Identification and Estimation of Nonstationary Dynamic Binary Choice Models

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Abstract

In a dynamic binary choice model that allows for general forms of nonstationarity, we transform the identification of the flow utility parameters into the solution of a (linear) system of equations. The identification of the parameters, therefore, follows the usual argument for linear GMM. In particular, we show that the state transition distribution is not essential for the identification and estimation of the parameters. We propose a three-step conditional-choice-probability-based semiparametric estimator that bypasses estimation of and simulating from the state transition distribution. Simulation experiments show that our estimator gives comparable or better estimates than a competitor estimator, yet it requires fewer assumptions in certain scenarios, is substantially easier to implement, and is computationally much less demanding. The asymptotic distribution of the estimator is provided, and the sensitivity of the estimator to a key assumption is also examined.

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1 Introduction

Structural dynamic discrete choice (DDC) models are a powerful framework for the economic analysis of inter-temporal choices of agents across a wide range of empirical studies. Although the conditional choice probability (CCP) estimators, proposed by Hotz and Miller (1993); Hotz et al. (1994) and further generalized and improved by many authors (e.g., Arcidiacono and Miller, 2011), simplify the estimation of DDC models by avoiding solving the full dynamic programming problem as in Rust (1987), the estimation of DDC models remains a daunting task. Among other complications CCP estimators require the estimation of the state transition distribution as input for a nonlinear generalized method of moments (GMM) estimation of the parameters in the flow utility function. Being a conditional distribution of all the state variables in the current period given all the state variables in the previous period, the state transition density is usually difficult to estimate nonparametrically, especially when there are many state variables and state variables are continuous. In consequence, researchers routinely use parametric specifications, but it is not always obvious what parametric specification is appropriate, and ad hoc choices do not always have sound economic justification. Moreover, the CCP estimators also require simulating from in order to integrate over the estimated state transition distribution. This can be very time-consuming as it puts enormous strain on computer memory (regardless whether the state transition distribution is nonparametric or parametric) if the decision horizon is long.

The main practical contribution of this paper is to develop a novel CCP-based semiparametric estimator of flow utility parameters that bypasses state transition distribution estimation and simulation. Instead, our three-step estimator uses preliminary nonparametric conditional mean estimates to obtain a linear system. The conditional mean estimates have a smaller dimension and better finite sample properties than the state transition estimates, yet our estimator is very flexible to allow the model primitives (e.g., flow utility functions, state transition distribution) to be time-varying, and the decision horizon to be finite, infinite or unknown. Although our estimator requires individual-level panel data, the length of the panel can be short ($T \geq 2$), and the sample terminal period could be before the decision terminal period. In our simulation experiments in Section 5, our estimator is a thousand times faster than a CCP estimator with nonparametric estimation of the state transition distribution.

Even for counterfactual analysis, where state transition distribution estimation and simulation is necessary, our state-transition-free estimator can still be useful in guiding the comparison and selection among different parametric specifications of the state transition distributions.

Our estimator builds on a rigorous and constructive identification analysis, which we believe is an important theoretical contribution to the DDC literature. This is made possible by our finding that the log odds ratio equations, which capture agents' optimal decision rule and involve iterated conditional means given the state variables in all future periods, can be simplified to a system that contains conditional means given the state variables only in the current period. Such a simplification eliminates the need for recursive simulation from the state transition distribution as is typically required for CCP estimators (e.g., Hotz and Miller, 1993; Hotz et al., 1994; Arcidiacono and Miller, 2011). It is worth emphasizing that such a simplification is achieved not by imposing stronger assumptions. We find that common assumptions that are usually made in the DDC literature (see, e.g., a summary by Aguirregabiria and Mira, 2010) implies a Markovian property for the observed state variables. That is, for the state variables in a future period, the current state variables are less informative than those in a future period that is closer to the target period. In consequence, the law of iterated expectations implies that the iterated conditional mean given the state variables in the current and the future periods is equal to the conditional mean given the current-period state variables only.

In our identification analysis, we start with this simplified system and transform it into a partially linear system under a linear flow utility assumption that is weaker than typical in the DDC literature. We avoid arbitrary "normalization" of the expected flow utility for one choice. (Norets and Tang, 2014; Aguirregabiria and Suzuki, 2014; Chou, 2016, demonstrate the bias induced by such a "normalization"). If one is willing to make an additional mild assumption about the sample-terminal-period integrated value function, then we show that the system can be further transformed into one that is fully linear in the parameters of interest, enabling clear and simple identification conditions as in linear GMM.[1] Sensitivity to this new assumption is also examined in the paper. We analyze various scenarios to underscore the versatility of our approach and note that excluded variables that do not affect the flow utility but affect future payoff, although not required for the identification, may help fulfilling the rank condition of the linear system in scenarios where they fail otherwise.

Based on the identification results, our CCP-based semiparametric estimator of the flow utility parameters proceeds in three steps. First, nonparametrically estimate the CCPs. Second, nonparametrically estimate conditional means, which have the CCPs as dependent variables. The third step is to plug the estimated CCPs and estimated conditional means into the linear system to estimate the parameters of interest via linear minimum distance (MD).

This paper is related to important earlier studies on the estimation of structural DDC models, especially the ones that leverage CCP estimates. Following Hotz and Miller (1993), various CCP estimators have been developed for a variety of settings. When the DDC model does not involve terminal or renewal choices, Hotz et al. (1994) develop a two-step CCP estimator. They first estimate CCPs and state transition distributions. They then estimate structural parameters by evaluating the choice-specific value functions using estimated CCPs and simulated future states

[1] Pesendorfer and Schmidt-Dengler (2008) point out the identification of dynamic discrete games in the infinite horizon stationary setting can be equivalently expressed as a linear GMM problem. We focus on single agent dynamic discrete choices, but in general nonstationary setting allowing for continuous state variables.

from estimated state transition distributions. Aguirregabiria and Mira (2002) find that the two-step nature of Hotz et al. (1994)'s CCP estimator can generate serious finite sample bias in the structural parameter estimates, and they reduce the bias by repeatedly updating the CCP estimates using the structural parameter estimates from the second step of Hotz et al. (1994)'s CCP estimator. Arcidiacono and Miller (2011) extend Hotz et al. (1994)'s CCP estimator by including a finite number of latent types. Leveraging the properties of infinite horizon stationary DDC models, Srisuma and Linton (2012) propose a simple CCP estimator when the state space includes continuous variables. Kalouptsidi, Scott and Souza-Rodrigues (2021) recently study the identification and estimation of DDC models with market-level observed and unobserved state variables that affect consumers' payoffs. Using individual-level panel data, they develop a new CCP estimator, of which the second step can be expressed as a linear regression. The key difference between our estimator and these CCP estimators is that we do not require estimating or simulating the state transition distributions.

This paper is also related to the growing literature on the identification of DDC models that follows the seminal work of Magnac and Thesmar (2002), which uses excluded variables in a limited setting.[2] Blevins (2014) shows how dynamic models with both discrete and continuous choice variables can be identified. Abbring and Daljord (2020) use excluded variables to identify the discount factor in the DDC models, which is usually assumed to be known in the literature. Arcidiacono and Miller (2020) explore the property of single action finite dependence, which is generalization of the terminal/renewal choices, to identify nonstationary DDC models when the decision horizon is beyond the data horizon ("short panel" data). Our paper establishes the identification of nonstationary dynamic binary choice models using short panel data in the absence of finite dependence, and excluded variables only play a non-essential, auxiliary role in our approach. To the best of our knowledge, no existing research shows the identification all structural primitives, including flow utility parameters and the discount factor, of a nonstationary DDC model using short panel data. Compared to the literature that investigates the identification of DDC models when there are serially correlated unobserved state variables (e.g. Kasahara and Shimotsu, 2009; Hu and Shum, 2012; Chou, Derdenger and Kumar, 2019; Kalouptsidi, Scott and Souza-Rodrigues, 2021), unobserved state variables being serially independent is one limitation of our paper.

Terminology and notation. In the rest of this paper, Hotz et al. (1994)'s two-step CCP estimator and its successors will be collectively referred to as "HM estimators" to avoid confusion, as our estimator also estimates CCPs in the first step but differs in how the CCP estimates are used in the estimation of the flow utility parameters. We use f as a generic symbol for (conditional) probability density/mass functions. The symbol "$\equiv$" means that the object on its left-hand side is

[2] They show that the exclusion restriction identifies the discount factor and the "difference between the expected values of two sequences of choices" (eq. (8) of their paper). But they did not show that the exclusion restriction can identify the flow utility functions themselves.

defined as the expression on its right-hand side.

Plan of paper. The rest of this paper proceeds as follows. Section 2 sets up a dynamic binary choice model that permits general nonstationarity and briefly reviews HM estimators. Section 3 transforms the identification problem into the solution of a linear system under common and mild new assumptions, analyzes the identification of the linear system, and provides a CCP-based semiparametric estimator of the model that bypasses the state transition distribution estimation. Section 4 discusses the bias induced by relaxing the new assumption made in our paper and how to reduce it. Section 5 compares the performance of our estimator and an HM estimator using simulated samples. We make concluding remarks in Section 6. All the proofs and certain related issues are in the appendices.

2 A Nonstationary Dynamic Binary Choice Model

In this section, we introduce a dynamic binary choice model that incorporates a general form of nonstationarity. Then, we briefly review the aspect of the HM estimators that we will simplify in Section 3.

2.1 Model

Each agent makes a binary choice $a_t \in \{0,1\}$ in each of a number of periods, denoted by $t \in \mathcal{T} \equiv \{T_{start}, T_{start}+1, \ldots, T_{end}\}$, where $T_{end} = \infty$ is allowed. Let $u_t(a_t, s_t) + \varepsilon_{a_t t}$ denote the flow utility that has an additively separable form, where the expected flow utility $u_t(a_t, s_t)$ depends on the choice a_t and s_t, a $d_s \times 1$ vector of observed state variables, through an unknown function u_t, and $\varepsilon_{a_t t}$ $(a_t = 0, 1)$ are unobserved (to researchers) scalar flow utility shocks. Let $\varepsilon_t = (\varepsilon_{0t}, \varepsilon_{1t})'$ denote the unobserved state variables and $\Omega_t = (s_t', \varepsilon_t')'$.

We maintain the following Assumptions 1 to 3 in this paper.

Assumption 1 (Controlled Markov process)**.** *For all t and $j \in \mathbb{N}^+$ such that t, $t+1$ and $t-j$ all belong to $\mathcal{T}$, assume that $\Omega_{t+1} \perp\!\!\!\perp (\Omega_{t-j}, a_{t-j}) \,|\, (\Omega_t, a_t)$.*

Assumption 2 (Flow utility shocks)**.** *For all t such that t and $t-1$ both belong to $\mathcal{T}$, assume: (i) $\varepsilon_t \perp\!\!\!\perp s_t$; (ii) $\varepsilon_t \perp\!\!\!\perp s_{t-1}$; (iii) ε_t is serially independent; and (iv) $\varepsilon_{0t} \perp\!\!\!\perp \varepsilon_{1t}$, and they both follow a type I extreme value distribution which is re-centered at zero.*

Assumption 3 (Conditional independence)**.** *For all t such that t and $t+1$ both belong to $\mathcal{T}$, assume $s_{t+1} \perp\!\!\!\perp \varepsilon_t \,|\, (s_t, a_t)$.*

Assumptions 1 to 3 are common assumptions in the literature of DDC models and have been made, explicitly or implicitly, in many applications. In particular, Assumptions 2(i) and 2(ii) are implied by Assumption IID in the survey by Aguirregabiria and Mira (2010) and are necessary for the conditional independence assumption in Rust (1987) (p. 1011).[3] Moreover, Assumptions 2(iii), 2(iv) and 3 correspond to Assumptions IID, CLOGIT and CI-X, respectively, in the survey by Aguirregabiria and Mira (2010).

Under Assumption 1, the agents solve a dynamic programming problem; that is, they observe Ω_t and choose a_t to maximize their expected lifetime payoff $\mathbb{E}(\sum_{j=0}^{T_{end}-t}\beta^j(u_{t+j}(a_{t+j},x_{t+j})+\varepsilon_{a_{t+j}t+j})\mid\Omega_t,a_t)$ in every period, where $\beta\in(0,1)$ is the discount factor. Note that Assumption 1 implies that the choice a_t in period t is completely determined by the current state variables Ω_t (see, for example, Aguirregabiria and Mira, 2010, p.39), so the value function in period t is also a function of Ω_t alone, which we denote as $V_t(\Omega_t)$. We also define the *integrated* value function in period t as $\bar{V}_t(s_t)=\mathbb{E}(V_t(\Omega_t)\mid s_t)$. Then under Assumptions 1 to 3, the agent's expected lifetime payoff if choosing $a_t=a$ can be shown (in Appendix A) to be

$$u_t(a,s_t)+\varepsilon_{at}+\beta\mathbb{E}(\bar{V}_{t+1}(s_{t+1})\mid s_t,a_t=a),\ \text{where } a=0 \text{ or } 1. \tag{1}$$

We use the following notation to denote the difference in the conditional means of a future random variable h_τ given s_t and between $a_t=1$ and $a_t=0$ ($\tau>t$):

$$\Delta\mathbb{E}(h_\tau\mid s_t)\equiv\mathbb{E}(h_\tau\mid s_t,a_t=1)-\mathbb{E}(h_\tau\mid s_t,a_t=0). \tag{2}$$

Then, the agent's optimal decision rule is

$$a_t=\mathbb{I}\{u_t(1,s_t)-u_t(0,s_t)+\beta\Delta\mathbb{E}(\bar{V}_{t+1}(s_{t+1})\mid s_t)>\varepsilon_{0t}-\varepsilon_{1t}\}. \tag{3}$$

We use $p_t(\cdot)$ to denote the CCP function in period t, that is,

$$p_t(s)=\Pr(a_t=1\mid s_t=s). \tag{4}$$

Note that the subscript t emphasizes that the CCP might be a different function in every period, and this is a result of the general nonstationarity (see Remark 1 below) allowed by the model in this paper.

In the rest of this paper, we will suppress the argument of the CCP functions and use p_t to denote $p_t(s_t)$ for notational conciseness, whenever it is not confusing. By Assumption 2(iv) and eq. (3), the log odds ratio has the expression

$$\ln\left(\frac{p_t}{1-p_t}\right)=u_t(1,s_t)-u_t(0,s_t)+\beta\Delta\mathbb{E}(\bar{V}_{t+1}(s_{t+1})\mid s_t), \tag{5}$$

[3] From the phrasing of Assumption IID in Aguirregabiria and Mira (2010), it is not obvious that Assumptions 2(i) and 2(ii) are implied, but they claim in the paragraph after Assumption DIS that, using our notation, Assumptions CI-X and IID together imply that $f(s_{t+1},\varepsilon_{t+1}\mid a_t,s_t,\varepsilon_t)=f_\varepsilon(\varepsilon_{t+1})f_s(s_{t+1}\mid a_t,s_t)$, which does not hold without Assumptions 2(i) or 2(ii).

which is similar to the log odds ratio in static logit models, but the difficulty of the dynamic model resides in the last term, which captures the difference in the discounted future lifetime payoffs between the two choices.[4]

Remark 1 (Nonstationarity)**.** *Nonstationarity in the context of DDC models means that the agent's decision problem in any period t is not an identical copy of that in other periods $t' \neq t$. The model in this paper permits several sources of nonstationarity: (i) the decision terminal period T_{end} may be finite and/or unknown; (ii) the function $u_t(a,s)$ that determines the non-stochastic part of the flow utility may vary with t; (iii) the state transition density $f_t(\Omega_{t+1} \mid \Omega_t)$ may vary with t. Any of these sources could result in time-varying CCPs and/or time-varying value functions.*

2.2 Brief Review of the HM Estimators

Our approach in Section 3 builds on the results in Hotz and Miller (1993) and Hotz et al. (1994), but we derive new results that allow for a substantial simplification of the analysis, under the same assumptions. Before detailing our approach, we briefly review in this subsection the aspect of the HM estimators that we will simplify.

The HM estimators are GMM estimators based on the moment conditions $\mathbb{E}(a_t - p_t \mid s_t) = 0$, in which, by eq. (5), the CCP equals $p_t = \Lambda(u_t(1,s_t) - u_t(0,s_t) + \beta\Delta\mathbb{E}(\bar{V}_{t+1}(s_{t+1}) \mid s_t))$ with the function $\Lambda(v) \equiv \exp(v)/(1+\exp(v))$. The key part of the HM estimators, therefore, is to evaluate $\Delta\mathbb{E}(\bar{V}_{t+1}(s_{t+1}) \mid s_t)$, the last term in eq. (5). They utilize the following recursive expression that can be shown (in Appendix A) to hold by Bellman's principle of optimality and Assumptions 1 to 3:

$$\bar{V}_{t+1}(s_{t+1}) = U^o_{t+1}(s_{t+1}) + \beta\mathbb{E}(\bar{V}_{t+2}(s_{t+2}) \mid s_{t+1}) \tag{6}$$

for t such that $t+1$ and $t+2$ both belong to $\mathcal{T}$. In eq. (6), the *integrated optimal* flow utility function $U^o_{t+1}(s_{t+1})$ incorporates the optimal choice in period $t+1$ and is a function of the CCP $p_{t+1}(s_{t+1})$ and of the expected flow utility $u_{t+1}(a_{t+1}, x_{t+1})$. $U^o_{t+1}(s_{t+1})$ has a closed form, as in Hotz and Miller (1993, eq. (3.8)). Then, by repeatedly applying eq. (6) to $t+2, t+3, \ldots, T_{end}$ and plugging into $\mathbb{E}(\bar{V}_{t+1}(s_{t+1}) \mid s_t, a_t = a)$, the last term in eq. (5) becomes the difference of the following expression between $a = 1$ and $a = 0$:

$$\begin{aligned} &\mathbb{E}(\bar{V}_{t+1}(s_{t+1}) \mid s_t, a_t = a) \\ =\ &\mathbb{E}(U^o_{t+1}(s_{t+1}) \mid s_t, a_t = a) + \beta\mathbb{E}(\mathbb{E}(\bar{V}_{t+2}(s_{t+2}) \mid s_{t+1}) \mid s_t, a_t = a) \end{aligned}$$

[4] Assumption 2(iv) is imposed for simplicity, and gives rise to the log odds ratio on the left-hand side of eq. (5). This assumption can be relaxed because Hotz and Miller (1993) show the existence of a one-to-one mapping between the right-hand side of eq. (5) and the CCP, and the mapping depends only on the joint distribution of ε_t. This is easy to see from the binary decision rule in eq. (3): let $F_{\varepsilon_0 - \varepsilon_1}$ be the cumulative distribution function of $\varepsilon_{0t} - \varepsilon_{1t}$, then the CCP function equals $p_t = F_{\varepsilon_0-\varepsilon_1}(u_t(1,x_t) - u_t(0,x_t) + \beta\Delta\mathbb{E}(\bar{V}_{t+1}(s_{t+1}) \mid s_t))$ by the decision rule, so we have $F^{-1}_{\varepsilon_0-\varepsilon_1}(p_t) = u_t(1,x_t) - u_t(0,x_t) + \beta\Delta\mathbb{E}(\bar{V}_{t+1}(s_{t+1}) \mid s_t)$, where the superscript $^{-1}$ indicates inverse function.

$$
\begin{aligned}
&= \mathbb{E}(U^o_{t+1}(s_{t+1}) \mid s_t, a_t = a) + \beta\mathbb{E}(\mathbb{E}(U^o_{t+2}(s_{t+2}) \mid s_{t+1}) \mid s_t, a_t = a) \\
&\quad + \beta^2\mathbb{E}(\mathbb{E}(\mathbb{E}(\bar{V}_{t+3}(s_{t+3}) \mid s_{t+2}) \mid s_{t+1}) \mid s_t, a_t = a) \\
&= \cdots \\
&= \mathbb{E}(U^o_{t+1}(s_{t+1}) \mid s_t, a_t = a) + \beta\mathbb{E}(\mathbb{E}(U^o_{t+2}(s_{t+2}) \mid s_{t+1}) \mid s_t, a_t = a) \\
&\quad + \beta^2\mathbb{E}(\mathbb{E}(\mathbb{E}(U^o_{t+3}(s_{t+3}) \mid s_{t+2}) \mid s_{t+1}) \mid s_t, a_t = a) + \cdots \\
&\quad + \beta^{T_{end}-t-2}\mathbb{E}(\mathbb{E}(\mathbb{E}(\cdots\mathbb{E}(U^o_{T_{end}-1}(s_{T_{end}-1}) \mid s_{T_{end}-2})\cdots \mid s_{t+2}) \mid s_{t+1}) \mid s_t, a_t = a) \\
&\quad + \beta^{T_{end}-t-1}\mathbb{E}(\mathbb{E}(\mathbb{E}(\cdots\mathbb{E}(\mathbb{E}(\bar{V}_{T_{end}}(s_{T_{end}}) \mid s_{T_{end}-1}) \mid s_{T_{tr}-2})\cdots \mid s_{t+2}) \mid s_{t+1}) \mid s_t, a_t = a). \qquad (7)
\end{aligned}
$$

This involves iterated conditional means given, sequentially, s_t, s_{t+1}, and up to $s_{T_{end}-1}$. When all the state variables in s_t are discrete, eq. (7) is just eq. (3.12) in Hotz and Miller (1993).

HM estimators evaluate eq. (7) by simulating many draws of entire time sequences of observed state variables and choices $s_t, a_t, s_{t+1}, a_{t+1}, \ldots, s_{T_{end}}, a_{T_{end}}$. Simulating from the transition distribution of the state variables is essential in this process.[5] The resulting CCPs under various hypothesized parameter values are then matched with the observed CCPs (i.e., observed choices proportions) in the data to solve for parameter estimates.

