non-compact; the removal of a single point causes the set of feasible thresholds to become non-closed. As a result, a restricted optimal mechanism may not exist in such cases. When it does not exist, V^{naw} is considered to be undefined.

Second, whenever V^{naw} exists, it must equal V^*. Assume, for contradiction, that V^{naw} exists but $V^* = V^{aw} > V^{naw}$. Let Δ denote the difference between V^{aw} and V^{naw}. Since the always-working mechanism is an extreme point of the compact set of feasible mechanisms and the principal's expected payoff is continuous with respect to each threshold, there must exist a non-always-working mechanism whose expected payoff is arbitrarily close to V^{aw}. Specifically, there must be some $\epsilon < \Delta$ such that a non-always-working mechanism achieves an expected payoff within ϵ of V^{aw}. This contradicts the assumption that V^{naw} is the maximum expected payoff among all non-always-working mechanisms. Therefore, V^{naw} must equal V^* whenever it exists.

Third, V^{naw} exists and equals V^* when $\alpha < \bar{\theta}$, since the always-working mechanism cannot be optimal in this case. Suppose, for contradiction, that the always-working mechanism is optimal. Consider decreasing the last threshold $c_N(\mathbf{1}_{N-1})$ from $\bar{\theta}$ to α. It can be verified that $U_1^*(\bar{\theta})$ remains unchanged at 0, while the principal's expected payoff strictly increases. [44] This contradicts the assumption that the always-working mechanism is op-

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[44]This is because the first-order condition of the principal's expected payoff with respect to $c_N(\mathbf{1}_{N-1})$ is $f(c_N)(\alpha - c_N)$, which indicates that the principal's expected payoff is maximized when $c_N = \alpha$.

timal. Therefore, since an optimal mechanism exists by Proposition 5, we conclude that $V^{naw} = V^*$ when $\alpha < \bar{\theta}$.

Figure 2 illustrates the principal's expected payoff under different mechanisms. The payoff under the always-working mechanism is given by $\alpha \cdot N - [\bar{\theta} + (N-1) \cdot E(\theta)]$, which is a linear and hence continuous function of α. According to Theorem 1, when $\alpha \geq \bar{\theta}$, the optimal non-always-working mechanism, if it exists, is a consecutive-working menu. This means that if $\alpha \geq \bar{\theta}$, the optimal mechanism is a consecutive-working menu whenever V^* exceeds V^{aw}, and it is an always-working mechanism when $V^* = V^{aw}$. Furthermore, Lemma 5 indicates that Assumption 2 guarantees that $V^{naw} = V^{cm}$ exists and is strictly greater than V^{aw} when $\alpha = \bar{\theta}$. Therefore, by the continuity of V^* established in Proposition 5, if the optimal mechanism is ever an always-working mechanism, this occurs for $\alpha > \bar{\theta}$. Consequently, there exists a range of α to the right of $\bar{\theta}$ such that the optimal mechanism is a consecutive-working menu.

To demonstrate that if the optimal mechanism switches to the always-working mechanism, it will not switch back to a consecutive-working menu, consider the following argument. Assume, for contradiction, that the optimal mechanism switches to the always-working mechanism at $\hat{\alpha}$ and then switches back to a consecutive-working menu at some $\alpha' > \bar{\theta}$, as depicted in Figure 5. By the continuity of V^*, we have $V^* = V^{aw}$ at α'. Since the optimal mechanism switches back to a consecutive-working menu at α', there must exist some $\alpha'' > \alpha'$ such that V^* is strictly greater than V^{aw} for $\alpha \in (\hat{\alpha}, \alpha'']$. By Proposition 5, V^* is absolutely continuous, and thus we have:

$$V^*(\alpha'') - V^*(\alpha') = \int_{\alpha'}^{\alpha''} \frac{dV^*}{d\alpha} d\alpha.$$

For V^{aw}, which is linear with respect to α, we have:

$$V^{aw}(\alpha'') - V^{aw}(\alpha') = \int_{\alpha'}^{\alpha''} N d\alpha.$$

63

Subtracting the second equation from the first gives:

$$
\begin{aligned}
&V^*(\alpha'') - V^*(\alpha') - (V^{aw}(\alpha'') - V^{aw}(\alpha')) \\
&= V^*(\alpha'') - V^{aw}(\alpha'') \\
&= \int_{\alpha'}^{\alpha''} \left(\frac{dV^*}{d\alpha} - N \right) d\alpha \\
&< 0
\end{aligned}
$$

