Equivalence between individual and group strategy-proofness under stability*

Pinaki Mandal†

Department of Economics, Arizona State University, USA

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Abstract

When policymakers implement mechanisms in real-world institutions, they often prefer strategy-proof mechanisms over manipulable ones. The Boston School Committee, for instance, replaced the Boston mechanism with the students-proposing deferred acceptance algorithm in July 2005 to eliminate students' incentives to misrepresent their preferences over schools. However, strategy-proof mechanisms are not always immune to manipulations by potential coalitions, even if these coalitions are small and easy to coordinate. This danger is avoided under group strategy-proof mechanisms.

In this paper, we study group strategy-proofness of stable matching mechanisms in two-sided matching markets when both sides of the market have strategic agents. In the context of a one-to-one matching market, we show that incorporating strategy-proofness for any stable matching mechanism not only removes the incentive for individual agents to manipulate, but also eliminates the incentive for any group of agents to manipulate (the group may include agents from both sides of the market), therefore implying group strategy-proofness. We establish this result assuming that the domain is sufficiently varied. We also explore the possibility of extending our findings to many-to-one matching markets.

Keywords: Two-sided matching; Stability; Strategy-proofness; Group strategy-proofness

JEL Classification: C78; D71; D82

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†E-mail: *pinaki.mandal@asu.edu*

1 Introduction

When designing mechanisms for strategic agents, it is crucial to ensure that the incentives are aligned. *Strategy-proofness* is the standard requirement, which requires that it should be in every agent's best interest to tell the truth. However, even if a mechanism is strategy-proof, it could still be vulnerable to manipulation by coalitions of agents, even if these coalitions are small and easy to coordinate. This is where *group strategy-proofness* comes in – it ensures that no group of agents can benefit from dishonesty, even if they coordinate their efforts. Therefore, while group strategy-proofness is a more stringent requirement than strategy-proofness, it is also more desirable. After all, what use would it be to guarantee that no single agent could manipulate if a few of them could jointly manipulate? This prompts some important questions. When is a mechanism that is strategy-proof also group strategy-proof? What types of preference domains ensure that these two requirements are equivalent? This paper provides a comprehensive solution to these questions for two-sided matching markets.

The theory of two-sided matching markets has captured the attention of researchers due to its real-world applications, such as assigning graduates to residency programs, students to colleges, or workers to firms. In this paper, we primarily focus on the well-known *marriage problem* (Gale and Shapley, 1962), which is a one-to-one matching market. This market comprises two finite and disjoint sets of agents, namely "men" and "women". Each agent on one side of the market has a (strict) preference over the agents on the other side and the *outside option*, where the outside option refers to the possibility of remaining unmatched. A matching between men and women is selected based on the agents' preferences, where each agent on one side of the market can only be matched with at most one agent on the other.

One important feature of our setting is that both strategy-proofness and group strategy-proofness are defined for the entire market, not just for one side of the market. To better understand this, let us consider the *deferred acceptance (DA) algorithm* (Gale and Shapley, 1962). While the DA algorithm is group strategy-proof for the proposers' side on the unrestricted domain (i.e., on the domain where every possible preference for an agent is admissible for that agent), it does not even guarantee strategy-proofness for all agents (see Dubins and Freedman (1981) and Roth (1982) for details). Henceforth, when we say a matching rule is (group) strategy-proof, we mean that the matching rule is (group) strategy-proof for *all* agents. Additionally, it is worth noting that a coalition may consist of both men and women.

1.1 Overview of our results

Before we delve into our results, let us first introduce the notion of *stability* (Gale and Shapley, 1962), which has been considered a desirable property to be satisfied by any matching.[1] A matching is stable if no individual

[1] In real-world applications, empirical studies have shown that stable mechanisms often succeed, while unstable ones fail. See Roth (2002) for a summary of this evidence.

agent prefers remaining unmatched to his/her current match, and no pair of agents – one on each side – would rather be matched to each other than to their present match.

Our paper showcases that strategy-proofness and group strategy-proofness hold the same significance for any stable matching rule, provided that the domain is sufficiently varied. We present two equivalence results (Theorems 1 and 2) to support our claim. Before discussing these equivalence results, we first review some results related to the DA algorithm, which establish the basis for the equivalence proofs. These results are also significant in their own right.

Recall that the *men-proposing DA (MPDA) algorithm* cannot be manipulated by any coalition of men, while it is not even strategy-proof for women on the unrestricted domain. Therefore, whenever a coalition manipulates the MPDA algorithm, it must include at least one woman. In Proposition 1, we show that the manipulative coalition not only contains at least one woman but must consist *only* of women, extending the result of Dubins and Freedman (1981).

We further explore how manipulation by a coalition affects the MPDA algorithm from the agents' perspective. Let us assume that a coalition of agents manipulates the MPDA algorithm. As per Proposition 1, we know that this manipulative coalition comprises only women, and therefore, these women will strictly benefit from the manipulation. In Proposition 2, we demonstrate a stronger result: every woman in the market, not just those in the manipulative coalition, weakly benefits, while every man weakly suffers from the manipulation. We also demonstrate that the manipulation does not affect the set of unmatched agents (Proposition 3). In other words, the set of unmatched agents before and after the manipulation remains the same. One key implication of Proposition 3 is that an unmatched agent cannot be part of a manipulative coalition for the DA algorithm.

We now focus on our main contribution – establishing the equivalence of strategy-proofness and group strategy-proofness for any stable matching rule on sufficiently varied domains. As we mentioned earlier, two equivalence results are presented to support this claim. The first equivalence result (Theorem 1) is obtained assuming the domain satisfies a richness condition, called *unrestricted tops with unique acceptability*, for both sides of the market. Unrestricted tops with unique acceptability for men roughly requires that for every man and every woman, the man has an admissible preference that ranks the woman first and the outside option second. For instance, if every possible preference is admissible for every man, the corresponding domain satisfies unrestricted tops with unique acceptability for men. Other examples of such domains include when the sets of admissible preferences for men comprise all *single-peaked preferences* (Black, 1948).

In our second equivalence result (Theorem 2), we require a stronger domain richness condition, called *unrestricted top pairs* (Alva, 2017), but only for one side of the market, not both. Unrestricted top pairs roughly requires that for every ordered pair of outcomes for an agent, there is an admissible preference that ranks them first and second. For instance, if every possible preference is admissible for every man, the corresponding domain satisfies unrestricted top pairs for men. Other examples of such domains include when the sets of admissible preferences for men comprise all *double-peaked preferences* (Flowers, 1975).

Our equivalence results are established on a characterization of the MPDA algorithm. In Theorem 3, we show that the MPDA algorithm is the *only* stable matching rule that satisfies strategy-proofness for men on a domain satisfying unrestricted tops with unique acceptability for men. To understand why this is the case, consider a different stable matching rule, φ, that matches a man to a different woman than the MPDA algorithm does. Due to the lattice structure of the stable matchings, the man in question prefers the match by the MPDA algorithm to the one by φ. Therefore, he can manipulate the stable matching rule φ by misreporting his preference by ranking his match by the MPDA algorithm first and the outside option second. It is worth noting that by interchanging the roles of men and women in the MPDA algorithm, using a similar argument, we can also infer that the *women-proposing DA (WPDA) algorithm* is the only stable matching rule that satisfies strategy-proofness for women on a domain satisfying unrestricted tops with unique acceptability for women.

We now provide informal outlines for the proofs of our equivalence results. To prove the first equivalence result, let us assume a stable and strategy-proof matching rule, φ, exists on a domain that satisfies unrestricted tops with unique acceptability for both sides of the market. Based on our characterization of the DA algorithm (Theorem 3), we can conclude that φ is equivalent to both the MPDA and WPDA algorithms on that domain. Moreover, since the manipulative coalitions for the MPDA algorithm can only consist of women, and the manipulative coalitions for the WPDA algorithm can only consist of men (Proposition 1), the fact that φ is equivalent to both the MPDA and WPDA algorithms implies that φ is group strategy-proof, thus concluding Theorem 1.

Our second equivalence result is established in two steps. Firstly, we prove that if a stable and strategy-proof matching rule exists on a domain satisfying unrestricted top pairs for men, it must be the MPDA algorithm. This result follows from Theorem 3, as unrestricted top pairs is a stronger domain richness condition than unrestricted tops with unique acceptability. Secondly, we demonstrate that if the MPDA algorithm is strategy-proof on a domain satisfying unrestricted top pairs for men, then it must also be group strategy-proof.

Notice that our equivalence results (Theorems 1 and 2) do not provide any information about the existence of a stable and group strategy-proof matching rule. For example, no stable matching rule is strategy-proof on the unrestricted domain (where the domain satisfies both richness conditions, i.e., unrestricted tops with unique acceptability and unrestricted top pairs, for both sides of the market), as demonstrated by Roth (1982). Therefore, our equivalence results are vacuously satisfied on the unrestricted domain.

To address this issue, we identify a domain condition that ensures the existence of a stable and group strategy-proof matching rule. Alcalde and Barberà (1994) introduce a restriction on the domain, called *top* *dominance*, and demonstrate that top dominance for women is a sufficient condition for the MPDA algorithm to be strategy-proof.[2] In Proposition 4, we show that it is also sufficient for the MPDA algorithm to be group strategy-proof, extending the result of Alcalde and Barberà (1994). Top dominance roughly requires that for every triplet of outcomes for an agent, if there exists an admissible preference that favors the first outcome over the second and the second over the third, then there cannot be another admissible preference that favors the first outcome over the third and the third over the second.