Remark 2 (Implementation issues of the HM estimators)**.** *The ideal draws of the state variables s_t requires estimation of the choice-specific state transition densities $f_{s_t|s_{t-1},a_{t-1}}$ for each t, and estimation of the CCPs p_t in eq. (4) is required to obtain the draws of the choice a_t for each t. Although these draws need only to be made once, they must be used to evaluate eq. (7) for every hypothesized parameter value. When the dimension of s_t is moderately large, such estimation, simulation and numerical integration can be tedious to perform and may require a huge simulation sample to deliver good approximation of $\mathbb{E}(\bar{V}_{t+1}(s_{t+1}) \mid s_t, a_t)$. Retaining these draws in the compute memory can put an enormous burden on it, significantly slowing down the implementation of the HM estimators, regardless whether the state transition distributions are parametric or nonparametric, or whether the model is stationary or nonstationary.*

In practice, a truncation period T_{tr} is often chosen to replace T_{end} in eq. (7).[6] If $f_{s_t|s_{t-1},a_{t-1}}$ varies with t, which generally implies that p_t also varies with t (i.e., a nonstationary model), then choosing T_{tr} can be tricky. On the one hand, if one chooses T_{tr} to be before the sample

[5]We are aware that with the aid of a renewal choice or a terminal choice, the above iteration and the complicated simulation can be avoided, but our discussion is about the general case. We are also aware that when the state variables are all discrete taking only a small number of values, there is no need to do simulation and the above iterated expectations can be computed analytically using state transition matrices. In addition, as eq. (7) involves choices only in the outmost conditional mean, it is not necessary to simulate the choice sequence $a_t, a_{t+1}, \ldots, a_{T_{tr}}$ if estimation is the sole purpose. Instead, one could simply estimate the state transition $f_{s_t|s_{t-1}}$, which embeds the optimal decision rule, and then simulate the state variable sequence sequence $s_t, s_{t+1}, \ldots, s_{T_{tr}}$ to evaluate eq. (7), but this is somehow not common in a typical implementation of the HM estimators.

[6]For example, when if T_{end} is infinite, then choosing a large but finite T_{tr} is necessary, and the impact of the truncation on the evaluation of eq. (7) is negligible since $\beta \in (0, 1)$.

terminal period, then $f_{s_t|s_{t-1},a_{t-1}}$ and p_t for $t \le T_{tr}$ can all be estimated from the data. The problem arises when only short panel data are available (i.e., T_{tr} very small), in this case the impact of the truncation is not negligible, so the estimate of $\mathbb{E}(\bar{V}_{t+1}(s_{t+1}) \mid s_t, a_t)$ can be substantially biased, leading to biased parameter estimates, and it is unclear how to systematically quantify such bias. On the other hand, if one chooses T_{tr} to be after the sample terminal period, then they must make additional assumptions regarding how the state variables and the flow utility parameters will evolve beyond the data horizon in this nonstationary model. Unfortunately, there is no agreed-upon way of making such assumptions, and it is unclear how large the bias is.

If the model is stationary, which implies that neither $f_{s_t|s_{t-1},a_{t-1}}$ nor $p_t(\cdot)$ depends on t, then T_{tr} can be later than the sample terminal period, but simulation and numerical integration could still be tedious to perform.

In practice, a parametric specification is often used when nonparametric estimation of $f_{s_t|s_{t-1},a_{t-1}}$ is difficult, such as when s_t has a large dimension or contains continuous variables (see, for example, Li and Racine, 2007, Chapter 5). It is not always obvious, however, what parametric specification is appropriate, or how sensitive the estimates or the counterfactuals are to such a specification.

3 Identification and Estimation

In this section, we show the identification of the structural parameters of the flow utility under common assumptions in the DDC literature (including Assumptions 1 to 3 above and Assumption 4 below) and a mild new assumption (Assumption 5 below). We also provide an CCP-based semiparametric estimator that bypasses the state transition distributions.

3.1 Identification

To show the identification of the parameters in the flow utility function $u_t(a, s)$, our analysis starts with eq. (5) and proceeds in three steps. Section 3.1.1 collapses the iterated conditional means in eq. (7) to a conditional mean that only conditions on s_t and a_t, by exploiting results for the DDC models under Assumptions 1 to 3. This substantially simplifies the analysis. The Markovian property of s_t (Lemma 1 below) is the most crucial, but appears to have been overlooked before. Section 3.1.2 transforms eq. (5) (one for each $t \in \{1, \dots, T-1\}$) into a system of linear equations in the structural parameters under a (weaker than) common Assumption 4 and a mild new Assumption 5. Section 3.1.3 provides a set of testable sufficient conditions for the identification of the resulting linear system using the usual linear GMM argument. Various implications, special cases and caveats are also mentioned in Section 3.1.3.

3.1.1 Collapse of Iterated Conditional Means

The difficulty of evaluating the log odds ratio for dynamic binary choice models, as opposed to static models, as explained in Section 2.2, stems from the iterated conditional means in the last term of eq. (5), which is the discounted future lifetime payoff difference. The first important theoretical result of this paper is to show that those iterated conditional means can be collapsed to a much simpler form under Assumptions 1 to 3.

Lemma 1 (Markovian observed state variables)**.** *Under Assumptions 1 and 2(i)-(iii), s_t is a first order Markov process; that is, $s_{t+1} \perp\!\!\!\perp s_{t-j} \mid s_t$ for t and $j \in \mathbb{N}^+$ such that t, $t+1$ and $t-j$ all belong to $\mathcal{T}$.*

Lemma 2 (Conditional independence)**.** *For t and $j \in \mathbb{N}^+$ such that t, $t+1$ and $t+j$ all belong to $\mathcal{T}$, suppose $g(\cdot)$ is a measurable function of s_{t+j}. Then, under Assumptions 1 and 2(i)-(iii),*

$$\mathbb{E}(\mathbb{E}(g(s_{t+j}) \mid s_{t+1}) \mid s_t, a_t) = \mathbb{E}(g(s_{t+j}) \mid s_t, a_t).$$

Theorem 1 (Collapse of iterated conditional means)**.** *Under Assumptions 1 to 3, the log odds ratio in eq. (5) simplifies to*

$$\ln\left(\frac{p_t}{1-p_t}\right) = u_t(1, x_t) - u_t(0, x_t) + \sum_{\tau=t+1}^{T^*-1} \beta^{\tau-t} \Delta\mathbb{E}(U^o_\tau(s_\tau) \mid s_t) \\ + \beta^{T^*-t} \Delta\mathbb{E}(\bar{V}_{T^*}(s_{T^*}) \mid s_t) \tag{8}$$

for all t and T^* both belonging to $\mathcal{T}$ such that $t < T^*$.

Proof. First note that the last term of eq. (5), by eq. (7), is the sum of multiple terms that have the form

$$\mathbb{E}(\mathbb{E}(\mathbb{E}(\cdots\mathbb{E}(g(s_{t+j}) \mid s_{t+j-1})\cdots \mid s_{t+2}) \mid s_{t+1}) \mid s_t, a_t), \tag{9}$$

where t and $j \in \mathbb{N}^+$ are such that $t, t+1, \ldots, t+j$ all belong to $\mathcal{T}$, and $g(\cdot)$ is a measurable function of s_{t+j}. As a result, the proof of this theorem will proceed in three steps. Step 1 is to use Lemma 1 to show that the expression in eq. (9) equals $\mathbb{E}(\mathbb{E}(g(s_{t+j}) \mid s_{t+1}) \mid s_t, a_t)$. Step 2 is to use Lemma 2 to further simplify the double conditional mean. Step 3 is to apply this general simplification to all relevant terms in eq. (5) to get the result of this theorem.

Step 1. Under Assumptions 1 and 2(i)-(iii), the expression in eq. (9) can be simplified as

$$\begin{aligned}
&\mathbb{E}(\mathbb{E}(\mathbb{E}(\mathbb{E}(\mathbb{E}(\cdots\mathbb{E}(g(s_{t+j}) \mid s_{t+j-1})\cdots \mid s_{t+4}) \mid s_{t+3}) \mid s_{t+2}) \mid s_{t+1}) \mid s_t, a_t) \\
&= \mathbb{E}(\mathbb{E}(\mathbb{E}(\mathbb{E}(\mathbb{E}(\cdots\mathbb{E}(g(s_{t+j}) \mid s_{t+j-1})\cdots \mid s_{t+4}) \mid s_{t+3}) \mid s_{t+2}, s_{t+1}) \mid s_{t+1}) \mid s_t, a_t) \\
&= \mathbb{E}(\mathbb{E}(\mathbb{E}(\mathbb{E}(\cdots\mathbb{E}(g(s_{t+j}) \mid s_{t+j-1})\cdots \mid s_{t+4}) \mid s_{t+3}) \mid s_{t+1}) \mid s_t, a_t) \\
&= \mathbb{E}(\mathbb{E}(\mathbb{E}(\mathbb{E}(\cdots\mathbb{E}(g(s_{t+j}) \mid s_{t+j-1})\cdots \mid s_{t+4}) \mid s_{t+3}, s_{t+1}) \mid s_{t+1}) \mid s_t, a_t)
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}(\mathbb{E}(\mathbb{E}(\cdots\mathbb{E}(g(s_{t+j}) \mid s_{t+j-1})\cdots \mid s_{t+4}) \mid s_{t+1}) \mid s_t, a_t) \\
&= \cdots \\
&= \mathbb{E}(\mathbb{E}(g(s_{t+j}) \mid s_{t+1}) \mid s_t, a_t),
\end{aligned} \tag{10}
$$

where the first and the third equalities hold by $s_{t+3} \perp\!\!\!\perp s_{t+1} \mid s_{t+2}$ and $s_{t+4} \perp\!\!\!\perp s_{t+1} \mid s_{t+3}$, respectively, both implied by Lemma 1, and the second and the fourth equalities hold by the law of iterated expectations. The rest of eq. (10) holds by repeatedly applying the argument used in the first four equalities to all the later periods.

Step 2. Under Assumptions 1 and 2(i)-(iii), combine eq. (10) and Lemma 2, we immediately get

$$
\mathbb{E}(\mathbb{E}(\mathbb{E}(\cdots\mathbb{E}(g(s_{t+j}) \mid s_{t+j-1})\cdots \mid s_{t+2}) \mid s_{t+1}) \mid s_t, a_t) = \mathbb{E}(g(s_{t+j}) \mid s_t, a_t). \tag{11}
$$

Step 3. Applying eq. (11) to every term in eq. (7) and plugging the resultant expressions to eq. (5), which holds under Assumptions 1 to 3, we get the result of this theorem. ∎

Remark 3 (Key of collapse)**.** *As is clear from eq. (10), the key to the collapse of the iterated conditional means in the log odds ratio is the Markovian property of s_t in Lemma 1, which appears to us to be a new result for DDC models characterized by Assumptions 1 to 3 and might be of independent interest. Intuitively, the Markovian property means that none of the observed state variables in earlier periods provides additional information about s_{t+j} beyond what is contained in s_{t+j-1}. At the same time, the law of iterated expectations essentially states that only the least informative information set matters. In consequence, the iterated conditional means given consecutive earlier observed state variables comes down to a conditional mean given the earliest state variable s_{t+1}, which is the least informative about s_{t+j}.*[7]

Theorem 1 is a crucial result that permits the identification analysis and the state-transition-free estimator in this paper, because it eliminates the need of simulating the whole sequence of s_t, hence the elimination of the need of estimating the state transition distributions of s_t (i.e., $f_{s_{t+1}|s_t}$). Instead, one only needs to estimate a few conditional means given s_t, as we will elaborate in the rest of this section.

3.1.2 Transformation into a Linear System

We conjecture that some progress could be made in the nonparametric identification of the expected flow utility function $u_t(a_t, s_t)$, since $U_\tau^o(s_\tau)$ can be written as a linear combination of $u_\tau(1, s_\tau)$ and $u_\tau(0, s_\tau)$ with the CCPs being the weights (see Lemma 3 below). We will leave this conjecture,

[7] Technically, let $\mathcal{F}_t$ denote the σ-algebra generated by s_t for all $t \in \mathcal{T}$. The Markovian property implies that for all relevant $j \in \mathbb{N}^+$, $(\mathcal{F}_{t+j} \cap \mathcal{F}_{t+j-1}) = (\mathcal{F}_{t+j} \cap (\mathcal{F}_{t+j-1} \cup \mathcal{F}_{t+1}))$, which, combined with the trivial relationship $(\mathcal{F}_{t+j} \cap \mathcal{F}_{t+1}) \subseteq (\mathcal{F}_{t+j} \cap (\mathcal{F}_{t+j-1} \cup \mathcal{F}_{t+1}))$, further implies $(\mathcal{F}_{t+j} \cap \mathcal{F}_{t+1}) \subseteq (\mathcal{F}_{t+j} \cap \mathcal{F}_{t+j-1})$, and the collapse of the iterated conditional means shown in eq. (10) follows the law of iterated expectations straightforwardly.

however, for future inquiry and in this paper make the following simplifying assumption that is general than typical in the DDC literature.

Assumption 4 (Expected flow utility)**.** *Let x_t denote a $d_x \times 1$ subvector of s_t, with $d_x \le d_s$. For each $t \in \mathcal{T}$, assume $u_t(0, x_t) = x_t'\delta_{0,t}$ and $u_t(1, x_t) = x_t'\delta_{1,t}$ for some $\delta_{0,t}$ and $\delta_{1,t}$. We normalize $\delta_{0,1}$ to an arbitrary $d_x \times 1$ vector of constants, denoted by c.*

Assumption 4 allows x_t to be s_t itself or a proper subvector of s_t. In the latter case, the coordinates of s_t that are not in x_t, denoted by a $d_z \times 1$ vector z_t (with $d_z \equiv d_s - d_x$), are observed state variables that do not affect the current flow utility but may affect the future lifetime payoff through their impact on the distribution of s_{t+1}. We call z_t "excluded variables", and their function will be discussed in Remark 9 below.

It is a common assumption in the DDC literature to assume that $u_t(0, x_t) = 0$ for all x_t values and all $t \in \mathcal{T}$ (or equivalently, $\delta_{0,t} = 0$ for all $t \in \mathcal{T}$), but we refrain from making such assumption because it has been illustrated to be arbitrary and to result in substantial bias in counterfactual analysis (Aguirregabiria and Suzuki, 2014; Norets and Tang, 2014; Chou, 2016). More importantly, such strong assumption is unnecessary, because in contrast to static binary choice models, dynamic models can leverage inter-temporal variation in x_t to separate $u_t(0, x_t)$ and $u_t(1, x_t)$, in addition to the difference between the two.[8] A normalization for the sample initial period, however, is necessary.[9]

The linear specification of $u_t(a, x_t)$, although common in the DDC literature, may appear restrictive. This concern could be partially alleviated as the state variables themselves and various functions of them (e.g., power series) are allowed to be included in x_t.[10] Under Assumption 4, the identification of the flow utility function, the key primitive structural object of this model, boils down to identifying δ_{0t} and δ_{1t} for $t \in \mathcal{T}$. The next lemma links the optimal flow utility function $U_t^o(s_t)$ in eq. (8) with these parameters.

Lemma 3 (Optimal flow utility)**.** *Under Assumptions 1 to 4, the expected optimal flow utility function $U_t^o(s_t)$ for $t \in \mathcal{T}$ can be written as*

$$U_t^o(s_t) = p_t x_t'\delta_{1,t} + (1 - p_t)x_t'\delta_{0,t} - p_t \ln(p_t) - (1 - p_t)\ln(1 - p_t). \tag{12}$$

[8] Remark 13 below is also related to this.

[9] One could normalize $\delta_{1,t}$ to an arbitrary constant vector c instead, but it will not affect the identification conditions or the values of the parameters δ and γ^K discussed in Section 3.1.3 below. Moreover, because this arbitrary value will be canceled in the difference in the sample initial period flow utility between the two choices and does not affect future payoff, it will not affect the counterfactual analysis either. In this sense, it is a real normalization.

[10] In fact, it is possible to relax Assumption 4 to something similar to Assumption 5 below. That is, suppose there exists a $K_u \times 1$ vector of known functions of x, denoted by $q^{K_u}(x) \equiv (q^{K_u,1}(x), \dots, q^{K_u,K_u}(x))'$, and for each $t \in \mathcal{T}$, there exist $K_u \times 1$ unknown vectors of parameters $\delta_{0t}^{K_u}$ and $\delta_{1t}^{K_u}$, such that $u_t(0, x_t) = q^{K_u}(x_t)'\delta_{0,t}^{K_u}$ and $u_t(1, x_t) = q^{K_u}(x_t)'\delta_{1,t}^{K_u}$. The identification of $\delta_{0,t}^{K_u}$ and $\delta_{1,t}^{K_u}$ follows the same argument as for $\delta_{0,t}$ and $\delta_{1,t}$ in Section 3.1.3 below, and the bias resulting from the violation of this specification can be quantified in the same way as in Section 4.1 below, both straightforwardly.

An conceptual distinction that is important for our identification analysis is between decision horizon and data horizon. Although the agent's *decision horizon* is $\mathcal{T}$ as introduced at the beginning of Section 2.1, the *data horizon*, denoted as $\mathcal{T}_{da} \equiv \{1, \ldots, T\}$, that is available for researchers to observe may only be a subset of it – that is, $T_{end} \geq T$ and $T_{start} \leq 1$. All the results we derive so far hold for $\forall t \in \mathcal{T}$; therefore, they also hold for $\forall t \in \mathcal{T}_{da}$. In particular, eq. (8) still holds if T^* is replaced by T. For identification analysis, however, only the data horizon should be utilized, because this paper allows for very general nonstationarity and avoids making additional assumptions regarding how the state variables and the flow utility parameters will evolve beyond the data horizon.[11]

Plugging the expression of $U_t^o(s_t)$ in Lemma 3 into eq. (8), replacing T^* with T, rearranging and defining

$$\Delta\bar{x}_{1,t}^{\tau} \equiv \Delta\mathbb{E}(p_\tau x_\tau \mid s_t) \tag{13a}$$

$$\Delta\bar{x}_{0,t}^{\tau} \equiv \Delta\mathbb{E}((1-p_\tau) x_\tau \mid s_t) \tag{13b}$$

$$\Delta\bar{x}_{t}^{\tau} \equiv \Delta\mathbb{E}(x_\tau \mid s_t) \tag{13c}$$

$$y_{T-1} \equiv \ln\Big(\frac{p_{T-1}}{1-p_{T-1}}\Big), \text{ and} \tag{13d}$$

$$y_t \equiv \ln\Big(\frac{p_t}{1-p_t}\Big) + \sum_{\tau=t+1}^{T-1} \beta^{\tau-t}\Delta\bar{\eta}_t^{\tau} \tag{13e}$$

for $t = 1, \ldots, T-2$ and $\tau > t$, where

$$\begin{aligned} \Delta\bar{\eta}_t^{\tau} &\equiv \Delta\mathbb{E}(\eta_\tau \mid s_t), \\ \eta_\tau &\equiv p_\tau \ln(p_\tau) + (1-p_\tau)\ln(1-p_\tau), \quad \text{for } \tau > t, \end{aligned} \tag{14}$$

we get the following equations

$$\begin{aligned} y_{T-1} &= x_{T-1}'\delta_{1,T-1} - x_{T-1}'\delta_{0,T-1} + \beta\Delta\mathbb{E}(\bar{V}_T(s_T) \mid s_{T-1}), \text{ and} \\ y_t &= x_t'\delta_{1,t} - x_t'\delta_{0,t} + \sum_{\tau=t+1}^{T-1}\beta^{\tau-t}\Delta\bar{x}_{1,t}^{\tau\prime}\delta_{1,\tau} + \sum_{\tau=t+1}^{T-1}\beta^{\tau-t}\Delta\bar{x}_{0,t}^{\tau\prime}\delta_{0,\tau} \\ &\quad + \beta^{T-t}\Delta\mathbb{E}(\bar{V}_T(s_T) \mid s_t) \end{aligned}$$

for $t = 1, \ldots, T-2$. We denote $\Delta_t \equiv \delta_{1,t} - \delta_{0,t}$ for $t = 1, \ldots, T$, then $\delta_{1,t} = \delta_{0,t} + \Delta_t$, and these equations can be rewritten as

$$y_{T-1} = x_{T-1}'\Delta_{T-1} + \beta\Delta\mathbb{E}(\bar{V}_T(s_T) \mid s_{T-1}), \text{ and} \tag{15a}$$

$$y_t = x_t'\Delta_t + \sum_{\tau=t+1}^{T-1}\beta^{\tau-t}\Delta\bar{x}_t^{\tau\prime}\delta_{0,\tau} + \sum_{\tau=t+1}^{T-1}\beta^{\tau-t}\Delta\bar{x}_{1,t}^{\tau\prime}\Delta_\tau$$

[11]Note that in contrast to the HM estimators, for which the choice of T_{tr} can be tricky and impactful (recall Remark 2 above and see Section 5.3 below for further details), our approach does not require the use of a truncation period T_{tr}, regardless of whether it is within or beyond the data horizon.

$$+\beta^{T-t}\Delta\mathbb{E}(\bar{V}_T(s_T)\,|\,s_t) \tag{15b}$$

for $t = 1,\ldots,T-2$, where eq. (15b) holds by the definitions of $\Delta\bar{x}^{\tau}_{1,t}$, $\Delta\bar{x}^{\tau}_{0,t}$ and $\Delta\bar{x}^{\tau}_{t}$ in eqs. (13a) to (13c). Note that c, the arbitrary normalized value of $\delta_{0,1}$, does not show up in eq. (15).