The first equality follows from the fact that $V^* = V^{aw}$ at α'. The inequality follows from Lemma 9, which establishes that $\frac{dV^*}{d\alpha} < N$ whenever $V^* \neq V^{aw}$. This inequality contradicts the fact that $V^* > V^{aw}$ at α''.[45] Therefore, once the optimal mechanism switches to the always-working mechanism, it cannot switch back to a consecutive-working menu.
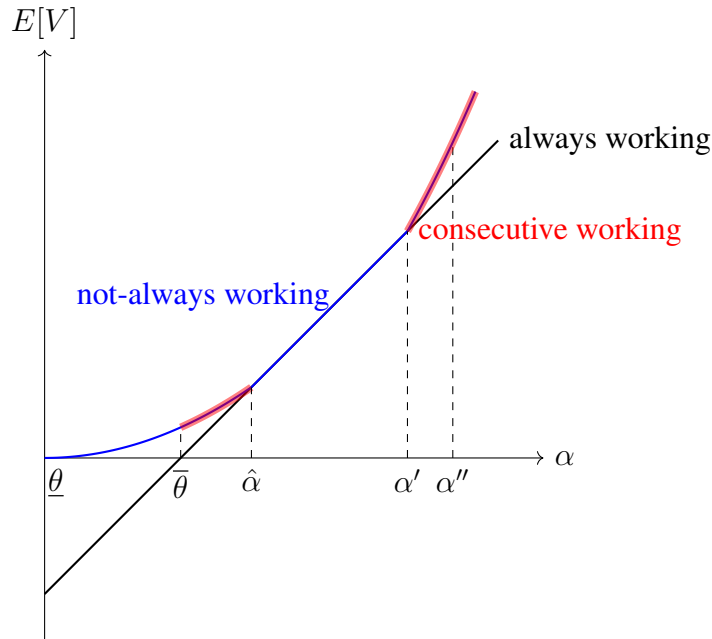

Figure 5: Counter example

Thus far, we have demonstrated that if the optimal mechanism transitions to the always-working mechanism, this transition occurs only once, and it does not revert to a consecutive-working menu. Next, we provide an example to illustrate that the optimal mechanism in-

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[45]Here, we do not directly compare the derivatives of V^{aw} and V^{cm} with respect to α at α' because V^{cm} might not exist at α'. Even if V^{cm} does exist at this point, its differentiability is not guaranteed. We only know that when it exists, it is differentiable almost everywhere.

deed switches to the always-working mechanism for sufficiently large α. According to Lemma 5, the derivative of V^* with respect to c_1 can be expressed as:

$$f(c_1)\Big[M - \frac{F(c_1)}{f(c_1)} - c_1\Big]$$

Here, M is a function of α and is defined as follows:

$$M = \alpha \cdot \big[N - T^*(0)\big] - (N-1)E(\theta) - u_1^*(\bar{\theta}) + P^*(0)$$

where $T^*(0)$ is the expected total working period when the agent shirks in the first period, so it is at most $N - 1$. Consequently, $N - T^*(0) \geq 1$. Thus, we have

$$
\begin{aligned}
M &\geq \alpha - (N-1)E(\theta) - u_1^*(\bar{\theta}) + P^*(0) \\
&> \alpha - (N-1)E(\theta) - \sum_{i=t}^{N} \int_{\underline{\theta}}^{c_t(\mathbf{0})} F(\theta)d\theta \\
&> \alpha - (N-1)E(\theta) - \sum_{i=t}^{N} \int_{\underline{\theta}}^{\bar{\theta}} F(\theta)d\theta \\
&= \alpha - (N-1)\cdot\bar{\theta}
\end{aligned}
$$

where the second inequality arises because $P^*(0) > 0$ and $u_1^*(\bar{\theta}) \leq \sum_{i=t}^{N} \int_{\underline{\theta}}^{c_t(\mathbf{0})} F(\theta)d\theta$. Specifically, $u_1^*(\bar{\theta}) = 0$ under the always-working mechanism, and $u_1^*(\bar{\theta}) = \sum_{i=t}^{N} \int_{\underline{\theta}}^{c_t(\mathbf{0})} F(\theta)d\theta$ under the consecutive-working menu. The last inequality follows from the fact that $c_t(\mathbf{0}) < \bar{\theta}$ for all $t \in \mathcal{N}$ under the consecutive-working menu. The last equality is due to $\int_{\underline{\theta}}^{\bar{\theta}} F(\theta)d\theta = \bar{\theta} - E(\theta)$. Thus, M is strictly greater than $\alpha - (N-1)\cdot\bar{\theta}$. Additionally, the function $\frac{F(c_1)}{f(c_1)} + c_1$ is continuous and increasing by Assumption 2. Therefore, we have