In order to ensure a stable and group strategy-proof matching rule, we provide another domain restriction, which is a constructive one. It is well-known that both the MPDA and WPDA algorithms are stable on the unrestricted domain (see Gale and Shapley (1962)), and there are preference profiles at which both these algorithms produce the same outcome. This means that only one stable matching exists at those preference profiles (see McVitie and Wilson (1971) for details). Conversely, at all preference profiles where only one stable matching exists, the MPDA and WPDA algorithms will produce the same outcome. Now, let us consider a domain of preference profiles where each preference profile has only one stable matching. In such cases, the MPDA algorithm, which is equivalent to the WPDA algorithm on the constructed domain, is both stable and group strategy-proof on that domain (Corollary 2). To understand why this holds, let us assume that the MPDA algorithm is not group strategy-proof on the constructed domain. This means that there exists a manipulative coalition, which, according to Proposition 1, must consist of only women. However, since the MPDA algorithm is equivalent to the WPDA algorithm on the constructed domain, by interchanging the roles of men and women in Proposition 1, we can infer that the manipulative coalition consists only of men, which contradicts the previous claim.

Finally, we explore the possibility of extending our findings to many-to-one matching markets. We consider the *college admissions problem* (Gale and Shapley, 1962), a well-studied market for our discussion. As usual, this market also comprises two finite and disjoint sets of agents: "colleges" and "students". In this setting, each student can only be matched with at most one college, while each college can be matched with more than one student.

Let us begin with a positive result that extends Theorem 3 to the college admissions problem: When the domain satisfies unrestricted tops with unique acceptability for students, the *students-proposing DA (SPDA) algorithm* is the *only* stable matching rule that satisfies strategy-proofness for students (Theorem 4). However, apart from this, extending the rest of the results to the college admissions problem is impossible. We demonstrate this impossibility through an example (Example 1). The reason behind this impossibility is the asymmetric nature of the college admissions problem; colleges can be matched with more than one student while students can be matched with at most one college.

[2] Top dominance for women is also a necessary domain restriction for the MPDA algorithm to be strategy-proof under two domain conditions (see Theorem 4 in Alcalde and Barberà (1994)).

To sum up, our main contribution in this paper is to identify conditions on domains where any stable matching rule satisfying strategy-proofness must also be immune to manipulations by groups. Our findings also highlight the advantages of implementing the DA algorithm in two-sided matching markets. Our research demonstrates that when the domain is sufficiently rich, the DA algorithm is the only stable matching rule that satisfies strategy-proofness for the proposers' side of the market.

1.2 Related literature

Barberà et al. (2016) show the equivalence of strategy-proofness and group strategy-proofness in general private good economies, which also encompasses the marriage problem. Their result is obtained under a richness condition of the domain and two requirements of the rule, neither of which is stability. For the social choice setting, Le Breton and Zaporozhets (2009) and Barberà et al. (2010) provide other domain conditions that ensure the equivalence of strategy-proofness and group strategy-proofness.

Strong group strategy-proofness is another well-studied group incentive compatibility requirement. Alva (2017) establishes that pairwise strong group strategy-proofness is equivalent to strong group strategy-proofness in a framework that accommodates public as well as private goods. His equivalence result is subject to a weaker richness condition than one of our richness conditions, which is unrestricted top pairs (there is no inclusion or exclusion relation between his richness condition and our other richness condition – unrestricted tops with unique acceptability). It is important to note that Alva's finding and our equivalence results (Theorems 1 and 2) are independent. This is because Alva's result does not consider individual strategy-proofness, and group strategy-proofness, combined with stability, does not imply (pairwise) strong group strategy-proofness.

1.3 Organization of the paper

The paper is structured as follows: In Section 2, we introduce basic concepts and notations that we use throughout the paper. We describe our model, define matching rules, and discuss their stability. Our main equivalence result is presented in Section 3. In Section 4, we present the DA algorithm, along with the results regarding this rule. Section 5 discusses the generalizability of our results to many-to-one matching markets. Finally, Section 6 concludes the paper. The Appendix contains the proofs.

2 Preliminaries

2.1 Basic notions and notations

For a finite set X, let $\mathbb{L}(X)$ denote the set of all strict linear orders over X.[3] An element of $\mathbb{L}(X)$ is called a ***preference*** over X. For a preference $P \in \mathbb{L}(X)$ and distinct $x, y \in X$, $x \; P \; y$ is interpreted as "x is preferred to y according to P". For $P \in \mathbb{L}(X)$, let R denote the weak part of P, i.e., for any $x, y \in X$, $x \; R \; y$ if and only if $\big[x \; P \; y \text{ or } x = y\big]$. Furthermore, for $P \in \mathbb{L}(X)$ and non-empty $X' \subseteq X$, let $\tau(P, X')$ denote the most preferred element in X' according to P, i.e., $\tau(P, X') = x$ if and only if $\big[x \in X' \text{ and } x \; P \; y \text{ for all } y \in X' \setminus \{x\}\big]$. For ease of presentation, we denote $\tau(P, X)$ by $\tau(P)$.

2.2 Model

There are two finite disjoint sets of agents, the set of *men* $M = \{m_1, \ldots, m_p\}$ and the set of *women* $W = \{w_1, \ldots, w_q\}$. Let $A = M \cup W$ be the set of all agents. Throughout this paper, we assume $p, q \geq 2$. Let $\emptyset$ denote the ***outside option*** – the null agent.

Each man m has a preference P_m over $W \cup \{\emptyset\}$, the set of all women and the outside option. The position in which he places the outside option in the preference has the meaning that the only women he is willing to be matched with are those whom he prefers to the outside option. Similarly, each woman w has a preference P_w over $M \cup \{\emptyset\}$. We say that woman w is ***acceptable*** to man m if $w \; P_m \; \emptyset$, and analogously, man m is *acceptable* to woman w if $m \; P_w \; \emptyset$.

We denote by $\mathcal{P}_a$ the set of admissible preferences for agent $a \in A$. Clearly, $\mathcal{P}_m \subseteq \mathbb{L}(W \cup \{\emptyset\})$ for all $m \in M$ and $\mathcal{P}_w \subseteq \mathbb{L}(M \cup \{\emptyset\})$ for all $w \in W$. A ***preference profile***, denoted by $P_A = (P_{m_1}, \ldots, P_{m_p}, P_{w_1}, \ldots, P_{w_q})$, is an element of the Cartesian product $\mathcal{P}_A := \prod_{i=1}^{p} \mathcal{P}_{m_i} \times \prod_{j=1}^{q} \mathcal{P}_{w_j}$, that represents a collection of preferences – one for each agent. Furthermore, as is the convention, P_{-a} denotes a collection of preferences of all agents except for a. Also, for $A' \subseteq A$, let $P_{A'}$ denote a collection of preferences of all agents in A' and $P_{-A'}$ a collection of preferences of all agents not in A'.

2.3 Matching rules and their stability

A ***matching*** (between M and W) is a function $\mu : A \to A \cup \{\emptyset\}$ such that

(i) $\mu(m) \in W \cup \{\emptyset\}$ for all $m \in M$,

(ii) $\mu(w) \in M \cup \{\emptyset\}$ for all $w \in W$, and

(iii) $\mu(m) = w$ if and only if $\mu(w) = m$ for all $m \in M$ and all $w \in W$.

[3] A ***strict linear order*** is a complete, asymmetric, and transitive binary relation.

Here, $\mu(m) = w$ means man m and woman w are matched to each other under the matching μ, and $\mu(a) = \emptyset$ means agent a is unmatched under the matching μ. We denote by $\mathcal{M}$ the set of all matchings.

A matching μ is ***individually rational*** at a preference profile P_A if for every $a \in A$, we have $\mu(a)\ R_a\ \emptyset$. A matching μ is *blocked* by a pair $(m, w) \in M \times W$ at a preference profile P_A if $w\ P_m\ \mu(m)$ and $m\ P_w\ \mu(w)$. A matching is ***stable*** at a preference profile if it is individually rational and is not blocked by any pair at that preference profile.

A ***matching rule*** is a function $\varphi : \mathcal{P}_A \to \mathcal{M}$. For a matching rule $\varphi : \mathcal{P}_A \to \mathcal{M}$ and a preference profile $P_A \in \mathcal{P}_A$, let $\varphi_a(P_A)$ denote the match of agent a by φ at P_A.

Definition 1. A matching rule $\varphi : \mathcal{P}_A \to \mathcal{M}$ is ***stable*** if for every $P_A \in \mathcal{P}_A$, $\varphi(P_A)$ is stable at P_A.

3 Equivalence between strategy-proofness and group strategy-proofness under stability

In practice, matching rules are often designed to satisfy incentive properties. Two well-studied such requirements are *strategy-proofness* and *group strategy-proofness*.

Definition 2. A matching rule $\varphi : \mathcal{P}_A \to \mathcal{M}$ is

(i) ***strategy-proof*** if for every $P_A \in \mathcal{P}_A$, every $a \in A$, and every $\tilde{P}_a \in \mathcal{P}_a$, we have $\varphi_a(P_A)\ R_a\ \varphi_a(\tilde{P}_a, P_{-a})$.

(ii) ***group strategy-proof*** if for every $P_A \in \mathcal{P}_A$, there do not exist a set of agents $A' \subseteq A$ and a preference profile $\tilde{P}_{A'}$ of the agents in A' such that $\varphi_a(\tilde{P}_{A'}, P_{-A'})\ P_a\ \varphi_a(P_A)$ for all $a \in A'$.

