Random Rationalizability*

Chris Shannon
UC Berkeley

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Abstract

This paper studies a general framework for revealed preference tests with errors in data. The paper develops techniques for estimation and inference in a general class of models. By adapting techniques from set estimation and topological data analysis, the main results construct several classes of estimators and show that these give consistent estimators of the model, or of some features of the model based on topological properties. These in turn can be used to estimate or test features of the model, including rationalizability and multiplicity of equilibria. Applications are given to demand, general equilibrium, and games with strategic complementarities.

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Contact: cshannon@econ.berkeley.edu

1 Introduction

Revealed preference theory provides a critical bridge between economic theory and empirical observations. Starting with classic work in consumer theory, these results determine the testable implications and empirical content of many central economic models. A key benchmark for designing general revealed preference tests comes from Afriat's foundational work on consumer demand. Afriat's Theorem provides a test allowing for imperfectly observed data, in particular a finite data set; gives a general nonparametric test, assuming only that preferences are locally nonsatiated rather than restricting to a fixed parametric class; and is constructive, constructing an explicit utility function rationalizing the observed data whenever this is possible. More recent work has vastly expanded the scope of revealed preference theory, deriving Afriat-style characterizations of a broad variety of models, including general equilibrium, matching, Cournot equilibrium, dynamic economies, stochastic choice, and Savage expected utility, among many others.[1]

Although much of this work is motivated by empirical tests of theories, many of these results remain exact, like Afriat's original theorem and test cyclical consistency or GARP. The exact nature of these results and tests can make them difficult to use or interpret. When data fails such a test, it is challenging to disentangle rejections of the theory from observational errors, choice mistakes, learning, or other approximations plausibly present in the data, even when data is derived from highly controlled environments like lab experiments. Designing general nonparametric revealed preference tests suitable for estimation or inference with noisy data has been a central challenge throughout this work.

This paper studies a general framework for revealed preference tests with errors in data. The paper develops techniques for estimation and inference in a general class of models. By adapting techniques from set estimation and topological data analysis, the main results construct several classes of estimators and show that these give consistent estimators. These in turn can be used to estimate the model or test features of the model, including ratio-

[1] Central contributions include Brown and Matzkin (1996), Carvajal, Deb, Fenske, and Quah (2013), Echenique and Saito (2015), Kubler (2004), and McFadden (2004); see the monograph of Chambers and Echenique (2016) for a detailed recent overview.

nalizability and multiplicity of equilibria. Applications are given to demand, general equilibrium, and games with strategic complementarities.

A main observation that underlies these results is that a wide variety of different applications give rise to testable models sharing common geometric properties, and that these geometric properties in turn can facilitate estimation in these models. The central geometric notion in these results is the reach of a set, introduced in Federer (1959). The reach roughly measures how much a set can be perturbed while retaining its essential structure. The key condition for the estimation results derived here is that sets have positive reach, which roughly guarantees that sufficiently small perturbations of the set are sufficiently similar, building on and extending some ideas from set estimation and topological data analysis. The applications in the paper illustrate that this positive reach condition is satisfied under natural assumptions in a number of different models, including consumer demand, market equilibrium, and games with strategic complementarities.

Many other papers consider errors in revealed preference tests, particularly in the setting of consumer choice. This includes early foundational work such as Afriat (1972, 1973), Varian (1985, 1990), and Houtman and Maks (1985), and more recent work such as Echenique, Lee, and Shum (2011), Aguiar and Serrano (2017), Halevy, Persitz, and Zrill (2018), and Dziewulski (2021). Many of these papers make important contributions by focusing on a specific problem, notably consumer demand, and designing more precise and nuanced notions of errors tailored to this problem. There is unavoidably a tradeoff between the scope of models and precision in the notion of errors. In contrast, here the focus is on estimation in a more general class of models, and the notion of error is correspondingly coarser and more abstract. As a consequence, the basic results are applicable in a wide variety of problems, and can accommodate many different sources for randomness in data, including noisy observations, random choice, learning, satisficing and approximation, and random interactions.

The paper is also similar in spirit to several recent important papers on estimation of preferences by Chambers, Echenique, and Lambert (2021) and Ugarte (2023). These papers focus on deriving consistent estimators of preferences from data on choices. Chambers, Echenique, and Lambert (2021) consider data on binary choices with errors, generated by an agent facing random draws of pairs of alternatives and choosing from each binary set according to their true preferences with some probability, and choosing uniformly randomly between the two alternatives otherwise. Their main results construct an estimator of the preference relation and show that it is consistent under suitable regularity conditions. Ugarte (2023) extends these ideas to choice from budget sets, in the classic Afriat problem, with data generated similarly by an agent facing random draws of prices and choosing from each corresponding budget set optimally according to their true preferences with some probability, and choosing uniformly randomly from the budget set otherwise. The results here complement theirs by considering a more general set of models, estimation problems, and data generating processes, and deriving techniques for consistent estimation of features of these models.

The paper proceeds as follows. Section 2 collects some basic definitions and notation, and develops the main examples that will be used throughout the paper. Section 3 defines the reach of a set, and gives some basic results characterizing the reach. Section 4 develops the general estimators and results on random rationalizability. Section 5 focuses on applications of these results in the leading examples. Additional results and proofs are collected in the appendix.

2 Preliminaries

We start with some preliminary definitions and notation to be used throughout the paper. A primary goal of these definitions is to provide a general framework in which to study the problem of rationalizability without needing to specify a particular model.

To that end, let X be a metric space with metric d. The set X is the domain for all observable information available to the modeler. Other than assuming this set is a metric space equipped with some metric, we impose no additional structure on X. The metric d might be chosen to reflect some aspects of a particular problem; we return to this point below. For the results of the paper, we take $X = \mathbf{R}^N$ and d to be the standard metric in $\mathbf{R}^N$.

Let $\mathcal{M} \subseteq 2^X$. The collection $\mathcal{M}$ is the set of models, and an element

$M \in \mathcal{M}$ is a model, so $M \subseteq X$. The set of models $\mathcal{M}$ gives the possible restrictions on observables in X that are consistent with a given theory. A particular model $M \in \mathcal{M}$ is one such set of restrictions consistent with the theory, for example corresponding to a particular parametric specification. We give several canonical examples below.

Let $O \subseteq X$ be a set of observations. Write $O := \{x_r : r \in R\}$. The set O is the data set observed by the analyst. For now R is a generic index set that could be finite or infinite. Given the set of models $\mathcal{M}$, the consistency of the data set O with the underlying theory generating the set of models $\mathcal{M}$ can then be defined naturally. A data set O is *rationalizable in* $\mathcal{M}$ if there exists $M \in \mathcal{M}$ such that $O \subseteq M$. For the results of the paper R is finite and will correspond to realizations of a collection of random variables generating the data. In that case, we will be interested in notions of rationalizability that allow for randomness in the data.

We consider several canonical examples next to illustrate the general framework. These will be motivating examples used throughout the paper for many of the main results.

Example 1: Consumer Demand To see the standard consumer demand problem as an example of this framework, let $X := \mathbf{R}^L_{++} \times \mathbf{R}_+ \times \mathbf{R}^L_+$. For the classic problems of integrability and rationalizability, in the line of results of Hurwicz and Uzawa (1971) and Afriat (1967), the analyst observes prices, income, and consumption bundles chosen by the consumer, thus observations of the form $((p, y), x) \in \mathbf{R}^L_{++} \times \mathbf{R}_+ \times \mathbf{R}^L_+$. Results by Houthakker (1950), Richter (1966), and Reny (2015) give conditions on the function $(p, y) \mapsto x$ such that it is a demand function generated by some preference relation, while results of Afriat give conditions on a finite set of such observations characterizing consistency with demand generated by some locally nonsatiated utility function.

Both problems can be seen as examples in this framework. Let

$$O := \{((p_r, y_r), x_r) \in \mathbf{R}^L_{++} \times \mathbf{R}_+ \times \mathbf{R}^L_+ : r \in R\}$$

be a set of observations. As above, R can be either finite or infinite. In the setting of Afriat's Theorem, R is finite. In the integrability problem, O

contains observations for every $p \in \mathbf{R}^L_{++}$, that is, for every $p \in \mathbf{R}^L_{++}$, there exists $r \in R$ and $x_r \in \mathbf{R}^L_+$ such that $p_r = p$, $p_r \cdot x_r = y_r$, and $((p_r, y_r), x_r) \in O$.

Consider the set of models $\mathcal{M}_d$ that are graphs of demand correspondences generated by some locally nonsatiated utility function. Thus $M \in \mathcal{M}_d$ if and only if

$$M = \left\{ ((p, y), x) \in \mathbf{R}^L_{++} \times \mathbf{R}_+ \times \mathbf{R}^L_+ : x \in \arg\max_{x' \in \mathbf{R}^L_+} U(x') \;\; \text{s.t.} \;\; p \cdot x' \leq y \right\}$$

where $U : \mathbf{R}^L_+ \to \mathbf{R}$ is locally nonsatiated. Both integrability and rationalizability correspond to the requirement that O is rationalizable in $\mathcal{M}_d$, that is, that $O \subseteq M_d$ for some $M \in \mathcal{M}_d$. $\diamondsuit$

Example 2: Exchange Equilibrium For this example, consider an exchange economy with m consumers and $L+1$ goods. Observations in this example are price vectors and profiles of initial endowment vectors. Brown and Matzkin (1996) characterize finite sets of such observations consistent with equilibrium in an exchange economy for some fixed collection of locally nonsatiated utility functions. This provides an equilibrium version of Afriat's Theorem, in which a key assumption is that individual consumption bundles are not observable. When only prices are observed, then any compact subset of $\mathbf{R}^L_{++}$ is rationalizable, by a result of Mas-Colell (1977), an important version of the Sonnenschein-Mantel-Debreu Theorem. In particular, every finite set of prices is rationalizable by this result. A key observation of Brown and Matzkin (1996) is then that with additional information at the individual level, some finite data sets can falsify the exchange equilibrium model.

To describe this problem, first normalize prices so that $p_{L+1} = 1$, and let $X := \mathbf{R}^L_{++} \times \mathbf{R}^{m(L+1)}_{++}$. Let $\mathcal{E} := \{U_1, \ldots, U_m\}$ where $U_i : \mathbf{R}^{L+1}_+ \to \mathbf{R}$ is locally nonsatiated for each $i = 1, \ldots, m$; $\mathcal{E}$ denotes the exchange economy in which the preferences of agent i are represented by U_i, and $\mathcal{E}_\omega$ denotes a particular specification of initial endowment vectors $\omega := (\omega_1, \ldots, \omega_m) \in \mathbf{R}^{m(L+1)}_{++}$ for these agents. Let $Z_{\mathcal{E}} : \mathbf{R}^L_{++} \times \mathbf{R}^{m(L+1)}_{++} \to 2^{\mathbf{R}^L}$ denote the aggregate excess demand correspondence for $\mathcal{E}$.

Let $O := \{(p_r, \omega_{1_r}, \ldots, \omega_{m_r}) \in \mathbf{R}^L_{++} \times \mathbf{R}^{m(L+1)}_{++} : r \in R\}$ be a set of such observations of prices and initial endowment profiles for a collection of

m agents. Consider the set of models $\mathcal{M}_e$ that are graphs of equilibrium correspondences generated by locally nonsatiated utility functions. Thus $M \in \mathcal{M}_e$ if and only if

$$M = \left\{(p, \omega) \in \mathbf{R}^L_{++} \times \mathbf{R}^{m(L+1)}_{++} : 0 \in Z_{\mathcal{E}}(p, \omega)\right\}$$

where as above $Z_{\mathcal{E}}$ is the aggregate excess demand correspondence generated by $\mathcal{E} = \{U_1, \ldots, U_m\}$. Note that with some abuse of notation, we use $Z_{\mathcal{E}}$ to denote aggregate excess demand for goods $1, \ldots, L$; Walras' Law implies this will be sufficient to characterize equilibria as above. $\diamondsuit$

Example 3: Finite Games with Strategic Complementarities Here consider finite player, finite action normal form games. The game has m players, and each player $i = 1, \ldots, m$ can take finitely many actions that affect N_i variables. Here $N := \sum_i N_i$ and profiles of players' actions $(a_1, \ldots, a_m) \in \mathbf{R}^N$ are the observable variables. Let $O := \{(a_{1r}, \ldots, a_{mr}) \in \mathbf{R}^N : r \in R\}$ be a set of observations of such action profiles. Let $N_{-i} := \sum_{j \neq i} N_j$ for each i. Specifying a payoff function $f_i : \mathbf{R}^{N_i} \times \mathbf{R}^{N_{-i}} \to \mathbf{R}$ and an action set $A_i \subseteq \mathbf{R}^{N_i}$ for each player i determines a corresponding game $G((f_1, A_1) \ldots, (f_m, A_m))$.

Suppose each such game is a finite game with strategic complementarities, so for each player i, A_i is a finite lattice in $\mathbf{R}^{N_i}$ and f_i is quasisupermodular in a_i and satisfies the single crossing property in (a_i, a_{-i}). For these games, the set of pure strategy Nash equilibria is a nonempty complete lattice, and these are the models for these games. Let $\mathcal{M}_{gsc}$ denote this collection, so $M \in \mathcal{M}_{gsc}$ if and only if

$$M = \{a \in \times_i A_i : a \text{ is a pure strategy Nash equilibrium of } G((f_1, A_1) \ldots, (f_m, A_m))\}$$

for the game $G((f_1, A_1) \ldots, (f_m, A_m))$. $\diamondsuit$

3 Reach

The central concept for these results is the reach of a set, introduced by Federer (1959). To define the reach, we start with some notation and preliminaries. Let $A \subseteq \mathbf{R}^N$. Let $d(x, A)$ denote the distance from the point x

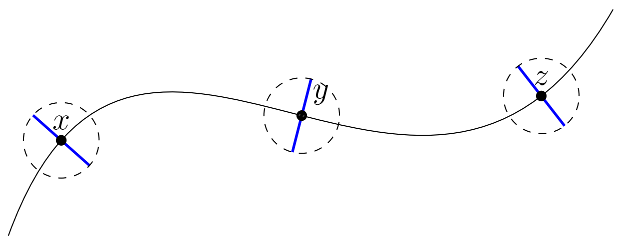

Figure 1: A set with positive reach at each point, and positive reach.

to A, so

$$d(x, A) := \inf_{a \in A} d(x, a)$$

Let $\mathrm{Unp}(A)$ be the set of points $x \in \mathbf{R}^N$ with a unique nearest point in A. Thus for $x \in \mathrm{Unp}(A)$, there is a unique point $\pi_A(x) \in A$ such that $d(x, A) = d(x, \pi_A(x))$. Let $\pi_A : \mathrm{Unp}(A) \to A$ be the function that maps each such point in $\mathrm{Unp}(A)$ to its unique nearest point in A. Thus for each $x \in \mathrm{Unp}(A)$, $\pi_A(x)$ is the unique point in A such that $d(x, A) = d(x, \pi_A(x))$.

For $a \in A$, define the *reach of A at the point $a \in A$* by

$$\mathrm{reach}(A, a) := \sup\{c \in \mathbf{R} : B_c(a) \subseteq \mathrm{Unp}(A)\}$$

where $B_c(a)$ denotes the open ball about a of radius c.

Finally, define the *reach* of A by

$$\mathrm{reach}(A) := \inf_{a \in A} \mathrm{reach}(A, a)$$

Roughly, $\mathrm{reach}(A)$ gives the largest amount by which A can be perturbed such that every point in the perturbed set has a unique nearest point in A. See Figures 1 and 2.

Sets with positive reach will play an important role in what follows. These sets can be thought of as generalizing central properties of both convex sets

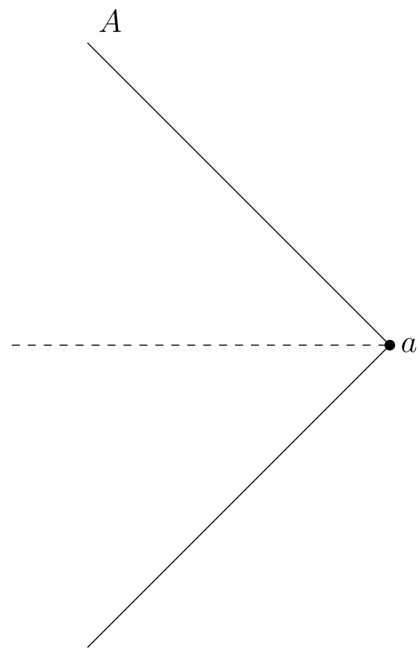

Figure 2: $\text{reach}(A, a) = 0$ and $\text{reach}(A) = 0$.

and smooth manifolds. In particular, it is straightforward to show that $\text{reach}(A) = \infty$ if and only if A is closed and convex. Similarly, if A is a compact manifold, then $\text{reach}(A) > 0$.

We start with a few preliminary observations and results. First, it is straightforward to see that $\text{reach}(A, a)$ is continuous in the point $a \in A$. Thus a compact set has positive reach if and only if it has positive reach at each point.

