On Game-Theoretic Underpinnings of the Gale-Nikaido-Kuhn-Debreu Lemma: Generalized Formulations*

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May 5, 2024

Abstract: In this paper, we view the Gale-Nikaido-Kuhn-Debreu (GNKD) lemma, a fundamental result in Walrasian general equilibrium theory, from the viewpoint of a qualitative or a generalized game, and provide two alternative proofs of the classical theorem. These stem from a reformulation of the Lemma that yields a three-fold generalization: it (i) weakens the convexity assumption on the excess demand correspondence, (ii) defines a new continuity concept that synthesizes both the majorization and inclusion approaches, (iii) shows that these generalizations are true for infinite dimensional commodity spaces. (94 words)

Journal of Economic Literature Classification Numbers: C00, D00, D01, D90, C62, O21.

2010 Mathematics Subject Classification Numbers: 91B55, 37E05.

Key Words: GNKD lemma, qualitative game, generalized game, non-convex, majorization, inclusion.

Running Title: The GNKD Lemma

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*This paper has been written for a potential presentation at the *XXXII European Workshop on Economic Theory (EWET 2024)* to take place in Manchester, June 19-21, 2024. The authors are pleased to dedicate this work to the memory of Monique Florenzano: it grew out of earlier work of Khan, McLean, and Uyanik (2024a,b) and an invited presentation at the *Memorial Event in Honour of Monique Florenzano,* held at Queen Mary College, 25 July 2015 and organized by Nizar Allouch. The authors thank Max Amarante, Simone Cerreia-Vioglio, Emma Moreno, Marcelo Fernandez, Leonidas Koutsougeras, Rajiv Vohra and Nicholas Yannelis for correspondence and conversation. Khan's work was supported by a contract from *Blue Green Future & Rebalance Earth* on "New Theoretical Paradigms for General Competitive Analysis."

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1 Introduction

The formal analogies between the theories of zero-sum two-person games, statistical decision functions, and resource allocation are valuable since a result obtained in any one of them can have an interesting counterpart in the two others; the difference between their philosophies should, however, by no means be overlooked in the resource allocation problem.

The central agency determining [the prices] is not inert and its behavior can be chosen precisely to conflict fully with the behavior of the various economic units.[1] Gerard Debreu (1951)[2]

Barring the paper of McKenzie (1954) and the book of Arrow and Hahn (1971), there are two main approaches to prove the existence of a competitive equilibrium in a private-ownership economy: the excess demand approach and the simultaneous optimization approach; see Debreu (1982) for details.[3] The substance and historical antecedents of this distinction have not been fully appreciated or articulated. The second was chronologically prior to the first, and it relied on Debreu's generalization of Nash's theorem for normal form games; see the comprehensive sketch of

[1]This central agency was to become in Arrow and Debreu (1954) a "fictitious participant who chooses and who may be termed the *market participant*." Independently, (Nikaido, 1956, p. 138) sets up a "fictitious negotiating procedure by letting a *referee* or a refereeing mechanism intervene into the negotiation, who may correspond to an auctioneer or price-manipulating authority." (Authors' italics, both here and elsewhere in the sequel.) Debreu (1956) limits himself to one simple sentence: "Given z in Z, let $\pi(z)$ be the set of maximizers of $p.z$ in P." This sentence was to be elaborated in a passive voice as a paragraph in (Debreu, 1959, p. 83) as the "central idea of the proof hinting at a tendency." We go into this issue in somewhat more detail in the third paragraph of the introduction to follow.

[2]See Debreu (1951, pp. 47 and 48) for this quotation; and Sections 11 and 12 for a detailed elaboration.

[3]For an explication and a comprehensive exegesis of McKenzie's contribution, see Khan (1993). For further elaboration of the approach of Arrow-Hahn, see Moore (1975, 2007) in addition to Arrow-Hahn themselves. In his report on their result with externalities, Sonnenschein (1975) also includes Starrett's name along with theirs. Florenzano (2009) singles out three main approaches, her third being the "so-called Negishi approach based ... on a fixed-point theorem applied in the utility space that is in the vector space whose dimension is equal to the finite number of consumers. This approach requires the preferences of each consumer to be represented by a utility function." What bears emphasis is that Negishi (1960) is forced by his method to assume concave rather than quasi-concave utility functions.

the proof in (Arrow and Debreu, 1954, Sections 2.3 and 2.4).[4] The main tool in the excess demand approach is the GNKD lemma, as provided by Gale (1955), Nikaido (1956), Kuhn (1956) and Debreu (1956),[5] and the classic treatment is available in the textbook explication of Debreu (1959) laying out a comprehensive proof of the existence of an equilibrium for an economy in which the individual production sets are neither convex nor closed as per Uzawa's seminal suggestion.[6]

The first proofs of the GKND lemma are based either on a fixed point theorem or the Knaster-Kuratowski-Mazurovich (KKM) lemma[7] but what needs emphasis is that the underlying problem in Debreu (1952) is not a game in the sense of Nash since the market player's actions affect the action sets of all the other players. It is this that leads Arrow-Debreu to formalize the distinctive notion of an *abstract economy,* and in its context, to call attention to two components, one of which concerns the payoff of the market participant.

> Suppose the market participant does not maximize instantaneously but, taking other participants' choices as given, adjusts his choice of prices so has to increase his payoff...; it can be increased by increasing p_h for those commodities for which $z_h > 0$, decreasing p_h if $z_h < 0$ (provided p_h is not already 0). But this is precisely the classical "law of supply and demand," and so the motivation of the market participant corresponds to one of the elements of a competitive equilibrium. This intuitive comment is not, however, the justification for this particular choice of a market pay-off, that justification will be found in [the sequel].

This intuition underlying the Debreu-Nikaido mapping is off the mark in that it totally ignores all but the market (or markets) where the excess demand is the largest: this bluff works.[8] In any case, the Arrow-Debreu passage has not been given as much of an emphasis as it ought. What is involved is a fixed point of a mapping and not an adjustment process: the equilibrium of the abstract economy is in the set-up of a one-shot simulatneous play. It needed (Hildenbrand and Kirman, 1976, p. 156) to re-emphasize to the profession that

> ... even though an adjustment process may not converge, nevertheless a fixed point of it exists. This would give us an equilibrium, since for such a *fixed point* p^* we clearly have to make no further adjustment. If we confine ourselves to a *fixed point of the adjustment process,* then

[4]It is not fully appreciated that the Arrow-Debreu paper did not rely on a fixed-point theorem, either Kakutani's or Eilenberg-Montgomery's, but instead stated a result, their Lemma 2.5, which the authors saw as a generalization of the theorem of Nash, but a special case of the principal result reported in Debreu (1952). The latter of course did rely on the Eilenberg-Montgomery fixed point theorem. Khan (2020) addresses this confusion; also see Maskin (2019) and Velupillai (2019). As we shall see below, this Lemma 2.5 in Arrow and Debreu (1954) is then taken up by Sonnenschein (1975); see Footnote 13 below.

[5]For the order of priority of the names in the lemma, see Debreu (1959, Note 2, p. 88) and Florenzano (2009).

[6]For Uzawa'a suggestion, see Note 1 on page 88 in Debreu (1959); also see Khan (2020) and Debreu (1962) for the emphasis on the need for the relaxation of the "free-disposal" assumption.

[7]Gale (1955) uses the KKM lemma, Kuhn (1956) uses the Eilenberg-Montgomery fixed point theorem, and Nikaido (1956) and Debreu (1956) use the Kakutani fixed point theorem.

[8]Khan (2020) misses Nikaido's name in his reference to what is being termed here as the Debreu-Nikaido mapping; see the discussion on page 615 of his article.

> this process as such has no real intrinsic economic content. We may thus arbitrarily choose a process to suit our purpose. The only criterion now is its mathematical convenience.[9]

This is in response to earlier hopes that the adjustment process suggested by the Nikaido-Debreu map, and as intuited by Arrow-Debreu, may be useful in understanding the price adjustment process towards equilibrium and using it to compute an equilibrium. This optimism regarding the Walrasian tâtonnement was punctured by the Gale-Scarf counterexample and finally put to rest by the Sonnenschein-Mantel-Debreu theorem.[10]

The contribution of this paper is four-fold: (i) to provide two alternative proofs for the classical GNKD lemma by using a formulation of a two-person qualitative game and also of a one-person generalized game, (ii) to provide a generalization of the GNKD lemma by weakening the convexity assumption on the excess demand correspondence, (iii) to provide a further generalization of the GNKD lemma by weakening the continuity assumption on the excess demand correspondence in a way that combines the majorization and inclusion assumptions that are used in the literature, and finally (iv) to provide a further generalization by taking argumentation to infinite-dimensional spaces.

Our first contribution can be viewed as a synthesis of the two approaches to existence theory in the sense that the GNKD lemma can itself be interpreted from the viewpoint of the simultaneous optimization approach as follows: there are two agents, one auxiliary market player (auctioneer) choosing the price to maximize the value of the excess demand and the other is a kind of "representative" consumer who chooses an excess demand vector at a given price profile. The consumer can be thought of as already best responding, and the equilibrium of this resulting game yields the conclusion of the lemma; see Reny (2016) for a detailed consideration.[11] In an alternative formulation one can interpret the lemma as a one-player generalized game, as for example, in Borglin and Keiding (1976, Corollary 2) which explicitly presents a result for a *single agent* generalized game. The agent's preference correspondence corresponds to the market player's preferences and the constrained correspondence corresponds to the excess demand correspondence. As above, the equilibrium of this

[9]The authors continue, "[t]he equilibrium concepts discussed in this book are essentially static in character. Once the economy is in equilibrium, then we gave good reasons as to why it should remain there. We have never said how an economy might get there." For an especially egregious example of the confusion between existence and stability of equilibrium, see Benetti, Nadal, and Salas (2004). At best, these authors can be seen as subscribing to a higher standard whereby a proof of a claim is to be informed by the substance of the claim. In the opinion of the authors, this represents a confusion of categories. It is to ask, for example, that any proof of Pythagoras' theorem exhibit some sort of right-angledness or perpendicularity in the execution of its proof, and little is accomplished by tying oneself and one's proofs with this siren-song.

[10]For the Gale-Scarf counterexamples, see Gale (1963) and Scarf (1960). For the Sonnenschein-Mantel-Debreu theorems, see Debreu (1974) and his references. For Arrow's later views on this work on adjustment processes stemming from Walrasian tâttonnement, see Dubra (2005).

[11]It should perhaps be noted here that Florenzano (2009) looks on the proof of the GNKD lemma presented in Debreu (1959) as already an "interesting mixture of both excess demand and simultaneous optimization approaches;" also see Florenzano (1987, 1991) and Florenzano and Moreno (2001).

game yields the conclusion of the lemma. It should be borne in mind that the construction of the qualitative game we mention above is different from the one that transforms a generalized game into a qualitative game.[12]

Our second contribution concerns the weakening of the convexity assumption in the classical setting of the lemma. The basic issue is ably laid out in Sonnenschein (1975): after dispelling potential and possible surprise at a paper on Walrasian general equilibrium theory "in a session devoted to "new developments", a subject that has has been well worked during the past twenty-five years," he takes Mas-Colell's breakthrough of 1974 as his point of departure, and introduces a diagram (reproduced as Figure 1 below) that tells all that needs to be told.[13] Non-ordered preferences yield non-convex-valued excess demand even when the preferences are strictly convex.