We assume that the discount factor β is known for conciseness and focus on the identification of Δ_t and $\delta_{0,t}$,[12] the key structural parameter of the model that specifies the flow utility functions in the first $T-1$ periods in the data.[13] Although y_t, $\Delta\bar{x}^{\tau}_{t}$ and $\Delta\bar{x}^{\tau}_{1,t}$ may appear to have complicated expressions, by their definitions in eqs. (13a) to (14), they can be regarded as known for the identification purpose, since all their components are either observed or identified from the data. Therefore, the identification of Δ_t and $\delta_{0,t}$ will take advantage of the linear structure of eq. (15), and the only outstanding complication is that $\Delta\mathbb{E}(\bar{V}_T(s_T)\,|\,s_t)$ $(t = 1,\ldots,T-1)$ is a different unknown function of s_t for each t, and yet the unknown inner function $\bar{V}_T(s_T)$ remains invariant.

Remark 4 (Infeasibility of Robinson (1988) approach)**.** *Although eq. (15) is a partially linear system, the approach developed by Robinson (1988) is not feasible to identify or to estimate the linear parameters Δ_t $(t = 1,\ldots,T-1)$ and $\delta_{0,t}$ $(t = 2,\ldots,T-1)$. This is because the non-linear terms in eq. (15) are unknown functions of s_t, which include all the regressors in the linear terms (x_t, as well as z_t that show up in $\Delta\bar{x}^{\tau}_{t}$ and $\Delta\bar{x}^{\tau}_{1,t}$) as subsets $(t = 1,\ldots,T-1)$, leading to the failure of the key condition (3.5) in Robinson (1988).*

Equation (15) is of primary importance to the rest of our identification analysis, because it resembles a system of partially linear regression equations ($T-1$ of them) with "dependent variables" y_t, "independent variables" x_t, $\beta^{\tau-t}\Delta\bar{x}^{\tau}_{t}$ and $\beta^{\tau-t}\Delta\bar{x}^{\tau}_{1,t}$, "non-linear term" $\beta^{T-t}\Delta\mathbb{E}(\bar{V}_T(s_T)|s_t)$, and "coefficients" Δ_τ and $\delta_{0,\tau}$ $(\tau \geq t)$.

The following assumption about the sample-terminal-period integrated value function $\bar{V}_T(s_T)$, new to the literature but mild, will help further transform eq. (15) into a familiar linear system, for which we are equipped with many tools from the canon.

Assumption 5 (Sample-terminal-period integrated value function)**.** *Assume that there exists a $K \times 1$ vector of known functions of s, denoted by $q^K(s) \equiv (q^{K,1}(s),\ldots,q^{K,K}(s))'$, and a $K \times 1$ unknown vector of parameters γ^K, such that*

$$\bar{V}_T(s_T) = q^K(s_t)'\gamma^K. \tag{16}$$

[12] Remark 12 below shows that the identification only needs minor modification to account for unknown β, and the corresponding estimation method follows straightforwardly by slightly modifying the estimator in Section 3.2 (omitted in this paper). Alternatively, β can be identified and estimated from a secondary data source, which is also common in the DDC literature.

[13] $\delta_{1,T}$ and $\delta_{0,T}$ are not identified without further assumptions if $T < T_{end}$, since there are no data to distinguish the flow utility in period T from the expected future payoffs after T. If $T = T_{end}$, then Remark 5 below show its identification.

Remark 5 (An important special case). *Assumption 5 may appear to be a very restrictive specification of the unknown sample-terminal-period integrated value function $\bar{V}_T(s_T)$, but it includes an important special case – when the sample terminal period is also the decision terminal period (i.e., $T = T_{end}$). In this case, the value function $\bar{V}_T(s_T)$ in the sample terminal period consists of the current flow utility only and by Lemma 3, it is*

$$\bar{V}_T(s_T) = x_T'\delta_{0,T} + p_T x_T'\Delta_T - \eta_T, \tag{17}$$

where η_T is defined in eq. (14), which is a function of

$$p_T = \frac{\exp(x_T'\Delta_T)}{1 + \exp(x_T'\Delta_T)}$$

due to $T = T_{end}$ and eq. (5). Note that Δ_T can be identified and estimated using only the cross-sectional data on the terminal period choices and state variables, therefore, Δ_T, p_T and η_T can all be regarded as known for identification purpose. In summary, this is a special case where Assumption 5 holds exactly with $K = 2d_x + 1$, $q^K(x_T, z_T) = (x_T', p_T x_T', -\eta_T)'$ and $\gamma^K = (\delta_{0,T}', \Delta_T', 1)'$.

It might be helpful to interpret Assumption 5 as approximating the $\bar{V}_T(s_T)$ using a series of basis functions $q^K(s)$, such as power series. In the rest of Section 3, we proceed with Assumption 5 holding exactly, but we will discuss the approximation perspective in Section 4.

Under Assumption 5, eq. (15) becomes a system of linear equations of Δ_t ($t = 1, \ldots, T-1$), $\delta_{0,t}$ ($t = 2, \ldots, T-1$) and γ^K:

$$y_{T-1} = x_{T-1}'\Delta_{T-1} + \beta\Delta\bar{q}_{T-1}^{K\prime}\gamma^K, \text{ and} \tag{18a}$$

$$\begin{aligned} y_t = {} & x_t'\Delta_t + \sum_{\tau=t+1}^{T-1}\beta^{\tau-t}\Delta\bar{x}_t^{\tau\prime}\delta_{0,\tau} + \sum_{\tau=t+1}^{T-1}\beta^{\tau-t}\Delta\bar{x}_{1,t}^{\tau\prime}\Delta_\tau \\ & + \beta^{T-t}\Delta\bar{q}_t^{K\prime}\gamma^K \end{aligned} \tag{18b}$$

for $t = 1, \ldots, T-2$, where

$$\Delta\bar{q}_t^K \equiv \Delta\mathbb{E}\big(q^K(s_t) \,\big|\, s_t\big), \tag{19}$$

whose randomness comes from both x_t and z_t. Like $\Delta\bar{x}_t^\tau$, $\Delta\bar{x}_{1,t}^\tau$ and y_t, we can regard $\Delta\bar{q}_t^K$ as known in the identification analysis, since they are directly identifiable from the data.

Equation (18) is the linear system that serves as the basis of our identification analysis in Section 3.1.3. The goal, therefore, is to identify $\delta \equiv (\Delta_1', \delta_{0,2}', \Delta_2', \ldots, \delta_{0,T-1}', \Delta_{T-1}')'$ and γ^K.

3.1.3 Identification of the Linear System

This subsection gives the sufficient and necessary condition for the identification of δ and γ^K in the linear system eq. (18). We also recognize that δ and γ^K are likely to be over-identified and give a few sets of sufficient conditions for that. The following features of eq. (18) are crucial for the identification analysis: (i) the "dependent variables" y_t and the "independent variables" x_t, $\Delta\bar{x}_t^\tau$, $\Delta\bar{x}_{1,t}^\tau$ and $\Delta\bar{q}_t^K$ ($t = 1, \ldots, T-1$ and $t < \tau \leq T-1$) consist of only observed variables or conditional mean functions of observed variables (which can be identified in a preliminary step) and can be regarded as known for the identification purpose, so the only unknowns are the "coefficients" δ and γ^K; (ii) eq. (18) does not contain any "error" or "disturbance" terms, so it should hold for *each and every* individual agent under Assumptions 1 to 5. The identification of δ and γ^K, therefore, boils down to under what conditions eq. (18) admits a unique solution. Whenever necessary, we will use the subscript $_{i,}$ ($i \in \{1, \ldots, N\}$), with a comma and in front of all the other subscripts, to indicate each agent in a size N random sample. In the rest of this subsection, we will first focus on the cases of $T = 2$ and $T = 3$ to highlight the key idea, then we extend to the general T case.

We start with the special case where $T = 2$. In this case, eq. (18b) does not exit, $\theta_2 \equiv (\Delta_1', \gamma^{K\prime})'$ are the only parameters, and eq. (18a) is equivalent to the following system of N equations with $d_x + K$ unknowns:

$$\mathbf{y}_{2,1} = \mathbf{X}_{2,1}\theta_2, \tag{20}$$

where $\mathbf{y}_{2,1}$ denotes the $N \times 1$ vector that stacks $y_{i,1}$ (i.e., agent i's copy of y_1), and $\mathbf{X}_{2,1}$ denotes the $N \times (d_x + K)$ matrix that stacks row vectors $X_{i,2,1} \equiv (x_{i,1}', \beta\Delta\bar{q}_{i,1}^{K\prime})'$ (i.e., agent i's copy of $X_{2,1}$) for $i = 1, \ldots, N$. An obvious sufficient condition for eq. (20) to admit a unique solution, therefore, is that $\mathbf{X}_{2,1}$ has rank $d_x + K$. That is, the vectors $X_{i,2,1}$ are linearly independent for some $d_x + K$ agents.

Equation (18a) for $T = 2$ also implies the following system of $d_x + K$ equations with $d_x + K$ unknowns:

$$\mathbb{E}(X_{2,1}y_1) = \mathbb{E}(X_{2,1}X_{2,1}')\theta_2. \tag{21}$$

The sufficient and necessary condition for eq. (21) to admit a unique solution is that its square Jacobian matrix $L_2 \equiv \mathbb{E}(X_{2,1}X_{2,1}')$ – also the second moment matrix of $(x_1', \beta\Delta\bar{q}_1^{K\prime})'$ – has full rank (i.e., $d_x + K$). This condition turns out to be also necessary for eq. (20) to admit a unique solution, because otherwise $X_{2,1}$ has a linear relationship among its variables and it cannot be linearly independent for any $d_x + K$ agents. Also note that this condition is equivalent to the second moment matrix of $(x_1', \Delta\bar{q}_1^{K\prime})'$ having rank $d_x + K$ since β is a constant.

The $T = 2$ case is special since there is no over-identification opportunity. Our discussion so far is summarized in the next proposition.

Proposition 1 (Identification when $T = 2$)**.** *When $T = 2$, Δ_1 and γ^K are the only parameters. They are identified if and only if the $(d_x + K) \times (d_x + K)$ matrix L_2 defined above has full rank. This condition is equivalent to that the second moment matrix of $(x_1', \Delta\bar{q}_1^{K\prime})'$ has full rank (i.e., $d_x + K$).*

Next, we consider the case where $T = 3$, which clearly illustrates the sources of identification and over-identification for $T > 2$ cases. When $T = 3$, eq. (18a) is equivalent to the following system of N equations with $d_x + K$ unknowns $\theta_{3,2} \equiv (\Delta_2', \gamma^{K\prime})'$:

$$\mathbf{y}_{3,2} = \mathbf{X}_{3,2}\theta_{3,2}, \tag{22}$$

where $\mathbf{y}_{3,2}$ denotes the $N \times 1$ vector that stacks $y_{i,2}$ (i.e., agent i's copy of y_2), and $\mathbf{X}_{3,2}$ denotes the $N \times (d_x + K)$ matrix that stacks row vectors $X_{i,3,2} \equiv (x_{i,2}', \beta\Delta\bar{q}_{i,2}^{K\prime})'$ (i.e., agent i's copy of $X_{3,2}$) for $i = 1, \ldots, N$; and eq. (18b) is equivalent to the following system of N equations with $3d_x + K$ unknowns $\theta_3 \equiv (\theta_{3,1}', \theta_{3,2}')'$:

$$\mathbf{y}_{3,1} = \mathbf{X}_{3,1}\theta_3, \tag{23}$$

where $\mathbf{y}_{3,1}$ denotes the $N \times 1$ vector that stacks $y_{i,1}$ (i.e., agent i's copy of y_1), $\mathbf{X}_{3,1}$ denotes the $N \times (3d_x + K)$ matrix that stacks row vectors $X_{i,3,1} \equiv (x_{i,1}', \beta\Delta\bar{x}_{i,1}^{2\prime}, \beta\Delta\bar{x}_{i,1,1}^{2\prime}, \beta^2\Delta\bar{q}_{i,1}^{K\prime})'$ (i.e., agent i's copy of $X_{3,1}$) for $i = 1, \ldots, N$, $\theta_{3,1} \equiv (\Delta_1', \delta_{0,2}')'$ and $\theta_{3,2}$ is defined above.

Just like the relationship between eq. (20) and eq. (21) for $T = 2$, whether eq. (22) and eq. (23) jointly pins down a unique value for θ_3 is equivalent to whether the following system of $4d_x + 2K$ equations with $3d_x + K$ unknowns,

$$\begin{bmatrix} \mathbb{E}(X_{3,2}y_2) \\ \mathbb{E}(X_{3,1}y_1) \end{bmatrix} = \begin{bmatrix} 0_{(d_x+K)\times 2d_x} & \mathbb{E}(X_{3,2}X_{3,2}') \\ \multicolumn{2}{c}{\mathbb{E}(X_{3,1}X_{3,1}')} \end{bmatrix} \cdot \begin{bmatrix} \theta_{3,1} \\ \theta_{3,2} \end{bmatrix}, \tag{24}$$

has a unique solution. The sufficient and necessary condition for this, just like for eq. (21), is that the Jacobian matrix of the linear system eq. (24), which is no longer a square matrix and contains more rows than columns, has full column rank (i.e., $3d_x + K$). Note that the first $d_x + K$ row of eq. (24) correspond to period $t = 2$, and $\mathbb{E}(X_{3,2}X_{3,2}')$ is the corresponding square Jacobian matrix; the last $3d_x + K$ row correspond to period $t = 1$, and $\mathbb{E}(X_{3,1}X_{3,1}')$ is the corresponding square Jacobian matrix.

Because the Jacobian matrix of eq. (24) has more rows ($4d_x + 2K$) than columns ($3d_x + K$), so there might be multiple ways for it to achieve full column rank. The key source of over-identification when $T = 3$ is the multitude of equations for periods $t = 1$ and $t = 2$. Our discussion of the $T = 3$ case is formally summarized as follows.

Proposition 2 (Identification and over-identification when $T = 3$)**.** *When $T = 3$, Δ_1, $\delta_{0,2}$, Δ_2 and γ^K are the only parameters.*

(i) The parameters are identified if and only if the $(4d_x + 2K) \times (3d_x + K)$ Jacobian matrix,

$$L_3 \equiv \begin{bmatrix} 0_{(d_x+K)\times 2d_x} & \mathbb{E}(X_{3,2}X_{3,2}') \\ \multicolumn{2}{c}{\mathbb{E}(X_{3,1}X_{3,1}')} \end{bmatrix},$$

has full column rank.

(ii) If more than one matrix that consists of $3d_x + K$ distinct rows from L_3 has full rank, then the parameters are over-identified.

The next two corollaries provide two sets of sufficient conditions for condition (i) in Proposition 2. Corollary 1 recognizes that the equations corresponding to $t = 1$ contain all the relevant parameters and have the potential of identifying them all. Corollary 2 takes advantage of the block-triangular structure of the Jacobian matrix L_3 to identify $\theta_{3,2}$ and $\theta_{3,1}$ sequentially. They could hold at the same time, leading to over-identification of the parameters.

Corollary 1 (Identification when $T = 3$)**.** *When $T = 3$, the parameters are identified if $\mathbb{E}(X_{3,1}X'_{3,1})$, or equivalently the second moment matrix of $(x'_1, \Delta\bar{x}_1^{2\prime}, \Delta\bar{x}_{1,1}^{2\prime}, \Delta\bar{q}_1^{K\prime})'$, has full rank.*

Corollary 2 (Identification when $T = 3$)**.** *When $T = 3$, the parameters are identified if: (i) the second moment matrix of $(x'_2, \Delta\bar{q}_2^{K\prime})'$ has full rank (i.e., d_x+K), and (ii) the second moment matrix of $(x'_1, \Delta\bar{x}_1^{2\prime})'$ has full rank (i.e, $2d_x$).*

The identification and over-identification conditions of the parameters for the general cases where $T > 3$ can be derived following the same argument. To formally state the results, we first define

$$\underbrace{X_{T,T-1}}_{(d_x+K)\times 1} \equiv \begin{bmatrix} x_{T-1} \\ \beta\Delta\bar{q}_{T-1}^K \end{bmatrix}, \qquad \underbrace{X_{T,t}}_{((2T-2t-1)d_x+K)\times 1} \equiv \begin{bmatrix} x_t \\ \beta\Delta\bar{x}_t^{t+1} \\ \beta\Delta\bar{x}_{1,t}^{t+1} \\ \beta^2\Delta\bar{x}_t^{t+2} \\ \beta^2\Delta\bar{x}_{1,t}^{t+2} \\ \vdots \\ \beta^{T-1-t}\Delta\bar{x}_t^{T-1} \\ \beta^{T-1-t}\Delta\bar{x}_{1,t}^{T-1} \\ \beta^{T-t}\Delta\bar{q}_t^K \end{bmatrix} \tag{25}$$

for $T \geq 2$ and $t = 1, 2, \ldots, T-2$, and let

$$\begin{aligned} \underbrace{\tilde{L}_{T,t}}_{((2T-2t-1)d_x+K)\times((2T-2t-1)d_x+K)} &\equiv \mathbb{E}(X_{T,t}X'_{T,t}) \text{ and} \\ \underbrace{L_{T,t}}_{((2T-2t-1)d_x+K)\times((2T-3)d_x+K)} &\equiv \begin{bmatrix} 0_{((2T-2t-1)d_x+K)\times(2(t-1)d_x)} & \tilde{L}_{T,t} \end{bmatrix} \end{aligned}$$

for $T \geq 2$ and $t = 1, \ldots, T-1$. Then the identification and over-identification conditions of the parameters, given in the next theorem, depends on the $((T-1)^2 d_x + (T-1)K) \times ((2T-3)d_x + K)$ matrix L_T that stacks $L_{T,t}$ for $t = T-1, \ldots, 1$ (in that order). It is obvious from the definition that

L_T has a block-triangular structure as follows:

$$L_T = \begin{bmatrix} 0_{(d_x+K)\times 2d_x} & 0_{(d_x+K)\times 2(T-5)d_x} & 0_{(d_x+K)\times 2d_x} & 0_{(d_x+K)\times 2d_x} & \mathbb{E}(X_{T,T-1}X'_{T,T-1}) \\ 0_{(3d_x+K)\times 2d_x} & 0_{(3d_x+K)\times 2(T-5)d_x} & 0_{(3d_x+K)\times 2d_x} & \mathbb{E}(X_{T,T-2}X'_{T,T-2}) & \\ 0_{(5d_x+K)\times 2d_x} & 0_{(5d_x+K)\times 2(T-5)d_x} & \mathbb{E}(X_{T,T-3}X'_{T,T-3}) & & \\ \vdots & & \iddots & & \\ 0_{((2T-5)d_x+K)\times 2d_x} & \mathbb{E}(X_{T,2}X'_{T,2}) & & & \\ \mathbb{E}(X_{T,1}X'_{T,1}) & & & & \end{bmatrix}. \quad (26)$$

Theorem 2 below encompasses Proposition 1 and Proposition 2 as special cases.

Theorem 2 (Identification and over-identification for general T)**.** *The parameters of interest are* $(\delta', \gamma^{K\prime})'$.

(i) The parameters are identified if and only if the L_T *matrix defined above has full column rank.*

(ii) If more than one matrix that consists of $(2T-3)d_x + K$ *distinct rows from* L_T *has full rank, then the parameters are over-identified.*

Again, condition (i) in Theorem 2 might hold under multiple distinct sets of sufficient conditions, and two of them are the next two corollaries, which are generalization of Corollary 1 and Corollary 2, respectively.

Corollary 3 (Identification for general T)**.** $(\delta', \gamma^{K\prime})'$ *is identified if the second moment matrix of* $(x'_1, \Delta\bar{x}_1^{2\prime}, \Delta\bar{x}_{1,1}^{2\prime}, \ldots, \Delta\bar{x}_1^{\tau\prime}, \Delta\bar{x}_{1,1}^{\tau\prime}, \ldots, \Delta\bar{x}_1^{T-1\prime}, \Delta\bar{x}_{1,1}^{T-1\prime}, \Delta\bar{q}_1^{K\prime})'$ *has full rank.*

Corollary 4 (Identification for general T)**.** $(\delta', \gamma^{K\prime})'$ *is identified if: (i) the second moment matrix of* $(x'_{T-1}, \Delta\bar{q}_{T-1}^{K\prime})'$ *has full rank (i.e.,* $d_x + K$*); and (ii) the second moment matrix of* $(x'_t, \Delta\bar{x}_t^{t+1\prime})'$ *has full rank (i.e,* $2d_x$*) for all* $t = 1, \ldots, T-2$.

Again, Corollary 3 recognizes that the equations corresponding to $t = 1$ contain all the relevant parameters and have the potential of identifying them all. Corollary 4 takes advantage of the block-triangular structure of L_T to sequentially identify the parameters. Condition (i) in Corollary 4 ensures that Δ_{T-1} and γ^K are identified, which can therefore be regarded as known for subsequent analysis. Condition (ii) in Corollary 4 then sequentially ensures the identification of Δ_t and $\delta_{0,t+1}$ for $t = T-2, \ldots, 1$ (in that order).

Remark 6 (Over-identification)**.** *A few observations are worth emphasizing here. First, the fundamental source of over-identification, as the discussion leading to Proposition 2 indicates, is the multitude of equations in eq. (18), one for each period* $t = 1, \ldots, T-1$*, for a given* $T \geq 3$*. Second, for all* $T \geq 2$*, the equation for period* $t = 1$ *in eq. (18) contains all the parameters of interest, so it is able to identify all the parameters if its Jacobian matrix has full rank (see Corollary 1 and Corollary 3). This observation, when* T *is large, can be utilized to reduce the sensitivity of our estimator* *to Assumption 5 (we will elaborate this in Section 4.2 below). Third, among all the equations in eq. (18), the one for period t contains all the parameters in the one for period t' if $t' > t$, so the matrix L_T has a block-triangular structure. This feature, as will be detailed in Remark 13 below, is useful in dealing with time-invariant variables in x_t.*

The following remarks help deepen the understanding of the source of (over-)identification by analyzing various scenarios. They also underscore that the transformation into a linear system greatly simplifies the identification analysis of DDC models.