$$
\begin{aligned}
f(c_1)&\Big[M - \frac{F(c_1)}{f(c_1)} - c_1\Big] \\
&> f(c_1)\Big[\alpha - (N-1)\cdot\bar{\theta} - \frac{F(\bar{\theta})}{f(\bar{\theta})} - \bar{\theta}\Big]
\end{aligned}
$$

When α exceeds $(N-1)\cdot\bar{\theta} + \frac{F(\bar{\theta})}{f(\bar{\theta})} + \bar{\theta}$, the derivative is positive for all $c_1 \in \Theta$. This indicates that the principal's expected payoff is strictly increasing in c_1 on the support, implying that $c_1^* = \bar{\theta}$, which corresponds to the always-working mechanism. Therefore, the optimal mechanism will eventually switch to the always-working mechanism for sufficiently large

values of α.

In conclusion, the optimal mechanism switches once and only once to the always-working mechanism at $\hat{\alpha}$. Specifically, the optimal mechanism is a consecutive-working menu when $\alpha \in [\bar{\theta}, \hat{\alpha})$, and it becomes an always-working mechanism when $\alpha \geq \hat{\alpha}$.

$\square$

A.18 Proof of Proposition 6

Proof. To start with, let's show that in the optimal deterministic mechanism, the threshold c_1^* is greater than x^{SB}, where x^{SB} is the optimal threshold when there is only one period [46]. When $N = 1$, then the optimization problem is:

$$\max_{x_1(\theta_1)} \int_{\underline{\theta}}^{\bar{\theta}} \left[\alpha - \theta_1 - \frac{F(\theta_1)}{f(\theta_1)} \right] x_1(\theta_1) f(\theta_1) d\theta_1$$

Denote $\theta + \frac{F(\theta)}{f(\theta)}$ by $G(\theta)$, which is increasing according to Assumption 2. The optimal action rule is:

$$x_1(\theta_1) = \begin{cases} 0, & \text{if } G(\theta_1) > \alpha \\ 1, & \text{if } G(\theta_1) \leq \alpha \end{cases}$$

Then, the optimal threshold is: $x^{SB} = G^{-1}(\alpha)$.

Let's return to the multiple-period case under deterministic rules. According to Lemma 5, $\frac{\partial V}{\partial c_1} = f(c_1)\left[M - \frac{F(c_1)}{f(c_1)} - c_1\right]$, where $M = \alpha \cdot N - \left[(N-1)E(\theta) + u_1^*(\bar{\theta})\right] - \left[\alpha \cdot T^*(0) - P^*(0)\right]$. When the optimal deterministic mechanism is a consecutive-working menu, then $c_1^* = G^{-1}(M)$, which is interior. We now aim to show that $c_1^* > x^{SB}$, which is equivalent to showing that $M > \alpha$. This is true because M can be rewritten as:

$$\begin{aligned} M &= \alpha + \left\{ \left[\alpha \cdot (N-1) - (N-1)E(\theta) - u_1^*(\bar{\theta})\right] - \left[\alpha \cdot T^*(0) - P^*(0)\right] \right\} \\ &= \alpha + (\hat{V}(1) - \hat{V}(0)) \\ &> \alpha \end{aligned}$$

The inequality holds because, conditional on the same $u_1^*(\bar{\theta})$, the principal can achieve the first best in every subsequent period if the agent works in the first period. The inequality is

---

[46]The abbreviation SB stands for "second best".

strict, because if the agent shirks in the first period, the principal's expected payoff is strictly less than the first best given that the threshold for starting to work in every subsequent period is interior. Thus, $c_1^* > x^{SB}$.