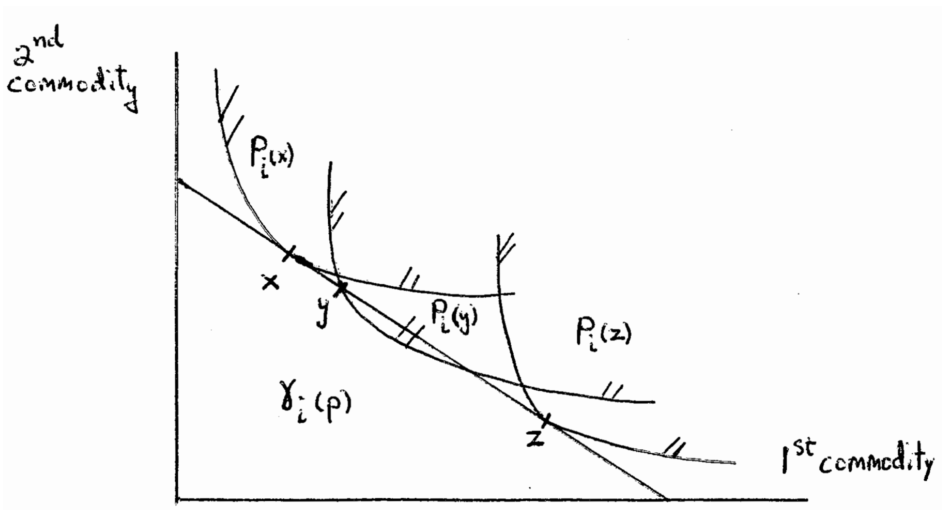

Figure 1: Figure 1 from Sonnenschein (1975)

[12]The subtext of this entire paragraph is the underpinning of the GKND lemma by non-cooperative game-theoretic arguments. As such, we ignore a large literature (pioneered by Hurwicz-Schmeidler on the one hand, and Shapley-Shubik on the other) that investigates Cournot-Bertrand game-theoretic underpinnings of Walrasian theory itself, and that goes much beyond the 1952-1954 remarks of Arrow-Debreu; see Hervés-Beloso and Moreno (2009b,a) for references both to the pioneering papers and to subsequent work. Also, the 2-person viewpoint on Walrasian theory articulated in Hervés-Beloso-Moreno concerns a "*society* game" in which one player pursues Paretian efficiency, and the other a version of "impartial and fair behavior" in the sense of Aubin.

[13]Sonnenschein (1975) was presented at the *Third World Congress of the Econometric Society* in August and published as Sonnenschein (1977) with a professionally-drawn figure, but with minimal changes. We reproduce the original figure in his own handwriting. Sonnenschein (1977) reports three results: two theorems, the first of which he attributes to Mas-Colell (1974), and the second to Arrow-Hahn-Starett; and a lemma which he attributes to "Debreu (1952) modified by Shafer and Sonnenschein (1975a)." It is the lemma that dovetails into the narrative laid out above; see Footnote 4 above. The paper presents a construction whereby an equilibrium of an economy without externalities but with non-ordered convex preferences is deduced from a corresponding result for an economy with externalities but with standard utility-representable preferences. The historians of the profession have surely registered the fact that we sight this paper twenty five years after it itself was written, as a neglected contribution of Sonnenschein's, perhaps as a result of several papers on the subject that he was involved in at the time [see Shafer and Sonnenschein (1975a,b, 1976)].

Leave alone convexity, Sonnenschein's Figure 1 shows that the excess demand correspondence may not even have connected values. He writes:

> As a result the aggregate excess demand function may have values which are not connected sets (and in fact it may not admit a selection which is both upper hemicontinuous and convex valued). Thus proofs of the existence of equilibrium which apply the Kakutani fixed point theorem to a modification of the aggregate excess demand function are likely to fail.

In hindsight, one can see these sentences as generating the distinction emphasized in Debreu (1982) as to the "simultaneous optimization" and "excess demand" approaches. But the point is that the latter does not fail, and the contribution of the work reported here brings out the sense in which it does not do so. To be sure there have been attempts already in Shafer and Sonnenschein (1975a) and Scapparone (2015) to relax the convexity assumption of preferences, but no one to our knowledge has considered the non-convexity of the excess demand correspondence.[14] The reason for our success relative to previous attempts is two-fold: (i) first, our reliance on what we have termed in ongoing work as a *majorization inclusion property (MIP)*, and making it a lynchpin for the existence theory, and (ii) secondly, on a reliance of a *reductio* rather than a direct argument.[15] Given the identification of the MIP as a fundamental property, .[16] We leave this as an open question. In this connection, see an example of an economy portrayed in Figure 2 below that this fully amenable to an application of our results.

Our third contribution concerns the weakening of the continuity assumption.[17] The technical aspects of our contribution can be highlighted through a further consideration of a hybrid continuity assumption that we term the *majorization-inclusion property* for an excess demand correspondence. This property allows a correspondence to be locally majorized by a "nice" correspondence at a point, or to have a local "nice" selection at some other point.[18] Hence, our continuity assumption weakens the majorization and inclusion-type continuity assumptions in that we allow for either local majorization or local continuous inclusion. Although continuous inclusion assumptions have been employed in the proof of the GNKD lemma, the majorization approach has to the best of our

[14]In the context of Scapparone (2015), it is worth pointing out in his interesting paper he keeps repeating that convex-valuedness and upper hemicontinuity of excess demands are indispensable, but as we show, they are not; also see Footnote 17 below.

[15]We elaborate on the MIP in the paragraphs belows. A first draft of ongoing work is circulated as Khan, McLean, and Uyanik (2023).

[16]See the paragraph below for a further discussion of the MIP.

[17]Scapparone (2015) writes: "A natural extension of [his] previous result would be the specification of properties of the preference relation which imply the existence of an upper hemi-continuous and convex-valued demand sub-correspondence." By a sub-correspondence, he means a selection, and as such makes contact with the sentence of Sonnenschein cited above.

[18]This type of continuity assumption is introduced in the context of generalized quasi-variational relation problems and constrained generalized games in ongoing work of the authors.

knowledge, not been used to generalize the GNKD lemma.[19]

Our fourth and final contribution can be simply put. It is that the generalizations of the antecedent literature that do not require a limitation to finite-dimensional commodity spaces. In this connection, we note that there is an alternative proof approach of the GNKD in the literature that uses both fixed point and separating hyperplane theorems. We also illustrate that this approach can be routinely used to provide a different proof of our generalizations.[20]

We now conclude this introduction by a cursory look at the antecedent literature in economics as well as in applied mathematics. The initial, now classical, papers of GNKD listed in the first paragraph of this introduction limited themselves to a finite-dimensional Euclidean space. There has been substantial work on generalizations to an infinite-dimensional space, a line of work that in the judgement of the authors has not yet found its final expression.[21] Beyond infinite-dimensionality, we note that the very formulation of the inward-pointing (Walras law) assumption, as well as the assumption of "free disposal" implicit in the use of the negative orthant in the conclusion of the lemma have been weakened.[22] Cornet and Florenzano work with larger price simplices, but we work with the unit simplex for expositional clarity. Finally, the GNKD Lemma has been used beyond the existence of equilibrium as a tool to prove results in other areas.[23] However, the thrust of our work is on weakening the continuity hypotheses of the GNKD, and it is important to note the McCabe-Yannelis mapping which necessitates the use of the separating hyperplane theorem in addition to a fixed point theorem.

2 Preliminaries

Assume X, Y are two topological spaces and $Q : X \twoheadrightarrow Y$ is a correspondence. Let $domQ$ denote the set $\{x \in X | Q(x) \neq \varnothing\}$, clQ denote the correspondence with values $cl(Q(x))$ where $cl(Q(x))$ denotes the closure of $Q(x)$, $intQ$ denote the correspondence with values $int(Q(x))$ where $int(Q(x))$ denotes the interior of $Q(x)$. If Y is a vector space, let coQ denote the correspondence with values $co(Q(x))$ where $co(Q(x))$ denotes the convex hull of $Q(x)$. Throughout the paper, we will use the abbreviation *TVS* for a topological vector space and use *LCTVS* for a TVS that is locally convex. Furthermore, we assume that every TVS is Hausdorff.

[19]See, for example, He and Yannelis (2017), Cornet (2020) and Khan and Uyanik (2021).

[20]See McCabe (1981), Geistdoerfer-Florenzano (1982), Florenzano (1983), Yannelis (1985), Mehta and Tarafdar (1987) and He and Yannelis (2017) for this alternative approach in finite and infinite dimensional settings.

[21]See for example, Florenzano (1983), Yannelis (1985), Mehta and Tarafdar (1987), Tan and Yu (1994), Kubota (2007), Park (2010), He and Yannelis (2017), Khan and Uyanik (2021).

[22]See, for example, Debreu (1956), Cornet (1975), Neuefeind (1980), Grandmont (1977), McCabe (1981), Geistdoerfer-Florenzano (1982), Florenzano and Le Van (1986), Florenzano (2009, 2013), Maskin and Roberts (2008) Musatov, Savvateev, and Weber (2016).

[23]See for example, Le, Le Van, Pham, and Saglam (2022), Gourdel, Le Van, Pham, and Viet (2023), Nikaidô and Uzawa (1960); Debreu (1962); Nishimura (1978); Bidard and Hosoda (1987).

Next, we define topological properties of a correspondence.

Definition 1. Let X, Y be two topological spaces and $Q : X \to Y$ be a correspondence.

(a) Q is **upper hemi-continuous (UHC) at** $x \in X$ if for all open set $V \subseteq Y$ with $Q(x) \subseteq V$, there exists an open set $U \subseteq X$ with $x \in U$ such that $Q(x') \subseteq V$ for all $x' \in U$. Q is **upper hemi-continuous** if Q is upper hemi-continuous at each $x \in X$.

(b) If Y is a topological vector space, then Q is **co-closed** if coQ has a closed graph.

(c) If Y is a topological vector space, then Q is KF**-majorized ($\mathcal{L}$-majorized) ($\mathcal{U}$-majorized) at** $x \in X$ if there exists an open neighborhood U^x of x and a correspondence $F^x : U^x \twoheadrightarrow Y$ with open graph (open fibers) (closed graph) and convex values such that for all $x' \in U^x$, $P(x') \subseteq F^x(x')$ and $x' \notin F^x(x')$. We say that Q is KF**-majorized ($\mathcal{L}$-majorized) ($\mathcal{U}$-majorized)** if Q is KF-majorized ($\mathcal{L}$-majorized) ($\mathcal{U}$-majorized) at every $x \in domQ$.

(d) If Y is a topological vector space, then Q has the **continuous inclusion property (CIP) at** $x \in X$ if there exists an open set $U(x)$ containing x and a non-empty valued co-closed correspondence $F : U(x) \to X$ such that $F(z) \subseteq Q(x')$ for every $z \in U(x)$. We say that Q has the **CIP** if Q has the continuous inclusion property at every $x \in domQ$.

By Aliprantis and Border (2006, Theorem 2.16), in a topological space, a point is a limit point of a net if and only if it is the limit of some subnet.

3 The Classical GNKD Lemma: Alternative Proofs

In this section, we present two new proofs of the classical GNKD lemma by representing the problem as a generalized game or a qualitative game whose equilibrium yields the conclusion of the GNKD lemma.

Let $\Delta = \{p \in \mathbb{R}^\ell | p_i \geq 0, \sum_{i=1}^{\ell} p_i = 1\}$ and $\psi : \Delta \twoheadrightarrow \mathbb{R}^\ell$ be a correspondence and $-\Omega = \{x \in \mathbb{R}^\ell | x_i \leq 0 \text{ for all } i = 1, \ldots, \ell\}$. Define $E_\psi = \{p \in \Delta | \psi(p) \cap -\Omega \neq \emptyset\}$, hence $E_\psi^c = \{p \in \Delta | \psi(p) \cap -\Omega = \emptyset\}$. Let $G : \Delta \twoheadrightarrow \mathbb{R}^\ell$ and $Z : \Delta \twoheadrightarrow \mathbb{R}^\ell$ be defined as $G(p) = \{z \in \mathbb{R}^\ell | p \cdot z \leq 0\}$ and $Z(p) = \psi(p) \cap G(p)$.

Theorem 1. *Suppose ψ is UHC and has non-empty, convex and compact values such that for all $p \in \Delta$, there exists $z \in \psi(p)$ such that $p \cdot z \leq 0$. Then, there exists $\bar{p} \in \Delta$ such that $\psi(\bar{p}) \cap -\Omega \neq \emptyset$.*

Next, we show that the GNKD Lemma can be proved by using a two-person qualitative game.

Proof of Theorem 1 by using Qualitive Games. Step 1. Since ψ is UHC and has compact values, there exists a compact set $D \subseteq \mathbb{R}^\ell$ such that $\psi(q) \subseteq D$ for all $q \in \Delta$ (Aliprantis and Border, 2006, Lemma 17.8). WLOG we can assume D is convex. Since D is compact and Hausdorff, and ψ is UHC and has closed values, therefore ψ has a closed graph. Note that G has a closed graph. Therefore, by Aliprantis and Border (2006, Theorem 17.25), the correspondence $Z = G \cap \psi$ has a closed graph. For all $p \in \Delta$, there exists $z \in \psi(p)$ such that $p \cdot z \leq 0$, hence $Z(p) = G(p) \cap \psi(p) \neq \varnothing$. For all $p \in \Delta$, since $\psi(p) \subseteq D$, $Z(p) \subseteq D$. Note that ψ and G have convex values. Therefore, Z has a closed graph and non-empty, convex values.