Lemma 1. *Let $A \subseteq \mathbf{R}^N$ be compact. Then reach$(A) > 0$ if and only if reach$(A, a) > 0$ for each $a \in A$.*

Proof. By definition, $\text{reach}(A, \cdot) : A \to \mathbf{R}_+ \cup \{+\infty\}$ is continuous and $\text{reach}(A, a) \geq \text{reach}(A)$ for each $a \in A$. The result follows. $\square$

Another useful observation is that positive reach is preserved by disjoint unions of compact sets.

Lemma 2. *Let $A, B \subseteq \mathbf{R}^N$ be compact and $A \cap B = \emptyset$. If reach$(A) > 0$ and reach$(B) > 0$, then reach$(A \cup B) > 0$ and*

$$\mathit{reach}(A \cup B) \geq \min\{\mathit{reach}(A), \mathit{reach}(B), \frac{1}{2}d(A, B)\}$$

Proof. Let $c < \min\{\text{reach}(A), \text{reach}(B), \frac{1}{2}d(A, B)\}$ and $x \in A \cup B$. Without loss of generality, suppose $x \in A$. Since $A \cap B = \emptyset$, this implies $x \notin B$. Then let $y \in B_c(x)$, and consider $\{z \in A \cup B : d(y, z) = d(y, A \cup B)\}$. Since $c < \frac{1}{2}d(A, B)$, any z in this set must be an element of A. To see this, note that if instead $z \in B$, then

$$\begin{aligned} d(x, z) \leq d(x, y) + d(y, z) &= d(x, y) + d(y, A \cup B) \\ &\leq d(x, y) + d(y, x) < 2c \leq d(A, B) \end{aligned}$$

which is a contradiction. Thus

$$\arg\min_{z \in A \cup B} d(y, z) = \arg\min_{z \in A} d(y, z)$$

Now since $c < \text{reach}(A)$, there is a unique such element of A, that is, $\arg\min_{z \in A} d(y, z)$ has a unique element. Thus $B_c(x) \subseteq \text{Unp}(A \cup B)$. Since $c < \min\{\text{reach}(A), \text{reach}(B), \frac{1}{2}d(A, B)\}$ was arbitrary,

$$\text{reach}(A \cup B, x) \geq \min\{\text{reach}(A), \text{reach}(B), \frac{1}{2}d(A, B)\}$$

Since $x \in A \cup B$ was arbitrary,

$$\text{reach}(A \cup B) = \inf_{x \in A \cup B} \text{reach}(A \cup B, x) \geq \min\{\text{reach}(A), \text{reach}(B), \frac{1}{2}d(A, B)\}$$

The result follows. $\square$

A key example of a set with positive reach is the graph of a suitably smooth function. This is a particularly useful observation for several of the examples of section 2.

Theorem 1. *Let $g : U \to \mathbf{R}^k$ where $U \subseteq \mathbf{R}^\ell$ is open, and write $g = (g_1, \ldots, g_k)$ with $g_i : U \to \mathbf{R}$ for each $i = 1, \ldots, k$. Suppose g is C^1 and $Dg_i : U \to \mathbf{R}^\ell$ is locally Lipschitz for each $i = 1, \ldots, k$. Let*

$$A := \{(x, y) \in \mathbf{R}^\ell \times \mathbf{R}^k : y = g(x),\ x^1 \leq x \leq x^2\}$$

for $x^1, x^2 \in U$ with $x^1 \ll x^2$. Then A is compact and reach$(A) > 0$.

Proof. First note that A is closed, and $A \subseteq [x^1, x^2] \times g([x^1, x^2])$. Since $[x^1, x^2]$ and $g([x^1, x^2])$ are compact, this implies A is compact. Since A is compact, it suffices to show that $\text{reach}(A, (x, y)) > 0$ for each $(x, y) \in A$. To that end, note that

$$A = \{(x, y) \in U \times \mathbf{R}^k : y = g(x)\} \cap \{(x, y) \in \mathbf{R}^\ell \times \mathbf{R}^k : x \in [x^1, x^2]\}$$

For each $i = 1, \ldots, k$, let $f_i : U \times \mathbf{R}^k \to \mathbf{R}$ be given by

$$f_i(x, y) = g_i(x) - y_i$$

Similarly, for each $j = 1, \ldots, \ell$, let $h_j^1, h_j^2 : \mathbf{R}^\ell \times \mathbf{R}^k \to \mathbf{R}$ be given by

$$h_j^1(x, y) = x_j^1 - x_j \quad \text{and} \quad h_j^2(x, y) = x_j - x_j^2$$

Then note that

$$\begin{aligned} A = \quad & \bigcap_{i=1}^{k} \{(x, y) \in U \times \mathbf{R}^k : f_i(x, y) = 0\} \\ & \cap \left(\bigcap_{j=1}^{\ell} \{(x, y) \in \mathbf{R}^\ell \times \mathbf{R}^k : h_j^1(x, y) \leq 0\} \right) \\ & \cap \left(\bigcap_{j=1}^{\ell} \{(x, y) \in \mathbf{R}^\ell \times \mathbf{R}^k : h_j^2(x, y) \leq 0\} \right) \end{aligned}$$

By construction, f_i, h_j^1, and h_j^2 are C^1 for each i and j. Moreover, for $(x, y) \in A$, $Df_i(x, y) = (Dg_i(x), -e_i)$, where e_i denotes the i^{th} standard basis vector in $\mathbf{R}^k$. Similarly, $Dh_j^1(x, y) = (-e_j, 0)$ and $Dh_j^2(x, y) = (e_j, 0)$, where with some abuse of notation e_j denotes the j^{th} standard basis vector in $\mathbf{R}^\ell$. In particular, Df_i, Dh_j^1, and Dh_j^2 are locally Lipschitz for each i and j.

Now fix $(x, y) \in A$. For each j, $x_j^1 < x_j^2$, so at most one of $h_j^1(x, y)$ and $h_j^2(x, y)$ is equal to zero. Then set

$$J_1 := \{j \in \{1, \ldots, \ell\} : h_j^1(x, y) = 0\} \quad \text{and} \quad J_2 := \{j \in \{1, \ldots, \ell\} : h_j^2(x, y) = 0\}$$

From the previous observation, $J_1 \cap J_2 = \emptyset$. Suppose $t_i \in \mathbf{R}$ for each $i = 1, \ldots, k$, $s_j^1 \in \mathbf{R}$ for each $j \in J_1$, and $s_j^2 \in \mathbf{R}$ for each $j \in J_2$ satisfy

$$\sum_{i=1}^{k} t_i Df_i(x, y) = \sum_{j \in J_1} s_j^1 Dh_j^1(x, y) + \sum_{j \in J_2} s_j^2 Dh_j^2(x, y)$$

This implies

$$\sum_{i=1}^{k} t_i(Dg_i(x,y), -e_i) = \sum_{j\in J_1} s_j^1(-e_j, 0) + \sum_{j\in J_2} s_j^2(e_j, 0)$$

Thus

$$-\sum_{i=1}^{k} t_i e_i = 0$$

which implies $t_i = 0$ for each i. Then equation above becomes

$$0 = \sum_{j\in J_1} s_j^1(-e_j, 0) + \sum_{j\in J_2} s_j^2(e_j, 0)$$

which implies

$$\sum_{j\in J_1} (-s_j^1)e_j + \sum_{j\in J_2} s_j^2 e_j = 0$$

Since $J_1 \cap J_2 = \emptyset$, $s_j^1 = 0$ for each $j \in J_1$ and $s_j^2 = 0$ for each $j \in J_2$. Then by Federer (1959, Theorem 4.12), reach$(A, (x,y)) > 0$. Since A is compact, the result follows. $\square$

The three examples from section 2 illustrate. In Example 1, models are graphs of demand correspondences for some locally nonsatiated utility function. Under several additional regularity conditions, such models will fit into this framework, which can be shown as an application of the previous result.

Example 1 (cont.): In the example of consumer demand, consider three additional restrictions. First, suppose that demand is unique for each $(p, y) \in \mathbf{R}^L_{++} \times \mathbf{R}_{++}$. This implies each model is the graph of a demand function generated by some appropriately restricted locally nonsatiated preference relation; for example, strict convexity is sufficient for this restriction. In addition, suppose demand is C^1 and its derivative is locally Lipschitz. Again this can be viewed as a consequence of sufficient convexity, smoothness, and regularity restrictions on preferences, and for simplicity we refer to these as C^2 regular utilities.[2] Finally, suppose observations are drawn from a

[2] For example, it is sufficient that preferences are strictly convex and strongly monotone, and represented by a utility function that is C^3 with everywhere nonsingular bordered Hessian.

fixed interval of prices and incomes $[(p^1, y^1), (p^2, y^2)]$ with $p^2 \gg p^1 \gg 0$ and $y^2 > y^1 > 0$.[3] With some abuse of terminology we refer to this as the set of "smooth demand" models, denoted $\mathcal{M}_{sd}$. Then $M \in \mathcal{M}_{sd}$ if and only if

$$M = \{((p, y), x) \in \mathbf{R}^L_{++} \times \mathbf{R}_{++} \times \mathbf{R}^L_+ : x = x_U(p, y),\ p^1 \leq p \leq p^2,\ y^1 \leq y \leq y^2\}$$

for a demand function x_U generated by some C^2 regular utility function U. Each $M \in \mathcal{M}_{sd}$ then has positive reach, by Theorem 1. $\diamondsuit$

In Example 2, models are graphs of equilibrium correspondences for some exchange economy with locally nonsatiated utility functions. Again under some regularity conditions, in particular that these are smooth economies in the sense of Debreu (1970), these models also fit into this framework.

Example 2 (cont.): In the example of exchange economies, again consider several additional assumptions. For simplicity, we will state some of these assumptions directly in terms of individual and aggregate excess demand; it is straightforward to derive these from suitable and well-understood primitive conditions on preferences. Suppose in each economy $\mathcal{E}$, individual preferences are strongly monotone and demands are unique, so each individual demand is a function. Suppose in addition aggregate excess demand $Z_{\mathcal{E}} : \mathbf{R}^L_{++} \times \mathbf{R}^{m(L+1)}_{++} \to \mathbf{R}^L_+$ is C^1 and has a locally Lipschitz derivative.[4] If these assumptions are satisfied, we will call $\mathcal{E}$ a *smooth economy*. In particular, under these assumptions classic results imply that $\mathcal{E}_\omega$ is a regular economy for almost all $\omega \in \mathbf{R}^{m(L+1)}_{++}$. Consider observations drawn from some fixed interval of endowments $[\omega^1, \omega^2]$ with $\omega^1, \omega^2 \in \mathbf{R}^{m(L+1)}_{++}$, $\omega^1 \leq \omega^2$. Again with some abuse of terminology, we will refer to this as the set of "smooth equilibrium" models, denoted $\mathcal{M}_{se}$. Then $M \in \mathcal{M}_{se}$ if and only if

$$M := \{(p, \omega) \in \mathbf{R}^L_{++} \times \mathbf{R}^{m(L+1)}_{++} : Z_{\mathcal{E}}(p, \omega) = 0, \quad \omega \in [\omega^1, \omega^2]\}$$

for some excess demand function $Z_{\mathcal{E}}$ generated by a smooth economy $\mathcal{E}$.

Given a smooth economy $\mathcal{E}$ and $\omega^1, \omega^2 \in \mathbf{R}^{m(L+1)}_{++}$, say $[\omega^1, \omega^2]$ is a *regular interval for* $\mathcal{E}$ if $\omega^1 \ll \omega^2$ and for all $\omega \in \partial([\omega^1, \omega^2])$, $\mathcal{E}_\omega$ is a regular economy.

[3] This interval could vary among models, but for simplicity we take a fixed such interval in this discussion.

[4] Again note by Walras' Law, it is sufficient to consider restrictions on $\{Z_1, \ldots, Z_L\}$.

Theorem 2. *Let $\mathcal{E}$ be a smooth economy and $[\omega^1, \omega^2]$ be a regular interval for $\mathcal{E}$, and let*

$$M := \{(p, \omega) \in \mathbf{R}^L_{++} \times [\omega^1, \omega^2] : Z_{\mathcal{E}}(p, \omega) = 0\}$$

Then M is compact and reach$(M) > 0$.

Proof. To simplify notation below, we omit the subscript $\mathcal{E}$ and write $Z := Z_{\mathcal{E}}$ for the aggregate excess demand function in $\mathcal{E}$. To show that M is closed, let $\{(p^n, \omega^n)\} \subseteq M$ such that $p^n \to p$ and $\omega^n \to \omega$. Then using the continuity of Z and the compactness of $[\omega^1, \omega^2]$, to show that $(p, \omega) \in M$ it suffices to show that $p \in \mathbf{R}^L_{++}$. Then to that end, suppose by way of contradiction that $p \not\gg 0$. For each n, define $\bar{p}^n \in \mathbf{R}^{L+1}_{++}$ by

$$\bar{p}^n_\ell = \begin{cases} \frac{p^n_\ell}{\sum_{k=1}^L p^n_k + 1} & \text{if } \ell = 1, \dots, L \\ \frac{1}{\sum_{k=1}^L p^n_k + 1} & \text{if } \ell = L + 1 \end{cases}$$

This just renormalizes so $\bar{p}^n \in \Delta(\mathbf{R}^{L+1}_+) = \{p \in \mathbf{R}^{L+1}_+ : \sum_{\ell=1}^{L+1} p_\ell = 1\}$. Let $\bar{p} \in \mathbf{R}^{L+1}_+$ be defined analogously. Then $\bar{p}^n \to \bar{p}$ by construction, and $\bar{p} \in \partial\Delta(\mathbf{R}^{L+1}_+)$ because $p \not\gg 0$. Since $\bar{p} \cdot \omega_i > 0$ for each $i = 1, \dots, m$ and individual utilities are strongly monotone, standard results imply

$$\max_{\ell=1,\dots,L+1} \|Z_\ell(\bar{p}^n, \omega^n)\| \to \infty$$

This is a contradiction, since $(p^n, \omega^n) \in M$ for each n, which implies

$$Z_\ell(\bar{p}^n, \omega^n) = Z_\ell(p^n, \omega^n) = 0 \quad \text{for each } n, \quad \text{for each } \ell = 1, \dots, L + 1$$

Thus $p \gg 0$. Since Z is continuous, $Z(p, \omega) = 0$, which implies $(p, \omega) \in M$. Thus M is closed.

Then to show that M is compact, it suffices to show that

$$\{p \in \mathbf{R}^L_{++} : Z(p, \omega) = 0,\ \omega \in [\omega^1, \omega^2]\} = \{p \in \mathbf{R}^L_{++} : (p, \omega) \in M \text{ for some } \omega \in [\omega^1, \omega^2]\}$$

is bounded. To that end, by way of contradiction suppose this set is not bounded. Then there exists $\{(p^n, \omega^n)\} \subseteq M$ such that $\{p^n\}$ is unbounded. For each n, define $\bar{p}^n \in \Delta(\mathbf{R}^{L+1}_+)$ as above. Because $\{p^n\}$ is unbounded, there is some subsequence $\{p^{n_k}\}$ of $\{p^n\}$ such that $\bar{p}^{n_k} \to \bar{p} \in \partial\Delta(\mathbf{R}_+^{L+1})$. Then $\{\omega^{n_k}\} \subseteq [\omega^1, \omega^2]$, so has a convergent subsequence; without loss of generality, taking a further subsequence of $\{p^n\}$ and relabeling if needed, $\omega^{n_k} \to \omega \in [\omega^1, \omega^2]$. As above, since $\bar{p} \cdot \omega_i > 0$ for each $i = 1, \ldots, m$ and individual utilities are strongly monotone, standard results imply

$$\max_{\ell=1,\ldots,L+1} \|Z_\ell(\bar{p}^{n_k}, \omega^{n_k})\| \to \infty$$

This is a contradiction, since $(p^{n_k}, \omega^{n_k}) \in M$ for each n_k, which implies

$$Z_\ell(\bar{p}^{n_k}, \omega^{n_k}) = Z_\ell(p^{n_k}, \omega^{n_k}) = 0 \quad \text{for each } n_k, \quad \text{for each } \ell = 1, \ldots, L+1$$

Thus M is bounded, and hence compact.