Next, to show that there exist stochastic mechanisms that outperform the optimal deterministic mechanism, consider alternating the optimal deterministic mechanism with the following stochastic one. The stochastic mechanism retains the same action and payment rules as the optimal deterministic mechanism for θ_1 below c_1^*. For θ_1 above the threshold c_1^*, the action rule in the first period is changed as follows:

$$q_1(\theta_1) = \begin{cases} 1, & \text{if } \theta_1 \in [\underline{\theta}, x^{SB}) \\ 1 - \epsilon, & \text{if } \theta_1 \in [x^{SB}, c_1^*] \end{cases}.$$

Here, we modify the recommended action from always working for types below c_1^* to always working for types below x^{SB} and working with probability $1 - \epsilon$ for types in $[x^{SB}, c_1^*]$, where ϵ is a small positive number. Let $u_1^*(\bar{\theta})$ denote the $u_1(\bar{\theta})$ in the optimal deterministic mechanism. [47] In this context, ϵ can be any number in $(0, \frac{u_1^*(\bar{\theta})}{\bar{\theta} - E(\theta)}]$. Furthermore, the action rules in subsequent periods for $\theta_1 \in [\underline{\theta}, c_1^*]$ remain the same as in the optimal deterministic mechanism for both realizations of $\tilde{x}_1$, i.e., working in every period ($q_t = 1$ for all t). The payment rule for $\theta_1 \in [\underline{\theta}, x^{SB})$ is adjusted to:

$$p_N = \theta_1 + \int_{\theta_1}^{\bar{\theta}} q_1(\theta_1) d\theta_1 + u_1^*(\bar{\theta}) + (N - 1)E(\theta).$$

For $\theta_1 \in [x^{SB}, c_1^*]$, since $\tilde{x}_1$ is a random variable, we need to specify the corresponding payments for both realizations of $\tilde{x}_1$. Let the payment rule be as follows:

$$p_N = \begin{cases} \theta_1 + (N - 1)E(\theta), & \text{if } x_1 = 1 \\ \frac{u_1^*(\bar{\theta}) + \int_{\theta_1}^{\bar{\theta}} q_1(\theta_1) d\theta_1}{\epsilon} + (N - 1)E(\theta), & \text{if } x_1 = 0 \end{cases}$$

We then verify the feasibility of this stochastic mechanism, i.e., the periodic IC and the periodic ex-post IR constraints are satisfied, by discussing the following three cases:

Case 1: $\theta_1 \notin [\underline{\theta}, x^{SB})$ and $t > 1$. Both the action rules and the payment rule remain the same as the optimal deterministic mechanism. Thus, the periodic IC and IR constraints are

---

[47] $u_1^*(\bar{\theta}) = \sum_{i=n}^{N} \int_{\underline{\theta}}^{c_i^*(\mathbf{0})} F(\theta_i) d\theta_i$, where $c_i^*(\mathbf{0})$ are from the original deterministic mechanism.

identical to those in the optimal deterministic mechanism and therefore are satisfied.

Case 2: $\theta_1 \in [\underline{\theta}, x^{SB})$ and $t > 1$. Here, the agent is prescribed to work for all cost realizations. Since the final payment does not depend on θ_t for all $t > 1$, thus, the periodic IC constraints are trivially satisfied. For the periodic IR constraint with $x_1 = 1$ in period t, we have:

$$
\begin{aligned}
-\theta_t - (N-t)E(\theta) + p_N \geq & -\theta_t - (N-t)E(\theta) + \theta_1 + (N-1)E(\theta) \\
= & \theta_1 + (t-1)E(\theta) - \theta_t \\
\geq & \underline{\theta} + E(\theta) - \bar{\theta} \\
\geq & 0
\end{aligned}
$$

The final inequality follows from Assumption 1. For the periodic IR constraint with $x_1 = 0$ in period t, we have

$$
-\theta_t - (N-t)E(\theta) + \frac{u_1^*(\bar{\theta}) + \int_{\theta_1}^{\bar{\theta}} q_1(\theta_1)d\theta_1}{\epsilon} + (N-1)E(\theta) \geq 0
$$

$$
\Leftrightarrow \quad \frac{u_1^*(\bar{\theta}) + \int_{\theta_1}^{\bar{\theta}} q_1(\theta_1)d\theta_1}{\epsilon} + (t-1)E(\theta) \geq \theta_t
$$

Given that ϵ is chosen from $(0, \frac{u_1^*(\bar{\theta})}{\bar{\theta} - E(\theta)}]$, it follows that $\frac{u_1^*(\bar{\theta}) + \int_{\theta_1}^{\bar{\theta}} q_1(\theta_1)d\theta_1}{\epsilon} > \bar{\theta} - E(\theta)$. Thus, the left-hand side of the inequality is greater than $\bar{\theta} + (t-2)E(\theta)$, which exceeds $\bar{\theta}$. Therefore, the periodic IR constraint for $x_1 = 0$ is also satisfied.