Step 2. Define a two-person qualitative game $\Gamma = (X_i, Q_i)_{i=1}^2$ as follows:

(a) $X_1 = \Delta$ and $X_2 = D$,

(b) $Q_1 : \Delta \times D \twoheadrightarrow \Delta$ is defined as $Q_1(p, z) = \{q \in \Delta | q \cdot z > p \cdot z\}$,

(c) $Q_2 : \Delta \times D \twoheadrightarrow D$ is defined as $Q_2(p, z) = \varnothing$ if $z \in Z(p)$ and $Q_2(p, z) = Z(p)$ if $z \notin Z(p)$.

By construction, Q_1 and Q_2 have convex values, and for all $(p, z) \in \Delta \times D$, $p \notin Q_1(p, z)$ and $z \notin Q_2(p, z)$.

Pick $(p, z) \in \Delta \times D$ such that $Q_i(p, z) \neq \varnothing$ for some $i = 1, 2$. If $Q_1(p, z) \neq \varnothing$, then since Q_1 has an open graph, there exists an open neighborhood $U_1(p, z)$ of (p, z) and $q \in \Delta$ such that $q \in Q_1(p', z')$ for all $(p', z') \in U_1(p, z)$.

If $Q_2(p, z) \neq \varnothing$, then $z \notin Z(p)$. Since Z has a closed graph, there exists an open neighborhood $U_2(p, z)$ of (p, z) such that $x' \notin Z(p')$ for all $(p', z') \in U_2(p, z)$. Therefore, $Z(p') \subseteq Q_2(p', z')$ for all $(p', z') \in U_2(p, z)$. Since any singleton set is closed in a Hausdorff space and Z has a closed graph and convex values, Γ satisfies the assumptions of Theorem 1 in Khan, McLean, and Uyanik (2024b), hence it has an equilibrium.

Step 3. Let $(\bar{p}, \bar{z}) \in \Delta \times D$ be an equilibrium of Γ. Then $Q_i(\bar{p}, \bar{z}) = \varnothing$ for all $i = 1, 2$. Since, $Q_2(\bar{p}, \bar{z}) = \varnothing$, $\bar{z} \in Z(\bar{p})$. Then $Z(\bar{p}) \subseteq G(\bar{p})$ implies that $\bar{p} \cdot \bar{z} \leq 0$. It follows from $Q_1(\bar{p}, \bar{z}) = \varnothing$ that $q \cdot \bar{z} \leq \bar{p} \cdot \bar{z}$ for all $q \in \Delta$. Combining these two inequalities, for all $q \in \Delta$, $q \cdot \bar{z} \leq 0$. This implies that $\bar{z} \leq 0$. (Otherwise, if $\bar{z}_i > 0$ for some i, then for $q \in \Delta$ with $q_i = 1$, $q \cdot \bar{z} > 0$.) ∎

Remark 1. Our construction of the qualitative game is motivated by but different from other constructions in the literature. The construction has the following interpretation: player 2 is best responding (excess demand is the best response correspondence) and player 1 has a nice preference relation, possibly discontinuous. This type of games is studied in Reny (2016).

Next, we provide an alternative proof of Theorem 1 that uses a formulation in terms of single player generalized game.

An Alternative Proof of Theorem 1. Step 1. Using exactly the same argument as that of the previous proof, we conclude that there exists a compact, convex nonempty set $D \subseteq \mathbb{R}^\ell$ such that $\psi(q) \subseteq D$ for all $q \in \Delta$. Defining $Z(p) = G(p) \cap \psi(p)$, it follows that $Z : \Delta \twoheadrightarrow D$ has a closed graph and non-empty, convex values.

Step 2. Define a correspondence $Q : \Delta \times D \twoheadrightarrow \Delta$ as $Q(p, z) = \{q \in \Delta | q \cdot z > p \cdot z\}$. Next, define correspondences $\hat{Q} : \Delta \times D \twoheadrightarrow \Delta \times D$ and $A : \Delta \times D \twoheadrightarrow \Delta \times D$ where $P(p, z) = Q(p, z) \times D$ and $A : \Delta \times D \twoheadrightarrow \Delta \times D$ is defined as $A(p, z) = \Delta \times Z(p)$. We must show that the single player generalized game $\Gamma = (X, P, A)$ where $X = \Delta \times D, A$ is the player's preference correspondence, and A is the player's action correspondence has an equilibrium. That that, we must show that there exists $(\bar{p}, \bar{z}) \in \Delta \times D$ such that $(\bar{p}, \bar{z}) \in A(\bar{p}, \bar{z})$ and $P(\bar{p}, \bar{z}) = \varnothing$.

Note that $P(p, z) \cap A(p, z) = Q(p, z) \times Z(p)$ for each $(p, z) \in \Delta \times D$. Since Z is a co-closed correspondence, it satisfies CIP. Since Q has an open graph and convex values, and D is Hausdorff, Q satisfies LIP and hence CIP. Therefore, by He and Yannelis (2017, Proposition 1(4)), $Q \times Z$ has the CIP. By Theorem 2 in Khan, McLean, and Uyanik (2024b), Γ has an equilibrium.

Step 3. Let $(\bar{p}, \bar{z}) \in \Delta \times D$ be an equilibrium of Γ. Therefore $(\bar{p}, \bar{z}) \in A(\bar{p}, \bar{z})$ and $P(\bar{p}, \bar{z}) = \varnothing$. Hence, $\bar{z} \in Z(\bar{p})$ and $Q(\bar{p}, \bar{z}) = \varnothing$ since $P(\bar{p}, \bar{z}) = Q(\bar{p}, \bar{z}) \times D$ and D is non-empty. It follows from $\bar{z} \in Z(\bar{p}) \subseteq G(\bar{p})$ that $\bar{p} \cdot \bar{z} \leq 0$, and from $Q(\bar{p}, \bar{z}) = \varnothing$ that $q \cdot \bar{z} \leq \bar{p} \cdot \bar{z}$ for all $q \in \Delta$. Combining these two inequalities, for all $q \in \Delta$, $q \cdot \bar{z} \leq 0$. This implies that $\bar{z} \leq 0$. ∎

Remark 2. The proofs we presented above derive the result of the GNKD lemma by transforming the problem into a qualitative game or a generalized game. An equilibrium of the derived qualitative/generalized game yields the result of the GNKD lemma directly. Therefore, we provide two new proofs of the GNKD lemma by using qualitative/generalized games.

Note that there are two main approaches in Walrasian equilibrium theory, one uses the GNKD lemma as the main technical tool (excess demand approach) and the other uses generalized games as the main technical tool (simultaneous optimization approach). Our new proofs provide a different connection between these two approaches.

4 The GNKD Lemma: Nonconvex-valued and Discontinuous Demand

Discontinuity in the preferences of the agents yields discontinuity in excess demand correspondence. Even if preferences are convex and continuous but non-transitive, the excess demand correspondence need not have convex values; see for example Sonnenschein (1977) for a discussion.[24]

In this section, we provide a generalization of the GNKD lemma by weakening the convexity and continuity assumptions on the excess demand correspondence. Our weakening of continuity is based on the majorization and inclusion concepts. To the best of our knowledge, majorization approach has not been used in the context of the GNKD lemma.[25]

[24] On this, see also Shafer (1974). Bergstrom, Parks, and Rader (1976) and Ghosh, Khan, and Uyanik (2023).

[25] Versions of the inclusion approach is used in He and Yannelis (2017) and Cornet (2020) to generalize the GNKD lemma.

We begin by introducing versions of majorization concepts for abstract correspondences in the context of an excess demand correspondence. The definitions below are related to GNKD lemma, the original versions are related to maximal elements (irreflexivity was assumed).

Definition 2. Let $\psi, Z : \Delta \twoheadrightarrow \mathbb{R}^\ell$ be correspondences defined as above. Then ψ is

(a) **$\mathcal{U}$-majorized at** $p \in \Delta$ if there exist an open set $U(p)$ containing p and a correspondence $F^p : U(p) \to \mathbb{R}^\ell$ with convex values such that F^p is UHC with compact values and for all $q \in U(p) \cap E^c_\psi$, $Z(q) \subseteq F^p(q)$ and $F^p(q) \cap -\Omega = \varnothing$.

(b) **KF-majorized at** $p \in \Delta$ if there exist an open set $U(p)$ containing p and a correspondence $F^p : U(p) \to \mathbb{R}^\ell$ with convex values such that F^p has open sections and for all $q \in U(p) \cap E^c_\psi$, $Z(q) \subseteq F^p(q)$ and $F^p(q) \cap -\Omega = \varnothing$.

Moreover, ψ is **$\mathcal{U}$-majorized** if it is $\mathcal{U}$-majorized at all $p \in E^c_\psi$, and it is **KF-majorized** if it is KF-majorized at all $p \in E^c_\psi$.

In this definition, the assumption $\psi(q) \subseteq F^p(q)$ implies $Z(q) \subseteq F^p(q)$. Hence, imposing majorization on the excess demand correspondence is stronger than the assumption above.[26] Moreover, in the literature there is another common majorization concept, $\mathcal{L}$-majorization, that replaces open sections with open fibers in the definiton of KF-majorization. See the discussion section for versions of the results in this section by using this majorization concept.

Proposition 1. *Let $\psi : \Delta \twoheadrightarrow \mathbb{R}^\ell$ be a correspondence with compact values as defined above. Then ψ is UHC at p implies it is $\mathcal{U}$-majorized at p, which implies it is KF-majorized at p.*

By definition, an UHC correspondence with compact values $\mathcal{U}$-majorizes itself, hence $\mathcal{U}$-majorization is weaker than UHC. The proof that ψ is $\mathcal{U}$-majorized at p implies it is KF-majorized at p is provided in Appendix.

Next, we define an inclusion property for excess demand correspondence following He and Yannelis (2017) and Cornet (2020).

Definition 3. Let $\psi, Z : \Delta \twoheadrightarrow \mathbb{R}^\ell$ be correspondences defined as above. Then ψ has the **Continuous Inclusion Property (CIP) at** $p \in \Delta$ if there exist an open set $U(p)$ containing p and an UHC correspondence $F^p : U(p) \to \mathbb{R}^\ell$ with compact values, and for all $q \in U(p) \cap E^c_\psi$, $F^p(q) \subseteq Z(q)$, $F^p(q) \neq \varnothing$, and $coZ(q) \cap -\Omega = \varnothing$. Moreover, ψ has the **CIP** if it has the CIP at all $p \in E^c_\psi$.

Next, we provide a weakening of the continuity assumption on the excess demand correspondence that combines the majorization and inclusion concepts defined above.

[26] This is not true for CIP. Moreover, whether CIP is imposed on ψ and Z requires weak and strong versions of Walras' law, respectively; see Remark 6 for details.

Definition 4. Let $\psi, Z : \Delta \twoheadrightarrow \mathbb{R}^\ell$ be correspondences defined as above. Then ψ satisfies the **Majorization-Inclusion Property (MIP) at** $p \in \Delta$ if there exist an open set $U(p)$ containing p and a correspondence $F^p : U(p) \to \mathbb{R}^\ell$ such that at least one of the following holds:

(a) F^p has open values and open lower sections, and for all $q \in U(p) \cap E^c_\psi$, $Z(q) \subseteq coF^p(q)$ and $coF^p(q) \cap -\Omega = \varnothing$.

(b) F^p is UHC with compact values and for all $q \in U(p) \cap E^c_\psi$, $F^p(q) \neq \varnothing$, $F^p(q) \subseteq Z(q)$, and $coZ(q) \cap -\Omega = \varnothing$.

Moreover, ψ satisfies the **MIP** if it has the MIP at each $p \in E^c_\psi$.

Part (a) imposes a majorization property on the excess demand correspondence. Part (b) is a version of CIP. It is usually stated by using co-closed local selections; see for example He and Yannelis (2017); Khan, McLean, and Uyanik (2024b). The version with UHC (with compact values of the selection, or the original correspondence) is used in Cornet (2020) and Podczeck and Yannelis (2022). Note that when the range is compact, the two definitions overlap. It is clear that each of part (a) and part (b) is weaker than the continuity assumption used in Theorem 1.