Now it suffices to show that reach$(M, (p, \omega)) > 0$ for each $(p, \omega) \in M$. To that end, note that

$$M = \{(p,\omega) \in \mathbf{R}_{++}^L \times \mathbf{R}_{++}^{m(L+1)} : Z(p,\omega) = 0\} \cap \{(p,\omega) \in \mathbf{R}_{++}^L \times \mathbf{R}_{++}^{m(L+1)} : \omega \in [\omega^1, \omega^2]\}$$

For each $i = 1, \ldots, m$ and $\ell = 1, \ldots, L+1$, let $h_{i\ell}^1, h_{i\ell}^2 : \mathbf{R}_{++}^L \times \mathbf{R}_{++}^{m(L+1)} \to \mathbf{R}$ be given by

$$h_{i\ell}^1(p,\omega) = \omega_{i\ell}^1 - \omega_{i\ell} \quad \text{and} \quad h_{i\ell}^2(p,\omega) = \omega_{i\ell} - \omega_{i\ell}^2$$

Then

$$\begin{aligned} M = \quad & \bigcap_{\ell=1}^{L} \{(p,\omega) \in \mathbf{R}_{++}^L \times \mathbf{R}_{++}^{m(L+1)} : Z_\ell(p,\omega) = 0\} \\ & \cap \left(\bigcap_{i,\ell} \{(p,\omega) \in \mathbf{R}_{++}^L \times \mathbf{R}_{++}^{m(L+1)} : h_{i\ell}^1(p,\omega) \le 0\} \right) \\ & \cap \left(\bigcap_{i,\ell} \{(p,\omega) \in \mathbf{R}_{++}^L \times \mathbf{R}_{++}^{m(L+1)} : h_{i\ell}^2(p,\omega) \le 0\} \right) \end{aligned}$$

By construction, $h_{i\ell}^1$ and $h_{i\ell}^2$ are C^1 for each (i, ℓ), and Z_ℓ is C^1 for each ℓ by assumption. Moreover, for each $(p,\omega) \in M$, $Dh_{i\ell}^1(p,\omega) = (0, -e_{i\ell})$ and $Dh_j^2(p,\omega) = (0, e_{i\ell})$, where $e_{i\ell}$ denotes the $i\ell^{th}$ standard basis vector in $\mathbf{R}^{m(L+1)}$. In particular, $Dh_{i\ell}^1$ and $Dh_{i\ell}^2$ are locally Lipschitz for each i and ℓ, and DZ_ℓ is locally Lipschitz for each ℓ by assumption. Since $\mathcal{E}$ is a smooth economy, the vectors $\{DZ_1(p,\omega),\ldots,DZ_L(p,\omega)\}$ are linearly independent for any $(p,\omega)\in M$ (0 is a regular value of Z – and in fact this is true for any (p,ω)).

Now fix $(p,\omega)\in M$. For each i and ℓ, $\omega^1_{i\ell}<\omega^2_{i\ell}$, so at most one of $h^1_{i\ell}(p,\omega)$ and $h^2_{i\ell}(p,\omega)$ is equal to zero. Then set

$$J_1 := \{(i,\ell) : h^1_{i\ell}(p,\omega)=0\} \quad \text{and} \quad J_2 := \{(i,\ell) : h^2_{i\ell}(x,y)=0\}$$

From the previous observation, $J_1\cap J_2=\emptyset$. Suppose $t_\ell\in\mathbf{R}$ for each $\ell=1,\ldots,L$, $s^1_{i\ell}\in\mathbf{R}$ for each $(i,\ell)\in J_1$, and $s^2_{i\ell}\in\mathbf{R}$ for each $(i,\ell)\in J_2$ satisfy

$$\sum_{\ell=1}^{L} t_\ell DZ_\ell(p,\omega) = \sum_{(i,\ell)\in J_1} s^1_{i\ell} Dh^1_{i\ell}(p,\omega) + \sum_{(i,\ell)\in J_2} s^2_{i\ell} Dh^2_{i\ell}(p,\omega)$$

This implies

$$\sum_{\ell=1}^{L} t_\ell DZ_\ell(p,\omega) = \sum_{(i,\ell)\in J_1} s^1_{i\ell}(0,-e_{i\ell}) + \sum_{(i,\ell)\in J_2} s^2_{i\ell}(0,e_{i\ell}) \qquad (*)$$

If $J_1=J_2=\emptyset$, then $(*)$ implies

$$\sum_{\ell=1}^{L} t_\ell DZ_\ell(p,\omega)=0$$

Since the vectors $\{DZ_1(p,\omega),\ldots,DZ_L(p,\omega)\}$ are linearly independent, this implies $t_\ell=0$ for each ℓ.

Now suppose either $J_1\neq\emptyset$ or $J_2\neq\emptyset$. This implies $\omega\in\partial([\omega^1,\omega^2])$ from the definition of J_1 and J_2. In this case, $(*)$ implies

$$\begin{aligned}
\sum_{\ell=1}^{L} t_\ell D_pZ_\ell(p,\omega) &= 0\\
\sum_{\ell=1}^{L} t_\ell D_\omega Z_\ell(p,\omega) &= \sum_{(i,\ell)\in J_1} s^1_{i\ell}(-e_{i\ell}) + \sum_{(i,\ell)\in J_2} s^2_{i\ell}e_{i\ell}
\end{aligned}$$

Since $[\omega^1, \omega^2]$ is a regular interval for $\mathcal{E}$ and $\omega \in \partial([\omega^1, \omega^2])$, $\mathcal{E}_\omega$ is a regular economy. Thus the vectors $\{D_p Z_1(p, \omega), \ldots, D_p Z_L(p, \omega)\}$ are linearly independent (note that $D_p Z(p, \omega)$ has full rank because $\mathcal{E}_\omega$ is a regular economy and $Z(p, \omega) = 0$). This implies $t_\ell = 0$ for each ℓ. Then $(*)$ implies

$$\sum_{(i,\ell) \in J_1} s^1_{i\ell}(-e_{i\ell}) + \sum_{(i,\ell) \in J_2} s^2_{i\ell} e_{i\ell} = 0$$

Since $J_1 \cap J_2 = \emptyset$, $s^1_{i\ell} = 0$ for each $(i, \ell) \in J_1$ and $s^2_{i\ell} = 0$ for each $(i, \ell) \in J_2$.

Then by Federer (1959, Theorem 4.12), $\mathrm{reach}(M, (p, \omega)) > 0$. Since M is compact, the result follows. □

◊

Example 3 (cont.): For finite games with strategic complementarities, each model $M \in \mathcal{M}_{gsc}$ is the set of pure strategy Nash equilibria in the corresponding game. Since these are games with strategic complementarities, each M is a nonempty finite lattice in $\mathbf{R}^N$. It is straightforward to see that these sets have positive reach. If $M := \{a^1, \ldots, a^J\} \subseteq \mathbf{R}^N$, then

$$\mathrm{reach}(M) = \frac{1}{2} \min_{a^j \neq a^k} d(a^j, a^k) > 0$$

In this case, note that the set of all action profiles A is a finite subset of $\mathbf{R}^N$, so already has positive reach by the same argument. Other natural solution concepts will also fit into this framework as a simple consequence, for example the dominance solution of all serially undominated pure strategy profiles. Similarly, restricting to some focal subset of equilibria, such as the smallest and largest equilibria, also fits in this framework. We discuss these points in more detail below. ◊

4 Random Rationalizability

In this section, we consider several explicit models of observations with random errors. We consider two main questions in this setting. First, can the model M be estimated from random noisy observations O_n for n sufficiently large? And second, can theories generating M be tested using O_n? The main results of this section show that for compact sets with positive reach, both questions can be answered. For such sets, a strongly consistent estimator for M can be constructed from O_n for n sufficiently large, provided the noise in the data is sufficiently small. Under weaker conditions on the noise, some features of the set M, in particular its homology, can be estimated from O_n for n sufficiently large. In the latter case, weaker conditions on the noise might not be sufficient to recover an estimate of the set M, but still might be sufficient to yield tests of some important features of the theory, such as rationalizability, or multiple equilibria.

The first result below shows that when M is a compact set with positive reach, then M can be estimated with a sufficiently rich data set under suitable regularity conditions. In this case the estimator approximates M with respect to Hausdorff distance and is strongly consistent. Recall if $A, B \subseteq \mathbf{R}^N$, the Hausdorff distance between A and B, denoted $d_H(A, B)$, is given by

$$\begin{aligned} d_H(A,B) &:= \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\} \\ &= \max\left\{\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(b,a)\right\} \end{aligned}$$

This result combines some ideas and techniques from set estimation, notably the Devroye-Wise estimator, with some ideas from topological data analysis and manifold estimation.

We start by giving the intuition behind the result and construction of the estimator. Here assume M is a compact set with positive reach, so $\text{reach}(M) > 0$. Given $\varepsilon > 0$, let

$$\mathcal{N}_\varepsilon(M) := \{y \in \mathbf{R}^N : d(y, M) < \varepsilon\} = \cup_{x\in M} B_\varepsilon(x)$$

The set $\mathcal{N}_\varepsilon(M)$ is the ε-neighborhood about M.

To construct the candidate for an estimator of M, let $\{\varepsilon_n\}$ be given with $\varepsilon_n > 0$ for each n, and first set

$$\hat{S}_{\varepsilon_n} := \bigcup_{i=1}^{n} \overline{B_{\varepsilon_n}(x_i)}$$

where $O_n = \{x_1, \ldots, x_n\}$. The set $\hat{S}_{\varepsilon_n}$ gives all the points a distance less than or equal to ε_n from some observation in O_n. This is a version of the Devroye-Wise estimator of an underlying set; see Devroye and Wise (1980). Next, set

$$\hat{r}_{\varepsilon_n} := \max_{x \in \hat{S}_{\varepsilon_n}} d(x, \partial \hat{S}_{\varepsilon_n})$$

Finally, set

$$\hat{M}_{\varepsilon_n} := \{x \in \hat{S}_{\varepsilon_n} : d(x, \partial \hat{S}_{\varepsilon_n}) \geq \hat{r}_{\varepsilon_n} - 2\varepsilon_n\}$$

The set $\hat{M}_{\varepsilon_n}$ gives all the points in $\hat{S}_{\varepsilon_n}$ that are a distance within $2\varepsilon_n$ of the maximum distance points in $\hat{S}_{\varepsilon_n}$ can be from the boundary $\partial \hat{S}_{\varepsilon_n}$.

This construction provides an estimator of M when reach$(M) > 0$ and when the noise in the observations in O_n is iid and sufficiently small, under some additional restrictions on this distribution. To formalize these regularity conditions we use two notions of standardness for a distribution and its support from Cuevas and Rodriguez-Casal (2004).

Definition 1. Let $A \subseteq \mathbf{R}^N$ be a Borel set and μ be a Borel measure on $\mathbf{R}^N$. The measure μ is *standard with respect to Lebesgue measure on A* if there exist $\delta, \gamma > 0$ such that

$$\mu(B_\alpha(x) \cap A) \geq \delta \lambda(B_\alpha(x) \cap A) \ \ \forall x \in A, \ \ \forall\ 0 < \alpha \leq \gamma$$

where λ is Lebesgue measure on $\mathbf{R}^N$.

Remark: If A is the support of μ and μ is absolutely continuous with respect to λ with a density bounded away from zero on A, then μ is standard with respect to λ on A.

Definition 2. A Borel set $A \subseteq \mathbf{R}^N$ is *Lebesgue standard* if there exist $\delta, \gamma > 0$ such that

$$\lambda(B_\alpha(x) \cap A) \geq \delta \lambda(B_\alpha(x)) \ \ \forall x \in A, \ \ \forall\ 0 < \alpha \leq \gamma$$

The set A is *standard with respect to* μ if there exist $\delta, \gamma > 0$ such that

$$\mu(B_\alpha(x) \cap A) \geq \delta \lambda(B_\alpha(x)) \ \ \forall x \in A, \ \ \forall\ 0 < \alpha \leq \gamma$$

Remark: Note that if A is Lebesgue standard and μ is standard with respect to Lebesgue measure on A, then A is standard with respect to μ.

If μ is standard with respect to Lebesgue measure on A with corresponding constants δ, γ, set

$$c(\mu, A) := 2\left(\frac{2}{\delta\lambda(B_1(0))}\right)^{\frac{1}{N}}$$

This constant $c(\mu, A)$ will provide an appropriate tuning constant in the sequence $\{\varepsilon_n\}$ used to construct an estimator based on μ.

For this result, suppose observations are drawn iid from a distribution with support $\overline{\mathcal{N}_\varepsilon(M)}$ that is also standard with respect to Lebesgue measure, as formalized below.

Definition 3. Let $M \subseteq \mathbf{R}^N$. $\mathcal{O}_n = \{X_1, \ldots, X_n\}$ is *ε-iid rationalizable for M* if for each i, $X_i \sim \mu$ iid, where μ is a Borel probability measure with support $\overline{\mathcal{N}_\varepsilon(M)}$, and μ is standard with respect to Lebesgue measure on $\overline{\mathcal{N}_\varepsilon(M)}$.

The key steps will then be to show that when the data is ε-iid rationalizable for M and M has positive reach, the sequence $\{\varepsilon_n\}$ can be chosen appropriately so that $\varepsilon_n \to 0$ while also ensuring that $\hat{S}_{\varepsilon_n}$ estimates a corresponding neighborhood of M and $\partial\hat{S}_{\varepsilon_n}$ estimates the boundary of this neighborhood, both to the desired order. In particular, ε_n can be chosen so that $\varepsilon_n = O(\frac{\log n}{n})$; this uses results in Cuevas and Rodriguez-Casal (2004). The final step shows that the corresponding estimator $\hat{M}_{\varepsilon_n}$ is strongly consistent, and converges to M in Hausdorff distance at the rate $O(\frac{\log n}{n})$. The arguments extend several related constructions from the manifold case, for example as in Niyogi, Smale, and Weinberger (2008) or Genovese, Perone-Pacifico, Verdinelli, and Wasserman (2012), to the more general environment of compact sets with positive reach; see the appendix for the proof.

Theorem 3. *Let $M \subseteq \mathbf{R}^N$ be a compact set with empty interior and positive reach. Suppose $\mathcal{O}_n = \{X_1, \ldots, X_n\}$ is ε-iid rationalizable for M where $\varepsilon <$*

reach(M). Let $\varepsilon_n = c(\frac{\log n}{n})^{\frac{1}{N}}$ for each n, where $c > c(\mu, \overline{\mathcal{N}_\varepsilon(M)})$. Then with probability one, for all sufficiently large n,

$$d_H(\hat{M}_{\varepsilon_n}, M) \leq 4\varepsilon_n$$

and $M \subseteq \hat{M}_{\varepsilon_n}$. In particular, with probability one, for all sufficiently large n, $d_H(\hat{M}_{\varepsilon_n}, M) = O((\frac{\log n}{n})^{\frac{1}{N}})$.

Remark: If ε is known a priori, this construction could be simplified, for example by replacing the random lower bound $\hat{r}_{\varepsilon_n} - 2\varepsilon_n$ in the definition of $\hat{M}_{\varepsilon_n}$ with the deterministic bound $\varepsilon - \varepsilon_n$. In this case, the assumption that M has empty interior is not necessary. Because M has positive reach and $\varepsilon < \text{reach}(M)$, $d(x, \partial S) \geq \varepsilon$ for each $x \in M$, while $d(x, \partial M) = \varepsilon$ if $x \in \partial M$. This implies $M = \{y \in S : d(y, \partial S) \geq \varepsilon\}$, and arguments in the proof can be adapted to show with probability one, for n sufficiently large $d_H(\hat{M}_{\varepsilon_n}, M) \leq 2\varepsilon_n$ in this case.

The second result relaxes the requirement that the support of μ is a neighborhood of the set M. For more general such distributions, estimation with respect to Hausdorff distance might break down. Instead, following a central theme in topological data analysis, a suitable estimator might still convey important topological information about M. In particular, we adapt the construction in the central work of Niyogi, Smale, and Weinberger (2008, 2011) to give a consistent estimator for the homology of M.

Intuitively, homology gives information about the number of holes or cycles of different dimensions in a set. We give a brief loose description here and discuss the aspects of homology that are the most important for the results and applications of this paper; for a good standard introduction see Hatcher (2002). Given a topological space X, the homology of X is a topological invariant $H(X)$ given by a sequence of abelian groups $H_0(X), H_1(X), H_2(X), \ldots$, where loosely the k^{th} homology group $H_k(X)$ describes the number of holes or cycles in X with a k-dimensional boundary. For $k = 0$, this corresponds to the path-connected components of X, and the dimension of $H_0(X)$ counts the number of these connected components of X. For example, if $X = \mathbf{R}^J$, then X has one connected component, and no holes with boundary of dimension 1 or higher. The corresponding homology groups are $H_0(\mathbf{R}^J) = \mathbf{Z}$ and $H_k(\mathbf{R}^J) = \{0\}$ for all $k \geq 1$, where $\{0\}$ represents the trivial group. A circle S^1 in $\mathbf{R}^2$ has one connected component, one hole with a one-dimensional boundary, and no holes with boundaries of dimension greater than 1, and the corresponding homology groups are $H_0(S^1) = H_1(S^1) = \mathbf{Z}$ and $H_k(S^1) = \{0\}$ for all $k \geq 2$. A ball in $\mathbf{R}^2$ (or $\mathbf{R}^J$ for any $J \geq 1$) also has one connected component, but in contrast has no holes with boundaries of dimension 1 or higher, so for example $H_0(B_1(0)) = \mathbf{Z}$ and $H_k(B_1(0)) = \{0\}$ for all $k \geq 1$, while a sphere S^2 in $\mathbf{R}^3$ has one connected component, no holes with one-dimensional boundaries, one hole with two-dimensional boundary, and no holes with boundaries of dimension greater than two. Homology is an important tool in topological data analysis for inferring some information about an underlying set from noisy observations it generates, even when it is not possible to accurately estimate the set. For the results in this paper, a few basic properties of homology will be sufficient, and while calculating homology groups for general topological spaces can be complicated, these calculations will be simple for the applications considered here. We discuss this in more detail below and in the applications in the next section.