If a matching rule φ on $\mathcal{P}_A$ is not group strategy-proof, then there exist $P_A \in \mathcal{P}_A$, a set of agents $A' \subseteq A$, and a preference profile $\tilde{P}_{A'}$ of the agents in A' such that $\varphi_a(\tilde{P}_{A'}, P_{-A'})\ P_a\ \varphi_a(P_A)$ for all $a \in A'$. In such cases, we say that φ *is manipulable at P_A by coalition A' via $\tilde{P}_{A'}$*, and we call such a coalition a *manipulative coalition*. Note that a coalition can be a singleton; thus, group strategy-proofness implies strategy-proofness. Additionally, a coalition may consist of both men and women.

Notice that all agents in the manipulative coalition must benefit from misreporting. We consider this requirement compelling, since it ensures that every member of the coalition has a clear incentive to participate in a collective deviation from truthful revelation.

In this paper, we establish a significant finding: any stable matching rule designed to eliminate individual manipulation is also immune to manipulations by coalitions, provided that the domain is sufficiently varied. We present two equivalence results in the following subsection to support our claim.

3.1 Equivalence results

Our first equivalence result is obtained assuming the domain satisfies a richness condition, called *unrestricted tops with unique acceptability*, for both sides of the market. Before presenting our equivalence result, we first introduce this required richness condition.

Definition 3 (Unrestricted tops with unique acceptability)**.** A domain of preference profiles $\mathcal{P}_A$ satisfies ***unrestricted tops with unique acceptability*** *for men* if for every $m \in M$ and every $w \in W$, there exists $\tilde{P} \in \mathcal{P}_m$ such that $w\ \tilde{P}\ \emptyset\ \tilde{P}\ z$ for all $z \in W \setminus \{w\}$.

Note that whenever the sets of admissible preferences for men are unrestricted, the corresponding domain satisfies unrestricted top pairs for men. Another instance of a domain satisfying unrestricted tops with unique acceptability for men is when the sets of admissible preferences for men include all *single-peaked preferences* (Black, 1948). We define ***unrestricted tops with unique acceptability*** *for women* in a similar way.

We now present the first main result of this paper. It shows the equivalence of strategy-proofness and group strategy-proofness for any stable matching rule when the domain satisfies unrestricted tops with unique acceptability for both sides of the market.

Theorem 1. *Let $\mathcal{P}_A$ satisfy unrestricted tops with unique acceptability for both sides of the market. Then, any stable matching rule on $\mathcal{P}_A$ is strategy-proof if and only if it is group strategy-proof.*

The proof of this theorem is relegated to Appendix A.4.

In our second equivalence result, we require a stronger domain richness condition, but only for one side of the market, not for both sides, unlike Theorem 1. We start by presenting *unrestricted top pairs* (Alva, 2017), which is the required richness condition.

Definition 4 (Unrestricted top pairs)**.** A domain of preference profiles $\mathcal{P}_A$ satisfies ***unrestricted top pairs*** *for men* if for every $m \in M$,

(i) for every $w, w' \in W$, there exists $P \in \mathcal{P}_m$ such that $w\ P\ w'\ P\ z$ for all $z \in (W \cup \{\emptyset\}) \setminus \{w, w'\}$,

(ii) for every $w \in W$, there exists $\tilde{P} \in \mathcal{P}_m$ such that $w\ \tilde{P}\ \emptyset\ \tilde{P}\ z$ for all $z \in W \setminus \{w\}$, and

(iii) there exists $P' \in \mathcal{P}_m$ such that $\tau(P') = \emptyset$.

As usual, when the sets of admissible preferences for men are unrestricted, the corresponding domain satisfies unrestricted top pairs for men. Another example of a domain that satisfies unrestricted top pairs for men is when men's admissible preferences include all *double-peaked preferences* (Flowers, 1975). We define ***unrestricted top pairs*** *for women* in a similar manner.

We now present our second equivalence result. It shows the equivalence of strategy-proofness and group strategy-proofness for any stable matching rule when the domain satisfies unrestricted top pairs for at least one side of the market.

Theorem 2. *Let $\mathcal{P}_A$ satisfy unrestricted top pairs for at least one side of the market. Then, any stable matching rule on $\mathcal{P}_A$ is strategy-proof if and only if it is group strategy-proof.*

The proof of this theorem is relegated to Appendix A.5.

Note 1. In our equivalence results (Theorems 1 and 2), the richness conditions of the domain are only sufficient to establish the equivalence of strategy-proofness and group strategy-proofness under stability. However, they do not guarantee the existence of a stable and (group) strategy-proof matching rule. For instance, on the unrestricted domain, where the domain satisfies both richness conditions (i.e., unrestricted tops with unique acceptability and unrestricted top pairs) for both sides of the market, no stable matching rule is strategy-proof, as demonstrated by Roth (1982). Therefore, our equivalence results are vacuously satisfied on the unrestricted domain. It is worth noting that Roth (1982) proves his result in a setting without outside options and with an equal number (at least three) of men and women. However, the result can be extended to our setting, i.e., with outside options and with arbitrary values (at least two) of the number of men and the number of women.

As mentioned in Note 1, our equivalence results do not provide any information about the existence of a stable and group strategy-proof matching rule, which is equally important. To address this, in Section 4.3, we establish a domain restriction that guarantees the existence of such a matching rule.

4 Deferred acceptance: Manipulative coalitions and group strategy-proofness

Deferred acceptance (DA) algorithm (Gale and Shapley, 1962) is the salient rule in two-sided matching markets for its theoretical appeal. This rule also plays an instrumental role in our equivalence results. In this section, we present a brief description of this rule, along with a few results that facilitate proving our equivalence results. These results are also significant in their own right.

4.1 A formal description

There are two types of the DA algorithm: the *men-proposing DA (MPDA) algorithm* – denoted by D^M, and the *women-proposing DA (WPDA) algorithm*. In the following, we provide a description of the MPDA algorithm at a preference profile P_A. The same of the WPDA algorithm can be obtained by interchanging the roles of men and women in the MPDA algorithm.

Step 1. Each man m proposes to his most preferred acceptable woman (according to P_m).[4] Every woman w, who has at least one proposal, tentatively keeps her most preferred acceptable man (according to P_w) among these proposals and rejects the rest.

Step 2. Every man m, who was rejected in the previous step, proposes to his next preferred acceptable woman. Every woman w, who has at least one proposal including any proposal tentatively kept from the earlier steps, tentatively keeps her most preferred acceptable man among these proposals and rejects the rest.

This procedure is then repeated from Step 2 till a step such that for each man, one of the following two happens: (i) he is accepted by some woman, (ii) he has proposed to all acceptable women. At this step, the proposal tentatively accepted by women becomes permanent. This completes the description of the MPDA algorithm.

Remark 1 (Gale and Shapley, 1962)**.** On the unrestricted domain $\mathbb{L}^p(W \cup \{\emptyset\}) \times \mathbb{L}^q(M \cup \{\emptyset\})$, both the DA algorithms are stable.

4.2 Structure of manipulative coalitions for the MPDA algorithm

Dubins and Freedman (1981) show that no coalition of men can manipulate the MPDA algorithm on the unrestricted domain, while Roth (1982) shows that no stable matching rule on the unrestricted domain is strategy-proof. Therefore, whenever a coalition manipulates the MPDA algorithm, it must include at least one woman. In Proposition 1, we show that the manipulative coalition not only contains at least one woman but must consist *only* of women.

Proposition 1. *Suppose a coalition $A' \subseteq A$ manipulates the MPDA algorithm at some preference profile. Then, $A' \subseteq W$.*

The result of Dubins and Freedman (1981) follows from Proposition 1 as a corollary.

Corollary 1 (Dubins and Freedman, 1981)**.** *On the unrestricted domain $\mathbb{L}^p(W \cup \{\emptyset\}) \times \mathbb{L}^q(M \cup \{\emptyset\})$, no coalition of men can manipulate the MPDA algorithm.*

We further explore how manipulation by a manipulative coalition affects the MPDA algorithm from the agents' perspective. In our following result, we show that whenever a coalition manipulates the MPDA algorithm, every woman in the market weakly benefits while every man in the market weakly suffers. Notice that Proposition 1 follows from this result.

[4] That is, if the most preferred woman of a man is acceptable to that man, he proposes to her. Otherwise, he does not propose to anybody.

Proposition 2. *On an arbitrary domain $\mathcal{P}_A$, suppose the MPDA algorithm D^M is manipulable at $P_A \in \mathcal{P}_A$ by coalition $A' \subseteq A$ via $\tilde{P}_{A'} \in \prod_{a \in A'} \mathcal{P}_a$. Then,*

(a) $D^M_m(P_A)\ R_m\ D^M_m(\tilde{P}_{A'}, P_{-A'})$ *for all* $m \in M$, *and*

(b) $D^M_w(\tilde{P}_{A'}, P_{-A'})\ R_w\ D^M_w(P_A)$ *for all* $w \in W$.

The proof of this proposition is relegated to Appendix A.1.

Our last result on manipulative coalitions demonstrates that the set of unmatched agents is not affected by manipulation.

Proposition 3. *On an arbitrary domain $\mathcal{P}_A$, suppose the MPDA algorithm D^M is manipulable at $P_A \in \mathcal{P}_A$ by coalition $A' \subseteq A$ via $\tilde{P}_{A'} \in \prod_{a \in A'} \mathcal{P}_a$. Then, for every $a \in A$,*

$$D^M_a(P_A) = \emptyset \iff D^M_a(\tilde{P}_{A'}, P_{-A'}) = \emptyset.$$

The proof of this proposition is relegated to Appendix A.2.