We need the following lemma in the proofs Theorems 2 and 4. We provide its proof in the Appendix. It assumes that the local majorizing correspondences have open fibers and the local inclusion correspondences are UHC, and derives a well-behaved global UHC correspondence.

Lemma 1. *Consider two correspondences $Q, T : X \to Y$ where X is a non-empty and compact subset of a topological space and Y is a non-empty and convex subset of a LCTVS. Suppose Q has non-empty values and T has open sections with convex values such that $Q(x) \subseteq clT(x)$ for all $x \in X$. Suppose also that for all $x \in X$, there exists an open set $U(x)$ containing x and a correspondence $F^x : U(x) \to Y$ with non-empty values such that at least one of the following holds:*

(a) *F^x has open sections and for each $z \in U(x)$, $Q(z) \subseteq coF^x(z)$,*

(b) *coF^x is UHC and has non-empty, compact values, and for each $z \in U(x)$, $F^x(z) \subseteq Q(z)$.*

Then there exist finitely many points $x_1, \dots, x_m$ in X, and an UHC correspondence $F : X \to Y$ with non-empty, convex, compact values such that for all $x \in X$, $F(x) \subseteq coQ(x)$ or $co(F(x) \cup coQ(x)) \subseteq coF^{x_k}(x) \cap clT(x)$ for some x_k with $x \in U(x_k)$.

The following theorem generalizes Theorem 1 by weakening the continuity assumption.

Theorem 2. *Suppose ψ has the MIP and non-empty values such that for all $p \in \Delta$, there exists $z \in \psi(p)$ such that $p \cdot z \leq 0$. Then, there exists $\bar{p} \in \Delta$ such that $\psi(\bar{p}) \cap -\Omega \neq \emptyset$.*

Proof of Theorem 2. Step 1. Assume towards a contradiction that $\psi(p) \cap -\Omega = \emptyset$ for all $p \in \Delta$. Hence, $E^c_\psi = \Delta$. It follows from the definition of G and the assumption that ψ has non-empty values that Z has non-empty values.

Step 2. We show that there exists an UHC correspondence $F : \Delta \twoheadrightarrow \mathbb{R}^\ell$ with non-empty, convex, compact values such that for all $p \in \Delta$, $F(p) \cap -\Omega = \varnothing$ and $F(p) \cap G(p) \neq \varnothing$. Since $E_\psi^c = \Delta$, and ψ satisfies MIP, there exists for each $p \in \Delta$ an open set $U(p)$ containing p and a correspondence $F^p : U(p) \to \mathbb{R}^\ell$ such that at least one of the following holds:

(a) F^p has open values and open lower sections, and for each $q \in U(p)$, $Z(q) \subseteq coF^p(q)$ and $coF^p(q) \cap -\Omega = \varnothing$.

(b) F^p is UHC with compact values and for each $q \in U(p)$, $F^p(q) \neq \varnothing$, $F^p(q) \subseteq Z(q)$, and $coZ(q) \cap -\Omega = \varnothing$.

Note that for part (b), since F^p is UHC with compact values, range of F^p is finite dimensional, by Aliprantis and Border (2006, Corollary 5.33 and Theorem 17.35(2)), coF^p is UHC with compact values.

Applying Lemma 1 implies that there exist finitely many points $q_1, \dots, q_m$ in Δ and an UHC correspondence $F : \Delta \twoheadrightarrow \mathbb{R}^\ell$ with non-empty, convex, compact values such that for all $p \in \Delta$, $F(p) \subseteq coZ(p)$ or $co(F(p) \cup Z(p)) \subseteq coF^{q_k}(p) \cap G(p)$ for some q_k with $p \in U(q_k)$. Pick $p \in \Delta$.

If $F(p) \subseteq coZ(p)$, then $F(p) \subseteq coZ(p) \subseteq G(p)$. Therefore, $F(p) \cap G(p) = F(p) \neq \varnothing$. By (a) and (b) above, $coZ(p) \cap -\Omega = \varnothing$, hence $F(p) \cap -\Omega = \varnothing$.

If $co(F(p) \cup Z(p)) \subseteq coF^{q_k}(p) \cap G(p)$ for some q_k with $p \in U(q_k)$, then by part (a) above $F^{q_k}(p) \cap -\Omega = \varnothing$, hence $F(p) \cap -\Omega = \varnothing$. Moreover, $F(p) \subseteq coF^{q_k}(p) \cap G(p)$ implies $F(p) \subseteq G(p)$ so $F(p) \cap G(p) \neq \varnothing$.

Step 3. By step 2, there exists an UHC correspondence $F : \Delta \twoheadrightarrow \mathbb{R}^\ell$ with non-empty, convex, compact values such that for all $p \in \Delta$, $F(p) \cap -\Omega = \varnothing$. Note that $-\Omega$ is non-empty, closed, and for all $p \in \Delta$, $F(p) \cap -\Omega = \varnothing$ and $F(p)$ is non-empty, compact and convex. By Aliprantis and Border (2006, Theorem 5.79), for all $p \in \Delta$, there exists $q \in \Delta$ such that $q \cdot F(p) > 0$.

Step 4. It follows from step 2 that F is UHC and has non-empty, convex and compact values and for all $p \in \Delta$, $F(p) \cap G(p) \neq \varnothing$, hence there exists $z \in F(p)$ such that $p \cdot z \leq 0$. Therefore, Theorem 1 implies that there there exists $\bar{p} \in \Delta$ and $\bar{z} \in F(\bar{p})$ such that $\bar{z} \leq 0$. Therefore, $q \cdot \bar{z} \leq 0$ for all $q \in \Delta$. By step 3, there exists $q^* \in \Delta$ such that $q^* \cdot z > 0$ for all $z \in F(\bar{p})$. In particular, $q^* \cdot \bar{z} > 0$ and we obtain the desired contradiction. ∎

The following example illustrates how non-convexity is allowed by MIP in Theorem 2.

Example 1. Consider the following two-consumer exchange economy with a bounded (truncated) consumption set and non-convex preferences as illustrated in Figure 2. Consumer A receives utility only form commodity 2 for low level of commodity 2, and for sufficiently high level of commodity 2, commodities 1 and 2 are perfect complements. For consumer B, commodities 1 and 2 are perfect complements when amounts of these commodities are sufficiently close, and they are perfect substitutes when the amounts are distinct.

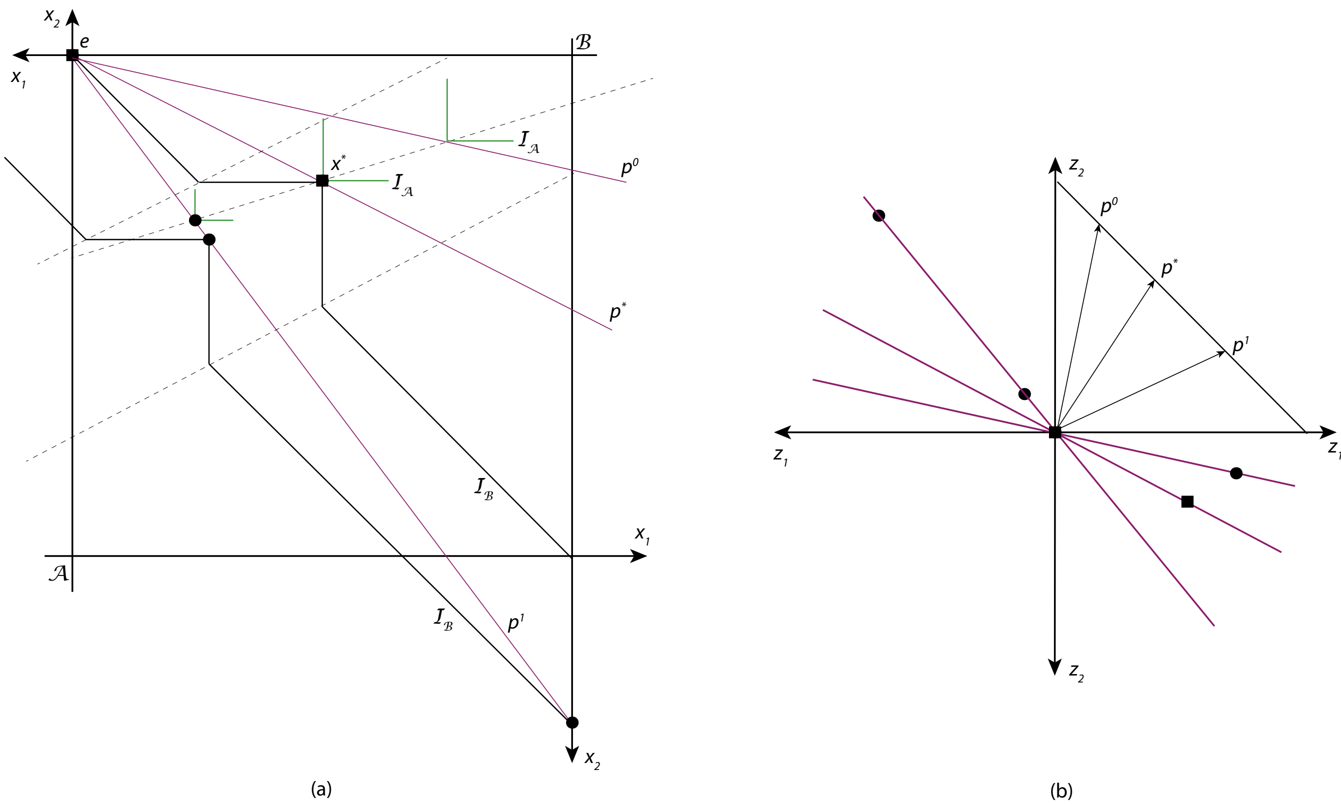

Figure 2: An Economy with Non-convex-valued Demand

This economy has a unique equilibrium: (x^*, p^*). It is clear that the excess demand correspondence does not have convex values at p^* and at p^1, though it is UHC. Moreover, the excess demand correspondence does not satisfy CIP at p^1. It satisfy KF-majorization (at all $p \neq p^*$, where p^* is equilibrium). To see this, for all $p \in \Delta$ such that $p \neq p^*$, let $U(p) = \{q \in \Delta | q \neq p^*\}$ and $F^p(q) = \{z \in \mathbb{R}^2 | p^* \cdot z > 0\}$. Therefore, ψ satisfies KF-majorization, and hence the MIP.

Remark 3. By using the arguments analogous to those in the proof of Theorem 2, part (a) of Definition 4 can be replaced with a version that imposes an open fibers assumption. Hence, we can obtain a generalization of the classical GNKD lemma by using a version of the hybrid continuity concept MIP that is based on CIP and $\mathcal{L}$-majorization.

Remark 4. The proof we presented above uses the classical result directly in step 4. As we show below, we can alternatively derive the result of the GNKD lemma by transforming the auxiliary problem into a generalized game in step 4. (Similarly, we can use the qualitative game transformation as above instead of a generalized game.) Note that, unlike the alternative proofs of the classical results, this is a proof by contradiction.

Alternative step 4 in the proof above by using a generalized game. Since F is UHC and has compact values, there exists a compact set $D \subseteq \mathbb{R}^\ell$ such that $F(q) \subseteq D$ for all $q \in \Delta$ (Aliprantis and Border, 2006, Lemma 17.8). WLOG, we can assume D is convex. Since D is compact and Hausdorff, and F is UHC and has closed values, therefore F has a closed graph.

Define a correspondence $Q : \Delta \times D \twoheadrightarrow \Delta$ as $Q(p, z) = \{q \in \Delta | q \cdot z > p \cdot z\}$. Let $\Gamma = (X, \hat{Q}, \hat{F})$ be a one-person generalized game where $X = \Delta \times D$ is the action set, $\hat{Q} : \Delta \times D \twoheadrightarrow \Delta \times D$ is the preference correspondence defined as $\hat{Q}(p, z) = Q(p, z) \times D$ and $\hat{F} : \Delta \times D \twoheadrightarrow \Delta \times D$ is defined as $\hat{F}(p, z) = \Delta \times F(p)$.