Niyogi, Smale, and Weinberger (2011) consider the problem of estimating homology when M is a manifold and observations are iid draws from random variables of the form $X_i = Y_i + \xi_i$, where Y_i has support M and ξ_i is normal with mean zero, and construct a consistent estimator for $H(M)$ in this case. Here we give an elementary version of their result, adapted to the more general setting where M is a set with positive reach but not necessarily a manifold.

For this result, observations are assumed to be iid draws from a distribution μ on $\mathbf{R}^N$ sufficiently concentrated around M. Roughly this requires that observations in some neighborhood around M always have positive probability, and observations sufficiently close to M are more likely than observations far from M, in a sense made precise below.

Definition 4. Let $M \subseteq \mathbf{R}^N$. $\mathcal{O}_n := \{X_1, \ldots, X_n\}$ is *iid rationalizable for* M if for each i, $X_i \sim \mu$ iid for a Borel probability measure μ on $\mathbf{R}^N$ such that for some $0 < s < \frac{1}{12}\text{reach}(M)$,

(i) for each $x \in M$, $\mu(B_{\frac{s}{2}}(x)) > 0$

(ii) there exists $r > 0$ such that

$$\inf_{x \in \mathcal{N}_s(M)} \mu(B_r(x)) > \sup_{y \in \mathbf{R}^N \setminus \mathcal{N}_{2s}(M)} \mu(B_r(y))$$

Remark: Note if μ has support contained in $\mathcal{N}_s(M)$, then $s < \frac{1}{6}\text{reach}(M)$ is sufficient for the general result in Theorem 4; we discuss this in more detail after sketching the proof. Niyogi, Smale, and Weinbeger (2011) show that an analogous property holds for the case of additive normal noise they consider; see also Genovese, Perone-Pacifico, Verdinelli, and Wasserman (2012) for other noise models in which a version of this condition holds in the manifold case.

The idea in Niyogi, Smale, and Weinberger (2011), adapted here, is to use the fact that observations close to M are sufficiently more likely than observations far from M to devise a cleaning procedure for the data, throwing away observations that are identified through this procedure as likely to be far from M. The tradeoff is losing some sharpness in estimating the set M. This is not necessarily enough to estimate M, but is strong enough to estimate the homology of M from the remaining sample, using ideas similar to those in the first result, and to guarantee that this estimate converges in probability.

The construction proceeds in several steps. First, let

$$A := \mathcal{N}_s(M) \quad \text{and} \quad B := \mathbf{R}^N \setminus \mathcal{N}_{2s}(M)$$

Then set

$$a := \inf_{x \in A} \mu(B_r(x)) \quad \text{and } b := \sup_{y \in B} \mu(B_r(y))$$

and set $h := \frac{a-b}{2}$. By assumption $a > b$, so $h > 0$. Consider a fixed n and associated data set $O_n = \{x_1, \ldots, x_n\}$. For each i, set

$$d_i := \sum_{j \neq i} \mathbf{1}(x_j \in B_r(x_i))$$

Thus d_i counts the number of other observations $j \neq i$ within distance r of x_i. To construct the cleaning procedure, set

$$O_n^f := \left\{ x_i \in O_n : \frac{d_i}{n-1} > \frac{a+b}{2} \right\}$$

The estimator $\hat{M}_n$ will be a version of the Devroye-Wise estimator using the cleaned data O_n^f, taking a union of ε balls around these remaining points. A key observation is that for appropriately chosen ε, the union of these balls will be sufficiently topologically similar to M to yield the desired estimate. Then to define the estimator, choose $\varepsilon > 0$ such that $6s < \varepsilon < \frac{1}{2}\text{reach}(M)$, and set

$$\hat{M}_n := \bigcup_{x \in O_n^f} B_\varepsilon(x)$$

When M has positive reach, $\hat{M}_n$ will be topologically similar to M with high probability for n sufficiently large, following an argument similar to Niyogi, Smale, and Weinberger (2011). In particular, with high probability M will be a deformation retract of $\hat{M}_n$, from which it follows that they have the same homology. Given a topological space X and subset $A \subseteq X$, a function $g : X \times [0,1] \to X$ is a *deformation retract* of X to A if g is continuous, $g(x,0) = x$ for all $x \in X$, $g(x,1) \in A$ for all $x \in X$, and $g(a,1) = a$ for all $a \in A$. If such a function g exists, A is a *deformation retract* of X, or equivalently, X *deformation retracts to* A.

We state this result next and then give a sketch of the proof.

Theorem 4. *Let $M \subseteq \mathbf{R}^N$ be a compact set with positive reach. Suppose $\mathcal{O}_n = \{X_1, \ldots, X_n\}$ is iid rationalizable for M. For each $\delta \in (0,1)$ there exists K such that for $n > K$, $H(\hat{M}_n) = H(M)$ with probability at least $1-\delta$, where $K = O(\beta(\delta))$, with $\beta(\delta) := \log\frac{1}{\delta}\log(\log\frac{1}{\delta})$.*

The proof of this result has three main steps. First, with high probability the cleaning procedure is accurate for all sufficiently large n, meaning it keeps all of the observations in $A = \mathcal{N}_s(M)$ and throws out all of the observations in $B = \mathbf{R}^N \setminus \mathcal{N}_{2s}(M)$. Second, with sufficiently many observations, $M \subseteq \cup_{x \in O_n^f} B_s(x) \subseteq \hat{M}_n$ with high probability. These both follow in turn from general results in Niyogi, Smale, and Weinberger (2011, Lemma 3; 2008, Lemma 5.1). Finally, for suitably chosen $\varepsilon > 0$, $\hat{M}_n = \cup_{x \in O_n^f} B_\varepsilon(x)$ deformation retracts to M. This step requires M to have positive reach, and shows in particular that this can be done using the natural mapping following the closest point function $(x,t) \mapsto (1-t)x + t\pi_M(x)$. These three steps are established in the following lemmas; see the appendix for proofs and additional discussion.

The first two steps are given in Lemmas 3 and 4.

Lemma 3. *Let $M \subseteq \mathbf{R}^N$ and suppose $\mathcal{O}_n = \{X_1, \ldots, X_n\}$ is iid rationalizable for M. For each $\delta \in (0,1)$ there exists K such that for $n > K$, with probability at least $1-\delta$,*

$$O_n^f \subseteq \mathcal{N}_{2s}(M) \quad \textit{and} \quad O_n \cap \mathcal{N}_s(M) \subseteq O_n^f$$

where $K = O(\beta(\delta))$, with $\beta(\delta) := \log \frac{1}{\delta} \log(\log \frac{1}{\delta})$.

Lemma 4. *(Niyogi, Smale and Weinberger (2008, Lemma 5.1)) Let μ be a probability measure on $\mathbf{R}^N$ and $A_1, \ldots, A_k \subseteq \mathbf{R}^N$ be measurable subsets such that $\mu(A_i) > \alpha > 0$ for all i. Let $O_n := \{x_1, \ldots, x_n\}$ be drawn iid according to μ. For each $\delta \in (0,1)$, if $n \geq \frac{1}{\alpha}(\log k + \log \frac{1}{\delta})$, then with probability at least $1-\delta$,*

$$O_n \cap A_i \neq \emptyset \;\; \forall\; i = 1, \ldots, k$$

The third step in the proof of Theorem 4 is Lemma 5 below, which gives general conditions under which M is a deformation retract of a set of the form $\cup_{i=1}^k B_\varepsilon(x_i)$. This adapts a similar result from Niyogi, Smale, and Weinberger (2011) for the case in which M is a manifold; here M is instead a general compact set with positive reach.[5] Here we give an elementary proof, and do not seek the tightest bounds.

Lemma 5. *Let $M \subseteq \mathbf{R}^N$ be compact and $\{x_1, \ldots, x_k\} \subseteq \mathbf{R}^N$. Suppose $M \subseteq \cup_{i=1}^k B_c(x_i)$ and $\{x_1, \ldots, x_k\} \subseteq \mathcal{N}_c(M)$ for some $c > 0$. If $3c < \varepsilon < \frac{1}{2}$reach$(M)$, then $\cup_{i=1}^k B_\varepsilon(x_i)$ deformation retracts to M.*

Putting these pieces together gives the proof of Theorem 4.

Proof of Theorem 4: Let $O_n = \{x_1, \ldots, x_n\}$, and fix $\delta \in (0,1)$. Let $A := \mathcal{N}_s(M)$. By Lemma 3, there exists K with $K = O(\beta(\delta))$ such that for $n > K$, with probability at least $1 - \frac{\delta}{2}$,

[5]See also, for example, Cucker, Krick, and Shub (2018) for a related result for the manifold case with different bounds.

(i) $O_n \cap A \subseteq O_n^f \subseteq \mathcal{N}_{2s}(M)$

Let $A_i = B_{\frac{s}{2}}(y_i)$ for $i = 1, \ldots, k$ where $y_i \in M$ for each i and $M \subseteq \cup_i A_i$. By construction, $\cup_i A_i \subseteq A$. By assumption $\min_i \mu(B_{\frac{s}{2}}(y_i)) > 0$, so let $\alpha > 0$ such that $\alpha < \min_i \mu(B_{\frac{s}{2}}(y_i))$. By Lemma 4, if $n > K' := \max\{K, \frac{1}{\alpha}(\log k + \log \frac{2}{\delta})\}$, then with probability at least $1 - \frac{\delta}{2}$,

(ii) $O_n \cap A_i \neq \emptyset$ for each $i = 1, \ldots, k$

Thus if $n > K'$, then with probability at least $1 - \delta$ both conditions (i) and (ii) hold. Then since $\cup_i A_i \subseteq A$, this implies

$$O_n \cap (\cup_i A_i) \subseteq O_n \cap A \subseteq O_n^f$$

Thus

$$M \subseteq \cup_i A_i = \cup_i B_{\frac{s}{2}}(y_i) \subseteq \cup_{x \in O_n^f} B_s(x)$$

Then $M \subseteq \cup_{x \in O_n^f} B_s(x) \subseteq \cup_{x \in O_n^f} B_{2s}(x)$, and by (i), $O_n^f \subseteq \mathcal{N}_{2s}(M)$. By Lemma 5, since $6s = 3(2s) < \varepsilon < \frac{1}{2}\text{reach}(M)$, $\hat{M}_n = \cup_{x \in O_n^f} B_\varepsilon(x)$ deformation retracts to M. Homology is preserved by deformation retract, which implies that $H(\hat{M}_n) = H(M)$. $\square$

Remark: If the support of μ is contained in $\mathcal{N}_s(M)$ for some $s > 0$, then the same argument can be used to show $s < \frac{1}{6}\text{reach}(M)$ will suffice to construct a suitable estimator for the homology of M. In this case, note that with probability one $O_n \subseteq \mathcal{N}_s(M)$, and thus $O_n^f \subseteq \mathcal{N}_s(M)$, for each n. Following the argument and notation in the proof of Theorem 4, given $\delta \in (0, 1)$, if $n > K'$ then $M \subseteq \cup_{x \in O_n^f} B_s(x)$ with probability at least $1 - \delta$. Then using Lemma 5, for $3s < \varepsilon < \frac{1}{2}\text{reach}(M)$, $\cup_{x \in O_n^f} B_\varepsilon(x)$ deformation retracts to M. Setting $\hat{M}_n := \cup_{x \in O_n^f} B_\varepsilon(x)$ then gives an appropriate estimator.

Remark: Similarly, if μ has support M, an even simpler estimator is available. In this case, let $0 < \varepsilon < \frac{1}{2}\text{reach}(M)$, and given $O_n = \{x, \ldots, x_n\}$, set $\hat{M}_n := \cup_{x \in O_n} B_\varepsilon(x)$. To see that $\hat{M}_n$ suffices in this case, let $\gamma > 0$ with $\gamma < \frac{\varepsilon}{2}$, and $\{y_1, \ldots, y_k\} \subseteq M$ such that $M \subseteq \cup_i B_{\frac{\gamma}{3}}(y_i)$. By Lemma 4, given

$\delta \in (0,1)$, there exists K such that if $n > K$, then with probability at least $1-\delta$,

$$O_n \cap B_{\frac{\gamma}{3}}(y_i) \neq \emptyset \text{ for each } i = 1, \ldots, k$$

Then by construction,

$$M \subseteq \cup_i B_{\frac{\gamma}{3}}(y_i) \subseteq \cup_{x \in O_n} B_{\frac{2\gamma}{3}}(x)$$

By Lemma 5, $\cup_{x \in O_n} B_\varepsilon(x)$ deformation retracts to M, since $\{x_1, \ldots, x_n\} \subseteq M \subseteq \mathcal{N}_{\frac{2\gamma}{3}}(M)$, $d(x_i, M) < \frac{2\gamma}{3}$ for each i, and $3(\frac{2\gamma}{3}) = 2\gamma < \varepsilon < \frac{1}{2}\text{reach}(M)$.

Remark: When the data is ε-iid rationalizable, the estimator $\hat{M}_{\varepsilon_n}$ might not recover the homology of M, even though the Hausdorff distance between $\hat{M}_{\varepsilon_n}$ and M goes to zero almost surely when M has empty interior.[6] An alternative estimator can provide a result for the almost sure convergence of homology in this setting, however. In this case, let $0 < \gamma < \frac{1}{2}\text{reach}(M)$, and set

$$\hat{M}^H_{\varepsilon_n} := \cup_{x \in \hat{M}_{\varepsilon_n}} B_\gamma(x)$$

Then for n sufficiently large so that $12\varepsilon_n = 3(4\varepsilon_n) < \gamma$, $\cup_{x \in \hat{M}_{\varepsilon_n}} B_\gamma(x)$ deformation retracts to M by an analogue of Lemma 5.[7] Thus with probability one, for n sufficiently large $H(\hat{M}^H_{\varepsilon_n}) = H(M)$.

Remark: The estimator $\hat{M}_n$ takes the form $\cup_{y \in Y} B_\varepsilon(y)$ for a fixed $\varepsilon > 0$ and finite set $Y \subseteq \mathbf{R}^N$, in particular where Y is the cleaned data set O^f_n. Algorithms for computing the homology groups of such a finite union of balls have been developed; for example see Cucker, Krick, and Shub (2018) for a detailed discussion and development of such an algorithm, as well as results on the complexity of these algorithms. Cucker, Krick, and Shub (2018) show that the homology groups of $\cup_{y \in Y} B_\varepsilon(y)$ can be computed for a general finite set $Y \subseteq \mathbf{R}^N$ with $|Y|^{O(N)}$ operations.

[6] Sets can be arbitrarily close in Hausdorff distance but have different homologies; for a simple example let $A_n := \{-\frac{1}{n}, \frac{1}{n}\}$ for each $n > 0$ and $A := \{0\}$.

[7] Here $\hat{M}_{\varepsilon_n}$ is a compact set that is not necessarily finite, but it is straightforward to see that the proof of Lemma 5 carries over for this case.

5 Applications

In this section we consider applications of the results of the previous section in the examples developed in sections 2 and 3.

Example 1 (cont.): In the example of consumer demand, suppose $M \in \mathcal{M}_{sd}$, so

$$M = \{((p, y), x) \in \mathbf{R}^L_{++} \times \mathbf{R}_{++} \times \mathbf{R}^L_+ : x = x_U(p, y),\ p^1 \leq p \leq p^2,\ y^1 \leq y \leq y^2\}$$

for a demand function x_U generated by some C^2 regular utility function U. From Theorem 1, M is a compact set with positive reach. It is straightforward to see that M also has empty interior: for any $(p, y) \in [(p^1, y^1), (p^2, y^2)]$, $((p, y), x) \in M$ if and only if $x = x_U(p, y)$, so no neighborhood of $((p, y), x)$ is contained in M.