A key implication of Proposition 3 is that an unmatched agent cannot be in a manipulative coalition. Note that by symmetry, Proposition 3 also holds for the WPDA algorithm.

Proposition 3 cannot be deduced from a result of McVitie and Wilson (1970), where they show that the set of unmatched agents remains the same across the stable matchings at a preference profile. To see this, note that in Proposition 3, the matching $D^M(P_A)$ is not stable at $(\tilde{P}_{A'}, P_{-A'})$, and the matching $D^M(\tilde{P}_{A'}, P_{-A'})$ is not stable at P_A in general.

4.3 Group strategy-proofness of the MPDA algorithm

Recall that our equivalence results (Theorems 1 and 2) do not provide any information about the existence of a stable and group strategy-proof matching rule. In order to address this issue, we identify two domain restrictions in this subsection that guarantee the existence of such a matching rule.

In their paper, Alcalde and Barberà (1994) introduce a restriction on the domain called *top dominance.* They show that if the domain satisfies top dominance for women, the MPDA algorithm becomes strategy-proof. Our following result extends their finding by proving that the domain satisfying top dominance for women is also sufficient for the MPDA algorithm to be group strategy-proof. Before stating this result, we first present the notion of top dominance.

Definition 5 (Top dominance)**.** A domain of preference profiles $\mathcal{P}_A$ satisfies ***top dominance*** *for women* if for every $w \in W$, $\mathcal{P}_w$ satisfies the following property: for every $x \in M$ and every $y, z \in M \cup \{\emptyset\}$, if there exists a preference $P \in \mathcal{P}_w$ with $x\ P\ y\ P\ z$ and $y\ R\ \emptyset$, then there is no preference $\tilde{P} \in \mathcal{P}_w$ such that $x\ \tilde{P}\ z\ \tilde{P}\ y$ and $z\ \tilde{R}\ \emptyset$.

Proposition 4. *Let $\mathcal{P}_A$ be an arbitrary domain of preference profiles. If $\mathcal{P}_A$ satisfies top dominance for women, then the MPDA algorithm is stable and group strategy-proof on $\mathcal{P}_A$.*

The proof of this proposition is relegated to Appendix B.

It is natural to wonder about the connection between the domain restriction of top dominance and the domain richness conditions of unrestricted top pairs and unrestricted tops with unique acceptability. As we have previously discussed, unrestricted top pairs is a stronger richness condition than unrestricted tops with unique acceptability. Moreover, if a domain satisfies unrestricted top pairs for women, it cannot satisfy top dominance for women. Finally, there is no inclusion or exclusion relation between top dominance and unrestricted tops with unique acceptability. Figure 1 summarizes a graphical representation of these relationships.

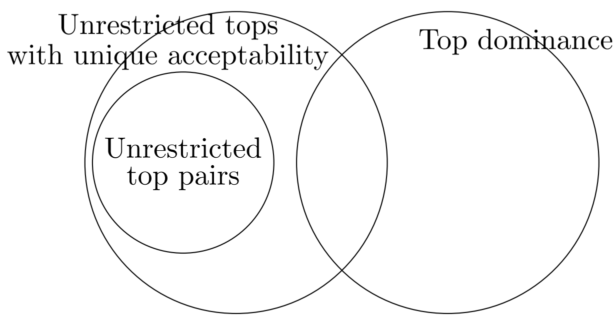

Figure 1: Venn diagram

In order to ensure a stable and group strategy-proof matching rule, we have a second domain restriction which is a constructive one. As mentioned earlier in Remark 1, both the MPDA and WPDA algorithms are stable on the unrestricted domain. It is also well-known that there exist preference profiles at which both these algorithms produce the same outcome. As a result, only one stable matching exists at those preference profiles (see McVitie and Wilson (1971) for details). Conversely, at all preference profiles where only one stable matching exists, the MPDA algorithm and WPDA algorithm will produce the same outcome, which is a result of Remark 1.

Let us consider a domain of preference profiles $\mathcal{P}_A$, where each preference profile in $\mathcal{P}_A$ has only one stable matching. In such cases, the MPDA algorithm is equivalent to the WPDA algorithm on $\mathcal{P}_A$. This equivalence, along with Proposition 1, proves that the MPDA algorithm is group strategy-proof on $\mathcal{P}_A$.

Corollary 2. *Let $\mathcal{P}_A$ be a domain of preference profiles such that every preference profile in $\mathcal{P}_A$ has only one stable matching. Then, the MPDA algorithm is stable and group strategy-proof on $\mathcal{P}_A$.*

To understand why Corollary 2 holds, let us assume that the MPDA algorithm is not group strategy-proof on $\mathcal{P}_A$. This means that there exists a manipulative coalition, which according to Proposition 1, must consist of only women. However, since the MPDA algorithm is equivalent to the WPDA algorithm on $\mathcal{P}_A$, by interchanging the roles of men and women in Proposition 1, we can infer that the manipulative coalition consists only of men, which contradicts the previous claim.

4.4 Uniqueness of the MPDA algorithm

In this subsection, we provide a crucial result that characterizes the MPDA algorithm, which also plays a vital role in our equivalence results. The result states that if the domain satisfies unrestricted tops with unique acceptability for men, then the MPDA algorithm is the *only* stable matching rule that satisfies strategy-proofness for men.

Theorem 3. *Let $\mathcal{P}_A$ satisfy unrestricted tops with unique acceptability for men. Then, the MPDA algorithm is the only stable matching rule on $\mathcal{P}_A$ that satisfies strategy-proofness for men.*

The proof of Theorem 3 is relegated to Appendix A.3; here we provide an outline of it. To understand why Theorem 3 holds, consider a different stable matching rule, φ, that matches a man to a different woman than the MPDA algorithm does. Due to the lattice structure of the stable matchings, the man in question prefers the match by the MPDA algorithm to the one by φ. Therefore, he can manipulate the stable matching rule φ by misreporting his preference by ranking his match by the MPDA algorithm first and the outside option second.

Alcalde and Barberà (1994) show that whenever the sets of admissible preferences for men are unrestricted, if a stable and strategy-proof matching rule exists, it must be the MPDA algorithm. Their result follows from Theorem 3.

Corollary 3 (Theorem 3 in Alcalde and Barberà (1994))**.** *Suppose $\mathcal{P}_m = \mathbb{L}(W \cup \{\emptyset\})$ for all $m \in M$. If a stable and strategy-proof matching rule exists on $\mathcal{P}_A$, it must be unique and the MPDA algorithm.*

The corollary below follows from the combination of Proposition 4 and Theorem 3.

Corollary 4. *Suppose $\mathcal{P}_A$ satisfies unrestricted tops with unique acceptability for men and top dominance for women. Then, the MPDA algorithm is the unique stable and strategy-proof matching rule on $\mathcal{P}_A$, which also satisfies group strategy-proofness on $\mathcal{P}_A$.*

5 Discussion: Many-to-one matching markets

In this section, we describe the *college admissions problem* (Gale and Shapley, 1962), a well-known many-to-one matching market, and discuss the possibility of extending our results to such a matching market.

5.1 Model

There are two finite disjoint sets of agents, the set of *colleges* C and the set of *students* S. Let $I = C \cup S$ be the set of all agents. Each college c has a *quota* $q_c \geq 1$ which represents the maximum number of students for which it has places. Let $\mathcal{S}_q := \{\tilde{S} \subseteq S \mid |\tilde{S}| \leq q\}$ be the set of subsets of S with cardinality at most q. Each college c has a preference P_c over $\mathcal{S}_{q_c}$ and each student s has a preference P_s over $C \cup \{\emptyset\}$. We say that student s is ***acceptable*** to college c if $\{s\}\ P_c\ \emptyset$, and analogously, college c is *acceptable* to student s if $c\ P_s\ \emptyset$. We denote by $\mathcal{P}_i$ the set of admissible preferences for agent $i \in I$. A ***preference profile***, denoted by $P_I = \big((P_c)_{c\in C}, (P_s)_{s\in S}\big)$, is an element of the Cartesian product $\mathcal{P}_I := \prod_{c\in C} \mathcal{P}_c \times \prod_{s\in S} \mathcal{P}_s$.

We impose a standard assumption on the preferences of colleges, called *responsiveness* (Roth, 1985). This property demonstrates a natural way to extend the preferences of colleges from individual students to sets of students.

Definition 6 (Responsiveness)**.** A college c's preference P_c satisfies ***responsiveness*** if for every $\tilde{S} \subseteq S$ with $|\tilde{S}| < q_c$,

(i) for every $s \in S \setminus \tilde{S}$,

$$(\tilde{S} \cup \{s\})\ P_c\ \tilde{S} \iff \{s\}\ P_c\ \emptyset, \text{ and}$$

(ii) for every $s, s' \in S \setminus \tilde{S}$,

$$(\tilde{S} \cup \{s\})\ P_c\ (\tilde{S} \cup \{s'\}) \iff \{s\}\ P_c\ \{s'\}.$$

A (many-to-one) ***matching*** (between C and S) is a function $\nu : I \to 2^S \cup C$ such that

(i) $\nu(c) \in \mathcal{S}_{q_c}$ for all $c \in C$,

(ii) $\nu(s) \in C \cup \{\emptyset\}$ for all $s \in S$, and

(iii) $\nu(s) = c$ if and only if $s \in \nu(c)$ for all $c \in C$ and all $s \in S$.