Note that $\hat{Q} \cap \hat{F} = Q \times F$. By step 2, F is a co-closed correspondence, hence it satisfies CIP. Since Q has an open graph and convex values, and D is Hausdorff, it satisfies LIP and hence CIP. Therefore, by He and Yannelis (2017, Proposition 1(4)), $Q \times F$ has the CIP. Then by Theorem 2 in Khan, McLean, and Uyanik (2024b), Γ has an equilibrium $(\bar{p}, \bar{z}) \in \Delta \times D$, hence $(\bar{p}, \bar{z}) \in \hat{F}(\bar{p}, \bar{z})$ and $\hat{Q}(\bar{p}, \bar{z}) = \varnothing$. Therefore $\bar{z} \in F(\bar{p})$ and $Q(\bar{p}, \bar{z}) = \varnothing$ since $\hat{Q}(\bar{p}, \bar{z}) = Q(\bar{p}, \bar{z}) \times D$ and D is non-empty.

It follows from $\bar{z} \in F(\bar{p}) \subseteq G(\bar{p})$ that $\bar{p} \cdot \bar{z} \leq 0$, and from $Q(\bar{p}, \bar{z}) = \varnothing$ that $q \cdot \bar{z} \leq \bar{p} \cdot \bar{z}$ for all $q \in \Delta$. Combining these two inequalities, for all $q \in \Delta$, $q \cdot \bar{z} \leq 0$. By step 3, there exists $q \in \Delta$ such that for all $z \in F(\bar{p})$, $q \cdot z > 0$, hence $q \cdot \bar{z} > 0$. This yields a contradiction.

Remark 5. It is possible to use the proof-construction in (Debreu, 1959, pp. 82 and 83) in step 4 above. In this construction, instead of deriving a generalized game, we use the following mapping. Let $\mu : D \twoheadrightarrow \Delta$ be a correspondence defined as $\mu(z) = \text{argmax}_{p \in \Delta} p \cdot z$, and $H : \Delta \times D \twoheadrightarrow \Delta$ be a correspondence defined as $H(p, z) = \mu(z) \times F(p)$. It is not difficult to show that H is an UHC correspondence with non-empty and convex values and hence by applying Kakutani's fixed point theorem, it has a fixed point. Analogous to the arguments in step 4 above, the fixed points of H yield a contradiction with the properties of F provided in step 3. A version of this method is used in the proof of He and Yannelis (2017, Theorem 4), instead of using the selection F, they use the correspondence itself in the construction.

Remark 6. The classical version of the GNKD lemma assumes a strong version of the Walras' law: for all $p \in \Delta$, $p \cdot \psi(p) \leq 0$. The antecedent literature weakens this assumption by the weak Walras' law: for all $p \in \Delta$, there exists $z \in \psi(p)$ such that $p \cdot z \leq 0$. Following this literature, Theorem 2 uses the weak Walras' law. Note that if the excess demand correspondence is UHC or it satisfies the majorization part of the MIP (part a) at all points, then weak Walras' law is enough. The method of proof does not work when inclusion property holds at some point. As in He and Yannelis (2017), we impose the inclusion property on $Z = \psi \cap G$ where $G(p) = \{z \in \mathbb{R}^\ell : p \cdot z \leq 0\}$ for all $p \in \Delta$. Note that this assumption is weaker than ψ satisfying CIP and the strong version of Walras' law as the latter implies $\psi(p) \subseteq G(p)$, hence $Z(p) = \psi(p)$ for all $p \in \Delta$. Finally, there is also another method to use the weak version of Walras' law in a finite-dimensional commodity space: first, prove the GNKD lemma by using the strong version of the Walras' law, and second, use an approximation technique to weaken Walras' law; see Cornet (2020) for details of this method.[27]

Next, in addition to the proofs given above and mentioned in Remark 4, we provide an alternative proof of Theorem 2 that uses a construction introduced by McCabe (1981) that has been widely used.[28] It requires a modification only in Step 4 as the other proofs we mention in the remarks above.

An Alternative Proof of Theorem 2. Repeating steps 1, 2, and 3 of the proof above implies that there exists a correspondence $F : \Delta \twoheadrightarrow \mathbb{R}^\ell$ satisfying the following assumptions: F is UHC, has non-empty and convex values, and for all $p \in \Delta$, there exists $q \in \Delta$ such that $q \cdot F(p) > 0$ and $F(p) \cap G(p) \neq \emptyset$.

Next, define a correspondence $\Psi : \Delta \twoheadrightarrow \Delta$ as $\Psi(p) = \{q \in \Delta : q \cdot F(p) > 0\}$. By construction of F, Ψ has nonempty values. It follows from McCabe (1981, Lemma) that Ψ has a continuous selection, hence there exists a continuous function $f : \Delta \to \Delta$ such that $f(q) \in \Psi(q)$ for all $q \in \Delta$. Since Δ is non-empty, compact and convex, Brouwer's theorem implies that there exists $\bar{p} \in \Delta$ such that $\bar{p} = f(\bar{p}) \in \Psi(\bar{p})$. Therefore, $\bar{p} \cdot F(\bar{p}) > 0$, that is, for all $z \in F(\bar{p})$, $\bar{p} \cdot z > 0$. Since $F(\bar{p}) \cap G(\bar{p}) \neq \varnothing$, there exists $z \in F(\bar{p})$ such that $\bar{p} \cdot z \leq 0$. This yields a contradiction. ∎

Remark 7. Instead of the MIP above, we can use a weakening of it that uses the domain of the correspondence. These will allow us to cover the continuity concepts provided in Prokopovych (2013, 2016).

5 The GNKD Lemma: Infinite Dimension

Let X be a LCTVS with topological dual X^*, C a closed convex cone in X having an interior point e, $C^* = \{p \in X^* | \ p(x) \leq 0 \text{ for all } x \in C\} \neq \{a\}$ be the dual cone of C, and $\Delta = \{p \in C^* | \ p(e) = -1\}$. Let $\psi : \Delta \twoheadrightarrow X$ be a correspondence. Define $E_\psi = \{p \in \Delta | \ \psi(p) \cap C \neq \emptyset\}$, hence $E_\psi^c = \{p \in \Delta | \ \psi(p) \cap C = \emptyset\}$. Let $G : \Delta \twoheadrightarrow X$ and $Z : \Delta \twoheadrightarrow X$ be defined as $G(p) = \{z \in X | \ p(z) \leq 0\}$ and $Z(p) = \psi(p) \cap G(p)$.

First, we provide an alternative proof of the infinite-dimensional version of the GNKD lemma by using qualitative games.

Theorem 3. *Suppose ψ is UHC, and has non-empty, convex and compact values such that for all $p \in \Delta$, there exists $z \in \psi(p)$ such that $p(z) \leq 0$. Then, there exists $\bar{p} \in \Delta$ such that $\psi(\bar{p}) \cap C \neq \emptyset$.*

[27]See Cornet (2020) and Podczeck (1997, Footnote 5) for a discussion and the use of the weak version of Walras' law, especially in infinite dimensional spaces.

[28]See for example Geistdoerfer-Florenzano (1982), Yannelis (1985), Mehta and Tarafdar (1987) and He and Yannelis (2017). The traces of this proof method can be found in Browder (1968) in the context of variational inequalities.

We provide a proof of this theorem that uses a formulation in terms of a two player qualitative game.

Proof of Theorem 3. Step 1. Since Δ is compact in (w*-topology), and ψ is UHC and has compact values, there exists a compact set $D \subseteq X$ such that $\psi(q) \subseteq D$ for all $q \in \Delta$ (Aliprantis and Border, 2006, Lemma 17.8). Since D is compact and Hausdorff, and ψ is UHC with closed values, it follows that ψ has a closed graph. Note that G has a closed graph. Therefore, by Aliprantis and Border (2006, Theorem 17.25), the correspondence $Z = G \cap \psi$ has a closed graph. For all $p \in \Delta$, there exists $z \in \psi(p)$ such that $p(z) \leq 0$, hence $Z(p) = G(p) \cap \psi(p) \neq \varnothing$. For all $p \in \Delta$, since $\psi(p) \subseteq D$, $Z(p) \subseteq D$. Note that ψ and G have convex values. Therefore, Z has a closed graph and non-empty, convex values.

Step 2. Define a two-person qualitative game $\Gamma = (A_i, Q_i)_{i=1}^2$ as follows:

(a) $A_1 = \Delta$ and $A_2 = X$,

(b) $Q_1 : \Delta \times X \twoheadrightarrow \Delta$ is defined as $Q_1(p, z) = \{q \in \Delta | q(z) > p(z)\}$,

(c) $Q_2 : \Delta \times X \twoheadrightarrow D$ is defined as $Q_2(p, z) = \varnothing$ if $z \in Z(p)$ and $Q_2(p, z) = Z(p)$ if $z \notin Z(p)$.

By construction, Q_1 and Q_2 have convex values, and for all $(p, z) \in \Delta \times X$, $p \notin Q_1(p, z)$ and $z \notin Q_2(p, z)$.

Pick $(p, z) \in \Delta \times X$ such that $Q_i(p, z) \neq \varnothing$ for some $i = 1, 2$. If $Q_1(p, z) \neq \varnothing$, then Q_1 has an open graph implies that there exists an open neighborhood $U_1(p, z)$ of (p, z) and $q \in \Delta$ such that $q \in Q_1(p', z')$ for all $(p', z') \in U_1(p, z)$.

If $Q_2(p, z) \neq \varnothing$, then $z \notin Z(p)$. Since Z has a closed graph, there exists an open neighborhood $U_2(p, z)$ of (p, z) such that $x' \notin Z(p')$ for all $(p', z') \in U_2(p, z)$. Therefore, $Z(p') \subseteq Q_2(p', z')$ for all $(p', z') \in U_2(p, z)$. Since any singleton set is closed in a Hausdorff space and Z has a closed graph and convex values, Γ satisfies the assumptions of Theorem 5 in Khan, McLean, and Uyanik (2024b), hence it has an equilibrium.

Step 3. Let $(\bar{p}, \bar{z}) \in \Delta \times X$ be an equilibrium of Γ. Then $Q_i(\bar{p}, \bar{z}) = \varnothing$ for all $i = 1, 2$. Since, $Q_2(\bar{p}, \bar{z}) = \varnothing$, $\bar{z} \in Z(\bar{p})$. Then $Z(\bar{p}) \subseteq G(\bar{p})$ implies that $\bar{p}(\bar{z}) \leq 0$. It follows from $Q_1(\bar{p}, \bar{z}) = \varnothing$ that $q(\bar{z}) \leq \bar{p}(\bar{z})$ for all $q \in \Delta$. Combining these two inequalities, for all $q \in \Delta$, $q(\bar{z}) \leq 0$.

It remains to show that $\bar{z} \in C$. Assume towards a contradiction that $\bar{z} \notin C$. Note that C is non-empty, closed, by Aliprantis and Border (2006, Corollary 5.80), there exists a nonzero continuous linear functional strongly separating C and $\bar{z}$. That is, there exist $\hat{q} \in X^*$, $\hat{q} \neq 0$, and $b \in \mathbb{R}$ such that $\sup_{z \in C} \hat{q}(z) < b < \hat{q}(\bar{z})$. Since $0 \in C$, $b > 0$. Since C is a cone, $\hat{q} \in C^*$. Without loss of generality, we can assume $\hat{q} \in \Delta$. (To see this, since $e \in intC$ and $\hat{q} \in C^*$ we have $\hat{q}(e) < 0$ and may define $\tilde{q} = -\hat{q}/\hat{q}(e) \in \Delta$ with the same separation property above.) Since $q(\bar{z}) \leq 0$ for all $q \in \Delta$, therefore $\hat{q}(\bar{z}) > 0$ yields a contradiction. Hence, $\bar{z} \in C$. ∎

Remark 8. Alternative proofs based on a generalized game, or a mapping used by McCabe (1981), Geistdoerfer-Florenzano (1982), Yannelis (1985), and Mehta and Tarafdar (1987) can be provided analogous to the finite-dimensional version of the GNKD lemma.

Next, we introduced the MIP for infinite dimensional spaces that allows a weakening of the continuity and convexity assumptions in the GNKD lemma.