Then if $\mathcal{O}_n$ is ε-iid rationalizable for M for $\varepsilon <$ reach(M), $\hat{M}_{\varepsilon_n}$ gives a strongly consistent estimator for M with respect to Hausdorff distance, by Theorem 3. This estimator can also be sharpened a bit, as follows. Given the data set $O_n = \{((p_1, y_1), x_1), \ldots, ((p_n, y_n), x_n)\}$ and $\hat{M}_{\varepsilon_n}$, let $\hat{x}_{\varepsilon_n} : [(p^1, y^1), (p^2, y^2)] \to \mathbf{R}^L_+$ be a function such that

$$\hat{x}_{\varepsilon_n}(p, y) = \begin{cases} x & \text{for some } x \text{ s.t. } p \cdot x = y \text{ and } ((p, y), x) \in \hat{M}_{\varepsilon_n} \\ \frac{y}{p_1} e_1 & \text{if } \hat{M}_{\varepsilon_n} \cap \{((p, y), x') : p \cdot x' = y\} = \emptyset \end{cases}$$

where e_1 denotes the first standard basis vector in $\mathbf{R}^L$. Call such a function a *budget-feasible selection* from $\hat{M}_{\varepsilon_n}$. Then note that with probability one, for all n sufficiently large, $M \subseteq \hat{M}_{\varepsilon_n}$, which implies $((p, y), \hat{x}_{\varepsilon_n}(p, y)) \in \hat{M}_{\varepsilon_n}$ for each (p, y), that is, $\hat{x}_{\varepsilon_n}(p, y)$ is a selection from $\{x \in \mathbf{R}^L_+ : ((p, y), x) \in \hat{M}_{\varepsilon_n}\}$, and in addition $p \cdot \hat{x}_{\varepsilon_n}(p, y) = y$, so $\hat{x}_{\varepsilon_n}(p, y)$ is an element of the budget set defined by (p, y). This implies $\hat{x}_{\varepsilon_n}$ converges uniformly to x_U, and moreover, for any $\gamma > 0$, $\hat{x}_{\varepsilon_n}$ is eventually γ-optimal, that is,

$$\hat{x}_{\varepsilon_n}(p, y) \in \{x \in \mathbf{R}^L_+ : p \cdot x = y,\quad U(x) > U(x_U(p, y)) - \gamma\}$$

We record these observations below.

Theorem 5. *Let $M \subseteq \mathcal{M}_{sd}$ and $\mathcal{O}_n$ be ε-iid rationalizable for M for $\varepsilon <$ reach(M). Let $\hat{x}_{\varepsilon_n} : [(p^1, y^1), (p^2, y^2)] \to \mathbf{R}^L_+$ be a budget-feasible selection from $\hat{M}_{\varepsilon_n}$ for each n. Then with probability one, $\hat{x}_{\varepsilon_n} \to x_U$ uniformly on $[(p^1, y^1), (p^2, y^2)]$, and for all $\gamma > 0$, $\hat{x}_{\varepsilon_n}$ is eventually γ-optimal.*

Proof. The set M is compact, has empty interior, and $\text{reach}(M) > 0$, so by Theorem 3, with probability one, for all n sufficiently large, $M \subseteq \hat{M}_{\varepsilon_n}$ and $d^H(\hat{M}_{\varepsilon_n}, M) \leq 4\varepsilon_n$. Then for all n sufficiently large, for each $(p, y) \in [(p^1, y^1), (p^2, y^2)]$, $((p, y), x_U(p, y)) \in M \subseteq \hat{M}_{\varepsilon_n}$, so $\hat{M}_{\varepsilon_n} \cap \{((p, y), x') : p \cdot x' = y\} \neq \emptyset$. This implies by construction that $((p, y), \hat{x}_{\varepsilon_n}(p, y)) \in \hat{M}_{\varepsilon_n}$. Thus with probability one, for all n sufficiently large,

$$d(\hat{x}_{\varepsilon_n}(p, y), x_U(p, y)) \leq d^H(\hat{M}_{\varepsilon_n}, M) \leq 4\varepsilon_n \ \ \forall\ (p, y) \in [(p^1, y^1), (p^2, y^2)]$$

Thus with probability one, $\hat{x}_{\varepsilon_n} \to x_U$ uniformly on $[(p^1, y^1), (p^2, y^2)]$.

For the second claim, note that with probability one, for all n sufficiently large, $\hat{M}_{\varepsilon_n} \subseteq \overline{\mathcal{N}_\varepsilon(M)}$, since $d^H(\hat{M}_{\varepsilon_n}, M) \leq 4\varepsilon_n$, and $\overline{\mathcal{N}_\varepsilon(M)}$ is compact. Since U is C^2 it is Lipschitz continuous on the compact set $\{x \in \mathbf{R}^L_+ : ((p, y), x) \in \overline{\mathcal{N}_\varepsilon(M)}$ for some $(p, y) \in [(p^1, y^1), (p^2, y^2)]\}$. Let K be a corresponding Lipschitz constant. Now let $\gamma > 0$. For all n sufficiently large,

$$\begin{aligned} |U(x_U(p, y)) - U(\hat{x}_{\varepsilon_n}(p, y))| &\leq Kd(x_U(p, y), \hat{x}_{\varepsilon_n}(p, y)) \\ &\leq K4\varepsilon_n < \gamma \ \ \forall\ (p, y) \in [(p^1, y^1), (p^2, y^2)] \end{aligned}$$

Thus for such n,

$$|U(x_U(p, y)) - U(\hat{x}_{\varepsilon_n}(p, y))| = U(x_U(p, y)) - U(\hat{x}_{\varepsilon_n}(p, y)) < \gamma \ \forall\ (p, y) \in [(p^1, y^1), (p^2, y^2)]$$

which implies $U(\hat{x}_{\varepsilon_n}(p, y)) > U(x_U(p, y)) - \gamma$ for all $(p, y) \in [(p^1, y^1), (p^2, y^2)]$. Thus with probability one, $\hat{x}_{\varepsilon_n}$ is eventually γ-optimal. Since $\gamma > 0$ was arbitrary, the result follows. □

Next suppose $\mathcal{O}_n$ is iid rationalizable for M. By Theorem 4, for each $\delta > 0$, for all n sufficiently large, $H(\hat{M}_n) = H(M)$ with probability $1 - \delta$. In this case, it is straightforward to see that $H(M)$ is particularly simple. Since M is the graph of a continuous function, M is homeomorphic to the domain of this function, $[(p^1, y^1), (p^2, y^2)]$. In addition, any closed convex subset of $\mathbf{R}^J$ is a deformation retract of $\mathbf{R}^J$ for any J. Homology is preserved by homeomorphisms and deformation retracts, so these observations imply $H(M) = H(\mathbf{R}^{L+1})$. Finally, $H(\mathbf{R}^{L+1})$ is simple: $H_0(\mathbf{R}^{L+1}) = \mathbf{Z}$ and $H_k(\mathbf{R}^{L+1}) = 0$ for all $k \geq 1$. We summarize these observations below.

Theorem 6. *Let $M \in \mathcal{M}_{sd}$ and $\mathcal{O}_n$ be iid rationalizable for M. Then for each $\delta > 0$, there exists K such that for $n > K$, $H(\hat{M}_n) = H(M) = H(\mathbf{R}^{L+1})$ with probability at least $1 - \delta$, where $K = O(\beta(\delta))$, with $\beta(\delta) := \log \frac{1}{\delta} \log(\log \frac{1}{\delta})$.*

$\diamondsuit$

Example 2 (cont.): In the example of exchange economies, suppose $\mathcal{E}$ is a smooth economy, $[\omega^1, \omega^2]$ is a regular interval for $\mathcal{E}$, and

$$M := \{(p, \omega) \in \mathbf{R}^L_{++} \times [\omega^1, \omega^2] : Z_{\mathcal{E}}(p, \omega) = 0\}$$

By Theorem 2, M is compact and reach$(M) > 0$, and it is again straightforward to see that M has an empty interior. For example, note that M is a subset of $\{(p, \omega) \in \mathbf{R}^L_{++} \times \mathbf{R}^{m(L+1)}_{++} : Z_{\mathcal{E}}(p, \omega) = 0\}$, which is a $m(L+1)$-dimensional C^1 submanifold of $\mathbf{R}^L \times \mathbf{R}^{m(L+1)}$ since $\mathcal{E}$ is a smooth economy, and thus has empty interior.

If $\mathcal{O}_n$ is ε-iid rationalizable for M for $\varepsilon <$ reach(M), then by Theorem 3, with probability one, for all n sufficiently large $d_H(\hat{M}_{\varepsilon_n}, M) \leq 4\varepsilon_n$ and $M \subseteq \hat{M}_{\varepsilon_n}$. For each $\omega \in [\omega^1, \omega^2]$, let

$$M(\omega) := \{p \in \mathbf{R}^L_{++} : (p, \omega) \in M\} \quad \text{and} \quad \hat{M}_{\varepsilon_n}(\omega) = \{p \in \mathbf{R}^L_{++} : (p, \omega) \in \hat{M}_{\varepsilon_n}\}$$

For each ω, $\hat{M}_{\varepsilon_n}(\omega)$ gives the set of estimated equilibrium prices for $\mathcal{E}_\omega$, while $M(\omega)$ is the set of true equilibrium prices for $\mathcal{E}_\omega$. Then Theorem 3 implies that with probability one, for all n sufficiently large, $M(\omega) \subseteq \hat{M}_{\varepsilon_n}(\omega)$ and $d_H(\hat{M}_{\varepsilon_n}(\omega), M(\omega)) \leq 4\varepsilon_n$ for all $\omega \in [\omega^1, \omega^2]$. We record these observations below.

Theorem 7. *Let $\mathcal{E}$ be a smooth economy and $[\omega^1, \omega^2]$ be a regular interval for $\mathcal{E}$. Let*

$$M := \{(p, \omega) \in \mathbf{R}^L_{++} \times [\omega^1, \omega^2] : Z_{\mathcal{E}}(p, \omega) = 0\}$$

and $\mathcal{O}_n$ be ε-iid rationalizable for M for $\varepsilon <$ reach(M). Then with probability one, for all n sufficiently large, $M(\omega) \subseteq \hat{M}_{\varepsilon_n}(\omega)$ and $d_H(\hat{M}_{\varepsilon_n}(\omega), M(\omega)) \leq 4\varepsilon_n$ for all $\omega \in [\omega^1, \omega^2]$, where $\varepsilon_n = O(\frac{\log n}{n})$.

Now suppose $\mathcal{O}_n$ is iid rationalizable for M. In this case, the homology of $\hat{M}_n$ can provide important information about M even though $\hat{M}_n$ might not approximate M sufficiently closely. We illustrate this point with two examples. First, suppose in addition that $\mathcal{E}_\omega$ has a unique equilibrium for each $\omega \in [\omega^1, \omega^2]$, that is, for each $\omega \in [\omega^1, \omega^2]$ there is a unique $p \in \mathbf{R}^L_{++}$ such that $(p, \omega) \in M$. Then in this case it is straightforward to see that the homology of M must be simple, as in the smooth demand case, because M is (homeomorphic to) the graph of a continuous function on $[\omega^1, \omega^2]$. Then this implies $H(M) = H([\omega^1, \omega^2]) = H(\mathbf{R}^{m(L+1)})$, as in the smooth demand case. In particular, $H_0(M) = \mathbf{Z}$; thus if $H_0(M) \neq \mathbf{Z}$ then $|M(\omega)| > 1$ for some $\omega \in [\omega^1, \omega^2]$, where $|A|$ denotes the cardinality of the set A. We record these observations next.

Theorem 8. *Let $\mathcal{E}$ be a smooth economy and $[\omega^1, \omega^2]$ be a regular interval for $\mathcal{E}$, and let*

$$M := \{(p, \omega) \in \mathbf{R}^L_{++} \times [\omega^1, \omega^2] : Z_{\mathcal{E}}(p, \omega) = 0\}$$

In addition, suppose for each $\omega \in [\omega^1, \omega^2]$ there exists a unique $p \in \mathbf{R}^L_{++}$ such that $(p, \omega) \in M$. If $\mathcal{O}_n$ is iid rationalizable for M, then for each $\delta \in (0, 1)$ there exists K such that for $n > K$, with probability at least $1 - \delta$,

$$H_0(\hat{M}_n) = \mathbf{Z} \quad \textit{and} \quad H_k(\hat{M}_n) = \{0\} \quad \forall k \geq 1$$

where $K = O(\beta(\delta))$, with $\beta(\delta) := \log \frac{1}{\delta} \log(\log \frac{1}{\delta})$.

More generally, under some additional regularity conditions making use of the features of a smooth economy, the homology of $\hat{M}_n$ can provide information about the number of equilibria for any $\omega \in [\omega^1, \omega^2]$. For this result, say $[\omega^1, \omega^2]$ is a *totally regular* interval for $\mathcal{E}$ if $\omega^1, \omega^2 \in \mathbf{R}^{m(L+1)}_{++}$, $\omega^1 \ll \omega^2$, and $\mathcal{E}_\omega$ is a regular economy for each $\omega \in [\omega^1, \omega^2]$. If $[\omega^1, \omega^2]$ is a totally regular interval for $\mathcal{E}$, then the number of equilibria in the economy $\mathcal{E}_\omega$ is the same for all $\omega \in [\omega^1, \omega^2]$; see Lemma 10 in the appendix. It is not difficult to show in this case that $H_0(M) = \mathbf{Z}^J$ and $H_k(M) = 0$ for all $k \geq 1$, where J is the common number of equilibria in each economy $\mathcal{E}_\omega$. We summarize this result below.

Theorem 9. *Let $\mathcal{E}$ be a smooth economy and $[\omega^1, \omega^2]$ be a totally regular interval for $\mathcal{E}$, and let*

$$M := \{(p, \omega) \in \mathbf{R}^L_{++} \times [\omega^1, \omega^2] : Z_{\mathcal{E}}(p, \omega) = 0\}$$

If $\mathcal{O}_n$ is iid rationalizable for M, then for each $\delta \in (0,1)$ there exists K such that for $n > K$, with probability at least $1-\delta$,

$$H_0(\hat{M}_n) = \mathbf{Z}^J \quad \text{and} \quad H_k(\hat{M}_n) = \{0\} \quad \forall k \geq 1$$

where $J = |\{p \in \mathbf{R}^L_{++} : (p,\omega) \in M\}|$ for each $\omega \in [\omega^1, \omega^2]$ and $K = O(\beta(\delta))$, with $\beta(\delta) := \log\frac{1}{\delta}\log(\log\frac{1}{\delta})$.

$\diamondsuit$

Example 3 (cont.): Here suppose $M \in \mathcal{M}_{gsc}$, so

$$M = \{a \in \times_i A_i : a \text{ is a pure strategy Nash equilibrium of } G((f_1, A_1), \ldots, (f_k, A_m))\}$$

for a game with strategic complementarities $G((f_1, A_1), \ldots, (f_m, A_m))$. In this case, M is a nonempty finite lattice in $\mathbf{R}^N$, so is a compact set with empty interior and positive reach.

By Theorem 3, if $\mathcal{O}_n$ is ε-iid rationalizable for M for $\varepsilon < \text{reach}(M)$, then with probability one, for all n sufficiently large, $d_H(\hat{M}_{\varepsilon_n}, M) \leq 4\varepsilon_n$ and $M \subseteq \hat{M}_{\varepsilon_n}$. As in Examples 1 and 2, this estimator can be sharpened making use of additional features of the models. First note that here M is finite, so write $M := \{a^1, \ldots, a^J\} \subseteq \mathbf{R}^N$, and without loss of generality take $a^1 := \inf M$ and $a^J := \sup M$. Then with probability one, for all n sufficiently large,

$$M \subseteq \hat{M}_{\varepsilon_n} \subseteq \cup_{j=1}^J B_{4\varepsilon_n}(a^j) \ \text{ and } B_{4\varepsilon_n}(a^j) \cap B_{4\varepsilon_n}(a^\ell) = \emptyset \ \forall \ j \neq \ell$$

A sharper estimator can be constructed using this observation, as follows. For each n, let $\hat{M}^{md}_{\varepsilon_n} \subseteq \hat{M}_{\varepsilon_n}$ be a maximal subset, with respect to set inclusion, with the property that $d(a, a') > 8\varepsilon_n$ for all $a, a' \in \hat{M}^{md}_{\varepsilon_n}$ with $a \neq a'$; say that such a subset is *maximally dispersed*. Then note that with probability one, for all n sufficiently large, $\hat{M}^{md}_{\varepsilon_n}$ is a finite lattice, $|\hat{M}^{md}_{\varepsilon_n}| = |M|$, and $\hat{M}^{md}_{\varepsilon_n}$ must contain exactly one element in each ball $B_{4\varepsilon_n}(a^j)$ for each $j = 1, \ldots, J$. In particular, $d_H(\hat{M}^{md}_{\varepsilon_n}, M) \leq 4\varepsilon_n$. Moreover, natural selections from $\hat{M}^{md}_{\varepsilon_n}$ approximate the smallest and largest equilibria in M. To see this, set

$$\hat{a}^1_{\varepsilon_n} = \begin{cases} \inf \hat{M}^{md}_{\varepsilon_n} & \text{if } \inf \hat{M}^{md}_{\varepsilon_n} \text{ exists} \\ a_n & \text{for some } a_n \in \hat{M}^{md}_{\varepsilon_n} \text{ otherwise} \end{cases}$$

and similarly, set

$$\hat{a}^J_{\varepsilon_n} = \begin{cases} \sup \hat{M}^{md}_{\varepsilon_n} & \text{if } \sup \hat{M}^{md}_{\varepsilon_n} \text{ exists} \\ a_n & \text{for some } a_n \in \hat{M}^{md}_{\varepsilon_n} \text{ otherwise} \end{cases}$$

Then again with probability one, $\hat{a}^1_{\varepsilon_n} \to a^1 = \inf M$ and $\hat{a}^J_{\varepsilon_n} \to a^J = \sup M$. We record these observations below.