A matching ν is *blocked by a college* c at a preference profile P_I if there exists $s \in \nu(c)$ such that $\emptyset\ P_c\ \{s\}$. A matching ν is *blocked by a student* s at a preference profile P_I if $\emptyset\ P_s\ \nu(s)$. A matching ν is ***individually rational*** at a preference profile P_I if it is not blocked by any college or student. A matching ν is *blocked by a pair* $(c, s) \in C \times S$ at a preference profile P_I if $c\ P_s\ \nu(s)$ and either (i) $\big[|\nu(c)| < q_c$ and $\{s\}\ P_c\ \emptyset\big]$, or (ii) $\big[$there exists $s' \in \nu(c)$ such that $\{s\}\ P_c\ \{s'\}\big]$. A matching ν is ***stable*** at a preference profile P_I if it is individually rational and is not blocked by any pair at that preference profile.

The DA algorithm naturally extends to the college admissions problem. In the following, we provide a description of the *students-proposing DA (SPDA) algorithm.*

Step 1. Each student s applies to her most preferred acceptable college. Every college c, which has at least one application, tentatively keeps its top q_c acceptable students among these applications and rejects the rest (if c has fewer acceptable applications than q_c, it tentatively keeps all of them).

Step 2. Every student s, who was rejected in the previous step, applies to her next preferred acceptable college. Every college c, which has at least one application including any applications tentatively kept from the earlier steps, tentatively keeps its top q_c acceptable students among these applications and rejects the rest.

This procedure is then repeated from Step 2 till a step such that for each student, one of the following two happens: (i) she is accepted by some college, (ii) she has applied to all acceptable colleges. At this step, the applications tentatively accepted by colleges become permanent. This completes the description of the SPDA algorithm.

Remark 2 (Gale and Shapley, 1962)**.** The SPDA algorithm produces a stable matching at every preference profile $P_I \in \mathcal{P}_I$.

5.2 (Im)possibility of extensions

Let us begin with a positive result that extends Theorem 3 to the college admissions problem: When the domain satisfies unrestricted tops with unique acceptability for students, the SPDA algorithm is the *unique* stable matching rule that satisfies strategy-proofness for students. To understand why this extension holds, note that colleges having responsive preferences is a sufficient condition for the stable SPDA algorithm to be strategy-proof for students (Roth, 1985).[5] Additionally, the uniqueness of the SPDA algorithm follows from a similar logic as for the proof of Theorem 3.

Theorem 4. *Let $\mathcal{P}_I$ satisfy unrestricted tops with unique acceptability for students. Then, the SPDA algorithm is the only stable matching rule on $\mathcal{P}_I$ that satisfies strategy-proofness for students.*

We will now discuss whether Propositions 1, 2, 3, and 4, as well as Theorem 2, can be extended to the college admissions problem. As previously mentioned, if colleges have responsive preferences, then the SPDA algorithm is strategy-proof for students. Not only that, even a coalition of students cannot manipulate the SPDA algorithm when colleges have responsive preferences.[6] But what happens if a group of students colludes with a non-empty group (possibly singleton) of colleges? Will that coalition be able to manipulate the SPDA algorithm? Recall that in the case of the marriage problem (a one-to-one matching market), we get a negative

[5] Roth (1985) also shows that colleges having responsive preferences is not a sufficient condition for the *colleges-proposing DA algorithm* to be strategy-proof for colleges.

[6] Martínez et al. (2004) extend this result further. They show that no coalition of students can manipulate the SPDA algorithm when the colleges' preferences satisfy both *substitutability* (Kelso Jr and Crawford, 1982) and *separability* (Barberà et al., 1991).

answer to such a question (Proposition 1). However, we obtain a different result for the college admissions problem; a group of students can indeed manipulate the SPDA algorithm by colluding with a non-empty group of colleges. Therefore, Proposition 1 cannot be extended to the college admissions problem. An example is provided below to illustrate this.

Example 1. Consider a market with three colleges $C = \{c_1, c_2, c_3\}$ and five students $S = \{s_1, \ldots, s_5\}$. College c_1 has a quota of 2 and other colleges have a quota of 1. Consider the preference profile P_I such that

$$P_{s_1} : c_3 c_1 \ldots, \quad P_{s_2} : c_1 c_3 \ldots, \quad P_{s_3} : c_1 \emptyset \ldots, \quad P_{s_4} : c_1 c_2 \ldots, \quad P_{s_5} : c_2 \emptyset \ldots,$$
$$P_{c_1} : \{s_1, s_2\}\ \{s_1, s_3\}\ \{s_1, s_4\}\ \{s_2, s_3\}\ \{s_2, s_4\}\ \{s_3, s_4\}\ \{s_1\}\ \{s_2\}\ \{s_3\}\ \{s_4\}\ \emptyset\ \{s_1, s_5\}\ \{s_2, s_5\}\ \{s_3, s_5\}\ \{s_4, s_5\}\ \{s_5\},$$
$$P_{c_2} : \{s_4\}\ \{s_5\}\ \emptyset\ \{s_1\}\ \{s_2\}\ \{s_3\}, \quad \text{and} \quad P_{c_3} : \{s_2\}\ \{s_5\}\ \{s_1\}\ \emptyset\ \{s_3\}\ \{s_4\}.$$

Clearly, the preferences of colleges satisfy responsiveness. The outcome of the SPDA algorithm at P_I is

$$\big[(c_1, \{s_2, s_3\}), (c_2, \{s_4\}), (c_3, \{s_1\}), (s_5, \emptyset)\big].$$

Let I' be a coalition of student s_5 and college c_1. Consider the preference profile $\tilde{P}_{I'}$ of this coalition such that

$$\tilde{P}_{s_5} : c_3 c_2 \ldots, \quad \text{and}$$
$$\tilde{P}_{c_1} : \{s_1, s_4\}\ \{s_2, s_4\}\ \{s_3, s_4\}\ \{s_1, s_2\}\ \{s_1, s_3\}\ \{s_2, s_3\}\ \{s_4\}\ \{s_1\}\ \{s_2\}\ \{s_3\}\ \emptyset\ \{s_4, s_5\}\ \{s_1, s_5\}\ \{s_2, s_5\}\ \{s_3, s_5\}\ \{s_5\}.$$

$\tilde{P}_{c_1}$ also satisfies responsiveness and the outcome of the SPDA algorithm at $(\tilde{P}_{I'}, P_{-I'})$ is

$$\big[(c_1, \{s_1, s_4\}), (c_2, \{s_5\}), (c_3, \{s_2\}), (s_3, \emptyset)\big].$$

Combining all these facts, it follows that the coalition of student s_5 and college c_1 manipulates the SPDA algorithm at P_I via $\tilde{P}_{I'}$. $\Diamond$

Example 1 also shows the impossibility of extending Propositions 2 and 3 to the college admissions problem. In fact, one important implication of Proposition 3 – *an unmatched agent cannot be in a manipulative coalition for the DA algorithm* – does not hold for the college admissions problem either. To see this, notice that in Example 1, student s_5, an unmatched agent, is in the manipulative coalition.

Lastly, the main takeaway of this paper – *the equivalence of strategy-proofness and group strategy-proofness under stability* (Theorem 2) – is also impossible to extend to the college admissions problem, and so is Proposition 4. A detailed explanation is provided below. We use an additional notion for this explanation. Given a preference P_c of a college c, let P^*_c denote the corresponding *induced preference* over $S \cup \{\emptyset\}$ where for every $s, s' \in S$, (i) $s \; P_c^* \; s' \iff \{s\} \; P_c \; \{s'\}$ and (ii) $s \; P_c^* \; \emptyset \iff \{s\} \; P_c \; \emptyset$.

Example 1 *(continued).* Construct a domain of preference profiles $\mathcal{P}_I$ for the given market such that $\mathcal{P}_I$ satisfies unrestricted top pairs for students and

$$\mathcal{P}_{c_1} = \{P_{c_1}, \tilde{P}_{c_1}\}, \;\; \mathcal{P}_{c_2} = \{P_{c_2}\}, \;\; \text{and} \;\; \mathcal{P}_{c_3} = \{P_{c_3}\}.$$

Notice that $\mathcal{P}_I$ satisfies top dominance for colleges when judged by the induced preferences. Because of this, and since colleges have responsive preferences, by Theorem 5 in Alcalde and Barberà (1994), the SPDA algorithm is stable and strategy-proof on $\mathcal{P}_I$.

Recall that the coalition I' of student s_5 and college c_1 manipulates the SPDA algorithm at P_I via $\tilde{P}_{I'}$. Moreover, by construction, both P_I and $(\tilde{P}_{I'}, P_{-I'})$ are admissible preference profiles. Combining all these facts, it follows that the SPDA algorithm is not group strategy-proof on $\mathcal{P}_I$, showing the impossibility of extending Theorem 2 and Proposition 4 to the college admissions problem. ♢

6 Concluding remarks

Our main contribution in this paper is to identify conditions on domains where any stable matching rule satisfying strategy-proofness must also be immune to manipulations by groups. Our findings also highlight the advantages of implementing the DA algorithm in two-sided matching markets. Our research demonstrates that when the domain is sufficiently rich, the DA algorithm is the only stable matching rule that satisfies strategy-proofness for the proposers' side of the market. Moreover, if the DA algorithm is strategy-proof for the entire market, group strategy-proofness is automatically achieved.

Appendix A Proofs of Theorems 1, 2, and 3 and Propositions 2 and 3

We prove Theorems 1 and 2 using Propositions 2 and 3, as well as Theorem 3. Therefore, we will first present the proofs of the two propositions and Theorem 3.

A.1 Proof of Proposition 2

To facilitate the proof of Proposition 2, we present a lemma formulated by J. S. Hwang and proved in Gale and Sotomayor (1985).