Definition 5. Let Δ, X, C be sets and $\psi, Z : \Delta \twoheadrightarrow X$ be correspondences defined as above. ψ satisfies the **Majorization-Inclusion Property (MIP)** if for all $p \in E^c_\psi$ there exist an open set $U(p)$ containing p and a correspondence $F^p : U(p) \to X$ such that at least one of the following holds:

(a) F^p has open values and open lower sections, and for all $q \in U(p) \cap E^c_\psi$, $Z(q) \subseteq coF^p(q) \subseteq G(q)$ and $coF^p(q) \cap C = \varnothing$,

(b) coF^p is UHC with compact values and for all $q \in U(p) \cap E^c_\psi$, $F^p(q) \neq \varnothing$, $F^p(q) \subseteq Z(q)$, and $coZ(q) \cap C = \varnothing$

Theorem 4. *Suppose ψ has the MIP and non-empty values such that for all $p \in \Delta$, there exists $z \in \psi(p)$ such that $p(z) \leq 0$. Then, there exists $\bar{p} \in \Delta$ such that $\psi(\bar{p}) \cap C \neq \emptyset$.*

Proof of Theorem 4. Step 1. Assume towards a contradiction that $\psi(p) \cap C = \emptyset$ for all $p \in \Delta$. Hence, $E^c_\psi = \Delta$. It follows from Walras' law and ψ has non-empty values that Z has non-empty values. Since $E^c_\psi = \Delta$, for all $p \in \Delta$, $Z(p) \subseteq \psi(p)$ implies that $Z(p) \cap C = \varnothing$.

Step 2. We show that there exists an UHC correspondence $F : \Delta \twoheadrightarrow X$ with non-empty, convex, compact values such that for all $p \in \Delta$, $F(p) \cap C = \varnothing$ and $F(p) \cap G(p) \neq \varnothing$. Setting $\Delta = X$, $X = Y$, $Z = Q$ and $intG = T$ in Lemma 1 implies that there exist finitely many points $q_1, \ldots, q_m$ in Δ and an UHC correspondence $F : \Delta \twoheadrightarrow X$ with non-empty, convex, compact values such that for all $p \in \Delta$, $F(p) \subseteq coZ(p)$ or $co(F(p) \cup Z(p)) \subseteq coF^{q_k}(p) \cap G(p)$ for some q_k with $p \in U(q_k)$. Pick $p \in \Delta$. If $F(p) \subseteq coZ(p)$, then $F(p) \subseteq coZ(p) \subseteq G(p)$. Therefore, $F(p) \cap G(p) = F(p) \neq \varnothing$. By parts (a) and (b) of Definition 5, $coZ(p) \cap C = \varnothing$, hence $F(p) \cap C = \varnothing$. If $co(F(p) \cup Z(p)) \subseteq coF^{q_k}(p) \cap G(p)$ for some q_k with $p \in U(q_k)$, then by part (a) of Definition 5, $F^{q_k}(p) \cap C = \varnothing$, hence $F(p) \cap C = \varnothing$. Moreover, $F(p) \subseteq coF^{q_k}(p) \cap G(p)$ implies $F(p) \subseteq G(p)$.

Step 3. By step 2, there exists an UHC correspondence $F : \Delta \twoheadrightarrow X$ with non-empty, convex, compact values such that for all $p \in \Delta$, $F(p) \cap C = \varnothing$. Note that C is non-empty, closed, and for all $p \in \Delta$, $F(p) \cap C = \varnothing$ and $F(p)$ is non-empty, compact and convex. Therefore, for all $p \in \Delta$, by Aliprantis and Border (2006, Theorem 5.79), there exists a nonzero continuous linear functional strongly separating C and $F(p)$. That is, for all $p \in \Delta$, there exist $q \in X^*$, $q \neq 0$, and $b \in \mathbb{R}$ such that

$$\sup_{z \in C} q(z) < b < \inf_{x \in F(p)} q(x).$$

Since $0 \in C$, $b > 0$. Since C is a cone, $q \in C^*$. Without loss of generality, we can assume $q \in \Delta$. (To see this, since $e \in intC$ and $q \in C^*$ we have $q(e) < 0$ and may define $\hat{q} = -q/q(e) \in \Delta$ with the same separation property above.)

Therefore, for all $p \in \Delta$, there exists $q \in \Delta$ such that $q(x) > 0$ for all $x \in F(p)$.

Step 4. It follows from step 2 that F is UHC and has non-empty, convex and compact values and for all $p \in \Delta$, $F(p) \cap G(p) \neq \varnothing$, hence there exists $z \in F(p)$ such that $p(z) \leq 0$. Therefore, Theorem 3 implies that there exists $\bar{p} \in \Delta$ and $\bar{z} \in X$ such that $\bar{z} \in F(\bar{p}) \cap C$. Since for all $q \in \Delta$, $q(x) \leq 0$ for all $x \in C$. Therefore, $\bar{z} \in C$ implies that for all $q \in \Delta$, $q(\bar{z}) \leq 0$. By step 3, there exists $q \in \Delta$ such that for all $z \in F(\bar{p})$, $q(z) > 0$, hence $q(\bar{z}) > 0$. This yields a contradiction. ■

6 Remarks on Some Further Directions

In the introduction, we already raised the question of easily-verifiable conditions on the preferences of the agents that yield excess demand correspondences with the MIP property identified and relied on in this work.

In this connection, it is important to understand that the argumentation, the specific method-of-proof, above does not work when the open sections assumption is replaced with assumption of open fibers in the definition of MIP. In fact, we do not know if a generalization of the Theorem 2 is true under this weaker assumption. This is the reason why we introduce various $\mathcal{L}$-majorization concepts for an excess demand correspondence that give rise to different MIP definitions.

We start with the standard version:

Definition 6. Let $\psi, Z : \Delta \twoheadrightarrow \mathbb{R}^\ell$ be correspondences defined as above. Then ψ is **$\mathcal{L}$-majorized** if for all $p \in E^c_\psi$, there exists an open set $U(p)$ containing p and a correspondence $F^p : U(p) \to \mathbb{R}^\ell$ with convex values such that F^p has open fibers, and for all $q \in U(p) \cap E^c_\psi$, $Z(q) \subseteq F^p(q)$ and $F^p(q) \cap -\Omega = \varnothing$.

Definition 6′. Let $\psi, Z : \Delta \twoheadrightarrow \mathbb{R}^\ell$ be correspondences defined as above. Then ψ is **$\mathcal{L}^0$-majorized** if there exists a function $g : \Delta \to \mathbb{R}^\ell$, and for all $p \in E^c_\psi$, an open set $U(p)$ containing p and a correspondence $F^p : U(p) \to \mathbb{R}^\ell$ with convex values such that $g(p) \in intG(p)$ and F^p has open fibers, and for all $q \in U(p) \cap E^c_\psi$, $g(q) \subseteq F^p(q)$ and $F^p(q) \cap -\Omega = \varnothing$.

Definition 6″. Let $\psi, Z : \Delta \twoheadrightarrow \mathbb{R}^\ell$ be correspondences defined as above. Then ψ is **$\mathcal{L}^1$-majorized** if for all $p \in E^c_\psi$, there exists an open set $U(p)$ containing p and a correspondence $F^p : U(p) \to \mathbb{R}^\ell$ with convex values such that $F^p \cap G$ has open fibers, and for all $q \in U(p) \cap E^c_\psi$, $Z(q) \subseteq F^p(q)$ and $F^p(q) \cap -\Omega = \varnothing$.

Definition 6 is a natural weakening of KF-majorization as it replaces the open graph assumption with the open fibers property. However, as mentioned above, the proof above does not work for this case as we cannot obtain a "nice" selection H from the global correspondence F in Step 3 that satisfies $H(q) \cap G(q) \neq \varnothing$ for all $q \in \Delta$ by using the step in the proof above. Definitions $6'$ and $6''$ provide two alternative versions that allow using $\mathcal{L}$-majorization. However, these definitions are not weaker than Definition 6 (we know the latter is not weaker and we do not have a proof that the former is weaker). Definition $6'$ allows us to use a proof that is almost identical to the one above. Definition $6''$ is the assumption we used previously and it is sufficient.

[We know that KF-maj does not imply CIP (see the example above), but we do not know if CIP is stronger than KF-maj. Moreover, we know that KF-maj implies L-maj, but in this context, the converse relation may also hold (hence they may be equivalent), but we do not know.]

[We have an example that satisfies $\mathcal{L}$-majorization and all the assumptions in Theorem 2 above, but the majorizing correspondence does not have an open graph. The conclusion of the GNKD lemma is true. In this example, the majorizing correspondence has a nice local open covering. This may be true in general. My **conjecture** is the following: a correspondence ψ as defined above (in the setting of excess demand correspondence) is $\mathcal{L}$-majorized if and only if it is KF-majorized. One direction is obvious, but I do not know whether the other direction is true or false.]

These are open questions. In addition, it is an open question to provide assumptions on preference relations that are sufficient for MIP. One reference dealing with non-convexity is Scapparone (2015).

Appendix A Omitted Proofs

We first provide the proof of Proposition 1 showing that $\mathcal{U}$-majorizaiton is stronger than KF-majorization.[29]

Proof of Proposition 1. Suppose that $\psi : \Delta \to \mathbb{R}^\ell$ is $\mathcal{U}$-majorized at $p \in \Delta$. Then, there exist an open set $V(p)$ containing p and a correspondence $H^p : V(p) \to \mathbb{R}^\ell$ with convex values such that H^p is UHC with compact values and for all $q \in V(p) \cap E_\psi^c$, $Z(q) \subseteq H^p(q)$ and $H^p(q) \cap -\Omega = \varnothing$.

Pick $p \in E_\psi^c$. Next, we construct an open set $U(p)$ containing p and a correspondence $F^p : U(p) \twoheadrightarrow \mathbb{R}^\ell$ with convex values such that F^p has open sections and for all $q \in U(p) \cap E_\psi^c$, $Z(q) \subseteq F^p(q)$ and $F^p(q) \cap -\Omega = \varnothing$.

Note that $H^p(p)$ is convex and compact, and $-\Omega$ is convex and closed. Using the proof of Theorem 9.2 of Schaefer (1971, p.65), we can find an open set V' containing Δ and a convex open set W containing $H^p(p)$ such that $V' \cap W = \varnothing$. Since $H^p : V(p) \to X$ is UHC, there exists an open set $V''(p) \subseteq V(p)$ containing p such that $H^p(q) \subseteq W$ for each $q \in V''(p)$. Moreover, for all $q \in V''(p)$, $V' \cap H^p(q) \subseteq V' \cap W = \varnothing$.

[29] The argument in the proof is motivated by those in Chang (2006, Lemma 2). Note that, as illustrated in this paper, the same argument applies also to infinite dimensional spaces.

Define $U(p) = V''(p)$ and $F^p(q) = W$ for each $q \in U(p)$. We claim that $F^p : U(p) \to X$ satisfies the conditions above.

Suppose that $q \in U(p) \cap E^c_\psi$. Then $q \in V(p) \cap E^c_\psi$. Next, $Z(q) \subseteq H^p(q) \subseteq W$ implies $Z(q) \subseteq W = F^p(q)$. Next, note that $V' \cap W = \varnothing$ so $F^p(q) \cap -\Omega = \varnothing$.

It is clear that F^p has open values. It remains to show that it has open lower sections. Note that $F^p(q) = W$ for every $q \in U(p)$. If $z \notin W$, then $(F^p)^{-1}(z) = \varnothing$, hence open. If $z \in W$, then $(F^p)^{-1}(z) = U(p)$, which is open in the subspace $U(p)$. Therefore, F^p has open lower sections. ■

Lemma 1 provide a result that yield an UHC correspondence that satisfies a majorization or an inclusion property. We provide the proof of the lemma in this Appendix.

Proof of Lemma 1. For all $x \in X$, there exists an open set $U(x)$ containing x and a correspondence $F^x : U(x) \to Y$ with non-empty values such that at least one of the following holds:

- F^x has open sections and for each $z \in U(x)$, $Q(x) \subseteq coF^x(z)$,
- coF^x is UHC and has non-empty, compact values, and for each $z \in U(x)$, $F^x(z) \subseteq Q(z)$.

Step 1. Since every TVS is regular, we can apply Willard (1970, Theorem 14.3, p.92) and conclude that, for each $x \in X$, there exists an open set $A(x)$ and a closed set $V(x)$ such that $x \in A(x) \subseteq V(x) \subseteq U(x)$. Since X is compact, there exist points $x^1, .., x^m \in X$, collections of sets $A^1, .., A^m$, $V^1, .., V^m$, and $U^1, .., U^m$ with $x^k \in A^k \subseteq V^k \subseteq U^k$, A^k and U^k are open and V^k for each $k \in M = \{1, .., m\}$, and $X = \cup_{k=1}^m A^k$, and for each $k \in M$ there exists a correspondence $F^k : U^k \to Y$ with non-empty values such that at least one of the following holds:

(a) F^k has open sections and for each $x \in U^k$, $Q(x) \subseteq coF^k(x)$,

(b) coF^k is UHC and has non-empty, compact values, and for each $x \in U^k$, $F^k(x) \subseteq Q(x)$.