Theorem 10. *Let $M \subseteq \mathcal{M}_{gsc}$ and $\mathcal{O}_n$ be ε-iid rationalizable for M for $\varepsilon <$ reach(M). For each n, let $\hat{M}^{md}_{\varepsilon_n} \subseteq \hat{M}_{\varepsilon_n}$ be a maximally dispersed subset. With probability one, for all n sufficiently large, $\hat{M}^{md}_{\varepsilon_n}$ is a finite lattice, $|\hat{M}^{md}_{\varepsilon_n}| = |M|$, and $d_H(\hat{M}^{md}_{\varepsilon_n}, M) \leq 4\varepsilon_n$. In addition, with probability one, $\hat{a}^1_{\varepsilon_n} \to a^1 = \inf M$ and $\hat{a}^J_{\varepsilon} \to a^J = \sup M$.*

If $\mathcal{O}_n$ is instead iid rationalizable for M, then again $\hat{M}_n$ might not approximate M, but its homology can provide important information. As in Example 2, it is straightforward to determine the homology of M in this example. Here M is a finite subset of $\mathbf{R}^N$ with J elements, so $H_0(M) = \mathbf{Z}^J$ and $H_k(M) = 0$ for all $k \geq 1$. By Theorem 4, for sufficiently large n, the dimension of $H_0(\hat{M}_n)$ will then match the number of pure strategy equilibria in M with high probability. We record these observations below.

Theorem 11. *Let $M \in \mathcal{M}_{gsc}$. If $\mathcal{O}_n$ is iid rationalizable for M, then for each $\delta \in (0,1)$ there exists K such that for $n > K$, with probability at least $1-\delta$,*

$$H_0(\hat{M}_n) = \mathbf{Z}^J \quad \textit{and} \quad H_k(\hat{M}_n) = \{0\} \quad \forall k \geq 1$$

where $J = |M|$ and $K = O(\beta(\delta))$, with $\beta(\delta) := \log \frac{1}{\delta} \log(\log \frac{1}{\delta})$.

$\diamondsuit$

6 Appendix

6.1 Preliminaries

We first give several preliminary results that are used throughout.

Lemma 6. *Let $M \subseteq \mathbf{R}^N$ be compact and reach$(M) > 0$.*

(i) Let $x \in M$ and $u \in \mathbf{R}^N$ with $\|u\| = 1$. Suppose

$$\bar{t} := \sup\{t \geq 0 : d(x + tu, M) = t\} > 0$$

Then $\bar{t} \geq$ reach(M) and for all $t <$ reach(M), $\pi_M(x + tu) = x$.

(ii) Suppose $d(y, M) <$ reach(M) and $x = \pi_M(y)$. If $x \neq y$, then for all $t <$ reach(M), $\pi_M(x + tu) = x$ where $u = \frac{y-x}{d(y,x)}$.

Proof. For (i), by Federer (1959, Theorem 4.8(6)),

$$\bar{t} \geq \text{reach}(M, x) \geq \text{reach}(M)$$

By construction, $\{t \geq 0 : d(x+tu, M) = t\}$ is an interval containing 0. Then note by definition, if $t < \bar{t}$ then $d(x + tu, M) = t$, and $d(x + tu, x) = t$ for all t. If $t < \text{reach}(M)$, then $t < \bar{t}$ and

$$d(x + tu, x) = t = d(x + tu, M) < \text{reach}(M)$$

Since $d(x+tu, M) < \text{reach}(M)$, there is a unique point in M closest to $x+tu$, that is, there is a unique point in M of distance $t = d(x+tu, M)$ from $x+tu$. But $x \in M$, which implies $\pi_M(x + tu) = x$.

For (ii), note that $y = x+cu$ where $u := \frac{y-x}{d(y,x)}$ and $c := d(y, x) > 0$. Then

$$d(x + cu, M) = d(y, M) = d(y, x) = c$$

by definition of x and c. Thus

$$\bar{t} := \sup\{t \geq 0 : d(x + tu, M) = t\} \geq c > 0$$

By part (i), for all $t < \text{reach}(M)$, $\pi_M(x + tu) = x$. □

Lemma 7. *Let $M \subseteq \mathbf{R}^N$ be a compact set with reach$(M) > 0$, and $0 < \varepsilon <$ reach(M).*

(i) For each $y \notin M$ there exists $z \in B_\varepsilon(y)$ such that $B_\varepsilon(z) \cap M = \emptyset$

(ii) For each $x \in \partial M$ there exists y such that $x \in \overline{B_\varepsilon(y)}$ and $B_\varepsilon(y) \cap M = \emptyset$.

(iii) For each $x \in \partial M$ there exists y such that $d(x, y) = \varepsilon$ and $\pi_M(y) = x$.

Proof. For (i), let $y \notin M$. If $d(y, M) \geq \varepsilon$, then $B_\varepsilon(y) \cap M = \emptyset$, so setting $z = y$ establishes the claim.

If $d(y, M) < \varepsilon$, let $x = \pi_M(y)$. Then $x \in M$ is well-defined since $\varepsilon <$ reach(M). Since $y \notin M$, $x \neq y$, and $d(y, M) = d(y, \pi_M(y)) = d(y, x) > 0$. Then let $z = x + \varepsilon \frac{y-x}{d(y,x)}$. By Lemma 6, since $\varepsilon <$ reach(M), $\pi_M(z) = x$ and $d(z, M) = \varepsilon$. Thus $B_\varepsilon(z) \cap M = \emptyset$. By construction, $d(y, z) = \varepsilon - d(x, y) < \varepsilon$, so $z \in B_\varepsilon(y)$.

For (ii), let $x \in \partial M$. Then there exists $\{x_n\} \subseteq M^c$ such that $x_n \to x$. By (i), for each x_n there exists $y_n \in B_\varepsilon(x_n)$ such that $B_\varepsilon(y_n) \cap M = \emptyset$. Then $\{y_n\}$ is bounded, so has a convergent subsequence $\{y_{n_k}\}$ with $y_{n_k} \to y$. Since $B_\varepsilon(y_{n_k}) \cap M = \emptyset$ for each n_k and $y_{n_k} \to y$, $B_\varepsilon(y) \cap M = \emptyset$. Since $d(y_{n_k}, x_{n_k}) < \varepsilon$ for each n_k and $x_{n_k} \to x$, $d(y, x) \leq \varepsilon$. Thus $x \in \overline{B_\varepsilon(y)}$.

Part (iii) follows immediately from (ii). □

6.2 Proof of Theorem 3

We start with several additional definitions that will be used in the proof of Theorem 3; see Walther (1997, 1999) and Cuevas and Rodriguez-Casal (2004) for additional discussion.

Definition 5. A bounded Borel subset $S \subseteq \mathbf{R}^N$ is *partly expandable* if there exist constants $r > 0$ and $C(S) \geq 1$ such that

$$d_H(\partial S, \partial(\cup_{x \in S} \overline{B_\delta(x)})) \leq C(S)\delta \qquad \forall \delta \in (0, r)$$

Definition 6. Let $S \subseteq \mathbf{R}^N$. A ball of radius $r > 0$ *rolls freely outside* S if for all $x \in \partial S$ there exists $y \in \mathbf{R}^N$ such that $x \in \overline{B_r(y)} \subseteq \overline{S^c}$.

Proof of Theorem 3: Set $S := \overline{\mathcal{N}_\varepsilon(M)}$, where $\varepsilon < \text{reach}(M)$. By assumption, S is the support of μ. The proof proceeds by first showing that the procedure sketched above provides an estimate of both S and ∂S of the required order under these assumptions. To that end, note that there exists $k > 0$ such that

$$\lambda(B_\alpha(x) \cap S) \geq k\lambda(B_\alpha(x)) \ \ \forall x \in S \text{ and } \forall \alpha \in (0, \varepsilon)$$

where λ denotes Lebesgue measure in $\mathbf{R}^N$. To see this, fix $\alpha \in (0, \varepsilon)$ and $x \in S$. If $\alpha \leq d(x, \partial S)$, then $B_\alpha(x) \subseteq S$. In this case,

$$\lambda(B_\alpha(x) \cap S) = \lambda(B_\alpha(x))$$

So suppose $\alpha > d(x, \partial S)$. Then let $y \in M$ be the unique closest point to x, so $y = \pi_M(x)$. If $y = x$, then $B_\alpha(x) \subseteq S$ since $\alpha < \varepsilon$. In that case, again as above,

$$\lambda(B_\alpha(x) \cap S) = \lambda(B_\alpha(x))$$

If $y \neq x$, let x^* be the point on the ray originating at x going through y of distance $\alpha/2$ from x, so $x^* = x + \frac{\alpha}{2}\frac{1}{d(y,x)}(y - x)$. Then note that $B_{\frac{\alpha}{2}}(x^*) \subseteq S$, since $d(x^*, y) < \frac{\alpha}{2}$ and $\alpha < \varepsilon$, and by construction $B_{\frac{\alpha}{2}}(x^*) \subseteq B_\alpha(x)$. Thus

$$B_{\frac{\alpha}{2}}(x^*) \subseteq B_\alpha(x) \cap S$$

which implies

$$\lambda(B_\alpha(x) \cap S) \geq \lambda(B_{\frac{\alpha}{2}}(x^*)) \geq \frac{1}{2^N}\lambda(B_\alpha(x))$$

Thus S is Lebesgue standard (using constants $\gamma = \varepsilon$ and $\delta = 2^{-N}$).

Now let $r \in (0, \text{reach}(M) - \varepsilon)$ and $x \in \partial S$. Let $\bar{x}$ be the point in M closest to x and let x^* be the point on the ray from $\bar{x}$ to x such that $x^* \notin S$ and $d(x, x^*) = r$. Then $\bar{x} \neq x$ and $x^* = \bar{x} + tu$ where $u = \frac{1}{d(x,\bar{x})}(x - \bar{x})$ and $t = d(x, \bar{x}) + r = \varepsilon + r$. Note by construction, $d(x^*, M) \leq \varepsilon + r < \text{reach}(M)$. This implies $\bar{x}$ must be the unique closest point to x^* in M by Lemma 6, and $d(x^*, \bar{x}) = \varepsilon + r$. Then $x \in \overline{B_r(x^*)} \subseteq \overline{S^c}$. To see this, let $y \in S$ and suppose by way of contradiction that $d(x^*, y) < r$. Let $\bar{y} \in M$ be the unique closest point to y. Then

$$d(x^*, \bar{y}) \leq d(y, \bar{y}) + d(x^*, y) < \varepsilon + r = d(x^*, \bar{x})$$

This implies $\bar{x}$ is not the closest point to x^* in M, a contradiction. Thus a ball of radius r rolls freely outside S. By Cuevas and Rodriguez-Casel (2004, Proposition 1), S is partly expandable with constant $C(S) = 1$.

Now by Cuevas and Rogriguez-Casel (2004, Theorem 3), with probability one, for all n sufficiently large,

$$d_H(O_n, S) \leq \varepsilon_n$$

where $O_n = \{x_1, \ldots, x_n\}$. Note that this implies with probability one, for all n sufficiently large, $S \subseteq \hat{S}_{\varepsilon_n}$. Then from Cuevas and Rodgriguez-Casel (2004, Theorem 2), with probability one, for all n sufficiently large,

$$d_H(\partial \hat{S}_{\varepsilon_n}, \partial S) \leq \varepsilon_n$$

Similarly, with probability one, for all n sufficiently large,

$$d_H(\hat{S}_{\varepsilon_n}, S) \leq \varepsilon_n$$

since $S \subseteq \hat{S}_{\varepsilon_n}$ and $O_n \subseteq S$.

Now claim with probability one, for all n sufficiently large,

$$\varepsilon - \varepsilon_n \leq \hat{r}_{\varepsilon_n} \leq \varepsilon + \varepsilon_n$$

To see this, first note by Lemma 7, if $x \in \partial M = M$, then there exists y such that $y \in \overline{B_\varepsilon(x)}$ and $B_\varepsilon(y) \cap M = \emptyset$. Then by definition, $y \in \partial S$. Since $B_\varepsilon(x) \subseteq S$, $d(x, \partial S) \geq \varepsilon$. Thus $d(x, \partial S) = \varepsilon$. Since with probability one, for all n sufficiently large $M \subseteq S \subseteq \hat{S}_{\varepsilon_n}$, this implies $\varepsilon - \varepsilon_n \leq \hat{r}_{\varepsilon_n}$ for each such n.

Then note for each $y \in S \setminus M$, $d(y, \partial S) \leq \varepsilon$. To see this, let $y \in S \setminus M$ and let $x = \pi_M(y)$. Then $0 < d(x, y) \leq \varepsilon < \text{reach}(M)$, so $x + t\frac{y-x}{d(y,x)} \in S$ for each $t \in [0, \varepsilon]$. Let $z := x + \varepsilon\frac{y-x}{d(y,x)}$. By Lemma 6, $\pi_M(z) = x$, and $d(z, x) = d(z, M) = \varepsilon$. This implies $z \in \partial S$, since if $x' \in M$ and $x' \neq x$, then $d(z, x') > d(z, x) = \varepsilon$. By construction,

$$d(x, y) + d(y, z) = d(x, z) = \varepsilon$$

which implies $d(y, z) = \varepsilon - d(x, y) \leq \varepsilon$. Since $z \in \partial S$ this implies $d(y, \partial S) \leq \varepsilon$. Also note that this implies

$$d(y, M) + d(y, z) = d(y, x) + d(y, z) = \varepsilon$$

so

$$d(y, M) + d(y, \partial S) \leq d(y, M) + d(y, z) = \varepsilon$$

Since $d(y, \partial S) \leq \varepsilon$ for all $y \in S$, if $d_H(\partial S, \partial \hat{S}_n) \leq \varepsilon_n$ then $\hat{r}_{\varepsilon_n} \leq \varepsilon + \varepsilon_n$ by definition. Thus putting these together, with probability one, for all n sufficiently large, $\varepsilon - \varepsilon_n \leq \hat{r}_{\varepsilon_n} \leq \varepsilon + \varepsilon_n$.

Finally, consider $d_H(\hat{M}_{\varepsilon_n}, M)$. With probability one, for all n sufficiently large, if $y \in \hat{M}_{\varepsilon_n}$,

$$\begin{aligned} d(y, \partial S) &\geq d(y, \partial \hat{S}_{\varepsilon_n}) - \varepsilon_n \\ &\geq \hat{r}_{\varepsilon_n} - 2\varepsilon_n - \varepsilon_n \\ &\geq \varepsilon - \varepsilon_n - 3\varepsilon_n = \varepsilon - 4\varepsilon_n \end{aligned}$$

where the first inequality follows from $d_H(\partial S, \partial \hat{S}_n) \leq \varepsilon_n$, the second from definition of $\hat{M}_{\varepsilon_n}$, and the third because $\hat{r}_{\varepsilon_n} \geq \varepsilon - \varepsilon_n$.

Then note that with probability one, for n sufficiently large, if $d(y, \partial S) \geq \varepsilon - 4\varepsilon_n$ then $y \in S$. This follows from the fact that $O_n \subseteq S$, and with probability one, for n sufficiently large, $S \subseteq \hat{S}_{\varepsilon_n}$, so by construction if $x \in \hat{S}_{\varepsilon_n} \setminus S$ then $d(x, \partial S) \leq \varepsilon_n$.

Thus with probability one, for n sufficiently large, if $y \in \hat{M}_{\varepsilon_n}$, then $y \in S$ and

$$\begin{aligned} d(y, M) &\leq \varepsilon - d(y, \partial S) \\ &\leq \varepsilon - (\varepsilon - 4\varepsilon_n) \\ &= 4\varepsilon_n \end{aligned}$$

where the first inequality follows from the fact that for all $y \in S$, $d(y, M) + d(y, S) \leq \varepsilon$, and the second follows from the previous argument.

Next, note that again with probability one, for all n sufficiently large, $M \subseteq \hat{M}_{\varepsilon_n}$, as if $y \in M$ then

$$\begin{aligned} d(y, \partial \hat{S}_{\varepsilon_n}) &\geq d(y, \partial S) - \varepsilon_n \\ &= \varepsilon - \varepsilon_n \\ &\geq \hat{r}_{\varepsilon_n} - 2\varepsilon_n \end{aligned}$$

where the first inequality follows from $d_H(\partial S, \partial \hat{S}_n) \leq \varepsilon_n$, the second from the fact that $d(x, \partial S) = \varepsilon$ for all $x \in M$, and the third from the fact that $\varepsilon \geq \hat{r}_{\varepsilon_n} - \varepsilon_n$. Then by definition, this implies $y \in \hat{M}_{\varepsilon_n}$. Thus with probability one, for all n sufficiently large, $M \subseteq \hat{M}_{\varepsilon_n}$ and $d_H(\hat{M}_{\varepsilon_n}, M) \leq 4\varepsilon_n$. □

6.3 Proof of Theorem 4

The proof of Theorem 4 has three main steps. First, with high probability for all sufficiently large n, the cleaning procedure keeps all of the observations in $A = \mathcal{N}_s(M)$ and throws out all of the observations in $B = \mathbf{R}^N \setminus \mathcal{N}_{2s}(M)$. Second, with sufficiently many observations, $M \subseteq \cup_{x \in O_n^f} B_s(x) \subseteq \hat{M}_n$ with high probability. These steps are established in Lemmas 3 and 4. We give the proof of Lemma 3 next; Lemma 4 is given in Niyogi, Smale, and Weinberger (2008, Lemma 5.1).