Lemma A.1 (Blocking Lemma)**.** *Consider a preference profile $P_A \in \mathbb{L}^p(W \cup \{\emptyset\}) \times \mathbb{L}^q(M \cup \{\emptyset\})$. Let μ be any individually rational matching at P_A and $M' := \{m \in M \mid \mu(m) \; P_m \; D_m^M(P_A)\}$ the set of men who are strictly* *better off under μ than under $D^M(P_A)$. If M' is non-empty, then there is a pair $(m,w) \in (M \setminus M') \times \mu(M')$ that blocks μ at P_A.*

Completion of the proof of Proposition 2. Since D^M is manipulable at P_A by coalition A' via $\tilde{P}_{A'}$, we have

$$D^M_a(\tilde{P}_{A'}, P_{-A'}) \; P_a \; D^M_a(P_A) \text{ for all } a \in A'. \tag{A.1}$$

For ease of presentation, we denote the matching $D^M(P_A)$ by μ and the matching $D^M(\tilde{P}_{A'}, P_{-A'})$ by $\tilde{\mu}$ in this proof.

Proof of part (a). We first show that $\tilde{\mu}$ is individually rational at P_A. Since $\tilde{\mu}$ is stable at $(\tilde{P}_{A'}, P_{-A'})$ (see Remark 1), we have

$$\tilde{\mu}(a) \; R_a \; \emptyset \text{ for all } a \in A \setminus A'. \tag{A.2}$$

Furthermore, μ being stable at P_A (see Remark 1) implies that $\mu(a) \; R_a \; \emptyset$ for all $a \in A'$. This, along with (A.1), yields

$$\tilde{\mu}(a) \; P_a \; \emptyset \text{ for all } a \in A'. \tag{A.3}$$

(A.2) and (A.3) together imply that $\tilde{\mu}$ is individually rational at P_A.

We now proceed to complete the proof of part (a). Let $M^+ := \{m \in M \mid \tilde{\mu}(m) \; P_m \; \mu(m)\}$ be the set of men who are strictly better off under $\tilde{\mu}$ than under μ. Assume for contradiction that M^+ is non-empty. Since $\tilde{\mu}$ is individually rational at P_A and M^+ is non-empty, by Lemma A.1, it follows that there is a pair $(m,w) \in (M \setminus M^+) \times \tilde{\mu}(M^+)$ that blocks $\tilde{\mu}$ at P_A.

Claim A.1. $m, w \in A \setminus A'$.

Proof of Claim A.1. Note that $M \cap A' \subseteq M^+$. Since $m \in M \setminus M^+$, this implies

$$m \in A \setminus A'. \tag{A.4}$$

Consider the man m' such that $\tilde{\mu}(m') = w$. Note that m' is well-defined since $w \in \tilde{\mu}(M^+)$. Clearly, $m' \in M^+$. The facts $m' \in M^+$ and $\tilde{\mu}(m') = w$ together imply

$$w \; P_{m'} \; \mu(m'). \tag{A.5}$$

Since μ is stable at P_A, (A.5) implies $\mu(w) \; P_w \; m'$. This, along with the fact $\tilde{\mu}(m') = w$ and (A.1), yields

$$w \in A \setminus A'. \tag{A.6}$$

(A.4) and (A.6) together complete the proof of Claim A.1. □

Recall that $\tilde{\mu}$ is stable at $(\tilde{P}_{A'}, P_{-A'})$. By Claim A.1, it follows that m and w do not change their preferences from P_A to $(\tilde{P}_{A'}, P_{-A'})$. This, together with the fact that (m, w) blocks $\tilde{\mu}$ at P_A, implies that (m, w) blocks $\tilde{\mu}$ at $(\tilde{P}_{A'}, P_{-A'})$, a contradiction to the fact that $\tilde{\mu}$ is stable at $(\tilde{P}_{A'}, P_{-A'})$. This completes the proof of part (a) of Proposition 2.

Proof of part (b). Assume for contradiction that there exists $w \in W$ such that

$$\mu(w)\ P_w\ \tilde{\mu}(w). \tag{A.7}$$

We first show that $\mu(w) \in M$. Note that (A.7) and (A.1) together imply $w \in A \setminus A'$, which means woman w does not change her preference from P_A to $(\tilde{P}_{A'}, P_{-A'})$. Since $\tilde{\mu}$ is stable at $(\tilde{P}_{A'}, P_{-A'})$ and woman w does not change her preference from P_A to $(\tilde{P}_{A'}, P_{-A'})$, we have $\tilde{\mu}(w)\ R_w\ \emptyset$, which, along with (A.7), yields $\mu(w)\ P_w\ \emptyset$. This, in particular, means $\mu(w) \in M$.

Consider the man m such that $\mu(w) = m$. By part (a) of this theorem, we have $w\ R_m\ \tilde{\mu}(m)$. Note also that (A.7) implies $\tilde{\mu}(m) \neq w$. Combining the facts $w\ R_m\ \tilde{\mu}(m)$ and $\tilde{\mu}(m) \neq w$, we have

$$w\ P_m\ \tilde{\mu}(m). \tag{A.8}$$

It follows from (A.7) and (A.8) that (m, w) blocks $\tilde{\mu}$ at P_A. Recall that $\tilde{\mu}$ is stable at $(\tilde{P}_{A'}, P_{-A'})$ and woman w does not change her preference from P_A to $(\tilde{P}_{A'}, P_{-A'})$. Also note that by Proposition 1, man m does not change his preference from P_A to $(\tilde{P}_{A'}, P_{-A'})$. Combining all these facts, it follows that (m, w) blocks $\tilde{\mu}$ at $(\tilde{P}_{A'}, P_{-A'})$, a contradiction to the fact that $\tilde{\mu}$ is stable at $(\tilde{P}_{A'}, P_{-A'})$. This completes the proof of part (b) of Proposition 2. ■

A.2 Proof of Proposition 3

For ease of presentation, we denote the matching $D^M(P_A)$ by μ and the matching $D^M(\tilde{P}_{A'}, P_{-A'})$ by $\tilde{\mu}$ in this proof.

Note that by Proposition 1, men do not change their preferences from P_A to $(\tilde{P}_{A'}, P_{-A'})$. Because of this, and since $\tilde{\mu}$ is stable at $(\tilde{P}_{A'}, P_{-A'})$ (see Remark 1), we have $\tilde{\mu}(m)\ R_m\ \emptyset$ for all $m \in M$. This, together with Proposition 2.(a), implies

$$\mu(m)\ R_m\ \tilde{\mu}(m)\ R_m\ \emptyset \text{ for all } m \in M,$$

which, in particular, means

$$\{m \in M \mid m \text{ is matched under } \tilde{\mu}\} \subseteq \{m \in M \mid m \text{ is matched under } \mu\}. \tag{A.9}$$

Similarly, since μ is stable at P_A (see Remark 1), we have $\mu(w)\ R_w\ \emptyset$ for all $w \in W$. This, together with Proposition 2.(b), implies

$$\tilde{\mu}(w)\ R_w\ \mu(w)\ R_w\ \emptyset \text{ for all } w \in W,$$

which, in particular, means

$$\{w \in W \mid w \text{ is matched under } \tilde{\mu}\} \supseteq \{w \in W \mid w \text{ is matched under } \mu\}. \tag{A.10}$$

Furthermore, by the definition of a matching,

$$\begin{aligned} &|\{m \in M \mid m \text{ is matched under } \tilde{\mu}\}| = |\{w \in W \mid w \text{ is matched under } \tilde{\mu}\}|, \text{ and} \\ &|\{m \in M \mid m \text{ is matched under } \mu\}| = |\{w \in W \mid w \text{ is matched under } \mu\}|. \end{aligned} \tag{A.11}$$

(A.9), (A.10), and (A.11) together complete the proof of Proposition 3. ■

A.3 Proof of Theorem 3

Before proving Theorem 3, we present two results: one from Gale and Shapley (1962) and the other from McVitie and Wilson (1970).

Remark 3 (Gale and Shapley, 1962)**.** Every man weakly prefers the match by the MPDA algorithm to the match under any other stable matching. Formally, for every $P_A \in \mathbb{L}^p(W \cup \{\emptyset\}) \times \mathbb{L}^q(M \cup \{\emptyset\})$, every stable matching μ at P_A, and every $m \in M$, we have $D^M_m(P_A)\ R_m\ \mu(m)$.

Remark 4 (McVitie and Wilson, 1970)**.** The set of unmatched agents remains the same across the stable matchings at a preference profile. Formally, for every $P_A \in \mathbb{L}^p(W \cup \{\emptyset\}) \times \mathbb{L}^q(M \cup \{\emptyset\})$, every stable matchings μ and ν at P_A, and every $a \in A$, $\mu(a) = \emptyset$ if and only if $\nu(a) = \emptyset$.

Completion of the proof of Theorem 3. First, note that on any arbitrary domain, the MPDA algorithm D^M is stable (see Remark 1) and strategy-proof for men (see Dubins and Freedman (1981)).

Let φ be a stable and strategy-proof matching rule on $\mathcal{P}_A$. Assume for contradiction that $\varphi \not\equiv D^M$ on $\mathcal{P}_A$. Then, there exist $P_A \in \mathcal{P}_A$ and $m \in M$ such that $\varphi_m(P_A) \neq D^M_m(P_A)$. Because of this, and since $\varphi(P_A)$ is stable at P_A, by Remark 3, we have $D^M_m(P_A) \in W$ and

$$D^M_m(P_A)\ P_m\ \varphi_m(P_A). \tag{A.12}$$

Consider the preference $\tilde{P}_m \in \mathcal{P}_m$ such that $D^M_m(P_A)\ \tilde{P}_m\ \emptyset\ \tilde{P}_m\ w$ for all $w \in W \setminus \{D^M_m(P_A)\}$. Note that $\tilde{P}_m$ is well-defined since $\mathcal{P}_A$ satisfies unrestricted tops with unique acceptability for men and $D^M_m(P_A) \in W$. Clearly, $D^M(P_A)$ is stable at $(\tilde{P}_m, P_{-m})$.