Let M^a denote those k for which (a) above holds and let $M^b = \{1, .., m\} \backslash M^a$. Define $U^a = \bigcup_{k \in M^a} U^k$, $V^a = \bigcup_{k \in M^a} V^k$, $A^a = \bigcup_{k \in M^a} A^k$, $A^b = \bigcup_{k \in M^b} A^k$.

Step 2. If $M^a \neq \varnothing$, then $U^a \neq \varnothing$ and for each $k \in M^a$, define a correspondence $\tilde{F}^k : V^a \twoheadrightarrow Y$ as

$$\tilde{F}^k(z) = \begin{cases} coF^k(z) & \text{if } z \in V^k \\ Y & \text{otherwise.} \end{cases}$$

It is clear that $\tilde{F}^k$ has convex values. Since F^k has non-empty values and Y is non-empty, $\tilde{F}^k$ has non-empty values. Next, we show that $\tilde{F}^k$ has open sections. Since the convex hull of an open set is open and F^k has open upper sections, $\tilde{F}^k$ has open upper sections. It remains to show that $\tilde{F}^k$ has open fibers. Towards this end, pick $y \in Y$. $\left(\tilde{F}^k\right)^{-1}(y) = \left(\left(coF^k\right)^{-1}(y) \cup \left(V^k\right)^c\right) \cap V^a$. By Yannelis and Prabhakar (1983, Lemma 5.1), F^k has open fibers implies that coF^k has open fibers.

Hence, $\left(coF^k\right)^{-1}(y)$ is open (in X). Since V^k is closed, its complement is open (in X). Therefore, $\left(\tilde{F}^k\right)^{-1}(y)$ is open in V^a, hence $\tilde{F}^k$ has open fibers.

Define a correspondence $\tilde{F}^a : V^a \to Y$ as

$$\tilde{F}^a(z) = \bigcap_{k \in M^a} \tilde{F}^k(z).$$

It is clear that $\tilde{F}^a$ has convex values, and by Yannelis and Prabhakar (1983, Fact 6.1), it has open fibers. Since the intersection of a finite number of open sets is open, $\tilde{F}^a$ has open upper sections. Next, we show that $\tilde{F}^a$ has non-empty values. To see this, pick $z \in V^a \subseteq U^a$. Then, $z \in V^k$ for some $k \in M^a$. Since $Q(z) \neq \varnothing$, and $Q(z) \subseteq coF^k(z) \subseteq \tilde{F}^k(z)$, therefore,

$$Q(z) \subseteq \bigcap_{k \in M^a} \tilde{F}^k(z) = \tilde{F}^a(z),$$

and we conclude that $\tilde{F}^a(z) \neq \varnothing$.

Define a correspondence $H^a : V^a \to Y$ as

$$H^a(z) = \tilde{F}^a(z) \cap T(z).$$

Next, we show that H^a has open fibers and non-empty and convex values. As the intersection of convex sets is convex, H^a has convex values. By Yannelis and Prabhakar (1983, Fact 6.1), H^a has open fibers. Pick $z \in V^a$. Since $Q(z) \subseteq clT(z)$ and $Q(z) \subseteq \tilde{F}^a(z)$, $Q(z) \subseteq \tilde{F}^a(z) \cap clT(z)$. Pick $y \in Q(z)$. It follows from $\tilde{F}^a$ has open upper sections that there exists an open set $U(y)$ containing y such that $U(y) \subseteq \tilde{F}^a(z)$. Since $y \in clT(z)$, there exists $y' \in U(y)$ such that $y' \in T(z) = intT(z)$, where the equality follows from the assumption that T has open upper sections. Hence, $y' \in \tilde{F}^a(z) \cap T(z) = H(z)$.

Since V^a is a closed subset of a compact set X, it is compact, and hence it is paracompact. By Yannelis and Prabhakar (1983, Theorem 3.1), there exists a continuous function $h^a : V^a \twoheadrightarrow Y$ such that $h^a(z) \in H^a(z)$ for all $z \in V^a$. Since V^a is compact, the set $Y^a = h^a(V^a) \subseteq Y$ is compact. Therefore, by Aliprantis and Border (2006, Lemma 17.6 and Theorem 17.11), the singleton-valued correspondence $F^a : V^a \twoheadrightarrow Y$ defined as $F^a(z) = \{h^a(z)\}$ is UHC, has a closed graph, and non-empty, convex, and compact values.

Step 3. We define a correspondence $F : X \to Y$ that is UHC and has nonempty, convex and compact values. Recall that $V^a = \cup_{k \in M^a} V^k$. Note that $X = \cup_{k \in \{a\} \cup M^b} V^k$ and define a correspondence

$F : X \to Y$ as

$$F(x) = co\left[\bigcup_{\substack{k\in\{\{a\}\cup M^b \\ :x\in V^k\}}} coF^k(x)\right]. \tag{1}$$

Step 3.1. $F : X \to Y$ is convex valued and nonempty valued. To see the latter, suppose that $x \in V^a$. Then $V^a \subseteq U^a$ and Step 2 establishes that $F^a(x) \neq \varnothing$. If $x \in V^k$ for some $k \in M^b$, then $V^k \subseteq U^k$ implies that $coF^k(x) \neq \varnothing$.

Step 3.2. $F : X \to Y$ is UHC and compact valued. First, for all $k \in \{a\} \cup M^b$, define $\hat{F}^k : X \twoheadrightarrow Y$ as follows:

$$\hat{F}^k(x) = \begin{cases} coF^k(x) & \text{if } x \in V^k \\ \varnothing & \text{if } x \notin V^k \end{cases}, \text{ and therefore, } F(x) = co\left[\bigcup_{\substack{k\in\{\{a\}\cup M^b: \\ x\in V^k\}}} \hat{F}^k(x)\right].$$

Since $coF^k : U^k \to Y$ is UHC and has compact values, and $V^k \subseteq U^k$, $coF^k : V^k \to Y$ is also UHC and has compact values. Next, we show that for all $k \in \{a\} \cup M^b$, $\hat{F}^k$ is UHC and has compact values. If $x \notin V^k$, then the UHC at x follows from V^k is closed. Pick $x \in V^k$. Since coF^k is UHC at x, for all open neighborhood U of $\hat{F}^k(x) = coF^k(x)$, there exists an open neighborhood V of x such that for all $x' \in V$, $coF^k(x') \subseteq U$. Note that $\hat{F}^k(x') \subseteq coF^k(x')$ for all $x' \in X$. Therefore, for all $x' \in V$, $\hat{F}^k(x') \subseteq U$. Hence, $\hat{F}^k$ is UHC at x. Moreover, for all $x' \in X$, either $\hat{F}^k(x') = coF^k(x')$ or $\hat{F}^k(x') = \varnothing$. Hence, $\hat{F}^k$ has compact values.

Define a correspondence $H : X \twoheadrightarrow Y$ as

$$H(x) = \bigcup_{\substack{k\in\{a\}\cup M^b \\ :x\in V^k}} \hat{F}^k(x).$$

Since for all $x \in X$, $\hat{F}^k$ is UHC and the union of a finite family of UHC correspondences is UHC (Aliprantis and Border, 2006, Theorem 17.27(2)), H is UHC. Since a finite union of compact sets is compact, H has compact values (Aliprantis and Border, 2006, p. 40). Since $H(x)$ is the union of finitely many compact, convex sets in a TVS, it follows that $F(x) = coH(x)$ is compact for each $x \in X$ (Aliprantis and Border (2006, Lemma 5.29)). Since Y is locally convex, H is UHC, coH has closed and compact values, $F = coH$ is also UHC (Aliprantis and Border (2006, Theorem 17.35)).

Step 4. The correspondence $F : X \twoheadrightarrow Y$ defined in (1) of step 3 is nonempty valued, convex valued, compact valued and UHC. To complete the proof of the theorem, we will show that for each $x \in X$, $F(x) \subseteq coQ(x)$ or $co(F(x) \cup coQ(x)) \subseteq coF^k(x)$ for some $k \in \{1, .., m\}$ with $x \in U^k$.

Pick $x \in X$. It follows form $X = \bigcup_{k=1}^{m} A^k = A^a \cup A^b$ that $x \in A^a \cup A^b$.

Step 4.1: Suppose that $x \in A^b$, hence $x \notin A^a$ (otherwise, $x \in V^k \subseteq U^k$ for some $k \in M^a$). Then $x \in A^{\hat{k}} \subseteq V^{\hat{k}}$ for some $\hat{k} \in M^b$. Applying condition (b) above, it follows that for all $j \in M^b$ and all $z \in V^j \subseteq U^j$, $F^j(z) \subseteq Q(z)$ implying that $coF^j(z) \subseteq coQ(z)$. In particular, $coF^k(x) \subseteq coQ(x)$ for each $j \in M^b$. Therefore,

$$F(x) = co\left[\bigcup_{\substack{j\in\{a\}\cup M^b\\ :x\in V^j}} coF^j(x)\right] = co\left[\bigcup_{\substack{j\in M^b\\ :x\in V^j}} coF^j(x)\right] \subseteq coQ(x). \tag{2}$$

Step 3.2: Suppose that $x \in A^a$. Then there exists $\hat{k} \in M^a$ such that $x \in A^{\hat{k}} \subseteq V^{\hat{k}} \subseteq U^{\hat{k}}$. Applying condition (a) above, the following holds: if $j \in M^a$ and $x \in V^j$, then $x \in U^j$ and $Q(x) \subseteq coF^j(x)$. In particular, $\mathrm{co}Q(x) \subseteq coF^j(x)$ if $j \in M^a$ and $x \in U^j$. Therefore, $x \in A^{\hat{k}} \subseteq V^{\hat{k}} \subseteq U^{\hat{k}}$ and $\hat{k} \in M^a$ imply that implying that

$$F^a(x) \subseteq \hat{F}^a(x) \cap T(x) = \left(\bigcap_{k\in M^a} \hat{F}^k(x)\right) \cap T(x) \subseteq \hat{F}^{\hat{k}}(x) \cap T(x) = coF^{\hat{k}}(x) \cap T(x). \tag{3}$$

By assumption $Q(x) \subseteq clT(x)$, where $clT(x)$ is convex. Recall that for all $k \in M^a$, $Q(x) \subseteq coF^k(x) = \hat{F}^k(x)$. Therefore, $coQ(x) \subseteq coF^k(x) \cap clT(x) = \hat{F}^k(x) \cap clT(x)$. Therefore, by Equation (3),

$$coQ(x) \subseteq co(F^a(x) \cup coQ(x)) \subseteq \hat{F}^a(x) \cap clT(x) \subseteq coF^{\hat{k}}(x) \cap clT(x).$$

Since

$$\bigcup_{\substack{k\in M^b\\ :x\in B^k}} coF^k(x) \subseteq coQ(x),$$

$$F(x) = co\left(coF^a(x) \cup \bigcup_{\substack{k\in M^b\\ :x\in B^k}} \left(coF^k(x)\right)\right) \subseteq co(F(x) \cup coQ(x)) \subseteq \hat{F}^a(x) \cap clT(x) \subseteq coF^{\hat{k}}(x) \cap clT(x).$$

■

7 Bibliography

Aliprantis, C. D., and K. C. Border (2006): *Infinite Dimensional Analysis: A Hitchhikers Guide*. Berlin: Springer-Verlag.

Arrow, K. J., and G. Debreu (1954): "Existence of an equilibrium for a competitive economy," *Econometrica*, 22(3), 265–290.

Arrow, K. J., and F. H. Hahn (1971): *General Competitive Analysis*. Amsterdam: North-Holland.

Benetti, C., A. Nadal, and C. Salas (2004): "The law of supply and demand in the proof of existence of general competitive equilibrium," in *The Flawed Foundations of General Equilibrium Theory*, pp. 84–101. London: Routledge.

Bergstrom, T. C. (1992): "When non-transitive relations take maxima and competitive equilibrium can't be beat," in *Economic Theory and International Trade: Essays in memoriam J. Trout Rader*, pp. 29–52. Springer.

Bergstrom, T. C., R. P. Parks, and T. Rader (1976): "Preferences which have open graphs," *Journal of Mathematical Economics*, 3(3), 265–268.