Case 3: $t = 1$. IC constraint in the first period requires that $u_1(\theta_1) = u_1(\bar{\theta}) + \int_{\theta_1}^{\bar{\theta}} q_1(\theta_1)d\theta_1$. When $\theta_1 \in (c_1^*, \bar{\theta}]$, by construction, $u_1(\theta_1) = u_1^*(\bar{\theta})$. Thus, we need to verify that for $\theta_1 \in [\underline{\theta}, c_1^*]$, $u_1(\theta_1) = u_1^*(\bar{\theta}) + \int_{\theta_1}^{\bar{\theta}} q_1(\theta_1)d\theta_1$. Specifically, for $\theta_1 \in [\underline{\theta}, x^{SB})$, we have:

$$
\begin{aligned}
u_1(\theta_1) &= p_N(h_N) - \theta_1 - (N-1)E(\theta) \\
&= \theta_1 + \int_{\theta_1}^{\bar{\theta}} q_1(\theta_1)d\theta_1 + u_1^*(\bar{\theta}) + (N-1)E(\theta) - \theta_1 - (N-1)E(\theta) \\
&= u_1^*(\bar{\theta}) + \int_{\theta_1}^{\bar{\theta}} q_1(\theta_1)d\theta_1
\end{aligned}
$$

For $\theta_1 \in [x^{SB}, c_1^*]$, we have:

$$
u_1(\theta_1) = q_1(\theta_1) \cdot \Big\{ \overbrace{\theta_1 + (N-1)E(\theta)}^{\text{final payment if } x_1 = 1} - \overbrace{\big[\theta_1 + (N-1)E(\theta)\big]}^{\text{expected total work cost if } x_1 = 1} \Big\} +
$$

$$
(1 - q_1(\theta_1)) \cdot \Big[\overbrace{\frac{u_1^*(\bar{\theta}) + \int_{\theta_1}^{\bar{\theta}} q_1(\theta_1)d\theta_1}{\epsilon} + (N-1)E(\theta)}^{\text{final payment if } x_1 = 0} - \overbrace{(N-1)E(\theta)}^{\text{expected total work cost if } x_1 = 0} \Big]
$$

$$
= (1 - q_1(\theta_1)) \cdot \frac{u_1^*(\bar{\theta}) + \int_{\theta_1}^{\bar{\theta}} q_1(\theta_1)d\theta_1}{\epsilon}
$$

$$
= \epsilon \cdot \frac{u_1^*(\bar{\theta}) + \int_{\theta_1}^{\bar{\theta}} q_1(\theta_1)d\theta_1}{\epsilon}
$$

$$
= u_1^*(\bar{\theta}) + \int_{\theta_1}^{\bar{\theta}} q_1(\theta_1)d\theta_1
$$

Thus, the IC constraints in the first period are satisfied. Next, we need to verify that the IR constraints in the first period are satisfied. This is straightforward because $u_1(\theta_1) = u_1^*(\bar{\theta}) + \int_{\theta_1}^{\bar{\theta}} x_1(\theta_1)d\theta_1$, which is always at least $u_1^*(\bar{\theta})$. Since $u_1^*(\bar{\theta})$ is positive, the IR constraints are met.

Lastly, we need to demonstrate that the principal's expected payoff strictly increases under this stochastic mechanism. The change in the principal's expected payoff is:

$$
\Delta = \overbrace{F(x^{SB}) \cdot \epsilon \cdot (c_1^* - x^{SB})}^{\text{Saved payment from types } \theta_1 \in [\underline{\theta}, x^{SB})} + \overbrace{(F(c_1^*) - F(x^{SB})) \cdot \epsilon \cdot c_1^*}^{\text{Saved payment from types } \theta_1 \in [x^{SB}, c_1^*]}
$$

$$
- \alpha \cdot \overbrace{\epsilon \cdot (F(c_1^*) - F(x^{SB}))}^{\text{Reduced working time}}
$$

$$
= \epsilon \cdot \Big\{ F(c_1^*) \cdot c_1^* - \alpha F(c_1^*) - \big[F(x^{SB}) \cdot x^{SB} - \alpha F(x^{SB}) \big] \Big\}
$$