Remark 7 (The simplest model)**.** *The simplest version of the model in Remark 5 is when $T = T_{end} = 2$. In this case, we have $\bar{V}_2(s_2) = x_2'\delta_{0,2} + p_2 x_2'\Delta_2 - \eta_2$ by eq. (17). Note that $p_2 x_2'\Delta_2 - \eta_2$ is known as discussed in Remark 5, so $\Delta\mathbb{E}(p_2 x_2'\Delta_2 - \eta_2 \mid s_1)$ can be moved to the left-hand side, and eq. (18a) becomes*

$$y_1 - \beta\Delta\mathbb{E}(p_2 x_2'\Delta_2 - \eta_2 \mid s_1) = x_1'\Delta_1 + \beta\Delta\mathbb{E}(x_2 \mid s_1)'\delta_{0,2}. \tag{27}$$

So, the sufficient and necessary condition in Proposition 1 boils down to none of the variables in the vector $(x_1', \Delta\mathbb{E}(x_2 \mid s_1)')'$ being perfectly linearly correlated with any linear combination of the other variables in the vector. Note that x_1 is a subvector of s_1, so this condition essentially requires that none of the variables in $\Delta\mathbb{E}(x_2 \mid s_1)$ is a purely linear function of x_1 alone.

Remark 8 (A closer look at condition (ii) in Corollary 4)**.** *Recall the definition in eq. (13c), we have $\Delta\bar{x}_t^{t+1} = \Delta\mathbb{E}(x_{t+1} \mid s_t)$ for $t = 1, \ldots, T-2$. So, condition (ii) in Corollary 4 is equivalent to none of the variables in the vector $(x_t', \Delta\mathbb{E}(x_{t+1} \mid s_t)')'$ being perfectly linearly correlated with any linear combination of the other variables in the vector. Similar to Remark 7, this essentially requires that none of the variables in $\Delta\mathbb{E}(x_{t+1} \mid s_t)$ is a purely linear function of x_t alone for $t = 1, \ldots, T-2$.*

Remark 9 (Function of excluded variables)**.** *Excluded variables z_t, when available, can make the identification conditions more likely to hold although, as is clear from Remark 7 and Remark 8, they are not essential for the identification. Consider the least favorable scenario where $\Delta\mathbb{E}(x_{t+1} \mid s_t)$ contains an additive component that is linear in x_t, denoted as $\rho_1 x_t$, with ρ_1 being a $d_x \times d_x$ matrix of constants. If z_t is empty, then $\Delta\mathbb{E}(x_{t+1} \mid s_t) = \rho_1 x_t$, and the identification conditions in Proposition 1 and Corollary 4 fail.*[14] *Suppose z_t is non-empty and there exists a $d_x \times 1$-vector-valued function of d_z arguments and a $d_x \times d_x$ matrix of constants, denoted as $\ell(\cdot) \equiv (\ell_1(\cdot), \ldots, \ell_{d_x}(\cdot))'$ and ρ_2,*[15] *respectively, such that*

$$\underbrace{\begin{bmatrix} x_t \\ \Delta\mathbb{E}(x_{t+1} \mid x_t, z_t) \end{bmatrix}}_{2d_x \times 1} = \underbrace{\begin{bmatrix} I_{d_x} & 0_{d_x \times d_z} \\ \rho_1 & \rho_2 \end{bmatrix}}_{2d_x \times 2d_x} \underbrace{\begin{bmatrix} x_t \\ \ell(z_t) \end{bmatrix}}_{2d_x \times 1}, \tag{28}$$

[14] Condition (i) in Corollary 4 can be analyzed using the same argument as in Remark 8 and this remark, and therefore is omitted for conciseness.

[15] ρ_1, $\ell(\cdot)$ and ρ_2 are allowed to depend on t, due to the nonstationary feature of the model, but it is unnecessary to make this dependence explicit here.

then the following is sufficient for the identification conditions in Proposition 1 and Corollary 4: (a) the second moment matrix of $(x_t', \ell(z_t)')'$ is invertible; and (b) ρ_2 has full rank (i.e., d_x).

Since x_{t+1}, x_t and z_t are all observed and $\Delta\mathbb{E}(x_{t+1} \mid x_t, z_t)$ can be nonparametrically estimated, the plausibility of (a) and (b) can be accessed empirically. It is notable that they do not necessarily restrict d_z, the number of excluded variables. If z_t enters $\Delta\mathbb{E}(x_{t+1} \mid x_t, z_t)$ linearly (i.e., $\ell(\cdot)$ are all linear functions in all arguments), then they require that $d_z \geq d_x$.[16] *The other extreme is that the variation of z_t and the non-linearity in $\ell(\cdot)$ combined are such that the variables in $(x_t', \ell(z_t)')'$ are mutually linearly independent, then one excluded variable suffices. Many cases in between the two extremes are also possible.*

Remark 10 (Consequences of (falsely) normalizing $\delta_{0,t}$)**.** *If one assumes, despite of resulting biased counterfactuals discussed after Assumption 4, that $\delta_{0,t} = 0$ for $t = 1, \ldots, T$, then they will find that the identification condition of Δ_t for $t = 1, \ldots, T-1$ becomes quite simple when $T = T_{end}$. First, because Δ_T is identified as shown in Remark 5, $\Delta\mathbb{E}(p_T x_T' \Delta_T - \eta_T \mid s_{T-1})$ can be regarded as known, then eq. (18a) becomes*

$$y_{T-1} - \beta\Delta\mathbb{E}(p_T x_T' \Delta_T - \eta_T \mid s_{T-1}) = x_{T-1}' \Delta_{T-1},$$

and therefore Δ_{T-1} is identified from this equation under the usual condition that the variables in x_{T-1} are not perfectly correlated. Similarly, eq. (18b) becomes

$$y_t - \beta\Delta\mathbb{E}(p_T x_T' \Delta_T - \eta_T \mid s_{T-1}) - \sum_{\tau=t+1}^{T-1} \beta^{\tau-t} \Delta\bar{x}_{1,t}^{\tau\prime} \Delta_\tau = x_t' \Delta_t$$

for $t = 1, \ldots, T-2$. Because the left-hand side is known from the data after period t, Δ_t is identified recursively under the usual condition that the variables in x_t are not perfectly correlated. Also note that excluded variables play no role in this setting.

Remark 11 (Stationary models)**.** *If the model is stationary, then $\Delta_t = \bar{\Delta}$ and $\delta_{0,t} = \bar{\delta}_0$ for all t. Due to such restrictions on the parameters, eq. (18) becomes*

$$y_{T-1} = x_{T-1}' \bar{\Delta} + \beta\Delta\bar{q}_{T-1}^{K\prime} \gamma^K, \text{ and} \tag{29a}$$

$$y_t = \left(x_t' + \sum_{\tau=t+1}^{T-1} \beta^{\tau-t} \Delta\bar{x}_{1,t}^{\tau\prime} \right) \bar{\Delta} + \left(\sum_{\tau=t+1}^{T-1} \beta^{\tau-t} \Delta\bar{x}_t^{\tau\prime} \right) \bar{\delta}_0 + \beta^{T-t} \Delta\bar{q}_t^{K\prime} \gamma^K \tag{29b}$$

for $t = 1, \ldots, T-2$. The (over-)identification of $\bar{\Delta}$, $\bar{\delta}_0$ and γ^K can therefore be clearly analyzed using eq. (29).

[16] To see this, first note that we can let $\ell(z_t) = z_t$ without loss of generality, then eq. (28) implies that we need $2d_x$ linearly independent random variables (the left-hand side) that are linear combinations of $d_x + d_z$ random variables (the last vector on the right-hand side).

A novel observation about stationary models can be made here: if $T \geq 3$, then stationary models may not even need the normalization $\delta_{0,1} = c$ we made in Assumption 4. To illustrate, consider the case when $T = 3$, in which eq. (29) becomes

$$y_2 = x_2' \bar{\Delta} + \beta \Delta \bar{q}_2^{K\prime} \gamma^K, \text{ and} \tag{30a}$$

$$y_1 = \left(x_1' + \beta \Delta \bar{x}_{1,1}^{2\prime}\right) \bar{\Delta} + \left(\beta \Delta \bar{x}_1^{2\prime}\right) \bar{\delta}_0 + \beta^2 \Delta \bar{q}_1^{K\prime} \gamma^K. \tag{30b}$$

Then $\bar{\Delta}$ and γ^K are identified from eq. (30a) if the variables in x_2 and $\Delta \bar{q}_2^K$ are not perfectly correlated, and then $\bar{\delta}_0$ is identified from eq. (30b) if the variables in $\Delta \bar{x}_1^2$ are not perfectly correlated. Since $\delta_{0,t} = \bar{\delta}_0$ for all t, $\delta_{0,1}$ is identified from the data rather than relying on an arbitrary normalization. The intuition, as we mentioned in the discussion after Assumption 4, is that the inter-temporal variation in dynamic models provides extra identification source than static models, even if the models are stationary.

Remark 12 (Identification with unknown discount factor)**.** *So far, our analysis assumes that the discount factor β is known, but in fact, β can be easily identified with a slightly stronger condition. For notational conciseness, we focus on the $T = 3$ case, and the key idea is clearly illustrated by strengthening the conditions for the sequential identification approach in Corollary 2.*

Under condition (i) in Corollary 2 (i.e., the second moment matrix of $(x_2', \Delta \bar{q}_2^{K\prime})'$ has full rank), $(\Delta_2', \beta \gamma^{K\prime})'$ are identified through eq. (18a) and can then be regarded as known. The unique solution of eq. (18b), on the other hand, is equivalent to the unique solution of

$$\begin{bmatrix} \mathbb{E}(x_1 y_1) \\ \mathbb{E}(\Delta \bar{x}_1^2 y_1) \\ \mathbb{E}(\Delta \bar{x}_{1,1}^2 y_1) \\ \mathbb{E}(\Delta \bar{q}_1^K y_1) \end{bmatrix} = \left[\begin{array}{cccc} \mathbb{E}(x_1 x_1') & \mathbb{E}(x_1 \Delta \bar{x}_1^{2\prime}) & \mathbb{E}(x_1 \Delta \bar{x}_{1,1}^{2\prime}) & \mathbb{E}(x_1 \Delta \bar{q}_1^{K\prime}) \\ \mathbb{E}(\Delta \bar{x}_1^2 x_1') & \mathbb{E}(\Delta \bar{x}_1^2 \Delta \bar{x}_1^{2\prime}) & \mathbb{E}(\Delta \bar{x}_1^2 \Delta \bar{x}_{1,1}^{2\prime}) & \mathbb{E}(\Delta \bar{x}_1^2 \Delta \bar{q}_1^{K\prime}) \\ \hdashline \mathbb{E}(\Delta \bar{x}_{1,1}^2 x_1') & \mathbb{E}(\Delta \bar{x}_{1,1}^2 \Delta \bar{x}_1^{2\prime}) & \mathbb{E}(\Delta \bar{x}_{1,1}^2 \Delta \bar{x}_{1,1}^{2\prime}) & \mathbb{E}(\Delta \bar{x}_{1,1}^2 \Delta \bar{q}_1^{K\prime}) \\ \mathbb{E}(\Delta \bar{q}_1^K x_1') & \mathbb{E}(\Delta \bar{q}_1^K \Delta \bar{x}_1^{2\prime}) & \mathbb{E}(\Delta \bar{q}_1^K \Delta \bar{x}_{1,1}^{2\prime}) & \mathbb{E}(\Delta \bar{q}_1^K \Delta \bar{q}_1^{K\prime}) \end{array}\right] \left[\begin{array}{c} \Delta_1 \\ \beta \delta_{0,2} \\ \hdashline \beta \Delta_2 \\ \beta(\beta \gamma^K) \end{array}\right], \tag{31}$$

where the last $d_x + K$ subvector of the unknown vector is proportional to the known parameters $(\Delta_2', \beta \gamma^{K\prime})'$ by a factor of β, and the first $2d_x$ subvector of the unknown vector is the unknown parameters $(\Delta_1', \beta \delta_{0,1}')'$. It is obvious, therefore, that β and $(\Delta_1', \beta \delta_{0,1}')'$ can be identified if the second moment matrix of the vector that consists of $(x_1', \Delta \bar{x}_1^{2\prime})'$ and any one single variable from $(\Delta \bar{x}_{1,1}^{2\prime}, \Delta \bar{q}_1^{K\prime})'$ has full rank, which is only slightly stronger than condition (ii) in Corollary 2 (i.e., the second moment matrix of $(x_1', \Delta \bar{x}_1^{2\prime})'$ has full rank).

A subtle identification requirement is made more explicit by our discussion in Remark 7 to Remark 12 – the choice a_t must exert impact on the next-period state variables x_{t+1}, otherwise at least some coordinates of $\Delta \mathbb{E}(x_{t+1} | s_t)$ will be zeros, and the corresponding coefficients in δ (or γ^K) cannot be identified.

This requirement will be violated if x_t contains an intercept or some variables that remain time-invariant for the entire sample duration, such as birthplace. Another perspective to understand this problem is by recalling the paragraph after Assumption 4, where we mentioned that the inter-temporal variation in the state variables x_t can be utilized to relax the common "normalization" $\delta_{0,t} = 0$ for all t in the DDC literature. Therefore, if the inter-temporal variation in x_t is absent, proper normalization is needed, which is characterized by the next remark.

Remark 13 (Identification with intercept or time-invariant variables in x_t)**.** *To accentuate the identification issue that arises due to time-invariant variables in x_t, here we focus on the case where the value of only the first coordinate of x_t, denoted by $x_{t,1}$, remains time-invariant for each agent for the entire sample duration. For conciseness, we also assume that none of the functions in $\Delta\mathbb{E}(q^K(s_t)\,|\,s_{T-1})$ is a function of $x_{T-1,1}$ alone so that Δ_{T-1} and γ^K are identified. The general case where multiple coordinates of x_t are time-invariant, as well as the minor modification needed to accommodate them in identification and estimation, is elaborated in Appendix D.1.*

A problem emerges when one tries to identify $\delta_{0,t}$ $(t = 2, \dots, T-1)$ in the presence of a time-invariant $x_{t,1}$. This problem is best understood via the sequential approach in Corollary 4. Take eq. (18b) for $t = T-2$ as an example, it becomes

$$y_{T-2} - \beta\Delta\bar{x}_{1,T-2}^{T-1\prime}\Delta_{T-1} - \beta^2\Delta\bar{q}_{T-2}^{K\prime}\gamma^K = x_{T-2}'\Delta_{T-2} + \beta\Delta\bar{x}_{T-2}^{T-1\prime}\delta_{0,T-1}, \tag{32}$$

where the left-hand side is known, and the first coordinate of $\Delta\bar{x}_{T-2}^{T-1}$, i.e., $\Delta\mathbb{E}(x_{T-1,1}|x_{T-2}, z_{T-2})$, is zero, because $x_{T-1,1} = x_{T-2,1}$ for both $a_{T-2} = 1$ and $a_{T-2} = 0$. In consequence, the first coordinate of $\delta_{0,T-1}$ cannot be identified. Note, however, that eq. (32) holds regardless of its value. On the other hand, all coordinates of Δ_{T-2} can be identified using eq. (32), including the coefficient for $x_{T-2,1}$ (just like how intercept can be identified in a linear regression).

Then proceed to eq. (18b) for $t = T-3$, and it becomes

$$\begin{aligned} &y_{T-3} - \beta\Delta\bar{x}_{1,T-3}^{T-2\prime}\Delta_{T-2} - \beta^2\Delta\bar{x}_{1,T-3}^{T-1\prime}\Delta_{T-1} - \beta^2\Delta\bar{x}_{T-3}^{T-1\prime}\delta_{0,T-1} - \beta^3\Delta\bar{q}_{T-3}^{K\prime}\gamma^K \\ &= x_{T-3}'\Delta_{T-3} + \beta\Delta\bar{x}_{T-3}^{T-2\prime}\delta_{0,T-2}, \end{aligned} \tag{33}$$

where the left-hand side is known. To see this, note that even though the first coordinate of $\delta_{0,T-1}$ is not identified, it does not matter since the corresponding coordinate of $\Delta\bar{x}_{T-3}^{T-1}$, i.e., $\Delta\mathbb{E}(x_{T-1,1}\,|\,x_{T-3}, z_{T-3})$, is zero, because $x_{T-1,1} = x_{T-2,1}$ for both $a_{T-3} = 1$ and $a_{T-3} = 0$, by essentially the same argument as before. This means that we are free to normalize the first coordinate of $\delta_{0,T-1}$ to any real value, and it will not affect the identification and estimation of the other parameters.[17]

This argument proceeds recursively for all earlier periods, and a few insights can already be drawn from our detailed analysis of the last three periods. First, Δ_t for $t = 1, \dots, T-1$ are identified even

[17] For the counterfactual scenario where $x_{t,1}$ remains the same value for the entire time, this normalization does not affect the counterfactual analysis either.

in the presence of time-invariant variable, and this is not surprising because only contemporaneous cross-sectional variation is required to identify the difference in flow utility between the two choices (recall static binary choice models). Second, the first coordinates of $\delta_{0,t}$ for $t = 2, \ldots, T-1$ cannot be identified if $x_{t,1}$ remains time-invariant for each agent for the entire sample duration (since condition (ii) in Corollary 4 fails), but their values do not affect the identification and estimation of the other parameters, and therefore they can be normalized to arbitrary values. More generally, if there are other coordinates in x_t that remain time-invariant for each agent for the entire sample duration, then by the same argument, the corresponding coordinates in $\delta_{0,t}$ for $t = 2, \ldots, T-1$ cannot be identified but can be arbitrarily normalized. Third, the coordinates of $\delta_{0,t}$ for $t = 2, \ldots, T-1$ that correspond to time-varying variables in x_t can be identified. Theorem 2' in Appendix D gives the modified sufficient and necessary condition for the identification of these parameters as well as Δ_t for $t = 1, \ldots, T-1$. It is worth pointing out that time-varying variables do not *have to change value for* every *agent in* every *period. As is clear from the above argument, the identification of the coordinates of $\delta_{0,t}$ ($t = 2, \ldots, T-1$) only requires the corresponding coordinates of x_t to change value for* some *agent in* every *period so that the corresponding coordinates of $\Delta\mathbb{E}(x_{T-1} \mid s_t)$ are non-zeros at the population level. As a result, the coefficients in $\delta_{0,t}$ ($t = 2, \ldots, T-1$) of both intermittently time-varying variables (e.g., marital status, highest degree) and continually time-varying variables (e.g., age, disposable income) are identified.*

3.2 Estimation

This subsection provides an estimation procedure of the flow utility parameters based on a random sample of size N, where for each agent $i \in \{1, \ldots, N\}$ we observe $D_{i,t} \equiv (a_{i,t}, s'_{i,t})' = (a_{i,t}, x'_{i,t}, z'_{i,t})'$ for every period $t \in \mathcal{T}_{da} = \{1, \ldots, T\}$. Let D_i denote all observed variables for agent i.

We use p to collectively denote all CCPs, use $\Delta\bar{\eta}$ to collectively denote all $\Delta\bar{\eta}_t^\tau$ ($t = 1, \ldots, T-2$, $t < \tau \leq T-1$), use $\Delta\bar{x}$ to collectively denote all $\Delta\bar{x}_{1,t}^\tau$ and $\Delta\bar{x}_t^\tau$ ($t = 1, \ldots, T-2$, $t < \tau \leq T-1$), and use $\Delta\bar{q}^K$ to collectively denote all $\Delta\bar{q}_t^K$ ($t = 1, \ldots, T-1$). In light of the identification analysis in Section 3.1.3, the parameters δ and γ^K can be estimated via minimum distance (MD) using the following moment functions:

$$m_{T-1}(D, \delta, \gamma^K, p, \Delta\bar{\eta}, \Delta\bar{x}, \Delta\bar{q}^K) \equiv -v_{T-1}(D, \delta, \gamma^K, p, \Delta\bar{\eta}, \Delta\bar{x}, \Delta\bar{q}^K) X_{T,T-1}, \text{ and} \tag{34a}$$

$$m_t(D, \delta, \gamma^K, p, \Delta\bar{\eta}, \Delta\bar{x}, \Delta\bar{q}^K) \equiv -v_t(D, \delta, \gamma^K, p, \Delta\bar{\eta}, \Delta\bar{x}, \Delta\bar{q}^K) X_{T,t}, \tag{34b}$$

where $v_{T-1}(D, \delta, \gamma^K, p, \Delta\bar{\eta}, \Delta\bar{x}, \Delta\bar{q}^K) \equiv y_{T-1} - x'_{T-1}\Delta_{T-1} - \beta\Delta\bar{q}_{T-1}^{K\prime}\gamma^K$, $v_t(D, \delta, \gamma^K, p, \Delta\bar{\eta}, \Delta\bar{x}, \Delta\bar{q}^K) \equiv y_t - x'_t\Delta_t - \sum_{\tau=t+1}^{T-1} \beta^{\tau-t}\Delta\bar{x}_t^{\tau\prime}\delta_{0,\tau} - \sum_{\tau=t+1}^{T-1} \beta^{\tau-t}\Delta\bar{x}_{1,t}^{\tau\prime}\Delta_\tau - \beta^{T-t}\Delta\bar{q}_t^{K\prime}\gamma^K$ for $t = 1, \ldots, T-2$, and $X_{T,t}$ for $t = 1, \ldots, T-1$ are defined in eq. (25). Let $m(D, \delta, \gamma^K, p, \Delta\bar{\eta}, \Delta\bar{x}, \Delta\bar{q}^K)$ be the stack of the moment functions $m_t(D, \delta, \gamma^K, p, \Delta\bar{\eta}, \Delta\bar{x}, \Delta\bar{q}^K)$ for $t = T-1, \ldots, 1$ (in that order), which is a $((T-1)^2 d_x + (T-1)K) \times 1$-vector-valued moment function that equals zeros under the true values of δ and γ^K. In this moment function, the nuisance parameters p, $\Delta\bar{\eta}$, $\Delta\bar{x}$ and $\Delta\bar{q}^K$, which are all conditional mean (difference) functions, need to be estimated in preliminary steps. Recall that $\Delta\bar{\eta}$, $\Delta\bar{x}$ and $\Delta\bar{q}^K$ all take the common form of $\Delta\mathbb{E}(h_\tau \mid x_t, z_t)$ ($t = 1, \dots, T-2$, $t < \tau \le T-1$) in eq. (2), where the generic notation h_τ denotes a random variable that is either directly observable or identified after knowing the CCP functions p_τ. The specific expressions of h_τ for $\Delta\bar{\eta}_t^\tau$, $\Delta\bar{x}_t^\tau$ and $\Delta\bar{q}_t^K$ are known and are given in eq. (14), eqs. (13a) to (13c), and eq. (19), respectively.