Proof of Lemma 3: This follows from Niyogi, Smale, and Weinberger (2011, Lemma 3), using Chernoff bounds and standard law of large numbers arguments. To see this, fix $\delta \in (0, 1)$. Let $A := \mathcal{N}_s(M)$ and $B := \mathbf{R}^N \setminus \mathcal{N}_{2s}(M)$. Let $O_n = \{x_1, \ldots, x_n\}$. Fix i, and for each $j \neq i$, set $y_{ij} := \mathbf{1}(x_j \notin B_r(x_i))$. Then the y_{ij}'s are $0-1$ random variables, iid with mean $\mu(B_r(x_i))$. Note that by construction, $d_i = \sum_{j \neq i} y_{ij}$. Then from Niyogi, Smale and Weinberger (2011, Lemma 3), by an argument using Chernoff bounds, if $n > 4\eta \log \eta$ where

$$\eta = \max\left(1 + \frac{2}{h^2} \log\left(\frac{2}{\delta}\right), 4\right)$$

then with probability greater than $1 - \delta$,

$$\frac{d_i}{n-1} > \frac{a+b}{2} \quad \forall x_i \in A$$

and

$$\frac{d_i}{n-1} < \frac{a+b}{2} \quad \forall x_i \in B$$

This implies first

$$O_n \cap A = O_n \cap \mathcal{N}_s(M) \subseteq O_n^f$$

and second

$$O_n^f \subseteq \mathbf{R}^N \setminus B = \mathcal{N}_{2s}(M)$$

Then provided $n > K := 4\eta \log \eta$ the result follows. Note by construction, $K = O\left(\log \frac{1}{\delta}\right)$. $\square$

The third step in the proof of Theorem 4 is Lemma 5, which gives general conditions under which M is a deformation retract of a set of the form $\cup_{i=1}^{k} B_\varepsilon(x_i)$. This adapts a similar result from Niyogi, Smale, and Weinberger (2011) for the case in which M is a manifold; here M is instead a general compact set with positive reach.[8] Here we give an elementary proof, and do not seek the tightest bounds.

Proof of Lemma 5: Let $g : \cup_i B_\varepsilon(x_i) \times [0,1] \to \mathbf{R}^N$ be given by

$$g(x,t) = (1-t)x + t\pi_M(x)$$

Then note that $\cup_i B_\varepsilon(x_i) \subseteq \mathcal{N}_{2\varepsilon}(M)$, and $2\varepsilon < \text{reach}(M)$, so π_M is well-defined and continuous on $\cup_i B_\varepsilon(x_i)$. Thus g is a well-defined continuous function. By construction, for each $x \in \cup_i B_\varepsilon(x_i)$, $g(x,0) = x$ and $g(x,1) = \pi_M(x) \in M$. If $x \in M$, then $g(x,1) = \pi_M(x) = x$. Then it suffices to show that $g(x,t) \in \cup_i B_\varepsilon(x_i)$ for each $x \in \cup_i B_\varepsilon(x_i)$ and each $t \in [0,1]$, that is, that $g : \cup_i B_\varepsilon(x_i) \times [0,1] \to \cup_i B_\varepsilon(x_i)$.

To that end, let $x \in \cup_i B_\varepsilon(x_i)$, and let $x_i \in \{x_1, \ldots, x_k\}$ such that $d(x, x_i) < \varepsilon$. Let $y = \pi_M(x)$, so $g(x,t) = (1-t)x + ty$ for each $t \in [0,1]$. If $d(y, x_i) < \varepsilon$, then we are done, since this implies $d((1-t)x + ty, x_i) = (1-t)d(x, x_i) + td(y, x_i) < \varepsilon$, that is, $g(x,t) \in B_\varepsilon(x_i)$.

Then suppose $d(y, x_i) \geq \varepsilon$. Let $\bar{t} \in (0,1]$ such that $d((1-\bar{t})x + \bar{t}y, x_i) = \varepsilon$, and set $w := (1-\bar{t})x + \bar{t}y$. By construction, for all $t < \bar{t}$, $d(g(x,t), x_i) < \varepsilon$, so $g(x,t) \in B_\varepsilon(x_i) \subseteq \cup_j B_\varepsilon(x_j)$ for all $t < \bar{t}$. So it suffices to show that $g(x,t) \in \cup_j B_\varepsilon(x_j)$ for all $t \geq \bar{t}$.

To that end, let $r = \frac{1}{3}\varepsilon$. Let z be the point on the ray originating at y and going through x of distance 2ε from y, so $z := y + 2\varepsilon \frac{x-y}{d(x,y)}$. By construction, $2\varepsilon = 6r < \text{reach}(M)$, so by Lemma 6, $\pi_M(z) = y$ and $d(z, M) = 2\varepsilon = 6r$. Let $u := \pi_M(x_i)$. Then $d(x_i, u) < c < r$, since $x_i \in \mathcal{N}_c(M)$, which implies

[8] See also, for example, Cucker, Krick, and Shub (2018) for a related result for the manifold case with different bounds.

$d(x_i, M) = d(x_i, u) < c$. From the triangle inequality,

$$d(z, x_i) \geq d(z, u) - d(x_i, u)$$

Then $d(z, u) \geq d(z, M) = 6r$, and $d(x_i, u) < c < r$, so this implies $d(z, x_i) > 5r$.

By assumption, for some $x_j \in \{x_1, \ldots, x_k\}$, $d(y, x_j) < c < r < \varepsilon$. Now it suffices to show that $d(w, x_j) < \varepsilon$, since for all $t \geq \bar{t}$, $g(x, t)$ is on the line segment between y and $g(x, \bar{t}) = w$ and $B_\varepsilon(x_j)$ is convex. From the triangle inequality,

$$d(w, x_j) \leq d(y, x_j) + d(y, w) < r + d(y, w)$$

By construction, w and z are both on the ray originating at y and going through x, and $d(y, w) = d(y, z) - d(z, w) = 6r - d(z, w)$. This implies

$$d(w, x_j) < r + 6r - d(z, w) = 7r - d(z, w)$$

Since $d(x, x_i) < \varepsilon = d(w, x_i)$, the law of cosines implies $\cos\theta > 0$, where θ is the angle between the vectors $w - x$ and $w - x_i$. By construction, θ is also the angle between $w - z$ and $w - x_i$. Then again by the law of cosines,

$$\begin{aligned} d(w, x_i)^2 + d(z, w)^2 &> d(z, x_i)^2 \\ \Rightarrow \varepsilon^2 + d(z, w)^2 &> d(z, x_i)^2 \\ \Rightarrow d(z, w)^2 &> d(z, x_i)^2 - \varepsilon^2 > (5r)^2 - \varepsilon^2 = (5r)^2 - (3r)^2 = 16r^2 \end{aligned}$$

Thus $d(z, w) > 4r$, which implies

$$d(w, x_j) < 7r - d(z, w) < 7r - 4r = 3r = \varepsilon$$

From the argument above, the result follows. $\square$

Putting these pieces together gives the proof of Theorem 4.

Proof of Theorem 4: Let $O_n = \{x_1, \ldots, x_n\}$, and fix $\delta \in (0, 1)$. Let $A := \mathcal{N}_s(M)$. By Lemma 3, there exists K with $K = O(\log \frac{1}{\delta})$ such that for $n > K$, with probability at least $1 - \frac{\delta}{2}$,

(i) $O_n \cap A \subseteq O_n^f \subseteq \mathcal{N}_{2s}(M)$

Let $A_i = B_{\frac{s}{2}}(y_i)$ for $i = 1, \dots, k$ where $y_i \in M$ for each i and $M \subseteq \cup_i A_i$. By construction, $\cup_i A_i \subseteq A$. By assumption $\min_i \mu(B_{\frac{s}{2}}(y_i)) > 0$, so let $\alpha > 0$ such that $\alpha < \min_i \mu(B_{\frac{s}{2}}(y_i))$. By Lemma 4, if $n > K' := \max\{K, \frac{1}{\alpha}(\log k + \log \frac{2}{\delta})\}$, then with probability at least $1 - \frac{\delta}{2}$,

(ii) $O_n \cap A_i \neq \emptyset$ for each $i = 1, \dots, k$

Thus if $n > K'$, then with probability at least $1 - \delta$ both conditions (i) and (ii) hold. Then since $\cup_i A_i \subseteq A$, this implies

$$O_n \cap (\cup_i A_i) \subseteq O_n \cap A \subseteq O_n^f$$

Thus

$$M \subseteq \cup_i A_i = \cup_i B_{\frac{s}{2}}(y_i) \subseteq \cup_{x \in O_n^f} B_s(x)$$

Then $M \subseteq \cup_{x \in O_n^f} B_s(x) \subseteq \cup_{x \in O_n^f} B_{2s}(x)$, and by (i), $O_n^f \subseteq \mathcal{N}_{2s}(M)$. By Lemma 5, since $6s = 3(2s) < \varepsilon < \frac{1}{2}\text{reach}(M)$, $\hat{M}_n = \cup_{x \in O_n^f} B_\varepsilon(x)$ deformation retracts to M. Homology is preserved by deformation retract, which implies that $H(\hat{M}_n) = H(M)$. $\square$

6.4 Proofs for Example 2

We use the following notation below. Let $\mathcal{E}$ be a smooth economy and $\omega^1, \omega^2 \in \mathbf{R}^{mL}_{++}$. We simplify notation by omitting the subscript $\mathcal{E}$ and writing $Z := Z_{\mathcal{E}}$ for the excess demand function in $\mathcal{E}$. Let

$$M := \{(p, \omega) \in \mathbf{R}^L_{++} \times [\omega^1, \omega^2] : Z(p, \omega) = 0\}$$

For a given $\omega \in [\omega^1, \omega^2]$, let

$$M(\omega) := \{p \in \mathbf{R}^L_{++} : Z(p, \omega) = 0\} = \{p \in \mathbf{R}^L_{++} : (p, \omega) \in M\}$$

In the economy $\mathcal{E}$, $M(\omega)$ is the set of equilibrium prices for the endowment profile ω. Given a set A, we write $|A|$ to denote the cardinality of A.

To prove Theorem 9, we first establish a series of lemmas.

Lemma 8. *Let $\mathcal{E}$ be a smooth economy and $[\omega^1, \omega^2]$ be a totally regular interval for $\mathcal{E}$. For each $\omega \in [\omega^1, \omega^2]$ there exists a neighborhood $U \ni \omega$ such that for all $\omega' \in U$,*

$$|M(\omega')| = |M(\omega)| = k_\omega$$

for some $k_\omega \in \mathbf{N}$.

Proof. Fix $\omega \in [\omega^1, \omega^2]$. Since $\mathcal{E}_\omega$ is a regular economy, it has a nonempty finite set of equilibria. Thus $|M(\omega)| = k_\omega$ for some $k_\omega \in \mathbf{N}$. Let $M(\omega) = \{p_1, \ldots, p_{k_\omega}\} \subseteq \mathbf{R}^L_{++}$. Moreover, again since $\mathcal{E}_\omega$ is a regular economy, there exists a neighborhood $U \ni \omega$, a neighborhood $V_i \ni p_i$ and a C^1 function $h_i : U \to V_i$ for each $i = 1, \ldots, k_\omega$ such that for all $(p', \omega') \in (\cup_i V_i) \times U$,

$$Z(p', \omega') = 0 \iff p' = h_i(\omega') \text{ for some } i = 1, \ldots, k_\omega$$

Moreover, since $p_i \neq p_j$ for each $i \neq j$, redefining using appropriate subsets if necessary, the neighborhoods U and V_i, $i = 1, \ldots, k$, can be chosen so that in addition $V_i \cap V_j = \emptyset$ for each $i \neq j$. This implies that for each $\omega' \in U$, $|M(\omega')| \geq k_w$.

Let $N > 0$ be sufficiently large so that $B_{\frac{1}{N}}(\omega) \subseteq U$. Now suppose by way of contradiction that the claim is false. Thus for each $n \geq N$, there exists $\omega_n \in B_{\frac{1}{n}}(\omega)$ such that $|M(\omega_n)| > k_\omega$. Thus for each ω_n there exists $p_n \in M(\omega_n)$ such that $p_n \notin \cup_i V_i$. The corresponding sequence $\{p_n\}$ is bounded (since $(p_n, \omega_n) \in M$ for each n and M is compact). Thus there exists a convergent subsequence $\{p_{n_k}\}$, with $p_{n_k} \to p$ for some $p \in \mathbf{R}^L_{++}$. By construction $\omega_n \to \omega$, so $\omega_{n_k} \to \omega$. Then

$$Z(p, \omega) = \lim_k Z(p_{n_k}, \omega_{n_k}) = 0$$

that is, $p \in M(\omega)$. Thus $p \in \cup_i V_i$. This is a contradiction, since $p_{n_k} \notin \cup_i V_i$ for each n_k. Thus there must exist some neighborhood $U_\omega \ni \omega$ such that for all $\omega' \in U_\omega$, $|M(\omega')| = k_\omega$. □

This lemma and the arguments in its proof immediately imply the following observation, which we record next for ease of reference.

Lemma 9. *Let $\mathcal{E}$ be a smooth economy and $[\omega^1, \omega^2]$ be a totally regular interval for $\mathcal{E}$. Let $\omega \in [\omega^1, \omega^2]$ and $M(\omega) = \{p_1, \ldots, p_{k_\omega}\}$. Then there exists a neighborhood $U_\omega \ni \omega$, a neighborhood $V_i \ni p_i$ and a C^1 function $h_i : U \to V_i$ for each $i = 1, \ldots, k_\omega$, with $V_i \cap V_j = \emptyset$ for all $i \neq j$, such that for all $\omega' \in U_\omega$,*

$$M(\omega') = \{p \in \mathbf{R}^L_{++} : Z(p, \omega') = 0\} = \{h_1(\omega'), \ldots, h_{k_\omega}(\omega')\}$$

and $h_i(\omega') \neq h_j(\omega')$ for each $i \neq j$.

Lemma 10. *Let $\mathcal{E}$ be a smooth economy and $[\omega^1, \omega^2]$ be a totally regular interval for $\mathcal{E}$. Then $|M(\omega)| = |M(\omega')|$ for each $\omega, \omega' \in [\omega^1, \omega^2]$.*

Proof. If there is a unique equilibrium for each $\omega \in [\omega^1, \omega^2]$ the claim clearly holds. Then suppose there exists $\omega \in [\omega^1, \omega^2]$ such that $k_\omega := |M(\omega)| > 1$. By Lemma 8, there exists a neighborhood U of ω such that $|M(\omega')| = k_\omega$ for all $\omega' \in U$. Let

$$U_1 := \{\omega' \in [\omega^1, \omega^2] : |M(\omega')| = k_\omega\}$$

By definition $U \cap [\omega^1, \omega^2] \subseteq U_1$, and $U_1 \neq \emptyset$.

Now it suffices to show that $U_1 = [\omega^1, \omega^2]$. To that end, first note that U_1 is relatively open, since by Lemma 8, for each $\omega' \in U_1$ there exists a neighborhood $U' \ni \omega'$ such that for all $\omega'' \in U'$,

$$|M(\omega'')| = |M(\omega')| = k_\omega$$

where the second equality follows from the definition of U_1. This implies $U' \cap [\omega^1, \omega^2] \subseteq U_1$. Thus U_1 is relatively open.

Moreover, U_1 is also closed. To see this, suppose $\{\omega_n\} \subseteq U_1$ and $\omega_n \to \omega'$. By way of contradiction, suppose $\omega' \notin U_1$, so $|M(\omega')| \neq k_\omega$. Then by Lemma 8, there exists a neighborhood $U' \ni \omega'$ such that for all $\omega'' \in U'$, $|M(\omega'')| = |M(\omega')|$. Since $\omega_n \to \omega'$, there exists N sufficiently large such that for all $n \geq N$, $\omega_n \in U'$. For such n, $|M(\omega_n)| = |M(\omega')| \neq k_\omega$. But this is a contradiction, since $\omega_n \in U_1$ for all n. Therefore U_1 is closed. Since U_1 is nonempty and $[\omega^1, \omega^2]$ is connected, this implies $U_1 = [\omega^1, \omega^2]$. □

Lemma 11. *Let $\mathcal{E}$ be a smooth economy and $[\omega^1, \omega^2]$ be a totally regular interval for $\mathcal{E}$. Let k be the common number of equilibria for each $\omega \in [\omega^1, \omega^2]$.*

Then there exist k continuous functions $h_i : [\omega^1, \omega^2] \to \mathbf{R}^L_{++}$, $i = 1, \ldots, k$, such that for each $\omega \in [\omega^1, \omega^2]$,

$$M(\omega) := \{p \in \mathbf{R}^L_{++} : Z(p, \omega) = 0\} = \{h_1(\omega), \ldots, h_k(\omega)\}$$

and $h_i(\omega) \neq h_j(\omega)$ for all $i \neq j$.