Bidard, C., and E. Hosoda (1987): "On consumption baskets in a generalized von Neumann model," *International Economic Review*, pp. 509–519.

Borglin, A., and H. Keiding (1976): "Existence of equilibrium actions and of equilibrium: A note on the 'new' existence theorems," *Journal of Mathematical Economics*, 3(3), 313–316.

Browder, F. E. (1968): "The fixed point theory of multi-valued mappings in topological vector spaces," *Mathematische Annalen*, 177(4), 283–301.

Chang, S.-Y. (2006): "Noncompact qualitative games with application to equilibria," *Nonlinear Analysis: Theory, Methods & Applications*, 65(3), 593–600.

Cornet, B. (1975): "Fixed point and surjectivity theorems for correspondences; applications," *Ronéotypé Université de Paris-Dauphine*.

Cornet, B. (2020): "The Gale–Nikaido–Debreu lemma with discontinuous excess demand," *Economic Theory Bulletin*, 8, 169–180.

Debreu, G. (1951): "The coefficient of resource utilization," *Econometrica*, pp. 273–292.

——— (1952): "A social equilibrium existence theorem," *Proceedings of the National Academy of Sciences*, 38(10), 886–893.

——— (1956): "Market equilibrium," *Proceedings of the National Academy of Sciences*, 42(11), 876–878.

——— (1959): *Theory of Value: An Axiomatic Analysis of Economic Equilibrium*, no. 17. Connecticut: Yale University Press.

——— (1962): "New concepts and techniques for equilibrium analysis," *International Economic Review*, 3(3), 257–273.

——— (1974): "Excess demand functions," *Journal of Mathematical Economics*, 1(1), 15–21.

——— (1982): "Existence of competitive equilibrium," in *Handbook of Mathematical Economics*, ed. by K. J. Arrow, and M. D. Intriligator, vol. 2, pp. 697–743. Elsevier.

Ding, X. P. (1998): "Equilibria of noncompact generalized games with $\mathcal{U}$-majorized preference correspondences," *Applied Mathematics Letters*, 11(5), 115–119.

Dubra, J. (2005): "Interview with Kenneth Arrow," *MPRA Paper No. 967, posted online at http://mpra.ub.uni-muenchen.de/967/07. November 2007*, 1, 1–24.

Florenzano, M. (1983): "On the existence of equilibria in economies with an infinite dimensional commodity space," *Journal of Mathematical Economics*, 12(3), 207–219.

——— (1987): "On an extension of the Gale-Nikaido-Debreu lemma," *Economics Letters*, 25, 51–53.

——— (1991): "Markets with countably many periods," *Economics Letters*, 36(3), 233–238.

——— (2009): "Two lemmas that changed general equilibrium theory," *Games and Economic Behavior*, 2(66), 603–605.

——— (2013): *General Equilibrium Analysis: Existence and Optimality Properties of Equilibria*. Boston: Kluwer Academic Publishers.

Florenzano, M., and C. Le Van (1986): "A note on the Gale-Nikaido-Debreu lemma and the existence of general equilibrium," *Economics Letters*, 22(2-3), 107–110.

Florenzano, M., and E. Moreno (2001): "Linear exchange economies with a continuum of agents," *Spanish Economic Review*, 3, 253–272.

Gale, D. (1955): "The law of supply and demand," *Mathematica Scandinavica*, 3(1), 155–169.

——— (1963): "A note on global instability of competitive equilibrium," *Naval Research Logistics Quarterly*, 10, 81–87.

Gale, D., and A. Mas-Colell (1975): "An equilibrium existence theorem for a general model without ordered preferences," *Journal of Mathematical Economics*, 2(1), 9–15.

Geistdoerfer-Florenzano, M. (1982): "The Gale-Nikaido-Debreu lemma and the existence of transitive equilibrium with or without the free-disposal assumption," *Journal of Mathematical Economics*, 9(1-2), 113–134.

Ghosh, A., M. A. Khan, and M. Uyanik (2023): "Separately convex and separately continuous preferences: On results of Schmeidler, Shafer, and Bergstrom-Parks-Rader," *arXiv: 2310.00531.*

Gourdel, P., C. Le Van, N.-S. Pham, and C. T. Viet (2023): "Hartman-Stampacchia theorem, Gale-Nikaido-Debreu lemma, and Brouwer and Kakutani fixed-point theorems," *Working Paper*.

Grandmont, J. M. (1977): "Temporary general equilibrium theory," *Econometrica*, pp. 535–572.

He, W., and N. C. Yannelis (2017): "Equilibria with discontinuous preferences: New fixed point theorems," *Journal of Mathematical Analysis and Applications*, 450(2), 1421–1433.

Hervés-Beloso, C., and E. Moreno (2009a): "Large economies and two-player games," *Journal of Mathematical Economics*, 45, 603–608.

——— (2009b): "Walrasian analysis via two-player games," *Games and Economic Behavior*, 65, 220–233.

Hildenbrand, W., and A. Kirman (1976): *Introduction to equilibrium analysis: variations on themes by Edgeworth and Walras*. New York: Elsevier.

Khan, M. A. (1993): "Lionel McKenzie on the existence of competitive equilibrium," in *General Equilibrium, Growth, and Trade*, ed. by R. Becker, M. Boldrin, R. Jones, and W. Thomson, pp. 3–50. New York: Academic Press.

——— (2020): "On the Finding of an Equilibrium: Düppe-Weintraub and the Problem of Scientific Credit," *Journal of Economic Literature*, online(1), 1–50.

Khan, M. A., R. P. McLean, and M. Uyanik (2023): "Majorization and inclusion in equilibrium theory: On generalized games and quasi-variational inequalities," *working paper*.

——— (2024a): "On constrained generalized games with action sets in non-locally-convex and non-Hausdorff topological vector spaces," *Journal of Mathematical Economics* (forthcoming).

——— (2024b): "On equilibria in constrained generalized games with the weak continuous inclusion property," *Journal of Mathematical Analysis and Applications* (forthcoming).

Khan, M. A., and M. Uyanik (2021): "The Yannelis-Prabhakar theorem on upper semi-continuous selections in paracompact spaces: extensions and applications," *Economic Theory*, 71(3), 799–840.

Kubota, H. (2007): "Gale-Nikaido's Lemma in Infinite Dimensional Spaces: Early Attempts by Prof. Nikaido," *Economic Journal of Hokkaido University*, 36, 1–22.

Kuhn, H. (1956): "A note on "The law of supply and demand"," *Mathematica Scandinavica*, 4, 143–146.

Le, T., C. Le Van, N.-S. Pham, and C. Saglam (2022): "A direct proof of the Gale–Nikaido–Debreu lemma using Sperner's lemma," *Journal of Optimization Theory and Applications*, 194(3), 1072–1080.

Mas-Colell, A. (1975): "A model of equilibrium with differentiated commodities," *Journal of Mathematical Economics*, 2(2), 263–295.

Maskin, E. S. (2019): "The Economics of Kenneth J. Arrow: A Selective Review," *Annual Review of Economics*, 11(1–26), 267–300.

Maskin, E. S., and K. W. Roberts (2008): "On the fundamental theorems of general equilibrium," *Economic Theory*, 35(2), 233–240.

McCabe, P. J. (1981): "On two market equilibrium theorems," *Journal of Mathematical Economics*, 8(2), 167–171.

McKenzie, L. W. (1954): "On equilibrium in Graham's model of world trade and other competitive systems," *Econometrica*, pp. 147–161.

Mehta, G., and E. Tarafdar (1987): "Infinite-dimensional Gale-Nikaido-Debreu theorem and a fixed-point theorem of Tarafdar," *Journal of Economic Theory*, 41(2), 333–339.

Moore, J. C. (1975): "The existence of 'compensated equilibrium' and the structure of the Pareto efficiency frontier," *International Economic Review*, 16, 267–300.

——— (2007): *General Equilibrium and Welfare Economics: An Introduction*. Heidleberg: Springer.

Musatov, D., A. Savvateev, and S. Weber (2016): "Gale–Nikaido–Debreu and Milgrom–Shannon: Communal interactions with endogenous community structures," *Journal of Economic Theory*, 166, 282–303.

Negishi, T. (1960): "Welfare economics and existence of an equilibrium for a competitive economy," *Metroeconomica*, 12(2-3), 92–97.

Neuefeind, W. (1980): "Notes on existence of equilibrium proofs and the boundary behavior of supply," *Econometrica*, pp. 1831–1837.

Nikaido, H. (1956): "On the classical multilateral exchange problem," *Metroeconomica*, 8(2), 135–145.

Nikaidô, H., and H. Uzawa (1960): "Stability and non-negativity in a Walrasian Tatonnement process," *International Economic Review*, 1(1), 50–59.

Nishimura, K. (1978): "A further remark on the number of equilibria of an economy," *International Economic Review*, pp. 679–685.

Park, S. (2010): "Further generalizations of the Gale–Nikaido–Debreu theorem," *Journal of Applied Mathematics and Computing*, 32, 171–176.

Podczeck, K. (1997): "Markets with infinitely many commodities and a continuum of agents with non-convex preferences," *Economic Theory*, 9, 385–426.

Podczeck, K., and N. C. Yannelis (2022): "Existence of Walrasian equilibria with discontinuous, non-ordered, interdependent and price-dependent preferences, without free disposal, and without compact consumption sets," *Economic Theory*, 73, 413–420.

Prokopovych, P. (2013): "The single deviation property in games with discontinuous payoffs," *Economic Theory*, 53(2), 383–402.

——— (2016): "Majorized correspondences and equilibrium existence in discontinuous games," *Economic Theory*, 61(3), 541–552.

Reny, P. J. (2016): "Nash equilibrium in discontinuous games," *Economic Theory*, 61(3), 553–569.

Scapparone, P. (2015): "Existence of an upper hemi-continuous and convex-valued demand sub-correspondence," *Mathematical Social Sciences*, 75, 123–129.

Scarf, H. (1960): "Some examples of global instability of the competitive equilibrium," *International Economic Review*, 1(3), 157–172.

Schaefer, H. H. (1971): *Locally Convex Topological Vector Spaces*. New York: Springer.

Shafer, W. (1974): "The nontransitive consumer," *Econometrica*, 42(5), 913–919.

Shafer, W., and H. Sonnenschein (1975a): "Equilibrium in abstract economies without ordered preferences," *Journal of Mathematical Economics*, 2(3), 345–348.

——— (1975b): "Some theorems on the existence of competitive equilibrium," *Journal of Economic Theory*, 11(1), 83–93.

——— (1976): "Equilibrium with externalities, commodity taxation, and lump sum transfers," *International Economic Review*, pp. 601–611.

Sonnenschein, H. (1975): "Some recent results on the existence of equilibrium in finite purely competitive economies," *Northwestern Discussion Paper No. 167*, pp. 1–10.

——— (1977): "Some recent results on the existence of equilibrium in finite purely competitive economies," in *Frontiers of Quantitative Economics, IIIA*, ed. by M. Intriligator, pp. 103–109. New York: Elsevier.

Tan, K.-K., and J. Yu (1994): "Minimax inequalities and generalisations of the Gale-Nikaido-Debreu Lemma," *Bulletin of the Australian Mathematical Society*, 49(2), 267–275.

Tan, K.-K., and X.-Z. Yuan (1993): "Equilibria of generalized games with $\mathcal{U}$-majorized preference correspondences," *Economics Letters*, 41(4), 379–383.

Tian, G. (2016): "On the existence of price equilibrium in economies with excess demand functions," *Economic Theory Bulletin*, 4, 5–16.

Toussaint, S. (1984): "On the existence of equilibria in economies with infinitely many commodities and without ordered preferences," *Journal of Economic Theory*, 33(1), 98–115.

Uyanik, M. (2016): "Existence and computation of equilibria in games and economies," Ph.D. thesis, Johns Hopkins University.

Velupillai, K. V. (2019): "Kenneth Joseph Arrow: 23 August 1921 – 21 February 2017," *Biographical Memoirs of Fellows of the Royal Society*, 67, 9–28.

Willard, S. (1970): *General Topology*. New York: Addison Wesley Publishing Co.

Yannelis, N. C. (1985): "On a market equilibrium theorem with an infinite number of commodities," *Journal of Mathematical Analysis and Applications*, 108(2), 595–599.

Yannelis, N. C., and N. D. Prabhakar (1983): "Existence of maximal elements and equilibria in linear topological spaces," *Journal of Mathematical Economics*, 12(3), 233–245.