Let $H(\theta) = F(\theta) \cdot \theta - \alpha F(\theta)$. Then, $\Delta = \epsilon(H(c_1^*) - H(x^{SB}))$. We have:

$$
\frac{\partial H(\theta)}{\partial \theta} = f(\theta)\Big(\theta + \frac{F(\theta)}{f(\theta)} - \alpha\Big) = f(\theta)(G(\theta) - \alpha)
$$

Since $G(\theta)$ is increasing and $G(x^{SB}) = \alpha$. It follows that $f(\theta)(G(\theta) - \alpha) \geq 0$ for $\theta \in (x^{SB}, c_1^*]$. Therefore, $H(\theta)$ is increasing in $(x^{SB}, c_1^*]$. Hence, $\Delta > 0$.[48] This completes the

---

[48]The inequality must be strict. If it were not, then $G(\theta) = \alpha$ for all $\theta \in (x^{SB}, c_1^*]$, which contradicts the

proof. □

A.19 Proof of Proposition 7

Proof. First, similar to the deterministic case, IC end envelope theorem imply that, for any given history h_{t-1} and w_{t-1}, we have:

$$u_t(h_{t-1}, w_{t-1}; \theta_t) = u_t(h_{t-1}, w_{t-1}; \bar{\theta}) + \int_{\theta_t}^{\bar{\theta}} q_t(h_{t-1}, w_{t-1}; \theta_t) d\theta_t$$

In addition, by definition, $u_t(h_{t-1}, w_{t-1}; \theta_t)$ can also be expressed as:

$$
\begin{aligned}
&u_t(h_{t-1}, w_{t-1}; \theta_t) \\
=&q_t(h_{t-1}, w_{t-1}; \theta_t) \cdot u_t(h_{t-1}, w_{t-1}; \theta_t, x_t = 1) + (1 - q_t(h_{t-1}, w_{t-1}; \theta_t)) \cdot u_t(h_{t-1}, w_{t-1}; \theta_t, x_t = 0) \\
=&q_t(h_{t-1}, w_{t-1}; \theta_t) \left[-\theta_t + u_{t+1}(h_t, (w_{t-1}, 1); \bar{\theta}) + \int_{\underline{\theta}}^{\bar{\theta}} \int_{\theta_{t+1}}^{\bar{\theta}} q_{t+1}(h_t, (w_{t-1}, 1); \theta_{t+1}) d\theta_{t+1} dF(\theta_{t+1}) \right] + \\
&(1 - q_t(h_{t-1}, w_{t-1}; \theta_t)) \left[u_{t+1}(h_t, (w_{t-1}, 0); \bar{\theta}) + \int_{\underline{\theta}}^{\bar{\theta}} \int_{\theta_{t+1}}^{\bar{\theta}} q_{t+1}(h_t, (w_{t-1}, 0); \theta_{t+1}) d\theta_{t+1} dF(\theta_{t+1}) \right] \\
=&q_t(h_{t-1}, w_{t-1}; \theta_t) \left[-\theta_t + u_{t+1}(h_t, (w_{t-1}, 1); \bar{\theta}) + \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_{t+1}) x_{t+1}(h_t, (w_{t-1}, 1); \theta_{t+1}) d\theta_{t+1} \right] + \\
&(1 - q_t(h_{t-1}, w_{t-1}; \theta_t)) \left[u_{t+1}(h_t, (w_{t-1}, 0), \bar{\theta}) + \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_{t+1}) x_{t+1}(h_t, (w_{t-1}, 0); \theta_{t+1}) d\theta_{t+1} \right]
\end{aligned}
$$

Therefore, we have:

$$
\begin{aligned}
&u_t(h_{t-1}, w_{t-1}; \bar{\theta}) + \int_{\theta_t}^{\bar{\theta}} q_t(h_{t-1}, w_{t-1}; \theta_t) d\theta_t = \\
&q_t(h_{t-1}, w_{t-1}; \theta_t) \left[-\theta_t + u_{t+1}(h_t, (w_{t-1}, 1); \bar{\theta}) + \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_{t+1}) x_{t+1}(h_t, (w_{t-1}, 1); \theta_{t+1}) d\theta_{t+1} \right] + \\
&(1 - q_t(h_{t-1}, w_{t-1}; \theta_t)) \left[u_{t+1}(h_t, (w_{t-1}, 0), \bar{\theta}) + \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_{t+1}) x_{t+1}(h_t, (w_{t-1}, 0); \theta_{t+1}) d\theta_{t+1} \right]
\end{aligned}
$$