Proof. Let $k = |M(\omega)|$ for each $\omega \in [\omega^1, \omega^2]$; by Lemma 10 k is well-defined.

If $k = 1$, so there is a unique equilibrium for each $\omega \in [\omega^1, \omega^2]$, then the result is immediate; see also Lemma 12 below.

Suppose $k > 1$. Then by Lemma 9, for each $\omega \in [\omega^1, \omega^2]$ there exists a neighborhood $U^\omega \ni \omega$ and C^1 functions $h_i^\omega : U^\omega \to \mathbf{R}^L_{++}$, $i = 1, \ldots, k$ such that for each $\omega' \in U^\omega$,

$$\{p \in \mathbf{R}^L_{++} : Z(p, \omega') = 0\} = \{h_1^\omega(\omega'), \ldots, h_k^\omega(\omega')\}$$

and $h_i^\omega(\omega') \neq h_j^\omega(\omega')$ for all $i \neq j$.

The collection $\{U^\omega : \omega \in [\omega^1, \omega^2]\}$ is an open cover of $[\omega^1, \omega^2]$, and $[\omega^1, \omega^2]$ is compact, so there exist $\omega^{r_1}, \ldots, \omega^{r_t} \in [\omega^1, \omega^2]$ such that

$$[\omega^1, \omega^2] \subseteq U^{\omega^{r_1}} \cup \cdots \cup U^{\omega^{r_t}} = \bigcup_{i=1}^{t} U^{\omega^{r_i}}$$

If $[\omega^1, \omega^2] \subseteq U^{\omega^{r_1}}$, then we are done. If not, then $[\omega^1, \omega^2] \setminus U^{\omega^{r_1}} \neq \emptyset$. Then there must exist some $r_i \neq r_1$ such that $U^{\omega^{r_1}} \cap U^{\omega^{r_i}} \neq \emptyset$. To see this, note that $[\omega^1, \omega^2] \cap U^{\omega^{r_1}} \neq \emptyset$, $[\omega^1, \omega^2] \cap (\cup_{r_i \neq r_1} U^{\omega^{r_i}}) \neq \emptyset$, and

$$[\omega^1, \omega^2] \subseteq \bigcup_i U^{\omega^{r_i}} = U^{\omega^{r_1}} \cup \left(\bigcup_{r_i \neq r_1} U^{\omega^{r_i}} \right)$$

Then since $U^{\omega^{r_1}}$ and $\cup_{r_i \neq r_1} U^{\omega^{r_i}}$ are open and $[\omega^1, \omega^2]$ is connected,

$$U^{\omega^{r_1}} \cap (\bigcup_{r_i \neq r_1} U^{\omega^{r_i}}) = \bigcup_{r_i \neq r_1} (U^{\omega^{r_1}} \cap U^{\omega^{r_i}}) \neq \emptyset$$

Then without loss of generality, suppose $U^{\omega^{r_1}} \cap U^{\omega^{r_2}} \neq \emptyset$. If $\omega \in [\omega^1, \omega^2] \cap (U^{\omega^{r_1}} \cap U^{\omega^{r_2}})$, then

$$\{p \in \mathbf{R}^L_{++} : Z(p, \omega) = 0\} = \{h_1^{\omega^{r_1}}(\omega), \ldots, h_k^{\omega^{r_1}}(\omega)\} = \{h_1^{\omega^{r_2}}(\omega), \ldots, h_k^{\omega^{r_2}}(\omega)\}$$

This implies, relabeling if necessary, $h_i^{\omega^{r_1}}(\omega) = h_i^{\omega^{r_2}}(\omega)$ for each $i = 1, \ldots, k$.

Now for each $i = 1, \ldots, k$, define $h_i : [\omega^1, \omega^2] \cap (U^{\omega^{r_1}} \cup U^{\omega^{r_2}}) \to \mathbf{R}^L_{++}$ by

$$h_i(\omega) := \begin{cases} h_i^{\omega^{r_1}}(\omega) & \text{if } \omega \in U^{\omega^{r_1}} \\ h_i^{\omega^{r_2}}(\omega) & \text{if } \omega \in U^{\omega^{r_2}} \end{cases}$$

By the above argument, h_i is well-defined and continuous, and by construction, for each $\omega \in [\omega^1, \omega^2] \cup (U^{\omega^{r_1}} \cup U^{\omega^{r_2}})$,

$$\{p \in \mathbf{R}^L_{++} : Z(p, \omega) = 0\} = \{h_1(\omega), \ldots, h_k(\omega)\}$$

where $h_i(\omega) \neq h_j(\omega)$ for all $i \neq j$.

Similarly, if $[\omega^1, \omega^2] \subseteq U^{\omega^{r_1}} \cup U^{\omega^{r_2}}$ then we are done. Otherwise, there must exist $r_i \notin \{r_1, r_2\}$ such that $(U^{\omega^{r_1}} \cup U^{\omega^{r_2}}) \cap U^{\omega^{r_i}} \neq \emptyset$. To see this, note that $[\omega^1, \omega^2] \cap (U^{\omega^{r_1}} \cup U^{\omega^{r_2}}) \neq \emptyset$, $[\omega^1, \omega^2] \cap \cup_{r_i \notin \{r_1, r_2\}} U^{\omega^{r_i}} \neq \emptyset$, and

$$[\omega^1, \omega^2] \subseteq \bigcup_i U^{\omega^{r_i}} = (U^{\omega^{r_1}} \cup U^{\omega^{r_2}}) \cup \left(\bigcup_{r_i \notin \{r_1, r_2\}} U^{\omega^{r_i}} \right)$$

Then since $U^{\omega^{r_1}} \cup U^{\omega^{r_2}}$ and $\cup_{r_i \notin \{r_1, r_2\}} U^{\omega^{r_i}}$ are open and $[\omega^1, \omega^2]$ is connected,

$$(U^{\omega^{r_1}} \cup U^{\omega^{r_2}}) \cap \left(\bigcup_{r_i \notin \{r_1, r_2\}} U^{\omega^{r_i}} \right) = \bigcup_{r_i \notin \{r_1, r_2\}} ((U^{\omega^{r_1}} \cup U^{\omega^{r_2}}) \cap U^{\omega^{r_i}}) \neq \emptyset$$

Then without loss of generality suppose $(U^{\omega^{r_1}} \cup U^{\omega^{r_2}}) \cap U^{\omega^{r_3}} \neq \emptyset$. If $\omega \in [\omega^1, \omega^2] \cap \left((U^{\omega^{r_1}} \cup U^{\omega^{r_2}}) \cap U^{\omega^{r_3}}\right)$, then

$$\{p \in \mathbf{R}^L_{++} : Z(p, \omega) = 0\} = \{h_1(\omega), \ldots, h_k(\omega)\} = \{h_1^{\omega^{r_3}}(\omega), \ldots, h_k^{\omega^{r_3}}(\omega)\}$$

Thus without loss of generality, relabeling if necessary, $h_i^{\omega^{r_3}}(\omega) = h_i(\omega)$ for each $i = 1, \ldots, k$.

Now for each $i = 1, \ldots, k$, extend the function h_i to $[\omega^1, \omega^2] \cap U^{\omega^{r_3}}$ by defining $h_i : [\omega^1, \omega^2] \cap (U^{\omega^{r_1}} \cup U^{\omega^{r_2}} \cup U^{\omega^{r_3}}) \to \mathbf{R}^L_{++}$ to be

$$h_i(\omega) := \begin{cases} h_i(\omega) & \text{if } \omega \in U^{\omega^{r_1}} \cup U^{\omega^{r_2}} \\ h_i^{\omega^{r_3}}(\omega) & \text{if } \omega \in U^{\omega^{r_3}} \end{cases}$$

By the above argument, h_i is well-defined and continuous on $[\omega^1, \omega^2] \cap (U^{\omega^{r_1}} \cup U^{\omega^{r_2}} \cup U^{\omega^{r_3}})$. By construction, for each $\omega \in [\omega^1, \omega^2] \cap (U^{\omega^{r_1}} \cup U^{\omega^{r_2}} \cup U^{\omega^{r_3}})$,

$$\{p \in \mathbf{R}^L_{++} : Z(p, \omega) = 0\} = \{h_1(\omega), \ldots, h_k(\omega)\}$$

where $h_i(\omega) \neq h_j(\omega)$ for all $i \neq j$.

Repeating this argument, eventually the construction terminates after finitely many steps, as there are finitely many sets $U^{\omega^{r_i}}$, $r_1, \ldots, r_t$, and $[\omega^1, \omega^2] \subseteq \cup_i U^{\omega^{r_i}}$. This construction yields a continuous function $h_i : [\omega^1, \omega^2] \to \mathbf{R}^L_{++}$ for each $i = 1, \ldots, k$ such that for each $\omega \in [\omega^1, \omega^2]$,

$$\{p \in \mathbf{R}^L_{++} : Z(p, \omega) = 0\} = \{h_1(\omega), \ldots, h_k(\omega)\}$$

and $h_i(\omega) \neq h_j(\omega)$ for all $i \neq j$. $\square$

Lemma 12. *Let $\mathcal{E}$ be a smooth economy and $\omega^1, \omega^2 \in \mathbf{R}^{mL}_{++}$ with $\omega^1 \leq \omega^2$. Suppose there is a unique equilibrium in $\mathcal{E}$ for each $\omega \in [\omega^1, \omega^2]$, and let*

$$M := \{(p, \omega) \in \mathbf{R}^L_{++} \times [\omega^1, \omega^2] : Z(p, \omega) = 0\}$$

Then there is a continuous function $h : [\omega^1, \omega^2] \to \mathbf{R}^L_{++}$ such that

$$M = \{(p, \omega) \in \mathbf{R}^L_{++} \times [\omega^1, \omega^2] : h(\omega) = p\}$$

Proof. For each $\omega \in [\omega^1, \omega^2]$, let $p_\omega \in \mathbf{R}^L_{++}$ be the unique equilibrium price vector for ω. Thus $Z(p, \omega) = 0$ if and only if $p = p_\omega$. Define $h : [\omega^1, \omega^2] \to \mathbf{R}^L_{++}$ by $h(\omega) = p_\omega$ for each $\omega \in [\omega^1, \omega^2]$. Then it suffices to show h is continuous. To that end, suppose $\{\omega^n\} \subseteq [\omega^1, \omega^2]$ and $\omega^n \to \omega$. Then $\omega \in [\omega^1, \omega^2]$. Let $p_n = h(\omega_n)$ for each n. Since $\{(p_n, \omega_n)\} \subseteq M$ and M is compact, $\{p_n\}$ is bounded. This implies $\{p_n\}$ has a convergent subsequence $\{p_{n_k}\}$, and $(p_{n_k}, \omega_{n_k}) \to (p, \omega) \in M$. Since $\omega \in [\omega^1, \omega^2]$ and $Z(p, \omega) = 0$, $p = p_\omega = h(\omega)$. Thus $h(\omega_{n_k}) = p_{n_k} \to p = h(\omega)$.

Now it suffices to show $p_n \to p$. To that end, by way of contradiction suppose $p_n \not\to p$. Then there exists $\varepsilon > 0$ and a subsequence $\{p_{n_j}\}$ of $\{p_n\}$ such that $p_{n_j} \notin B_\varepsilon(p)$ for each j. Passing to a further subsequence and relabeling if necessary, without loss of generality $p_{n_j} \to p'$ for some $p' \in \mathbf{R}^L_{++}$. So $(p_{n_j}, \omega_{n_j}) \to (p', \omega)$. By definition, $p_{n_j} = h(\omega_{n_j})$, so $(p_{n_j}, \omega_{n_j}) \in M$ for each j. Since M is closed, this implies $(p', \omega) \in M$, that is, $Z(p', \omega) = 0$. Thus $p' = p$. This is a contradiction, since by construction $p_{n_j} \notin B_\varepsilon(p)$ for all n_j. Therefore $p_n \to p$, that is, $h(\omega_n) \to h(\omega)$. Since $\omega \in [\omega^1, \omega^2]$ was arbitrary, the result follows. $\square$

Lemma 13. *Let $\mathcal{E}$ be a smooth economy and $\omega^1, \omega^2 \in \mathbf{R}^{mL}_{++}$ with $\omega^1 \leq \omega^2$. Suppose there is a unique equilibrium in $\mathcal{E}$ for each $\omega \in [\omega^1, \omega^2]$, and let*

$$M := \{(p, \omega) \in \mathbf{R}^L_{++} \times [\omega^1, \omega^2] : Z(p, \omega) = 0\}$$

Then

$$\begin{aligned} H_0(M) &= \mathbf{Z} \\ H_s(M) &= 0 \quad \forall s \geq 1 \end{aligned}$$

Proof. By Lemma 12, there is a continuous function $h : [\omega^1, \omega^2] \to \mathbf{R}^L_{++}$ such that

$$M = \{(p, \omega) \in \mathbf{R}^L_{++} \times [\omega^1, \omega^2] : h(\omega) = p\}$$

Thus M is homeomorphic to the graph of the continuous function h on $[\omega^1, \omega^2]$, which implies M is homeomorphic to $[\omega^1, \omega^2]$. Since $[\omega^1, \omega^2] \subseteq \mathbf{R}^{mL}$ is closed and convex, it is a deformation retract of $\mathbf{R}^{mL}$. Since homology is preserved by homeomorphism and by deformation retract, this implies

$$\begin{aligned} H_0(M) = H_0([\omega^1, \omega^2]) = H_0(\mathbf{R}^{mL}) = \mathbf{Z} \\ H_s(M) = H_s([\omega^1, \omega^2]) = H_s(\mathbf{R}^{mL}) = 0 \qquad \forall s \geq 1 \end{aligned}$$

where the final equalities follow from the fact $\mathbf{R}^{mL}$ is contractible. $\square$

Lemma 14. *Let $\mathcal{E}$ be a smooth economy and $[\omega^1, \omega^2]$ be a totally regular interval for $\mathcal{E}$. Let k be the common number of equilibria for each $\omega \in [\omega^1, \omega^2]$. Let*

$$M := \{(p, \omega) \in \mathbf{R}^L_{++} \times [\omega^1, \omega^2] : Z(p, \omega) = 0\}$$

Then

$$\begin{aligned} H_0(M) &= \mathbf{Z}^k \\ H_s(M) &= 0 \quad \forall s \geq 1 \end{aligned}$$

Proof. By Lemma 11, there exist continuous functions $h_i : [\omega^1, \omega^2] \to \mathbf{R}^L_{++}$ for $i = 1, \ldots, k$ such that

$$M = \bigcup_{i=1}^{k} \{(p, \omega) \in \mathbf{R}^L_{++} \times [\omega^1, \omega^2] : h_i(\omega) = p\}$$

and such that $h_i(\omega) \neq h_j(\omega)$ for all $i \neq j$ and for all ω.

Set $M_i := \{(p, \omega) \in \mathbf{R}^L_{++} \times [\omega^1, \omega^2] : h_i(\omega) = p\}$ for each $i = 1, \ldots, k$. Then since $h_i(\omega) \neq h_j(\omega)$ for all $i \neq j$ and for all ω,

$$\{(p, \omega) \in \mathbf{R}^L_{++} \times [\omega^1, \omega^2] : h_i(\omega) = p\} \cap \{(p, \omega) \in \mathbf{R}^L_{++} \times [\omega^1, \omega^2] : h_j(\omega) = p\} = \emptyset \ \forall i \neq j$$

that is, $M_i \cap M_j = \emptyset$ for all $i \neq j$.

Fix i. Since $[\omega^1, \omega^2]$ is closed and h_i is continuous, M_i is closed. By Theorem 2 M is compact, and $M_i \subseteq M$, thus M_i is also compact. In addition, M_i is homeomorphic to the graph of the continuous function h_i on $[\omega^1, \omega^2]$, so M_i is homeomorphic to $[\omega^1, \omega^2]$. Since $[\omega^1, \omega^2] \subseteq \mathbf{R}^{mL}$ is closed and convex, it is a deformation retract of $\mathbf{R}^{mL}$. Since homology is preserved by homeomorphism and by deformation retract, this implies

$$\begin{aligned} H_0(M_i) &= H_0([\omega^1, \omega^2]) = H_0(\mathbf{R}^{mL}) = \mathbf{Z} \\ H_s(M_i) &= H_s([\omega^1, \omega^2]) = H_s(\mathbf{R}^{mL}) = 0 \qquad \forall s \geq 1 \end{aligned}$$

where the final equalities follow from the fact $\mathbf{R}^{mL}$ is contractible.

Since i was arbitrary, this holds for each $i = 1, \ldots, k$. Then $M = \cup_{i=1}^{k} M_i$ and $M_i \cap M_j = \emptyset$ for all $i \neq j$, so the homology of M is the direct sum of the homologies of $M_1, \ldots, M_k$. That is, $H_0(M) = \mathbf{Z}^k$ and $H_s(M) = 0$ for each $s \geq 1$. □

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