To summarize, the estimation proceeds in the following three steps.

(I) For each $t = 1, \dots, T-1$, use the data $\{D_{i,t}\}_{i=1}^N$ to estimate the CCP function p_t in eq. (4) and obtain the estimated CCP values $\hat{p}_{i,t}$ for each agent $i \in \{1, \dots, N\}$.[18]

(II) Obtain $\hat{h}_{i,\tau}$ by substituting unknown $p_{i,\tau}$ with $\hat{p}_{i,\tau}$ whenever applicable. Then, for each $t = 1, \dots, T-2$ and $t < \tau \le T-1$, use the data $\{\hat{h}_{i,\tau}, D_{i,t}\}_{i=1}^N$ to estimate the corresponding conditional mean difference functions in $\widehat{\Delta\bar{\eta}}$, $\widehat{\Delta\bar{x}}$ and $\widehat{\Delta\bar{q}}^K$ and obtain their estimated values for each agent $i \in \{1, \dots, N\}$.

(III) Let $\bar{m}_N(\delta, \gamma^K) \equiv \frac{1}{N}\sum_{i=1}^N \hat{m}(D_i, \delta, \gamma^K)$, where $\hat{m}(D_i, \delta, \gamma^K) \equiv m(D_i, \delta, \gamma^K, \hat{p}, \widehat{\Delta\bar{\eta}}_i, \widehat{\Delta\bar{x}}_i, \widehat{\Delta\bar{q}}_i^K)$. Then, $(\hat{\delta}', \hat{\gamma}_K')'$ is the MD estimator as follows:

$$(\hat{\delta}', \hat{\gamma}^{K\prime})' \equiv \underset{\delta\in\mathbb{R}^{(2T-3)d_x}, \gamma^K\in\mathbb{R}^K}{\arg\min} \bar{m}_N(\delta, \gamma^K)' W_N \bar{m}_N(\delta, \gamma^K), \tag{35}$$

where W_N is a $\big((T-1)^2 d_x + (T-1)K\big) \times \big((T-1)^2 d_x + (T-1)K\big)$ symmetric weighting matrix that converges in probability to a positive definite matrix W as $N \to \infty$.

Although the unique solution to eq. (35) has an explicit-form solution due to the linearity of $\bar{m}_N$ in δ and γ^K, we skip it for conciseness, since the the convexity of the objective function in eq. (35) makes it easy to solve numerically.

Remark 14 (Advantages of our estimator)**.** *The advantages of our CCP-based semiparametric estimator, compared to the HM estimators recounted in Section 2.2, are threefold. First, it avoids the estimation of the state transition distributions $f_{s_t|s_{t-1},a_{t-1}}$. Estimating the conditional means is a problem with smaller dimension and better finite-sample properties than the state transition distributions, which can be difficult to estimate or simulate nonparametrically, especially when s_t has a large dimension or contains both continuous and discrete variables (e.g., spouse income and number of young children). The fact that any finite-order Markov process of finite number of variables can be rewritten as a first-order Markov process by expanding the state vector exacerbates the large-dimension problem even further. Even if a parametric specification is used, specifying a conditional*

[18] The CCP functions could be parametrically or nonparametrically, and for generality we assume this is done nonparametrically. The same goes for other unknown functions below unless indicated otherwise.

mean model is easier than specifying the entire conditional distribution. Second, unlike the HM estimators, which require simulating from the estimated state transition distributions, our estimator only uses the observed data, and therefore is much less demanding computationally. (In the simulation experiments in Section 5 below, the HM estimator is a thousand times more time-consuming than our estimator.) Third, although estimating, simulating from state transition distributions is unavoidable for counterfactual analysis, our state-transition-distribution-free estimator could still serve as a benchmark to guide the choice among parametric state transition models.

Remark 15 (Estimation step (II))**.** *The conditional mean difference functions $\widehat{\Delta\bar{\eta}}$, $\widehat{\Delta\bar{x}}$ and $\widehat{\Delta\bar{q}}^K$, taking the common form of $\Delta\mathbb{E}(h_\tau \mid x_t, z_t) = \mathbb{E}(h_\tau \mid x_t, z_t, a_t = 1) - \mathbb{E}(h_\tau \mid x_t, z_t, a_t = 0)$, resemble the average treatment effect in the program evaluation and the missing data literature.*[19] *Many estimators from that literature can therefore be used, and here we briefly discuss two of them.*

A "inverse probability weighted" (IPW) estimator is based on a useful identity $\Delta\mathbb{E}(h_\tau \mid x_t, z_t) = \mathbb{E}(\phi(h_\tau, a_t, p_t) \mid x_t, z_t)$, where

$$\phi(h_\tau, a_t, p_t) \equiv \frac{a_t h_\tau}{p_t} - \frac{(1-a_t)h_\tau}{1-p_t}, \tag{36}$$

and the function $\Phi(x_t, z_t) \equiv \mathbb{E}(\phi(h_\tau, a_t, p_t) \mid x_t, z_t)$ can be estimated by regressing $\phi(\hat{h}_{i,\tau}, a_{i,t}, \hat{p}_{i,t})$ on $x_{i,t}$ and $z_{i,t}$.[20] *Then, the estimated value of the corresponding function for agent i equals to $\widehat{\Phi}(x_{i,t}, z_{i,t})$.*

A "conditional mean projection" (CEP) estimator (or "imputation" estimator) estimates the function $\Phi_1(x_t, z_t) \equiv \mathbb{E}(h_\tau \mid x_t, z_t, a_t = 1)$ using the subsample $\{\hat{h}_{i,\tau}, x_{i,t}, z_{i,t}\}_{a_{i,t}=1}$ and estimates the function $\Phi_0(x_t, z_t) \equiv \mathbb{E}(h_\tau \mid x_t, z_t, a_t = 0)$ using the subsample $\{\hat{h}_{i,\tau}, x_{i,t}, z_{i,t}\}_{a_{i,t}=0}$ for $t = 1, \ldots, T-2$, $t < \tau \leq T-1$. Then, the estimated value of the conditional mean difference for agent i equals to $\widehat{\Phi}_1(x_{i,t}, z_{i,t}) - \widehat{\Phi}_0(x_{i,t}, z_{i,t})$.

[19] It is worth pointing out that the goal of step (II) is simply to estimate the conditional means of *realized* (directly observed or previously identified) h_τ given x_t, z_t and a_t, not the (unconditional or conditional) means of "potential outcomes" like in the program evaluation literature. As a result, the "ignorability" condition, which serves the purpose of equating the conditional means of realized variables to structural parameters in the potential outcome model, is *not* required to hold. On the other hand, one can conceptualize the realized h_τ in a future period τ ($\tau > t$) as $h_\tau = a_t h_\tau^{(1)} + (1-a_t)h_\tau^{(0)}$, where $h_\tau^{(1)}$ and $h_\tau^{(0)}$ are "potential outcomes" for scenarios $a_t = 1$ and $a_t = 0$, respectively. Then the DDC model studied in this paper (particularly, Assumptions 1 to 3) permits "ignorability" in the sense that Assumptions 1 to 3 do not necessarily imply or exclude $(h_\tau^{(1)}, h_\tau^{(0)}) \perp\!\!\!\perp a_t \mid x_t, z_t$, and vice versa.

[20] Equation (36) holds due to the Law of Total Probability. To see this, note that

$$\begin{aligned}\mathbb{E}(a_t h_\tau \mid x_t, z_t) &= \mathbb{E}(a_t h_\tau \mid x_t, z_t, a_t = 1)\Pr(a_t = 1 \mid x_t, z_t) + \mathbb{E}(a_t h_\tau \mid x_t, z_t, a_t = 0)\Pr(a_t = 0 \mid x_t, z_t) \\ &= \mathbb{E}(h_\tau \mid x_t, z_t, a_t = 1)p_t(x_t, z_t).\end{aligned}$$

Because p_t is a function of x_t, z_t only, it can be moved to the left-hand side of the equation and into the conditional expectation, implying $\mathbb{E}(a_t h_\tau / p_t \mid x_t, z_t) = \mathbb{E}(h_\tau \mid x_t, z_t, a_t = 1)$. Similarly, it can be shown that $\mathbb{E}((1-a_t)h_\tau/(1-p_t) \mid x_t, z_t) = \mathbb{E}(h_\tau \mid x_t, z_t, a_t = 0)$. Again, note that the "ignorability" condition is *not* required for eq. (36) to hold.

Both the IPW and the CEP estimators are shown to be $\sqrt{N}$-consistent and asymptotically normal (Hirano, Imbens and Ridder, 2003; Chen, Hong and Tamer, 2005), but the CEP estimator requires weaker regularity conditions. In particular, it does not require the CCPs to be uniformly bounded away from zero and one. Suppose the IPW estimator is used in step (II) and fulfills the regularity conditions in Section 5 of Newey (1994*a*), then Proposition 3 below gives the asymptotic distribution of $\hat{\delta}$ by characterizing its influence function. Its derivation heavily uses Newey (1994*a*)'s pathwise-derivative-based characterization of the influence function of semiparametric estimators, which is relegated to Appendix C. Meanwhile, Chen, Hong and Tarozzi (2008) show that the IPW and the CEP estimators are asymptotically equivalent (eqs. (7) and (12), as well as the discussion on p.819), so if the CEP estimator is used in step (II), the asymptotic distribution of $\hat{\delta}$ will remain the same as in Proposition 3.[21]

To state the proposition, we let $m_{t,0}(D) \equiv m_t(D, \delta_0, \gamma_0^K, p_0, \Delta\bar{\eta}_0, \Delta\bar{x}_0, \Delta\bar{q}_0^K)$ for $t = 1, \ldots, T-1$ and stack them to $m_0(D) \equiv (m_{T-1,0}(D), \ldots, m_{1,0}(D))'$. Here and in the rest of this paper, we use the subscript "$_{,0}$", with a comma in the front and after all the other subscripts (if applicable), to emphasize that all the parameters (potentially infinite-dimensional) that the object involves take the true values. The influence function of $(\hat{\delta}', \hat{\gamma}^{K\prime})'$ takes the form

$$\psi(D) \equiv -(L_T' W L_T)^{-1} L_T' W \left(m_0(D) + \alpha(D)\right), \tag{37}$$

where

$$\alpha(D) \equiv \alpha_\eta(D) + \alpha_x(D) + \alpha_q(D) + \alpha_{p,direct}(D) + \alpha_{p,indirect}(D),$$

L_T is defined in eq. (26), and $\alpha_\eta(D)$, $\alpha_x(D)$, $\alpha_q(D)$, $\alpha_{p,direct}(D)$ and $\alpha_{p,indirect}(D)$ are defined in eq. (C.2), eq. (C.3), eq. (C.4), eq. (C.5) and eq. (C.6) in Appendix C, respectively.

Proposition 3 (Asymptotic distribution of $\hat{\delta}$)**.** *Under Assumptions 1 to 5 and the regularity conditions in Section 5 of Newey (1994a), the three-step CCP-based semiparametric estimator $\hat{\delta}$ defined in eq. (35) has the asymptotic distribution*

$$\sqrt{N}\left(\hat{\delta} - \delta\right) \xrightarrow{d.} \mathcal{N}\left(0, V\right),$$

where $V \equiv \mathbb{E}(\psi_\delta(D_i)\psi_\delta'(D_i))$, and $\psi_\delta(\cdot)$ is the first $(2T-3)d_x$ coordinates of the influence function $\psi(D)$ in eq. (37).

To use Proposition 3 for inference, a consistent estimator of the asymptotic variance V is necessary. We follow Lemma 5.4 of Newey (1994*a*) to give a general approach to $\widehat{V}$, although for specific nonparametric conditional mean estimators (e.g., kernel or series) used in steps (I) and (II), simpler estimators are often available (Newey, 1994*a*,*b*). Let

$$\widehat{\widetilde{L}}_{T,T-1} \equiv \begin{bmatrix} \frac{1}{N}\sum_{i=1}^N x_{i,T-1}x_{i,T-1}' & \frac{\beta}{N}\sum_{i=1}^N x_{i,T-1}\widehat{\Delta\bar{q}}_{i,T-1}^{K\prime} \\ \frac{\beta}{N}\sum_{i=1}^N \widehat{\Delta\bar{q}}_{i,T-1}^{K} x_{i,T-1}' & \frac{\beta^2}{N}\sum_{i=1}^N \widehat{\Delta\bar{q}}_{i,T-1}^{K}\widehat{\Delta\bar{q}}_{i,T-1}^{K\prime} \end{bmatrix},$$

[21] We will use the CEP estimator in step (II) in our simulation experiments (Section 5).

where $\widehat{\Delta\bar{q}}_{i,T-1}^K \equiv \widehat{\Delta\mathbb{E}}\left(q^K(x_T, z_T) \mid x_{i,T-1}, z_{i,T-1}\right)$. Define $\widehat{\tilde{L}}_{T,t}$ similarly for $t = 1, \ldots, T-2$ so that we obtain $\widehat{L}_T$. In addition, let $\hat{\alpha}_\eta(D_i)$, $\hat{\alpha}_x(D_i)$, $\hat{\alpha}_q(D_i)$, $\hat{\alpha}_{p,direct}(D_i)$ and $\hat{\alpha}_{p,indirect}(D_i)$ be estimators of $\alpha_\eta(D_i)$, $\alpha_x(D_i)$, $\alpha_q(D_i)$, $\alpha_{p,direct}(D_i)$ and $\alpha_{p,indirect}(D_i)$, respectively, with unknown parameters and functions in the latter be replaced by the corresponding estimates obtained via our estimator. Define

$$\begin{aligned}
\hat{\alpha}(D_i) &\equiv \hat{\alpha}_\eta(D_i) + \hat{\alpha}_x(D_i) + \hat{\alpha}_q(D_i) + \hat{\alpha}_{p,direct}(D_i) + \hat{\alpha}_{p,indirect}(D_i),\\
\hat{\varphi}(D_i) &\equiv m(D_i, \hat{\delta}, \hat{\gamma}^K, \hat{p}, \widehat{\Delta\bar{\eta}}, \widehat{\Delta\bar{x}}, \widehat{\Delta\bar{q}}) + \hat{\alpha}(D_i),\\
\widehat{\Sigma} &\equiv \frac{1}{N}\sum_{i=1}^N \hat{\varphi}(D_i)\hat{\varphi}(D_i)', \text{ and}\\
\widehat{V} &\equiv (\widehat{L}_T' W_N \widehat{L}_T)^{-1}\widehat{L}_T' W_N \widehat{\Sigma} W_N \widehat{L}_T (\widehat{L}_T' W_N \widehat{L}_T)^{-1}.
\end{aligned} \tag{38}$$

Proposition 4 (Consistent estimator of asymptotic variance)**.** *Under Assumptions 1 to 5 and the regularity conditions in Section 5 of Newey (1994a), $\widehat{V} \xrightarrow{p.} V$.*

The results in this subsection premise that none of the variables in x_t is time-invariant, and, according to Remark 13, all coordinates of δ and γ^K are identified. To accommodate time-invariant variables in x_t, we merely need to slightly revise the notation to exclude the unidentified (and unnecessary to estimate) coordinates of $\delta_{0,t}$, and all the results in this subsection essentially remain unchanged for the other coordinates. We elaborate this in Appendix D.2.

4 Sensitivity of Estimation to Assumption 5

The sample-terminal-period integrated value function $\bar{V}_T(s_T)$ oftentimes has a different form from that in Assumption 5. In this section, we will first quantify the estimation bias of the flow utility parameter induced by imposing Assumption 5 when it might be violated. Then, we will discuss how to utilize the over-identification opportunity in the linear system eq. (18) to reduce the impact of Assumption 5 on estimation.

4.1 Bias in Estimation from Imposing Assumption 5

The parametric form $\bar{V}_T(s_T) = q^K(x_T, z_T)'\gamma^K$ in Proposition 3 can be interpreted as an approximation of $\bar{V}_T(s_T)$, especially when one uses common series basis functions, such as power series, as $q^K(x_T, z_T)$. Define the approximation error

$$r^K(x_T, z_T) \equiv \bar{V}_T(s_T) - q^K(x_T, z_T)'\gamma^K$$

and its expected difference in period t ($t = 1, \ldots, T-1$)

$$\Delta\bar{r}_t^K \equiv \Delta\mathbb{E}(r^K(x_T, z_T) \mid x_t, z_t).$$

Next lemma shows that the expected difference of the approximation error diminishes rapidly with K, the dimension of the approximation of $\bar{V}_T(s_T)$.

Lemma 4 (Approximation error $\Delta\bar{r}_t^K$ due to approximating $\bar{V}_T(s_T)$ by power series)**.** *Suppose that*

(i) $q^K(x_T, z_T)$ *is a triangular sequence of powers of* (x_T, z_T);[22]

(ii) $K = (k+1)^{d_s}$, *where* $d_s = d_x + d_z$ *is the dimension of* s_T, *so that* $q^K(x_T, z_T)$ *has powers in all state variables at least up to* k;

(iii) the support $\mathcal{S}$ *of* $s_T = (x_T, z_T)$ *is a compact subset of* $\mathbb{R}^{d_s}$;

(iv) $\bar{V}_T(s_T)$ *is* m *times continuously differentiable.*

Then there exists γ^K *such that the resulting approximation error satisfies*

$$\mathbb{E}\big((\Delta\bar{r}_t^K)^2\big) = O\left(K^{-\frac{2m}{d_s}}\right), \text{ for } t = 1, \dots, T-1.$$

Let δ_{pseudo}^K denote the probability limit of our estimator $\hat{\delta}$ in eq. (35). Because the $\bar{m}_N$ in eq. (35) is linear in δ and γ^K, we know that δ_{pseudo}^K exists, is unique and has an explicit-form solution under the conditions of Theorem 2 (see, for example, Newey and McFadden, 1994).[23] These properties help translate the approximation error of $\bar{V}_T(s_T)$ to the asymptotic bias $\delta_{pseudo}^K - \delta$ straightforwardly.

Theorem 3 (Asymptotic bias bound of $\hat{\delta}$ due to approximating $\bar{V}_T(s_T)$ by power series)**.** *Suppose Assumptions 1 to 4 and the conditions (i) to (iv) in Lemma 4 hold. Also suppose that* $q^K(x_T, z_T)$ *and* x_t *for* $t = 1, \dots, T-1$ *all have finite second moments,* $0 < \lambda_{min}(W) \le \lambda_{max}(W) < \infty$, *and* $0 < \lambda_{min}(L'L) \le \lambda_{max}(L'L) < \infty$, *where* λ_{min} *and* λ_{max} *denote the smallest and the largest eigenvalues, respectively. Let* $\|\cdot\|$ *denote the Frobenius norm, then we have*

$$\|\delta_{pseudo}^K - \delta\| = O\left(K^{-\frac{m}{d_s}}\right).$$

4.2 Reducing Sensitivity of Estimation to Assumption 5

When the sample horizon is long, an advantage of the HM estimators is that the sample-terminal-period integrated value function $\bar{V}_T(s_T)$ has a negligible impact on estimation. This is visible from eq. (7), where only the last term contains $\bar{V}_{T_{tr}}(s_{T_{tr}})$ and the pre-multiplied factor $\beta^{T_{tr}-t-1}$ quickly approaches zero as $T_{tr} - t$ increases. In fact, a typical implementation of the HM estimators would truncate and ignore the last term of eq. (7) for this reason. The same idea, combine with the

[22] That is, power of each state variable is added to $q^K(x_T, z_T)$ one by one, and $q^K(x_T, z_T)$ includes all lower order powers before adding any higher order power. See eq. (22) in Hirano, Imbens and Ridder (2003) for details.

[23] In addition, Proposition 3 continues to hold with δ replaced by δ_{pseudo}^K.

over-identification opportunity recognized in Remark 6, can be leveraged to reduce the sensitivity of our estimator to Assumption 5.

Start by considering eq. (15b) for $t = 1$. It contains the entire parameter vector δ and is linear in δ; only the last term of it contains $\bar{V}_T(s_T)$ and the pre-multiplied factor β^{T-1} quickly approaches zero as T increases. Truncating the last term, eq. (15b) for $t = 1$ becomes a linear equation

$$y_1 = x_1'\Delta_1 + \sum_{\tau=2}^{T-1} \beta^{\tau-1}\Delta\bar{x}_t^{\tau\prime}\delta_{0,\tau} + \sum_{\tau=2}^{T-1} \beta^{\tau-1}\Delta\bar{x}_{1,1}^{\tau\prime}\Delta_\tau,$$

which is capable of identifying the entire vector δ if the usual rank condition holds.[24] Similarly, eq. (15b) for $t = 2$, when truncating the last term, is capable of identifying the last $(2T-5)d_x$ coordinates of δ; and the same logic applies to those t for which β^{T-t} is reasonably small. Therefore, the truncated version of eq. (15b) for the first few periods constitute a linear system that is insensitive to Assumption 5, but at the same time still provides ample over-identification opportunity.