However, since $D^M(P_A)$ is stable at $(\tilde{P}_m, P_{-m})$, by Remark 4 and the construction of $\tilde{P}_m$, we have $\varphi_m(\tilde{P}_m, P_{-m}) = D^M_m(P_A)$, which, together with (A.12), contradicts strategy-proofness of φ on $\mathcal{P}_A$. This completes the proof of Theorem 3. ■

A.4 Proof of Theorem 1

Before proving Theorem 1, we present two remarks. The remarks follow from Proposition 1 and Theorem 3, respectively, by interchanging the roles of men and women in the MPDA algorithm.

Remark 5. Suppose a coalition $A' \subseteq A$ manipulates the WPDA algorithm at some preference profile. Then, $A' \subseteq M$.

Remark 6. Let $\mathcal{P}_A$ satisfy unrestricted tops with unique acceptability for women. Then, the WPDA algorithm is the only stable matching rule on $\mathcal{P}_A$ that satisfies strategy-proofness for women.

Completion of the proof of Theorem 1. The "if" part of the theorem follows from the respective definitions. We proceed to prove the "only-if" part. Suppose there exists a stable and strategy-proof matching rule, φ, on $\mathcal{P}_A$. Since $\mathcal{P}_A$ satisfies unrestricted tops with unique acceptability for both sides of the market, by Theorem 3 and Remark 6, φ must be equivalent to both the MPDA and WPDA algorithms on $\mathcal{P}_A$.

Assume for contradiction that φ is not group strategy-proof on $\mathcal{P}_A$. Then, there exists a non-empty set of agents $A' \subseteq A$ that manipulates φ at some preference profile $P_A \in \mathcal{P}_A$. Since φ is equivalent to the MPDA algorithm on $\mathcal{P}_A$ and A' manipulates φ at P_A, by Proposition 1, we have $A' \subseteq W$. Similarly, since φ is equivalent to the WPDA algorithm on $\mathcal{P}_A$ and A' manipulates φ at P_A, by Remark 5, we have $A' \subseteq M$. However, the facts $A' \subseteq W$ and $A' \subseteq M$ together contradict the fact that A' is a non-empty set. This completes the proof of Theorem 1. ■

A.5 Proof of Theorem 2

We first prove a lemma that we use in the proof of Theorem 2.

A.5.1 Lemma A.2 and its proof

Lemma A.2 identifies a richness condition of the domain for women under which stability and strategy-proofness become incompatible whenever the domain satisfies unrestricted top pairs for men.

Lemma A.2. *Let $\mathcal{P}_A$ satisfy unrestricted top pairs for men. Suppose there exists an alternating sequence $m^1, w^1, m^2, w^2, \ldots, m^k, w^k$ of distinct men and women such that*

(i) for every $i = 2, \ldots, k$, there exists $P_{w^i} \in \mathcal{P}_{w^i}$ with $m^i \; P_{w^i} \; m^{i-1} \; P_{w^i} \; \emptyset$, and

(ii) there exist $P_{w^1}, \tilde{P}_{w^1} \in \mathcal{P}_{w^1}$ with $m^1 \; P_{w^1} \; m^k \; P_{w^1} \; \emptyset$ such that for some $z \in M \cup \{\emptyset\}$,

(a) $m^k \; P_{w^1} \; z$, and

(b) $z \; \tilde{R}_{w^1} \; \emptyset$ and $m^1 \; \tilde{P}_{w^1} \; z \; \tilde{P}_{w^1} \; m^k$.

Then, no stable matching rule on $\mathcal{P}_A$ is strategy-proof.

Proof of Lemma A.2. Suppose there exists an alternating sequence $m^1, w^1, m^2, w^2, \ldots, m^k, w^k$ of distinct men and women such that

(i) for every $i = 2, \ldots, k$, there exists $P_{w^i} \in \mathcal{P}_{w^i}$ with $m^i \; P_{w^i} \; m^{i-1} \; P_{w^i} \; \emptyset$, and

(ii) there exist $P_{w^1}, \tilde{P}_{w^1} \in \mathcal{P}_{w^1}$ with $m^1 \; P_{w^1} \; m^k \; P_{w^1} \; \emptyset$ such that for some $z \in M \cup \{\emptyset\}$,

(a) $m^k \; P_{w^1} \; z$, and

(b) $z \; \tilde{R}_{w^1} \; \emptyset$ and $m^1 \; \tilde{P}_{w^1} \; z \; \tilde{P}_{w^1} \; m^k$.

Assume for contradiction that there exists a stable and strategy-proof matching rule on $\mathcal{P}_A$. Note that since $\mathcal{P}_A$ satisfies unrestricted top pairs for men, by Theorem 3, it must be the MPDA algorithm. We distinguish the following two cases.

Case 1: Suppose $z = \emptyset$.

Since $\mathcal{P}_A$ satisfies unrestricted top pairs for men, we can construct a collection of preferences $P_{-\{w^1,\ldots,w^k\}}$ of all agents except for women $w^1, \ldots, w^k$ such that

$$\begin{aligned} &P_{m^i} : w^{i+1} w^i \ldots \text{ for all } i = 1, \ldots, k-1, \\ &P_{m^k} : w^1 w^k \ldots, \text{ and} \\ &\tau(P_m) = \emptyset \text{ for all } m \notin \{m^1, \ldots, m^k\}. \end{aligned}$$

(Recall that $w^1 w^2 \ldots$ denotes a preference that ranks w^1 first and w^2 second.)

It is straightforward to verify the following facts.

$$\begin{aligned} D^M(P_{w^1}, P_{w^2}, \ldots, P_{w^k}, P_{-\{w^1,\ldots,w^k\}}) &= \begin{bmatrix} (m^k, w^1), (m^i, w^{i+1}) \;\; \forall \;\; i = 1, \ldots, k-1, \\ (a, \emptyset) \;\; \forall \;\; a \notin \{m^1, \ldots, m^k, w^1, \ldots, w^k\} \end{bmatrix} \text{ and} \\ D^M(\tilde{P}_{w^1}, P_{w^2}, \ldots, P_{w^k}, P_{-\{w^1,\ldots,w^k\}}) &= \begin{bmatrix} (m^i, w^i) \;\; \forall \;\; i = 1, \ldots, k, \\ (a, \emptyset) \;\; \forall \;\; a \notin \{m^1, \ldots, m^k, w^1, \ldots, w^k\} \end{bmatrix}. \end{aligned} \tag{A.13}$$

However, (A.13) implies that w^1 can manipulate the MPDA algorithm at $(P_{w^1}, P_{w^2}, \ldots, P_{w^k}, P_{-\{w^1,\ldots,w^k\}})$ via $\tilde{P}_{w^1}$, a contradiction to the fact that the MPDA algorithm is strategy-proof on $\mathcal{P}_A$. This completes the proof for Case 1.

Case 2: Suppose $z = \tilde{m}$ for some $\tilde{m} \in M$.

(i) Suppose $\tilde{m} \notin \{m^1, \ldots, m^k\}$.

Since $\mathcal{P}_A$ satisfies unrestricted top pairs for men, we can construct a collection of preferences $P_{-\{w^1,\ldots,w^k\}}$ of all agents except for women $w^1, \ldots, w^k$ such that

$$\begin{aligned}
&P_{m^i} : w^{i+1} w^i \ldots \text{ for all } i = 1, \ldots, k-1, \\
&P_{m^k} : w^1 w^k \ldots, \\
&P_{\tilde{m}} : w^1 \emptyset \ldots, \text{ and} \\
&\tau(P_m) = \emptyset \text{ for all } m \notin \{m^1, \ldots, m^k, \tilde{m}\}.
\end{aligned}$$

Using a similar argument as for Case 1, it follows that w^1 can manipulate the MPDA algorithm at $(P_{w^1}, P_{w^2}, \ldots, P_{w^k}, P_{-\{w^1,\ldots,w^k\}})$ via $\tilde{P}_{w^1}$, a contradiction to the fact that the MPDA algorithm is strategy-proof on $\mathcal{P}_A$.

(ii) Suppose $\tilde{m} \in \{m^1, \ldots, m^k\}$.