It implies that, for any given history, $u_t(h_{t-1}, w_{t-1}; \bar{\theta})$ is no longer uniquely determined by the action rules and $u_{t-1}(h_{t-2}, w_{t-2}; \bar{\theta})$. Specifically, we no longer have a one-to-

---

fact that $G(\bar{\theta}) > M > \alpha$ as ensure by Assumption 2.

one mapping between $u_t(h_{t-1}, \bar{\theta})$ and $u_{t+1}(h_t, \bar{\theta})$ as in the deterministic case. Instead, $u_t(h_{t-1}, w_{t-1}; \bar{\theta})$ maps to two constants: $u_{t+1}(h_t, (w_{t-1}, 1); \bar{\theta})$ and $u_{t+1}(h_t, (w_{t-1}, 0); \bar{\theta})$. As long as the above relationship holds, and all the periodic IC and IR constraints are satisfied, any set of $\left\{ u_{t+1}(h_t, (w_{t-1}, 1); \bar{\theta}), u_{t+1}(h_t, (w_{t-1}, 0); \bar{\theta}) \right\}$ can be implementable. We argue that the optimal way to implement the action rule is to set the following equal to zero:

$$u_t(h_{t-1}, w_{t-1}; \theta_t, \tilde{x}_t = 1) = -\theta_t + u_{t+1}(h_t, (w_{t-1}, 1), \bar{\theta}) + \int_{\underline{\theta}}^{\bar{\theta}} F(\theta_{t+1}) q_{t+1}(h_t, (w_{t-1}, 1), \theta_{t+1}) d\theta_{t+1}$$

(17)

First, let's verify that setting (17) to zero is without loss. This is because the principal can still achieve the first best in all subsequent periods when $x_t = 1$.[49] Specifically, the principal provides a final payment of $\theta_t + (N - t)E(\theta)$ to the agent and assigns him to work for sure in all subsequent periods. This approach is feasible as all the periodic IR constraints hold under Assumption 1. It can be seen as the principal achieving the first best by paying the agent the average cost and ensuring that he works for sure in all subsequent periods. By setting (17) to zero, the principal effectively transfers all the informational rent needed to induce the agent to work in periods before (and including) period t to the case where the agent shirks in period t, he information rent in this scenario equals:

$$\frac{u_t(h_{t-1}, w_{t-1}; \bar{\theta}) + \int_{\theta_t}^{\bar{\theta}} q_t(h_{t-1}, w_{t-1}; \theta_t) d\theta_t}{1 - q_t(h_{t-1}, w_{t-1}; \theta_t)}$$

(18)

We can add this additional amount to the final payment for all future realizations after $x_t = 0$. It is intuitive that the action and payment rules that were previously implementable remain implementable after increasing the final payments. Specifically, all the IC constraints continue to hold as before, since the incremental payment applies to all future realizations and does not affect future incentives. Moreover, the IR constraints are relaxed when we increase the final payments. Therefore, it is without loss of generality to transfer all the information rent to the case when the agent shirks in period t.

Next, we need to show that the principal can sometimes be strictly better off when $u_t(h_{t-1}, w_{t-1}; \theta_t, x_t = 1)$ is set to zero. If the first-best outcome is already achieved after $x_t = 0$, then transfering the promised information rents won't have a bite. However, if the first-best outcome is not achieved, doing so can strictly increase the principal's expected

---

[49]We use the term "still" here because it's straightforward to verify that the principal can more easily achieve the first best when $u_t(h_{t-1}, w_{t-1}; \theta_t, \tilde{x}_t = 1) \geq 0$.

payoff. In this scenario, there must exist some θ_{t+1} such that $q_{t+1}(h_t, (w_t, 0); \theta_t) \neq 1$. Let Δ denote the increase in payment for the case when $x_t = 0$ after setting (17) to zero. We can then consider adjusting q_{t+1} as illustrated by the following graph.
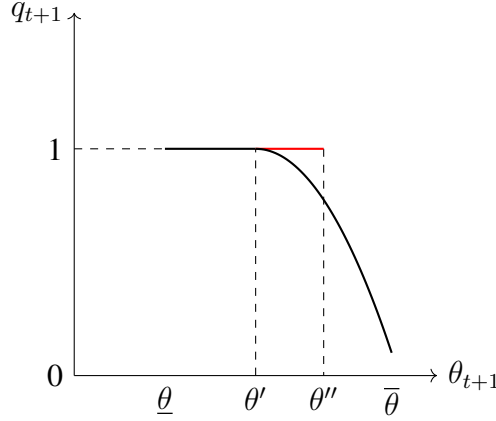
We extend the largest type such that $q_{t+1} = 1$ from θ' to a larger type θ'' such that $\int_{\theta'}^{\theta''} (1 - q_{t+1}) F(\theta_{t+1}) d\theta_{t+1} = \Delta$.[50] We keep all future action rules unchanged and leave the final payments the same for $\theta_{t+1} \in (\theta'', \bar{\theta}]$. For $\theta_{t+1} \in [\underline{\theta}, \theta'']$, we increase the final payments by $(1 - q_{t+1}(\theta_{t+1})) \cdot \theta_{t+1} + \int_{\theta_{t+1}}^{\theta''} (1 - q_{t+1}(\theta_{t+1})) d\theta_{t+1}$. After this adjustment, the increase in the principal's expected payment is:

$$\int_{\underline{\theta}}^{\theta''} \left[(1 - q_{t+1}(\theta_{t+1})) \cdot \theta_{t+1} + \int_{\theta_{t+1}}^{\theta''} (1 - q_{t+1}(\theta_{t+1})) d\theta_{t+1} \right] dF(\theta_{t+1})$$

$$= \int_{\underline{\theta}}^{\theta''} (1 - q_{t+1}(\theta_{t+1})) \cdot \theta_{t+1} dF(\theta_{t+1}) + \int_{\underline{\theta}}^{\theta''} \int_{\theta_{t+1}}^{\theta''} (1 - q_{t+1}(\theta_{t+1})) d\theta_{t+1} dF(\theta_{t+1})$$

$$= \int_{\underline{\theta}}^{\theta''} (1 - q_{t+1}(\theta_{t+1})) \cdot \theta_{t+1} f(\theta_{t+1}) d\theta_{t+1} + \int_{\underline{\theta}}^{\theta''} (1 - q_{t+1}(\theta_{t+1})) F(\theta_{t+1}) d\theta_{t+1}$$

$$= \int_{\underline{\theta}}^{\theta''} (1 - q_{t+1}(\theta_{t+1})) \cdot \theta_{t+1} f(\theta_{t+1}) d\theta_{t+1} + \Delta$$

The term Δ is covered by the promised information rent shifted from the case when $x_t = 1$ to the case when $x_t = 0$. Therefore, the out-of-pocket extra payment is $\int_{\underline{\theta}}^{\theta''} (1 - q_{t+1}(\theta_{t+1})) \cdot \theta_{t+1} f(\theta_{t+1}) d\theta_{t+1}$. Meanwhile, the principal's expected gain from the agent's additional working time is

$$\alpha \cdot \int_{\underline{\theta}}^{\theta''} (1 - q_{t+1}(\theta_{t+1})) f(\theta_{t+1}) d\theta_{t+1}$$

---

[50]If such θ' does not exist, then we extend from $\underline{\theta}$.

Since $\alpha \geq \bar{\theta}$, this gain exceeds the expected extra payment, making the principal strictly better off.

Next, we need to demonstrate the feasibility of this modification. This adjustment maintains all future IC constraints, as it does not affect future incentives. For period $t + 1$, the payments are adjusted so that the envelope expression for the expected payoff still holds. For periods before $t + 1$, IC constraints remain unaffected because the agent's expected payoffs involving the path of h_t and w_t remain unchanged. Specifically, after the adjustment, the agent's expected cost increases by $\int_{\underline{\theta}}^{\theta''} (1 - q_{t+1}(\theta_{t+1})) \cdot \theta_{t+1} f(\theta_{t+1}) d\theta_{t+1}$, and he is compensated for exactly this amount.

For participation constraints. The current and all future periodic IR constraints for $\theta_{t+1} \in (\theta'', \bar{\theta}]$ remain unchanged. For $\theta_{t+1} \in [\underline{\theta}, \theta'']$, the agent is assigned to work with higher probability but is compensated for the additional cost $(1 - x_{t+1}(\theta_{t+1})) \cdot \theta_{t+1}$. In fact, the agent receives extra information rent amounting to $\int_{\theta_{t+1}}^{\theta''} (1 - x_{t+1}(\theta_{t+1})) d\theta_{t+1}$. Consequently, both the current and all future periodic IR constraints are relaxed. The IR constraints before period $t + 1$ are unaffected since the agent's expected payoffs in earlier periods remain unchanged. Thus, the modification is feasible. This completes the proof.

$\square$