5 Simulation Experiments

Using a simple three-period model, we compare the performance of our three-step semiparametric estimator in Section 3.2, referred to as the CRS estimator henceforth, with that of a generic nonparametric implementation of the HM estimator. The simulation results below substantiate our discussion in Remark 14 – the CRS estimator presents comparable or better estimates than the HM estimator, yet it requires fewer assumptions in certain scenarios and is substantially easier to implement than the HM estimator.

5.1 Model Specification and Parameterization

We consider a three-period model (i.e., $T_{start} = 1$ and $T_{end} = 3$) with two time-varying observed state variables $x_{t,1}$ and $x_{t,2}$ and an intercept (i.e., $d_x = 3$), as well as one observed excluded variable z_t, for $t = 1, 2, 3$. Let the flow utility function be

$$u_t(a, x_t) = \delta_{a,t,0} + \delta_{a,t,1}x_{t,1} + \delta_{a,t,2}x_{t,2}, \tag{39}$$

with the parameters for $a = 1$ being $(\delta_{1,t,0}, \delta_{1,t,1}, \delta_{1,t,2}) = (-1, 0.4, 0.5)$ and the parameters for $a = 0$ being $(\delta_{0,t,0}, \delta_{0,t,1}, \delta_{0,t,2}) = (2.9, -0.5, -0.8)$, all remaining invariant for all periods $t = 1, 2, 3$.

The time-varying state variables $s_t = (x_{1t}, x_{2t}, z_t)'$ follow time-invariant choice-specific VAR(1) transition processes:

$$(x_{t,1}, x_{t,2}, z_t)' = c_a + A_a(x_{t-1,1}, x_{t-1,2}, z_{t-1})' + w_{at},$$

[24] Precisely, if the upper-left $((2T-3)d_x) \times ((2T-3)d_x)$ submatrix of $\tilde{L}_1$ has full rank.

where A_a is a 3×3 matrix, c_a is a 3×1 vector, and $w_{at} \sim N(0_{3\times1}, \Sigma_a)$ and independent across $a = 0, 1$ and $t = 1, 2, 3$. We let

$$c_0 = (2, 1, 0)', \qquad A_0 = \begin{bmatrix} 0.2 & 0 & 0.2 \\ 0 & 0.2 & 0.2 \\ 0 & 0 & 0.5 \end{bmatrix}, \qquad \text{and} \qquad \Sigma_0 = \frac{1}{2} \cdot I_{3\times3}.$$

We let $a = 1$ be a "reset" choice in the sense that $A_1 = 0_{3\times3}$. To ensure that the stationary distribution of s_t is the same whether $a_t = 0$ or 1,[25] we let

$$c_1 = (I_{3\times3} - A_0)^{-1} c_0 = (2.5, 1.25, 0)', \qquad \mathrm{vec}(\Sigma_1) = (I_{9\times9} - A_0 \otimes A_0)^{-1} \mathrm{vec}(\Sigma_0).$$

We let the discount factor $\beta = 0.9$ and assume that it is known to researchers.

To generate a simulation sample from this model, we must know the CCP function $p_t(x_{t,1}, x_{t,2}, z_t)$ for any value of the continuous vector $(x_{t,1}, x_{t,2}, z_t)$ by solving the dynamic programming (DP) problem. This is not a trivial task, and we proceed in three steps. First, we discretize the continuous state space of s_t into an efficient grid and obtain the corresponding state transition processes using the "EDS" method (Maliar and Maliar, 2015; Gordon, 2021). This allows us to draw from a much smaller number of state grid points but still approximate well the stochastic behavior of the state variables. Second, we solve the DP problem backwardly with the state variables on the discrete EDS grid, which was shown to well approximate the solution of the original DP problem (Maliar and Maliar, 2015). This step gives the CCP value on the discrete EDS grid, from which we obtain the CCP function of continuous state variables by polynomial interpolation. Third, we simulate the three periods of the state variables s_t and the choices a_t for N agents and R Monte Carlo repetitions. The details of these three steps, which might be of independent interest, can be found in Appendix F. We acknowledge that this "discretization-interpolation" approach to generating simulation samples will inevitably result in bias in final parameter estimation, but because the CRS and the HM estimators are based on the same simulation samples, any approximation errors in generating the samples should affect both estimators equally.

In all of our simulation experiments, we use all the three periods of data for estimation (i.e., $T = 3$).[26] Although the flow utility parameters and the state transition processes are in fact time-invariant, we assume that researchers do not have this piece of information and allow them to vary with t. With three periods of data available, the implementation of the HM estimator depends crucially on whether researchers know that the sample terminal period is the decision terminal period. The next two subsections will report, separately, the simulation results in the scenarios where researchers know or do not know $T = T_{end}$.

[25] As detailed in Appendix F.1, this property makes the model easier to discretize.

[26] Note that $T = 3$ is a case that is less favorable to the CRS estimator because, as argued in Section 4.2, the longer the data horizon (i.e. T larger), the less sensitive the CRS estimator is to Assumption 5. On the other hand, the shorter the data horizon, the faster it is for the HM estimator to simulate the state variables and the choices if researchers know $T = T_{end}$.

5.2 Results If Researchers Know $T = T_{end} = 3$

If researchers know that $T = T_{end}$, then Assumption 5 in this paper holds with $q^K(x_T, z_T)$ taking a known form (recall Remark 5). At the same time, the HM estimator does not need to simulate the state transition distributions and the choices beyond the third period when evaluating the moment functions.

In this scenario, $\delta_{0,1,k}$ ($k = 0,1,2$) are normalized to the true value $(2.9, -0.5, -0.8)$ (recall Assumption 4), $\delta_{0,t,0}$ ($t = 1,2,3$) are not identified but can be normalized to its true value 2.9 (recall Remark 13), and $\delta_{0,3,k}$ ($k = 1,2$) and $\Delta_{3,k}$ ($k = 0,1,2$) are a part of γ^K (recall Remark 5, so $\delta_{1,3,k}$ for $k = 0,1,2$ are also identified and estimated). In total, 13 out of 18 flow utility parameters are identified and estimated by both the CRS and the HM estimators. We let $N = 250$ and $R = 1000$.

Table 1: Flow Utility Parameter Estimates If $T = T_{end} = 3$ Is Known

			CRS estimator			HM estimator		
	$\delta_{a,t,k}$	Truth	Bias	Std Dev	MSE	Bias	Std Dev	MSE
$t = 1$	$\delta_{1,1,0}$	−1.0	−0.107	0.602	0.374	−0.228	0.790	0.676
	$\delta_{1,1,1}$	0.4	0.020	0.201	0.041	0.055	0.239	0.060
	$\delta_{1,1,2}$	0.5	0.045	0.293	0.088	0.085	0.270	0.080
$t = 2$	$\delta_{0,2,1}$	−0.5	0.032	0.195	0.039	−0.088	0.997	1.002
	$\delta_{0,2,2}$	−0.8	0.066	0.278	0.082	0.003	0.956	0.914
	$\delta_{1,2,0}$	−1.0	−0.201	0.605	0.406	−0.237	0.796	0.690
	$\delta_{1,2,1}$	0.4	0.070	0.251	0.068	−0.035	0.993	0.987
	$\delta_{1,2,2}$	0.5	0.170	0.339	0.144	0.097	0.954	0.920
$t = 3$	$\delta_{0,3,1}$	−0.5	0.007	0.191	0.037	−0.002	0.768	0.590
	$\delta_{0,3,2}$	−0.8	0.029	0.261	0.069	0.058	0.756	0.575
	$\delta_{1,3,0}$	−1.0	−0.172	0.635	0.433	−0.136	0.683	0.485
	$\delta_{1,3,1}$	0.4	0.041	0.235	0.057	0.025	0.770	0.594
	$\delta_{1,3,2}$	0.5	0.103	0.311	0.107	0.110	0.758	0.587

† There are $T = 3$ periods and $N = 250$ agents in Monte Carlo repetition. The results are based on $R = 1000$ Monte Carlo repetitions.

‡ In this simple scenario, we show that the flow utility parameters in the terminal period (i.e., $t = 3$) are also identified because $\gamma^K = (\delta_{0,3}', \Delta_3', 1)'$ (recall Remark 5) and $\delta_{1,3} = \delta_{0,3} + \Delta_3$.

Table 1 reports the true values, the simulation biases, the simulation standard deviations and the simulation mean squared errors (MSEs) for both the CRS and the HM estimators. The CRS estimator exhibits smaller variances (except $\delta_{1,1,2}$) and smaller MSEs (except $\delta_{1,1,2}$) than the HM estimator, and for many parameters, much smaller. The CRS estimator has smaller biases for about half of the parameters. For this simple scenario, the performance of the CRS and the HM estimators are comparable, with the CRS estimator appearing slightly better in terms of MSE.

It is worth emphasizing that the CRS estimator is much faster than the HM estimator to implement – it took our four-core laptop four minutes to complete $R = 1000$ repetitions of the CRS estimator on $T = 3$ and $N = 250$ samples, but about 26 hours on a latest 12-core Mac Studio to compute the HM estimator using the same simulation samples – a thousand-time difference.[27]

Moreover, the HM estimator results reported in Table 1 are based on a commercial optimization software (Knitro, see Byrd, Nocedal and Waltz, 2006, for details), which we used to enhance the performance of the HM estimator, while the CRS estimator only uses the core packages of R to solve a linear problem. In a separate and unreported experiment, we used the `optim` function (Nelder-Mead algorithm) in the basic `stats` package of R to solve the nonlinear optimization problem of the HM estimator. It also took the same computer about a whole day to complete $R = 1000$ repetitions using the same simulation samples, but for many of the samples the HM estimator did not converge.[28] It is not surprising, then, that the resultant HM estimator exhibits worse performance than the HM estimator results reported here, with larger biases, variances and MSEs than the CRS estimator for almost all parameters (for many parameters, much larger).[29]

5.3 Results If Researchers Do Not Know $T = T_{end}$

If researchers have three periods of data but do not know whether or not the sample terminal period is the decision terminal period, or if they know $T < T_{end}$ but do not know the value of T_{end}, the implementation of both the CRS and the HM estimators becomes more complex. In this scenario, the functions $q^K(x_T, z_T)$ used in the CRS estimator no longer have a known form, and we use a power series, as described in Lemma 4, as our $q^K(x_T, z_T)$ functions. The HM estimator, in contrast, must make additional assumptions about the value of T_{end} and the flow utility parameters and the state transition distributions beyond the data horizon, because the state variables and the choices need to be simulated till T_{end} (or a large enough T_{tr} satisfying $T < T_{tr} \leq T_{end}$, so that the impacts of the truncation is negligible, recall Remark 2) in evaluating the moment functions of the

[27] For the CRS estimator, the unknown conditional mean functions are estimated nonparametrically using a power series described in Lemma 4. For the HM estimator, the state transition distributions are estimated by kernel density estimators, and we draw 500 sequences of state variables and choices for three periods when evaluating the moment functions. In a preliminary experiment, we tried to draw 1500 sequences instead of 500. This attempt, while still taking minutes to complete the CRS estimator and delivering similar results as 500 draws for both estimators, appeared to take unreasonably longer time to run even a few repetitions of the HM estimator, suggesting that the memory space might be the binding resource when implementing the HM estimator. So, we chose to use 500 draws for the HM estimator for all the reported results.

[28] After finding that the HM estimator did not converge for many simulation samples, we increased the maximum number of iterations of the Nelder-Mead algorithm from 500 to 1000, and the non-convergent issue did not improve much.

[29] These results are not reported but available upon request.

HM estimator. We assume that researchers, in this scenario, assume that the decision horizon is infinite, that the flow utility parameters beyond the third period are the same as those in the third period, and that the state transition distributions beyond the third period are the same as those from $t = 2$ to $t = 3$. We draw 500 sequences of state variables and choices for 50 periods when evaluating the moment functions of the HM estimator.[30]

In this scenario, the flow utility parameters in the third period are no longer a part of γ^K. Because the CRS estimator is agnostic about and robust to the value of T_{end}, the identification result in Theorem 2 only covers periods $t = 1$ and $t = 2$, so the flow utility parameters in only the first two periods are estimated by the CRS estimator. On the other hand, due to additional assumptions about T_{end}, the flow utility and the state transition beyond period $t = 3$, which we must make for its implementation, the HM estimator also delivers estimates of flow utility parameters in period $t = 3$. We still let $N = 250$ and $R = 1000$.

Table 2 reports the true values, the simulation biases, the simulation standard deviations and the simulation MSEs for both the CRS and the HM estimators. Again, the CRS estimator exhibits smaller biases for more than half of the parameters and much smaller variances for all parameters than the HM estimator, leading to significantly advantageous MSEs of the CRS estimator. A little more caution, however, is needed when interpreting these results for this scenario, because the performance of the HM estimator depends on the additional assumptions that researchers make. If the additional assumptions that researchers make in this scenario differ from the above additional assumptions that we assume researchers make in implementing the HM estimator, then the HM estimator *may* deliver different results (and possibly more advantageous than the CRS estimator).[31] So the results in Table 2 do not guarantee that the CRS estimator *will* perform better than the HM estimator under *all* possible additional assumptions that researchers make when implementing the HM estimator. The upside is that the CRS estimator frees researchers from having to make arbitrary additional assumptions beyond the data horizon, and therefore delivers estimates that are less sensitive to the assumptions.

When it comes to computing time, the two estimators, again, exhibit a thousand-time difference.[32] It is even more obvious in this scenario that the simulated state variable and choice

[30]The discount factor is $\beta = 0.9$, so the impact of the unknown value functions beyond 50 periods is $0.9^{50} \approx 0.005$ and negligible, although the decision horizon is assumed to be infinite.

[31]For example, if researchers assume that $T_{end} = T = 3$, which is allowed in this scenario, then the HM estimator results will be the same as those in Table 1, which still compare disadvantageously to the CRS estimator in Table 2. On the other hand, although $q^K(x_T, z_T)$ takes a known form in the scenario where $T = T_{end} = 3$ is known, researchers can choose to ignore this piece of information and still use a power series as $q^K(x_T, z_T)$. Then, the comparison should be between the CRS results in Table 1 and the HM results in Table 2, where the advantages of the CRS estimator are even more visible, possibly due the dependence between steps (II) and (III) of the CRS estimation procedure.

[32]The unknown conditional mean functions for the CRS estimator and the state transition distributions are estimated using the same methods, respectively, as in Section 5.2. Again, we used the commercial optimization software

Table 2: Flow Utility Parameter Estimates If $T = T_{end}$ Is Unknown

	$\delta_{a,t,k}$	Truth	CRS estimator Bias	CRS estimator Std Dev	CRS estimator MSE	HM estimator Bias	HM estimator Std Dev	HM estimator MSE
$t = 1$	$\delta_{1,1,0}$	−1.0	−0.068	0.447	0.204	−0.182	0.931	0.900
	$\delta_{1,1,1}$	0.4	0.008	0.159	0.025	0.046	0.267	0.073
	$\delta_{1,1,2}$	0.5	0.049	0.242	0.061	0.055	0.273	0.078
$t = 2$	$\delta_{0,2,1}$	−0.5	0.027	0.109	0.013	−0.066	0.952	0.911
	$\delta_{0,2,2}$	−0.8	0.033	0.162	0.027	−0.001	1.056	1.115
	$\delta_{1,2,0}$	−1.0	−0.103	0.452	0.215	−0.179	0.674	0.486
	$\delta_{1,2,1}$	0.4	0.039	0.172	0.031	−0.037	0.970	0.942
	$\delta_{1,2,2}$	0.5	0.116	0.226	0.065	0.078	1.053	1.115
$t = 3$	$\delta_{0,3,2}$	−0.5	–	–	–	0.013	0.297	0.088
	$\delta_{0,3,3}$	−0.8	–	–	–	0.129	0.320	0.119
	$\delta_{1,3,1}$	−1.0	–	–	–	−0.125	0.698	0.503
	$\delta_{1,3,2}$	0.4	–	–	–	0.046	0.333	0.113
	$\delta_{1,3,3}$	0.5	–	–	–	0.168	0.337	0.142

† There are $T = 3$ periods and $N = 250$ agents in each Monte Carlo repetition. The results are based on $R = 1000$ Monte Carlo repetitions.

‡ In this scenario, $\delta_{a,3,k}$ ($a = 0, 1$ and $k = 0, 1, 2$) are not identified under Assumptions 1 to 5, and therefore are not estimated by the CRS estimator. Under additional (and required) assumptions about T_{end}, the flow utility and the state transition beyond period $t = 3$ (detailed in the paper), which we made for its implementation, the HM estimator delivers estimates of them.

sequences of the HM estimator can put a strain on the memory, because using 12 cores turned out to be much slower than using 10 cores on our computer as the former depleted the memory.

6 Concluding Remarks

A few lines of future inquiry are worth exploring. First, identifying the expected flow utility function $u_t(a_t, x_t)$ while relaxing Assumption 4 (linearity) might be possible, as alluded to at the beginning of Section 3.1.2. Second, generalizing the analysis in this paper to permit unobserved heterogeneity among the agents, as characterized by latent types, appears to be feasible in light of some previous research (e.g., Kasahara and Shimotsu, 2009; Arcidiacono and Miller, 2011). Third, to allow for serially correlated flow utility shocks, while challenging, will widen the scope of applicability of

Knitro (Byrd, Nocedal and Waltz, 2006) to solve for the HM estimator, but for this scenario we did not try the basic optimization solvers in R.

nonstationary DDC models.

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Online Appendices
for "Identification and Estimation of Nonstationary Dynamic Binary Choice Models"
by Cheng Chou, Geert Ridder and Ruoyao Shi

A Proofs of eq. (1) and eq. (6) in Section 2

Equation (1) and eq. (6), or close variations of them, underlie most econometric analysis of DDC models. Despite their ubiquity, rigorous proofs under clear assumptions are hard to find in the DDC literature. For completeness and as a preparation for our analysis, this appendix proves them under the model setup and Assumptions 1 to 3 in Section 2.1.

Lemma A.1. *Under Assumptions 1 and 2(i)-(iii), $\varepsilon_{t+1} \perp\!\!\!\perp \varepsilon_t \,|\, (s_t, a_t)$ for t such that t and $t+1$ both belong to $\mathcal{T}$.*

Proof. First note the choice a_t is completely determined by s_t and ε_t under Assumption 1 (see Aguirregabiria and Mira, 2010, p.39, and it can easily be shown), then by Assumptions 2(ii) and 2(iii), we have $\varepsilon_{t+1} \perp\!\!\!\perp a_t$. As a result,

$$f(\varepsilon_{t+1} \,|\, \varepsilon_t, s_t, a_t) = f(\varepsilon_{t+1}) = f(\varepsilon_{t+1} \,|\, s_t, a_t).$$

Then we have

$$\begin{aligned} f(\varepsilon_{t+1}, \varepsilon_t \,|\, s_t, a_t) &= f(\varepsilon_t \,|\, s_t, a_t) f(\varepsilon_{t+1} \,|\, \varepsilon_t, s_t, a_t) \\ &= f(\varepsilon_t \,|\, s_t, a_t) f(\varepsilon_{t+1} \,|\, s_t, a_t); \end{aligned}$$

that is, $\varepsilon_{t+1} \perp\!\!\!\perp \varepsilon_t \,|\, (s_t, a_t)$. ∎

Lemma A.2. *Under Assumptions 1 and 2(i)-(iii), $\varepsilon_t \perp\!\!\!\perp (s_{t-j}, a_{t-j}) \,|\, s_t$ for t and $j \in \mathbb{N}^+$ such that t and $t-j$ all belong to $\mathcal{T}$.*

Proof. To prove this, we separate the cases $j = 1$ and $j \geq 2$. For $j = 1$, recall that Assumption 1 implies that a_{t-1} is completely determined by s_{t-1} and ε_{t-1}, then $\varepsilon_t \perp\!\!\!\perp a_{t-1}$ by Assumptions 2(ii) and 2(iii). Together with Assumption 2(i) and 2(ii), this further implies

$$f(\varepsilon_t \,|\, s_{t-1}, a_{t-1}, s_t) = f(\varepsilon_t) = f(\varepsilon_t \,|\, s_t). \tag{A.1}$$

In consequence, we have

$$\begin{aligned} f(\varepsilon_t, s_{t-1}, a_{t-1} \,|\, s_t) &= f(s_{t-1}, a_{t-1} \,|\, s_t) f(\varepsilon_t \,|\, s_{t-1}, a_{t-1}, s_t) \\ &= f(s_{t-1}, a_{t-1} \,|\, s_t) f(\varepsilon_t \,|\, s_t). \end{aligned}$$

For $j \geq 2$, we have

$$f(\varepsilon_t, s_{t-j}, a_{t-j} \,|\, s_t)$$

$$
\begin{aligned}
&= \int f(\varepsilon_t, \Omega_{t-1}, a_{t-1}, s_{t-j}, a_{t-j} \mid s_t) d\Omega_{t-1} da_{t-1} \\
&= f(s_{t-j}, a_{t-j} \mid s_t) \int f(\varepsilon_t, \Omega_{t-1}, a_{t-1} \mid s_{t-j}, a_{t-j}, s_t) d\Omega_{t-1} da_{t-1} \\
&= f(s_{t-j}, a_{t-j} \mid s_t) \int f(\Omega_{t-1}, a_{t-1} \mid s_{t-j}, a_{t-j}, s_t) f(\varepsilon_t \mid \Omega_{t-1}, a_{t-1}, s_{t-j}, a_{t-j}, s_t) d\Omega_{t-1} da_{t-1}.
\end{aligned} \tag{A.2}
$$

Note that

$$
\begin{aligned}
& f(\varepsilon_t \mid \Omega_{t-1}, a_{t-1}, s_{t-j}, a_{t-j}, s_t) \\
&= \frac{f(\varepsilon_t \mid \Omega_{t-1}, a_{t-1}, s_{t-j}, a_{t-j}, s_t) f(s_t \mid \Omega_{t-1}, a_{t-1})}{f(s_t \mid \Omega_{t-1}, a_{t-1})} \\
&= \frac{f(\varepsilon_t \mid \Omega_{t-1}, a_{t-1}, s_{t-j}, a_{t-j}, s_t) f(s_t \mid \Omega_{t-1}, a_{t-1}, s_{t-j}, a_{t-j})}{f(s_t \mid \Omega_{t-1}, a_{t-1})} \\
&= \frac{f(\varepsilon_t, s_t \mid \Omega_{t-1}, a_{t-1}, s_{t-j}, a_{t-j})}{f(s_t \mid \Omega_{t-1}, a_{t-1})} \\
&= \frac{f(\varepsilon_t, s_t \mid \Omega_{t-1}, a_{t-1})}{f(s_t \mid \Omega_{t-1}, a_{t-1})} \\
&= f(\varepsilon_t \mid \Omega_{t-1}, a_{t-1}, s_t) \\
&= f(\varepsilon_t \mid s_t),
\end{aligned} \tag{A.3}
$$

where the second and the fourth equalities hold by Assumption 1, and the last equality holds by Assumptions 1, 2(i)-(iii) and an argument similar to the one used to show eq. (A.1). Plug eq. (A.3) into eq. (A.2), we get

$$
\begin{aligned}
& f(\varepsilon_t, s_{t-j}, a_{t-j} \mid s_t) \\
&= f(s_{t-j}, a_{t-j} \mid s_t) f(\varepsilon_t \mid s_t) \int f(\Omega_{t-1}, a_{t-1} \mid s_{t-j}, a_{t-j}, s_t) d\Omega_{t-1} da_{t-1} \\
&= f(s_{t-j}, a_{t-j} \mid s_t) f(\varepsilon_t \mid s_t).
\end{aligned}
$$

This completes the proof. ∎

Lemma A.3. *Under Assumptions 1 to 3,*

$$\bar{V}_t(s_t) = U_t^o(s_t) + \beta \mathbb{E}(\bar{V}_{t+1}(s_{t+1}) \mid s_t)$$

for t such that t and $t+1$ both belong to $\mathcal{T}$.