Since $\tilde{m} \in \{m^1, \ldots, m^k\}$, the fact $m^1 \; \tilde{P}_{w^1} \; \tilde{m} \; \tilde{P}_{w^1} \; m^k$ implies $\tilde{m} \in \{m^2, \ldots, m^{k-1}\}$. Let $\tilde{m} = m^{k^*}$ for some $k^* \in \{2, \ldots, k-1\}$. Since $\mathcal{P}_A$ satisfies unrestricted top pairs for men, we can construct a collection of preferences $P_{-\{w^1,\ldots,w^k\}}$ of all agents except for women $w^1, \ldots, w^k$ such that

$$\begin{aligned}
&P_{m^i} : w^{i+1} w^i \ldots \text{ for all } i = 1, \ldots, k^*-1, \\
&P_{m^{k^*}} : w^1 w^{k^*} \ldots, \\
&P_{m^k} : w^1 \emptyset \ldots, \text{ and} \\
&\tau(P_m) = \emptyset \text{ for all } m \notin \{m^1, \ldots, m^{k^*}, m^k\}.
\end{aligned}$$

It is straightforward to verify the following facts.

$$\begin{aligned}
&D^M(\tilde{P}_{w^1}, P_{w^2}, \ldots, P_{w^k}, P_{-\{w^1,\ldots,w^k\}}) = \begin{bmatrix} (m^{k^*}, w^1), (m^i, w^{i+1}) \; \forall \; i = 1, \ldots, k^*-1, \\ (a, \emptyset) \; \forall \; a \notin \{m^1, \ldots, m^{k^*}, w^1, \ldots, w^{k^*}\} \end{bmatrix} \text{ and} \\
&D^M(P_{w^1}, P_{w^2}, \ldots, P_{w^k}, P_{-\{w^1,\ldots,w^k\}}) = \begin{bmatrix} (m^i, w^i) \; \forall \; i = 1, \ldots, k^*, \\ (a, \emptyset) \; \forall \; a \notin \{m^1, \ldots, m^{k^*}, w^1, \ldots, w^{k^*}\} \end{bmatrix}.
\end{aligned} \tag{A.14}$$

However, (A.14) implies that w^1 can manipulate the MPDA algorithm at $(\tilde{P}_{w^1}, P_{w^2}, \ldots, P_{w^k}, P_{-\{w^1,\ldots,w^k\}})$ via P_{w^1}, a contradiction to the fact that the MPDA algorithm is strategy-proof on $\mathcal{P}_A$. This completes the proof for Case 2.

Since Cases 1 and 2 are exhaustive, this completes the proof of Lemma A.2. ■

A.5.2 Completion of the proof of Theorem 2

The "if" part of the theorem follows from the respective definitions. We proceed to prove the "only-if" part. Without loss of generality, assume that $\mathcal{P}_A$ satisfies unrestricted top pairs for men. By definition, $\mathcal{P}_A$ also satisfies unrestricted tops with unique acceptability for men. Suppose there exists a stable and strategy-proof matching rule on $\mathcal{P}_A$. By Theorem 3, it must be the MPDA algorithm. Assume for contradiction that the MPDA algorithm D^M is not group strategy-proof on $\mathcal{P}_A$. Then, there exist $P_A \in \mathcal{P}_A$, a set of agents $A' \subseteq A$, and a preference profile $\tilde{P}_{A'}$ of the agents in A' such that

$$D^M_a(\tilde{P}_{A'}, P_{-A'})\ P_a\ D^M_a(P_A) \text{ for all } a \in A'. \tag{A.15}$$

For ease of presentation, we denote the matching $D^M(P_A)$ by μ and the matching $D^M(\tilde{P}_{A'}, P_{-A'})$ by $\tilde{\mu}$ in this proof. Furthermore, let $\tilde{\mu}^s$ denote the (tentative) matching at the end of some step s of the MPDA algorithm at $(\tilde{P}_{A'}, P_{-A'})$.

Note that by Proposition 1 and Proposition 2.(a), men do not change their preferences from P_A to $(\tilde{P}_{A'}, P_{-A'})$ and each man weakly prefers μ to $\tilde{\mu}$ where $\mu \neq \tilde{\mu}$. Let s^* be first step (of the MPDA algorithm) at $(\tilde{P}_{A'}, P_{-A'})$ when some man, say m, gets rejected by $\mu(m)$. Consider the woman w^1 such that $\mu(m) = w^1$. Clearly, $\tilde{\mu}(w^1) \neq m$. Construct an alternating sequence $m^1, w^1, m^2, w^2, \ldots, m^k, w^k$ of distinct men and women with $m^k \equiv m$ such that

(i) $\tilde{\mu}(w^i) = m^i$ for all $i = 1, \ldots, k$, and

(ii) $\mu(m^i) = w^{i+1}$ for all $i = 1, \ldots, k-1$.

Since both the number of men and the number of women are finite, by Proposition 3, it follows that the constructed sequence is well-defined.

Since μ is stable at P_A (see Remark 1), it follows from the construction of the sequence and Proposition 2.(b) that

$$m^1\ P_{w^1}\ m^k\ P_{w^1}\ \emptyset, \text{ and} \tag{A.16a}$$

$$m^i\ P_{w^i}\ m^{i-1}\ P_{w^i}\ \emptyset \text{ for all } i = 2, \ldots, k. \tag{A.16b}$$

We distinguish the following two cases.

Case 1: Suppose $\tilde{\mu}^{s^*}(w^1) = \emptyset$.

Since m^k gets rejected by w^1 in Step s^* (of the MPDA algorithm) at $(\tilde{P}_{A'}, P_{-A'})$, the fact $\tilde{\mu}^{s^*}(w^1) = \emptyset$, together with (A.16a), implies $w^1 \in A'$ and

$$\emptyset \; \tilde{P}_{w^1} \; m^k. \tag{A.17}$$

Moreover, since $\tilde{\mu}$ is stable at $(\tilde{P}_{A'}, P_{-A'})$ (see Remark 1), $\tilde{\mu}(w^1) = m^1$ implies $m^1 \; \tilde{P}_{w^1} \; \emptyset$, which, along with (A.17), yields

$$m^1 \; \tilde{P}_{w^1} \; \emptyset \; \tilde{P}_{w^1} \; m^k. \tag{A.18}$$

However, since $\mathcal{P}_A$ satisfies unrestricted top pairs for men, by Lemma A.2, (A.16) and (A.18) together contradict the fact that the MPDA algorithm is both stable and strategy-proof on $\mathcal{P}_A$. This completes the proof for Case 1.

Case 2: Suppose $\tilde{\mu}^{s^*}(w^1) \in M$.

Consider the man $\tilde{m}$ such that $\tilde{\mu}^{s^*}(w^1) = \tilde{m}$. Since m^k gets rejected by w^1 in Step s^* (of the MPDA algorithm) at $(\tilde{P}_{A'}, P_{-A'})$, the fact $\tilde{\mu}^{s^*}(w^1) = \tilde{m}$ implies $\tilde{m} \neq m^k$. Moreover, since man $\tilde{m}$ does not change his preference from P_A to $(\tilde{P}_{A'}, P_{-A'})$, by the assumption of Step s^* being the first step at $(\tilde{P}_{A'}, P_{-A'})$ where some man gets rejected by his match under μ, $\tilde{\mu}^{s^*}(w^1) = \tilde{m}$ implies $w^1 \; R_{\tilde{m}} \; \mu(\tilde{m})$. This, along with the facts $\mu(m^k) = w^1$ and $\tilde{m} \neq m^k$, yields $w^1 \; P_{\tilde{m}} \; \mu(\tilde{m})$. Because of this, and since μ is stable at P_A with $\mu(m^k) = w^1$, we have

$$m^k \; P_{w^1} \; \tilde{m}. \tag{A.19}$$

Furthermore, since m^k gets rejected by w^1 in Step s^* at $(\tilde{P}_{A'}, P_{-A'})$, the fact $\tilde{\mu}^{s^*}(w^1) = \tilde{m}$, together with (A.19) implies $w^1 \in A'$ and

$$\tilde{m} \; \tilde{P}_{w^1} \; \emptyset, \text{ and} \tag{A.20a}$$

$$\tilde{m} \; \tilde{P}_{w^1} \; m^k. \tag{A.20b}$$

Note that (A.16a) and (A.19) together imply $\tilde{m} \neq m^1$.

By the definition of the MPDA algorithm, we have $\tilde{\mu}(w^1) \; \tilde{R}_{w^1} \; \tilde{\mu}^{s^*}(w^1)$. Because of this, and since $\tilde{\mu}(w^1) = m^1$, $\tilde{\mu}^{s^*}(w^1) = \tilde{m}$, and $\tilde{m} \neq m^1$, we have $m^1 \; \tilde{P}_{w^1} \; \tilde{m}$, which, along with (A.20b), yields

$$m^1 \; \tilde{P}_{w^1} \; \tilde{m} \; \tilde{P}_{w^1} \; m^k. \tag{A.21}$$

However, since $\mathcal{P}_A$ satisfies unrestricted top pairs for men, by Lemma A.2, (A.16), (A.19), (A.20a), and (A.21) together contradict the fact that the MPDA algorithm is both stable and strategy-proof on $\mathcal{P}_A$. This completes the proof for Case 2.

Since Cases 1 and 2 are exhaustive, this completes the proof of Theorem 2. ■

Appendix B Proof of Proposition 4

Construct a domain of preference profiles $\tilde{\mathcal{P}}_A$ such that $\tilde{\mathcal{P}}_m = \mathbb{L}(W \cup \{\emptyset\})$ for all $m \in M$ and $\tilde{\mathcal{P}}_w = \mathcal{P}_w$ for all $w \in W$. Clearly, $\tilde{\mathcal{P}}_A$ satisfies unrestricted top pairs for men and top dominance for women.

Since $\tilde{\mathcal{P}}_A$ satisfies top dominance for women, by the corollary in Alcalde and Barberà (1994), the MPDA algorithm is stable and strategy-proof on $\tilde{\mathcal{P}}_A$. Because of this, and since $\tilde{\mathcal{P}}_A$ satisfies unrestricted top pairs for men, by Theorem 2, it follows that the MPDA algorithm is also group strategy-proof on $\tilde{\mathcal{P}}_A$.

However, since the MPDA algorithm is stable and group strategy-proof on $\tilde{\mathcal{P}}_A$, it also satisfies stability and group strategy-proofness on the smaller domain $\mathcal{P}_A$. This completes the proof of Proposition 4. ■

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