Proof. By the well-known expectation maximization of the logit model (e.g. Theorem 1 of Arcidiacono and Miller, 2011),

$$\bar{V}_t(s_t) = u_t(0, x_t) + \beta \mathbb{E}(\bar{V}_{t+1}(s_{t+1}) \mid s_t, a_t = 0) - \ln(1 - p_t). \tag{A.4}$$

The next step is to express $\beta \mathbb{E}(\bar{V}_{t+1}(s_{t+1}) \mid s_t, a_t = 0)$ in terms of $\beta \mathbb{E}(\bar{V}_{t+1}(s_{t+1}) \mid s_t)$. Note that the CCPs incorporate the equilibrium optimal decision rule given the current s_t, and by the law of total probabilities, we have

$$\mathbb{E}(\bar{V}_{t+1}(s_{t+1}) \mid s_t) = p_t \mathbb{E}(\bar{V}_{t+1}(s_{t+1}) \mid s_t, a_t = 1) + (1 - p_t) \mathbb{E}(\bar{V}_{t+1}(s_{t+1}) \mid s_t, a_t = 0)$$

$$= \mathbb{E}(\bar{V}_{t+1}(s_{t+1}) \mid s_t, a_t = 0) + p_t \Delta \mathbb{E}(\bar{V}_{t+1}(s_{t+1}) \mid s_t),$$

which immediately implies

$$\beta \mathbb{E}(\bar{V}_{t+1}(s_{t+1}) \mid s_t, a_t = 0) = \beta \mathbb{E}(\bar{V}_{t+1}(s_{t+1}) \mid s_t) - p_t \beta \Delta \mathbb{E}(\bar{V}_{t+1}(s_{t+1}) \mid s_t). \tag{A.5}$$

Substituting $\beta \Delta \mathbb{E}(\bar{V}_{t+1}(s_{t+1}) \mid s_t, a_t = 0)$ in eq. (A.4) with its expression in eq. (A.5), we have

$$\bar{V}_t(s_t) = u_t(0, x_t) - \ln(1 - p_t) - p_t \beta \Delta \mathbb{E}(\bar{V}_{t+1}(s_{t+1}) \mid s_t) + \beta \mathbb{E}(\bar{V}_{t+1}(s_{t+1}) \mid s_t). \tag{A.6}$$

By eq. (5),

$$\beta \Delta \mathbb{E}(\bar{V}_{t+1}(s_{t+1}) \mid s_t) = \ln\left(\frac{p_t}{1 - p_t}\right) - [u_t(1, x_t) - u_t(0, x_t)]. \tag{A.7}$$

Substituting $\beta \Delta \mathbb{E}(\bar{V}_{t+1}(s_{t+1}) \mid s_t)$ in eq. (A.6) with its expression in eq. (A.7), we have

$$\begin{aligned} \bar{V}_t(s_t) = {} & p_t u_t(1, x_t) + (1 - p_t) u_t(0, x_t) - p_t \ln p_t - (1 - p_t) \ln(1 - p_t) \\ & + \beta \mathbb{E}(\bar{V}_{t+1}(s_{t+1}) \mid s_t), \end{aligned}$$

where the sum $p_t u_t(1, x_t) + (1 - p_t) u_t(0, x_t) - p_t \ln p_t - (1 - p_t) \ln(1 - p_t)$ is just $U_t^o(s_t)$ according to eq. (3.8) of Hotz and Miller (1993), for which Assumptions 1 to 3 are sufficient conditions. ∎

Proof of eq. (1). By Assumption 1, the expected payoff is $u_t(a, x_t) + \varepsilon_{at} + \beta \mathbb{E}(V_{t+1}(s_{t+1}, \varepsilon_{t+1}) | s_t, \varepsilon_t, a_t)$. Assumption 3 and Lemma A.1 together imply that $\Omega_{t+1} \perp\!\!\!\perp \varepsilon_t | (s_t, a_t)$, and therefore $\mathbb{E}(V_{t+1}(s_{t+1}, \varepsilon_{t+1}) | s_t, \varepsilon_t, a_t) = \mathbb{E}(V_{t+1}(s_{t+1}, \varepsilon_{t+1}) \mid s_t, a_t)$. Moreover, because Lemma A.2 states that $\varepsilon_{t+1} \perp\!\!\!\perp (s_t, a_t) \mid s_{t+1}$, we have $\mathbb{E}(V_{t+1}(s_{t+}, \varepsilon_{t+1}) | s_t, a_t) = \mathbb{E}(\mathbb{E}(V_{t+1}(s_{t+1}, \varepsilon_{t+1}) | s_{t+1}, s_t, a_t) | s_t, a_t) = \mathbb{E}(\mathbb{E}(V_{t+1}(s_{t+1}, \varepsilon_{t+1}) | s_{t+1}) | s_t, a_t) = \mathbb{E}(\bar{V}_{t+1}(s_{t+1}) \mid s_t, a_t)$. ∎

Equation (6) holds simply by increasing the time index in Lemma A.3 by one.

B Proofs of the Results in Section 3.1

This appendix provides proofs of the results in Section 3.1, as well as auxiliary intermediary lemmas used in their proofs.

Lemma B.1 (Markovian state variables)**.** *Under Assumption 1, Ω_t is a first order Markov process; that is, $\Omega_{t+1} \perp\!\!\!\perp \Omega_{t-j} \mid \Omega_t$ for t and $j \in \mathbb{N}^+$ such that t, $t+1$ and $t-j$ all belong to $\mathcal{T}$.*

Proof. First note that $f(\Omega_{t+1} \mid a_t, \Omega_t, \Omega_{t-j}) = f(\Omega_{t+1} \mid a_t, \Omega_t)$ by Assumption 1. Also recall that the choice a_t is completely determined by Ω_t under Assumption 1 (see Aguirregabiria and Mira, 2010, p.39, and it can easily be shown), so $\Pr(a_t \mid \Omega_t, \Omega_{t-j}) = \Pr(a_t \mid \Omega_t)$. These two equalities, together with the law of total probabilities, lead to

$$f(\Omega_{t+1} \mid \Omega_t, \Omega_{t-j}) = \sum_{a_t = 0,1} f(\Omega_{t+1} \mid a_t, \Omega_t, \Omega_{t-j}) \Pr(a_t \mid \Omega_t, \Omega_{t-j})$$

$$
\begin{aligned}
&= \sum_{a_t=0,1} f(\Omega_{t+1} \mid a_t, \Omega_t) \Pr(a_t \mid \Omega_t) \\
&= f(\Omega_{t+1} \mid \Omega_t),
\end{aligned}
$$

which is the result of this lemma. ∎

Lemma B.2. *Under Assumptions 1 and 2(i)-(iii), $s_{t+j} \perp\!\!\!\perp s_t \mid s_{t+1}$ and $\Omega_{t+j} \perp\!\!\!\perp \Omega_t \mid \Omega_{t+1}$ for t and $j \in \mathbb{N}^+$ such that t, $t+1$ and $t+j$ all belong to $\mathcal{T}$.*

Proof. We consider

$$
\begin{aligned}
& f(s_{t+j}, s_t \mid s_{t+1}) \\
&= \int f(s_{t+j}, s_{t+j-1}, \ldots, s_{t+2}, s_t \mid s_{t+1}) ds_{t+j-1} \cdots ds_{t+2} \\
&= f(s_t \mid s_{t+1}) \int f(s_{t+j}, s_{t+j-1}, \ldots, s_{t+2} \mid s_t, s_{t+1}) ds_{t+j-1} \cdots ds_{t+2} \\
&= f(s_t \mid s_{t+1}) \int f(s_{t+j}, s_{t+j-1}, \ldots, s_{t+2} \mid s_{t+1}) ds_{t+j-1} \cdots ds_{t+2} \\
&= f(s_t \mid s_{t+1}) f(s_{t+j} \mid s_{t+1}),
\end{aligned}
$$

where the third equality holds by Lemma 1. The result $f(\Omega_{t+j}, \Omega_t | \Omega_{t+1}) = f(\Omega_t | \Omega_{t+1}) f(\Omega_{t+j} | \Omega_{t+1})$ can be shown by the same argument using Lemma B.1, and this completes the proof of this lemma. ∎

Lemma B.3. *Under Assumptions 1 and 2(i)-(iii), $s_{t+j} \perp\!\!\!\perp \varepsilon_t \mid s_{t+1}$ for t and $j \in \mathbb{N}^+$ such that t, $t+1$ and $t+j$ all belong to $\mathcal{T}$.*

Proof. We consider

$$
\begin{aligned}
& f(s_{t+j}, \varepsilon_t \mid s_{t+1}) \\
&= \int f(s_{t+j}, \varepsilon_{t+1}, \varepsilon_t \mid s_{t+1}) d\varepsilon_{t+1} \\
&= \int f(\varepsilon_{t+1} \mid s_{t+1}) f(s_{t+j}, \varepsilon_t \mid s_{t+1}, \varepsilon_{t+1}) d\varepsilon_{t+1} \\
&= \int f(\varepsilon_{t+1} \mid s_{t+1}) f(s_{t+j} \mid s_{t+1}, \varepsilon_{t+1}) f(\varepsilon_t \mid s_{t+1}, \varepsilon_{t+1}) d\varepsilon_{t+1} \\
&= \int f(s_{t+j}, \varepsilon_{t+1} \mid s_{t+1}) f(\varepsilon_t \mid s_{t+1}, \varepsilon_{t+1}) d\varepsilon_{t+1}, \qquad \text{(B.1)}
\end{aligned}
$$

where the third equality holds by $s_{t+j} \perp\!\!\!\perp \varepsilon_t \mid (s_{t+1}, \varepsilon_{t+1})$ implied by Lemma B.2. Note that in eq. (B.1),

$$
\begin{aligned}
f(\varepsilon_t \mid s_{t+1}, \varepsilon_{t+1}) &= \frac{f(\varepsilon_t, s_{t+1}, \varepsilon_{t+1})}{f(s_{t+1}, \varepsilon_{t+1})} \\
&= \frac{f(\varepsilon_t, s_{t+1}) f(\varepsilon_{t+1})}{f(s_{t+1}) f(\varepsilon_{t+1})} = \frac{f(\varepsilon_t, s_{t+1})}{f(s_{t+1})} = f(\varepsilon_t \mid s_{t+1}),
\end{aligned}
$$

where the second equality holds by Assumptions 2(i) and 2(iii). As a result, eq. (B.1) becomes

$$\begin{aligned} f(s_{t+j}, \varepsilon_t \mid s_{t+1}) &= \int f(s_{t+j}, \varepsilon_{t+1} \mid s_{t+1}) f(\varepsilon_t \mid s_{t+1}) d\varepsilon_{t+1} \\ &= f(\varepsilon_t \mid s_{t+1}) \int f(s_{t+j}, \varepsilon_{t+1} \mid s_{t+1}) d\varepsilon_{t+1} \\ &= f(\varepsilon_t \mid s_{t+1}) f(s_{t+j} \mid s_{t+1}), \end{aligned}$$

which is the result of this lemma. ■

Proof of Lemma 1. First note that by the first order Markovian property of Ω_t shown in Lemma B.1, we have $s_{t+1} \perp\!\!\!\perp s_{t-j} \mid (s_t, \varepsilon_t)$, which indicates $f(s_{t+1}, s_{t-j} \mid s_t, \varepsilon_t) = f(s_{t+1} \mid s_t, \varepsilon_t) f(s_{t-j} \mid s_t, \varepsilon_t)$. Moreover, the result of Lemma A.2 further simplifies the last term $f(s_{t-j} \mid s_t, \varepsilon_t)$ in this equality to $f(s_{t-j} \mid s_t)$, implying $f(s_{t+1}, s_{t-j} \mid s_t, \varepsilon_t) = f(s_{t+1} \mid s_t, \varepsilon_t) f(s_{t-j} \mid s_t)$. Finally, applying the integral operator $\int \cdot dF(\varepsilon_t \mid s_t)$ to both sides of this equality gives

$$\begin{aligned} \int f(s_{t+1}, s_{t-j} \mid s_t, \varepsilon_t) dF(\varepsilon_t \mid s_t) &= \int f(s_{t+1} \mid s_t, \varepsilon_t) f(s_{t-j} \mid s_t) dF(\varepsilon_t \mid s_t) \\ \Longrightarrow f(s_{t+1}, s_{t-j} \mid s_t) &= f(s_{t+1} \mid s_t) f(s_{t-j} \mid s_t), \end{aligned}$$

which is the result of this lemma. ■

Proof of Lemma 2. First recall that a_t is completely determined by s_t and ε_t under Assumption 1, then Lemmas B.2 and B.3 imply that $s_{t+j} \perp\!\!\!\perp (s_t, a_t) \mid s_{t+1}$. Then, we have

$$\begin{aligned} &\mathbb{E}(\mathbb{E}(g(s_{t+j}) \mid s_{t+1}) \mid s_t, a_t) \\ = &\int \int g(s_{t+j}) f(s_{t+j} \mid s_{t+1}) f(s_{t+1} \mid s_t, a_t) ds_{t+j} ds_{t+1} \\ = &\int \int g(s_{t+j}) f(s_{t+j} \mid s_{t+1}, s_t, a_t) f(s_{t+1} \mid s_t, a_t) ds_{t+j} ds_{t+1} \\ = &\int \int g(s_{t+j}) f(s_{t+j}, s_{t+1} \mid s_t, a_t) ds_{t+j} ds_{t+1} \\ = &\int g(s_{t+j}) f(s_{t+j} \mid s_t, a_t) ds_{t+j} \\ = &\mathbb{E}(g(s_{t+j}) \mid s_t, a_t). \end{aligned}$$

■

Proof of Lemma 3. As mentioned in Lemma A.3, Assumptions 1 to 3 are sufficient for eq. (3.8) of Hotz and Miller (1993), which states that

$$U_t^o(s_t) = p_t u_t(1, x_t) + (1 - p_t) u_t(0, x_t) - p_t \ln p_t - (1 - p_t) \ln(1 - p_t).$$

The result of this lemma, therefore, follows by plugging in the expression of $u_t(1, x_t)$ and $u_t(0, x_t)$ in Assumption 4. ■

C Influence Function $\psi_\delta(D)$ in Proposition 3

In this section, we use Proposition 5 of Newey (1994*a*) to compute the influence function $\psi(D)$ of $(\hat{\delta}', \hat{\gamma}^{K\prime})'$. The first $(2T-3)d_x$ coordinates of $\psi(D)$, denoted as $\psi_\delta(D)$, are the influence function of $\hat{\delta}$.

Proposition 3 and Proposition 4 hold by Lemma 5.3 and Lemma 5.4 of Newey (1994*a*) under the regularity conditions in Section 5 of Newey (1994*a*), and therefore the proofs are omitted here.

C.1 Basic Terms in the Influence Function

In eq. (37), L_T is the Jacobian matrix of the moment functions (with respect to δ and γ^K); $\alpha_\eta(D)$, $\alpha_x(D)$ and $\alpha_q(D)$ are the respective adjustment terms for the estimation of $\Delta\bar{\eta}$, $\Delta\bar{x}$ and $\Delta\bar{q}^K$ in step (II) of the estimation strategy in Section 3.2, $\alpha_{p,direct}(D)$ is the adjustment term for the estimation of the CCPs in step (I) that appears directly in the moment function, and $\alpha_{p,indirect}(D)$ is the adjustment term for the estimation of the CCPs in step (II) that appears in the "dependent variables" of the nonparametric regressions leading to $\Delta\bar{\eta}$, $\Delta\bar{x}$ and $\Delta\bar{q}^K$ in step (II).

Note that by eq. (18), as well as the definition of the moment function m in eq. (34), we have $m_0(D) = 0$, because $v_t(D, \delta_0, \gamma_0^K, p_0, \Delta\bar{\eta}_0, \Delta\bar{x}_0, \Delta\bar{q}_0^K) = 0$ for $t = 1, \ldots, T-1$. In other words, the influence function consists only the adjustment terms.

In the rest of this subsection, we verify that the L_T matrix define in eq. (26) indeed equals to the Jacobian matrix, denoted by M. We know that

$$\underset{((T-1)^2 d_x + (T-1)K)\times((2T-3)d_x + K)}{M} \equiv \begin{bmatrix} M_{T-1,\delta} & M_{T-1,\gamma} \\ \vdots & \vdots \\ M_{1,\delta} & M_{1,\gamma} \end{bmatrix},$$

where

$$\underset{((2T-2t-1)d_x + K)\times(2T-3)d_x}{M_{t,\delta}} \equiv \frac{\partial}{\partial \delta'} \mathbb{E}(m_t(D, \delta, \gamma_0^K, p_0, \Delta\bar{\eta}_0, \Delta\bar{x}_0, \Delta\bar{q}_0^K))\Big|_{\delta=\delta_0},$$

and

$$\underset{((2T-2t-1)d_x + K)\times K}{M_{t,\gamma}} \equiv \frac{\partial}{\partial \gamma^{K\prime}} \mathbb{E}(m_t(D, \delta_0, \gamma^K, p_0, \Delta\bar{\eta}_0, \Delta\bar{x}_0, \Delta\bar{q}_0^K))\Big|_{\gamma^K=\gamma_0^K}.$$

So, it is easy to see that, assuming interchangeability of differentiation and integral,

$$M = L_T. \tag{C.1}$$

C.2 Adjustment Terms in the Influence Function

Note that the moment functions m depend on the CCP functions in step (I) and the nonparametric regression functions in step (II) only through their values at given argument values, so Proposition 5 of Newey (1994*a*) applies. In the rest of this subsection, we use $\tilde{p}$, $\widetilde{\Delta\bar{\eta}}$, $\widetilde{\Delta\bar{x}}$ and $\widetilde{\Delta\bar{q}^K}$ to denote given values that these unknown functions take, and their coordinates are denoted in a similar way. These values are taken as real variables, not functions, in the partial derivatives shown in the rest of this subsection.

C.2.1 Adjustment Term for Estimated $\Delta\bar{\eta}$ Functions

To compute the adjustment term for estimated $\Delta\bar{\eta}$ functions, we first need to compute

$$M_{t,\Delta\bar{\eta}^{\tau}_{t'}} \equiv \frac{\partial}{\partial \widetilde{\Delta\bar{\eta}}^{\tau}_{t'}} m_t(D, \delta_0, \gamma_0^K, p_0, \widetilde{\Delta\bar{\eta}}^{\tau}_{t'}, \Delta\bar{\eta}^{\tau,c}_{t',0}, \Delta\bar{x}_0, \Delta\bar{q}_0^K)\Big|_{\widetilde{\Delta\bar{\eta}}^{\tau}_{t'}=\Delta\bar{\eta}^{\tau}_{t',0}},$$

where $\Delta\bar{\eta}^{\tau,c}_{t'}$ denotes all the coordinates of $\Delta\bar{\eta}$ other than $\Delta\bar{\eta}^{\tau}_{t'}$.

Through basic algebra, we get $M_{t,\Delta\bar{\eta}^{\tau}_{t'}} = 0_{((2T-2t-1)d_x+K)\times 1}$ for $t = 1,\ldots,T-2$, $t' \neq t$ and $\tau = t'+1,\ldots,T-1$, $M_{T-1,\Delta\bar{\eta}^{\tau}_{t'}} = 0_{(d_x+K)\times 1}$ for $t' = 1,\ldots,T-1$ and $\tau = t'+1,\ldots,T-1$, and $M_{t,\Delta\bar{\eta}^{\tau}_{t}} = -\beta^{\tau-t}X_{T,t,0}$ for $t = 1,\ldots,T-2$ and $\tau = t+1,\ldots,T-1$. As a result,

$$\alpha_\eta(D) = -\begin{bmatrix} 0_{(d_x+K)\times 1} \\ \beta X_{T,T-2,0}\left[\left(\frac{a_{T-2}}{p_{T-2,0}} - \frac{1-a_{T-2}}{1-p_{T-2,0}}\right)\eta_{T-1,0} - \Delta\bar{\eta}^{T-1}_{T-2,0}\right] \\ \vdots \\ \sum_{\tau=t+1}^{T-1}\beta^{\tau-t}X_{T,t,0}\left[\left(\frac{a_t}{p_{t,0}} - \frac{1-a_t}{1-p_{t,0}}\right)\eta_{\tau,0} - \Delta\bar{\eta}^{\tau}_{t,0}\right] \\ \vdots \\ \sum_{\tau=2}^{T-1}\beta^{\tau-1}X_{T,1,0}\left[\left(\frac{a_1}{p_{1,0}} - \frac{1-a_1}{1-p_{1,0}}\right)\eta_{\tau,0} - \Delta\bar{\eta}^{\tau}_{1,0}\right] \end{bmatrix}. \tag{C.2}$$

C.2.2 Adjustment Term for Estimated $\Delta\bar{x}$ Functions

Note that $\Delta\bar{x}$ contains two types of conditional mean functions, $\Delta\bar{x}^{\tau}_{t}$ defined in eq. (13c) and $\Delta\bar{x}^{\tau}_{1,t}$ define in eq. (13a). To compute the adjustment term for estimated $\Delta\bar{x}$ functions, therefore, we first need to compute

$$M_{t,\Delta\bar{x}^{\tau}_{t'}} \equiv \frac{\partial}{\partial \widetilde{\Delta\bar{x}}^{\tau}_{t'}} m_t(D, \delta_0, \gamma_0^K, p_0, \Delta\bar{\eta}_0, \widetilde{\Delta\bar{x}}^{\tau}_{t'}, \Delta\bar{x}^{\tau,c}_{t',0}, \Delta\bar{q}_0^K)\Big|_{\widetilde{\Delta\bar{x}}^{\tau}_{t'}=\Delta\bar{x}^{\tau}_{t',0}}$$

and

$$M_{t,\Delta\bar{x}^{\tau}_{1,t'}} \equiv \frac{\partial}{\partial \widetilde{\Delta\bar{x}}^{\tau}_{1,t'}} m_t(D, \delta_0, \gamma_0^K, p_0, \Delta\bar{\eta}_0, \widetilde{\Delta\bar{x}}^{\tau}_{1,t'}, \Delta\bar{x}^{\tau,c}_{1,t',0}, \Delta\bar{q}_0^K)\Big|_{\widetilde{\Delta\bar{x}}^{\tau}_{1,t'}=\Delta\bar{x}^{\tau}_{1,t',0}},$$

where $\Delta\bar{x}^{\tau,c}_{t'}$ denotes all the coordinates of $\Delta\bar{x}$ other than $\Delta\bar{x}^{\tau}_{t'}$, and $\Delta\bar{x}^{\tau,c}_{1,t'}$ denotes all the coordinates of $\Delta\bar{x}$ other than $\Delta\bar{x}^{\tau}_{1,t'}$.

Through basic algebra, we get $M_{t,\Delta\bar{x}^{\tau}_{t'}} = 0_{((2T-2t-1)d_x+K)\times d_x}$ for $t = 1,\ldots,T-2$, $t' \neq t$ and $\tau = t'+1,\ldots,T-1$, $M_{T-1,\Delta\bar{x}^{\tau}_{t'}} = 0_{(d_x+K)\times d_x}$ for $t' = 1,\ldots,T-1$ and $\tau = t+1,\ldots,T-1$, and $M_{t,\Delta\bar{x}^{\tau}_{t}} = \beta^{\tau-t}X_{T,t,0}\delta'_{0,\tau,0}$ for $t = 1,\ldots,T-2$ and $\tau = t+1,\ldots,T-1$. In addition, $M_{t,\Delta\bar{x}^{\tau}_{1,t'}} = 0_{((2T-2t-1)d_x+K)\times d_x}$ for $t = 1,\ldots,T-2$, $t' \neq t$ and $\tau = t'+1,\ldots,T-1$, $M_{T-1,\Delta\bar{x}^{\tau}_{1,t'}} = 0_{(d_x+K)\times d_x}$ for $t' = 1,\ldots,T-1$ and

$\tau = t+1,\ldots,T-1$, and $M_{t,\Delta\bar{x}^{\tau}_{1,t}} = \beta^{\tau-t} X_{T,t,0}\Delta'_{\tau,0}$ for $t = 1,\ldots,T-2$ and $\tau = t+1,\ldots,T-1$. As a result,

$$\alpha_x(D) = \begin{bmatrix} 0_{(d_x+K)\times 1} \\ \beta X_{T,T-2,0}\delta'_{0,T-2,0}\left[\left(\frac{a_{T-2}}{p_{T-2,0}} - \frac{1-a_{T-2}}{1-p_{T-2,0}}\right)x_{T-1} - \Delta\bar{x}^{T-1}_{T-2,0}\right] \\ \vdots \\ \sum_{\tau=t+1}^{T-1}\beta^{\tau-t}X_{T,t,0}\delta'_{0,\tau,0}\left[\left(\frac{a_t}{p_{t,0}} - \frac{1-a_t}{1-p_{t,0}}\right)x_{\tau} - \Delta\bar{x}^{\tau}_{t,0}\right] \\ \vdots \\ \sum_{\tau=2}^{T-1}\beta^{\tau-1}X_{T,1,0}\delta'_{0,\tau,0}\left[\left(\frac{a_1}{p_{1,0}} - \frac{1-a_1}{1-p_{1,0}}\right)x_{\tau} - \Delta\bar{x}^{\tau}_{1,0}\right] \end{bmatrix}$$

$$+ \begin{bmatrix} 0_{(d_x+K)\times 1} \\ \beta X_{T,T-2,0}\Delta'_{T-2,0}\left[\left(\frac{a_{T-2}}{p_{T-2,0}} - \frac{1-a_{T-2}}{1-p_{T-2,0}}\right)p_{T-1,0}x_{T-1} - \Delta\bar{x}^{T-1}_{1,T-2,0}\right] \\ \vdots \\ \sum_{\tau=t+1}^{T-1}\beta^{\tau-t}X_{T,t,0}\Delta'_{\tau,0}\left[\left(\frac{a_t}{p_{t,0}} - \frac{1-a_t}{1-p_{t,0}}\right)p_{\tau,0}x_{\tau} - \Delta\bar{x}^{\tau}_{1,t,0}\right] \\ \vdots \\ \sum_{\tau=2}^{T-1}\beta^{\tau-1}X_{T,1,0}\Delta'_{\tau,0}\left[\left(\frac{a_1}{p_{1,0}} - \frac{1-a_1}{1-p_{1,0}}\right)p_{\tau,0}x_{\tau} - \Delta\bar{x}^{\tau}_{1,1,0}\right] \end{bmatrix}. \tag{C.3}$$

C.2.3 Adjustment Term for Estimated $\Delta\bar{q}^K$ Functions

To compute the adjustment term for estimated $\Delta\bar{q}^K$ functions, we first need to compute

$$M_{t,\Delta\bar{q}^K_{t'}} \equiv \frac{\partial}{\partial\widetilde{\Delta\bar{q}}^K_{t'}} m_t(D,\delta_0,\gamma^K_0,p_0,\Delta\bar{\eta}_0,\Delta\bar{x}_0,\widetilde{\Delta\bar{q}}^K_{t'},\Delta\bar{q}^{K,c}_{t',0})\Big|_{\widetilde{\Delta\bar{q}}^K_{t'}=\Delta\bar{q}^K_{t',0}},$$

where $\Delta\bar{q}^{K,c}_{t'}$ denotes all the coordinates of $\Delta\bar{q}^K$ other than $\Delta\bar{q}^K_{t'}$.

Through basic algebra, we get $M_{t,\Delta\bar{q}^K_{t'}} = 0_{((2T-2t-1)d_x+K)\times K}$ for $t = 1,\ldots,T-1$ and $t' \neq t$, and $M_{t,\Delta\bar{q}^K_t} = \beta^{T-t}X_{T,t,0}\gamma^{K\prime}_0$ for $t = 1,\ldots,T-1$. As a result,

$$\alpha_q(D) = \begin{bmatrix} \beta X_{T,T-1,0}\gamma^{K\prime}_0\left[\left(\frac{a_{T-1}}{p_{T-1,0}} - \frac{1-a_{T-1}}{1-p_{T-1,0}}\right)q^K(x_T,z_T) - \Delta\bar{q}^K_{T-1,0}\right] \\ \vdots \\ \beta^{T-t}X_{T,t,0}\gamma^{K\prime}_0\left[\left(\frac{a_t}{p_{t,0}} - \frac{1-a_t}{1-p_{t,0}}\right)q^K(x_T,z_T) - \Delta\bar{q}^K_{t,0}\right] \\ \vdots \\ \beta^{T-1}X_{T,1,0}\gamma^{K\prime}_0\left[\left(\frac{a_1}{p_{1,0}} - \frac{1-a_1}{1-p_{1,0}}\right)q^K(x_T,z_T) - \Delta\bar{q}^K_{1,0}\right] \end{bmatrix}. \tag{C.4}$$

C.2.4 Adjustment Terms for Estimated CCPs

We first compute $\alpha_{p,direct}(D)$, which captures the impact of estimated CCPs that appear directly in the moment functions. To do so, we need to compute

$$M_{t,p_{t'}} \equiv \frac{\partial}{\partial\widetilde{p_{t'}}} m_t(D,\delta_0,\gamma^K_0,\widetilde{p_{t'}},p^c_{t',0},\Delta\bar{\eta}_0,\Delta\bar{x}_0,\Delta\bar{q}^K_0)\Big|_{\widetilde{p_{t'}}=p_{t',0}},$$

where $p_{t'}^c$ denotes all the coordinates of p other than $p_{t'}$. Note that the CCPs only appear directly as the first term of the contemporaneous y_t, in the form of $\ln\left(\frac{p_t}{1-p_t}\right)$, so $M_{t,p_{t'}} = 0_{((2T-2t-1)d_x+K)\times 1}$ for $t = 1,\dots,T-1$ and $t' \neq t$, and $M_{t,p_t} = -X_{T,t,0}/(p_{t,0}(1-p_{t,0}))$ for $t = 1,\dots,T-1$. As a result,

$$\alpha_{p,direct}(D) = -\begin{bmatrix} \frac{X_{T,T-1,0}}{p_{T-1,0}(1-p_{T-1,0})}\left(a_{T-1} - p_{T-1,0}\right) \\ \vdots \\ \frac{X_{T,t,0}}{p_{t,0}(1-p_{t,0})}\left(a_t - p_{t,0}\right) \\ \vdots \\ \frac{X_{T,1,0}}{p_{1,0}(1-p_{1,0})}\left(a_1 - p_{1,0}\right) \end{bmatrix}. \tag{C.5}$$

We then compute $\alpha_{p,indirect}$, which captures the impact of estimated CCPs that appear as the "dependent variables" of the unknown functions $\Delta\bar{\eta}$ and $\Delta\bar{x}$. Due to eq. (36), p_τ is part of the numerators of the "dependent variables" of the nonparametric regressions for $\Delta\bar{\eta}_t^\tau$ and $\Delta\bar{x}_{1,t}^\tau$, and p_t is part of their denominators. For $\tau > t$ and $t = 1,\dots,T-2$, define

$$\mathcal{H}_{t,p_\tau}^\tau \equiv \left(\frac{a_t}{p_{t,0}} - \frac{1-a_t}{1-p_{t,0}}\right)\ln\frac{p_{\tau,0}}{1-p_{\tau,0}}, \quad \mathcal{H}_{t,p_t}^\tau \equiv -\left(\frac{a_t}{p_{t,0}^2} + \frac{1-a_t}{(1-p_{t,0})^2}\right)\eta_{\tau,0},$$
$$\mathcal{X}_{t,p_\tau}^\tau \equiv \left(\frac{a_t}{p_{t,0}} - \frac{1-a_t}{1-p_{t,0}}\right)x_\tau, \text{ and } \mathcal{X}_{t,p_t}^\tau \equiv -\left(\frac{a_t}{p_{t,0}^2} + \frac{1-a_t}{(1-p_{t,0})^2}\right)p_{\tau,0}x_\tau,$$

where we recall that η_τ is defined in eq. (14). By chain rule, we combine these with $M_{t,\Delta\bar{\eta}_t^\tau}$ and $M_{t,\Delta\bar{x}_{1,t}^\tau}$ for $t = 1,\dots,T-2$ and $\tau = t+1,\dots,T-1$ to get

$$\begin{aligned}\alpha_{p,indirect}(D) = &-\begin{bmatrix} 0_{(d_x+K)\times 1} \\ \beta\ln\frac{p_{T-1,0}}{1-p_{T-1,0}}\mathbb{E}\left(X_{T,T-2,0}\left(\frac{a_{T-2}}{p_{T-2,0}} - \frac{1-a_{T-2}}{1-p_{T-2,0}}\right)|x_{T-1}, z_{T-1}\right)\left(a_{T-1} - p_{T-1,0}\right) \\ \vdots \\ \sum_{\tau=t+1}^{T-1}\beta^{\tau-t}\ln\frac{p_{\tau,0}}{1-p_{\tau,0}}\mathbb{E}\left(X_{T,t,0}\left(\frac{a_t}{p_{t,0}} - \frac{1-a_t}{1-p_{t,0}}\right)|x_\tau, z_\tau\right)\left(a_\tau - p_{\tau,0}\right) \\ \vdots \\ \sum_{\tau=2}^{T-1}\beta^{\tau-1}\ln\frac{p_{\tau,0}}{1-p_{\tau,0}}\mathbb{E}\left(X_{T,1,0}\left(\frac{a_1}{p_{1,0}} - \frac{1-a_1}{1-p_{1,0}}\right)|x_\tau, z_\tau\right)\left(a_\tau - p_{\tau,0}\right) \end{bmatrix} \\ &+\begin{bmatrix} 0_{(d_x+K)\times 1} \\ \beta X_{T,T-2,0}\left(\frac{a_{T-2}}{p_{T-2,0}^2} + \frac{1-a_{T-2}}{(1-p_{T-2,0})^2}\right)\mathbb{E}(\eta_{T-1,0}|x_{T-2}, z_{T-2})\left(a_{T-2} - p_{T-2,0}\right) \\ \vdots \\ \sum_{\tau=t+1}^{T-1}\beta^{\tau-t}X_{T,t,0}\left(\frac{a_t}{p_{t,0}^2} + \frac{1-a_t}{(1-p_{t,0})^2}\right)\mathbb{E}(\eta_{\tau,0}|x_t, z_t)\left(a_t - p_{t,0}\right) \\ \vdots \\ \sum_{\tau=2}^{T-1}\beta^{\tau-1}X_{T,t,0}\left(\frac{a_1}{p_{1,0}^2} + \frac{1-a_1}{(1-p_{1,0})^2}\right)\mathbb{E}(\eta_{\tau,0}|x_1, z_1)\left(a_1 - p_{1,0}\right) \end{bmatrix}\end{aligned}$$

$$+\begin{bmatrix} 0_{(d_x+K)\times 1} \\ \beta x_{T-1}\mathbb{E}\Big(X_{T,T-2,0}\left(\frac{a_{T-2}}{p_{T-2,0}}-\frac{1-a_{T-2}}{1-p_{T-2,0}}\right)|x_{T-1},z_{T-1}\Big)\left(a_{T-1}-p_{T-1,0}\right) \\ \vdots \\ \sum_{\tau=t+1}^{T-1}\beta^{\tau-t}x_\tau\mathbb{E}\Big(X_{T,t,0}\left(\frac{a_t}{p_{t,0}}-\frac{1-a_t}{1-p_{t,0}}\right)|x_\tau,z_\tau\Big)\left(a_\tau-p_{\tau,0}\right) \\ \vdots \\ \sum_{\tau=2}^{T-1}\beta^{\tau-1}x_\tau\mathbb{E}\Big(X_{T,1,0}\left(\frac{a_1}{p_{1,0}}-\frac{1-a_1}{1-p_{1,0}}\right)|x_\tau,z_\tau\Big)\left(a_\tau-p_{\tau,0}\right) \end{bmatrix}$$

$$-\begin{bmatrix} 0_{(d_x+K)\times 1} \\ \beta X_{T,T-2,0}\left(\frac{a_{T-2}}{p_{T-2,0}^2}+\frac{1-a_{T-2}}{(1-p_{T-2,0})^2}\right)\mathbb{E}(p_{T-1,0}x_{T-1}|x_{T-2},z_{T-2})\left(a_{T-2}-p_{T-2,0}\right) \\ \vdots \\ \sum_{\tau=t+1}^{T-1}\beta^{\tau-t}X_{T,t,0}\left(\frac{a_t}{p_{t,0}^2}+\frac{1-a_t}{(1-p_{t,0})^2}\right)\mathbb{E}(p_{\tau,0}x_\tau|x_t,z_t)\left(a_t-p_{t,0}\right) \\ \vdots \\ \sum_{\tau=2}^{T-1}\beta^{\tau-1}X_{T,t,0}\left(\frac{a_1}{p_{1,0}^2}+\frac{1-a_1}{(1-p_{1,0})^2}\right)\mathbb{E}(p_{\tau,0}x_\tau|x_1,z_1)\left(a_1-p_{1,0}\right) \end{bmatrix}. \tag{C.6}$$

D Accommodating Time-Invariant Variables in x_t

In this section, we modify the theorems in Section 3.1.3 and Section 3.2 to accommodate the possibility that x_t contains time-invariant variables. Because the proofs are essentially the same as in the main text with only modified notation, we omit them.

Without loss of generality, suppose the last d_{x^*} coordinates of x_t, where $t = 1, \dots, T$ and $0 \le d_{x^*} \le d_x$, are time-varying. Denote these variables as x_t^*. So, the first $d_x - d_{x^*}$ coordinates of x_t are time-invariant variables. Recall that Remark 13 shows that the corresponding first $d_x - d_{x^*}$ coordinates of $\delta_{0,t}$ for $t = 2, \dots, T-1$ are not identified and can be normalized to arbitrary values without affecting the identification and estimation of the other parameters in δ and γ^K. Let $\delta_{0,t}^*$ denote the last d_{x^*} coordinates of $\delta_{0,t}$ for $t = 2, \dots, T-1$. Therefore, our goal becomes to identify and to estimate $\delta^* \equiv (\Delta_1', \delta_{0,2}^{*\prime}, \Delta_2', \dots, \delta_{0,T-1}^{*\prime}, \Delta_{T-1}')'$ and γ^K.

D.1 Identification

We let $\Delta\bar{x}_t^{*\tau}$ denote the last d_{x^*} coordinates of $\Delta\bar{x}_t^\tau$ for $t = 1, \dots, T-2$ and $t < \tau \le T-1$.[33] eq. (18) becomes a system of linear equations of Δ_t $(t = 1, \dots, T-1)$, $\delta_{0,t}^*$ $(t = 2, \dots, T-1)$ and γ^K:

$$y_{T-1} = x_{T-1}'\Delta_{T-1} + \beta\Delta\bar{q}_{T-1}^{K\prime}\gamma^K, \text{ and}$$
$$y_t = x_t'\Delta_t + \sum_{\tau=t+1}^{T-1}\beta^{\tau-t}\Delta\bar{x}_t^{*\tau\prime}\delta_{0,\tau}^* + \sum_{\tau=t+1}^{T-1}\beta^{\tau-t}\Delta\bar{x}_{1,t}^{\tau\prime}\Delta_\tau$$

[33] That is, $\Delta\bar{x}_{1,t}^{*\tau} \equiv \Delta\mathbb{E}(p_\tau x_\tau^* \mid s_t)$. Note that the conditioning variables are the entire vector s_t, including the time-invariant variables in x_t.

$$+\beta^{T-t}\Delta\bar{q}_t^{K\prime}\gamma^K$$

for $t = 1, \dots, T-2$.

Proposition 1 remains unchanged. Other parallel results are stated below with an apostrophe to the original theorem numbers.

Proposition 2' (Identification and over-identification with time-invariant variables in x_t when $T = 3$)**.** *When $T = 3$, Δ_1, $\delta^*_{0,2}$, Δ_2 and γ^K are the identifiable parameters.*

(i) The parameters are identified if and only if the $(3d_x + d_{x^*} + 2K) \times (2d_x + d_{x^*} + K)$ Jacobian matrix,

$$L_3^* \equiv \left[\begin{array}{cc} 0_{(d_x+K)\times(d_x+d_{x^*})} & \mathbb{E}(X_{3,2}X_{3,2}') \\ \hdashline \multicolumn{2}{c}{\mathbb{E}(X^*_{3,1}X^{*\prime}_{3,1})} \end{array}\right],$$

has full column rank, where $X^*_{3,1} \equiv (x_1', \beta\Delta\bar{x}_1^{*2\prime}, \beta\Delta\bar{x}_{1,1}^{2\prime}, \beta^2\Delta\bar{q}_1^{K\prime})'$ and $X_{3,2}$ is defined immediately after eq. (22).

(ii) If more than one matrix that consists of $2d_x + d_{x^*} + K$ distinct rows from L_3^* has full rank, then the parameters are over-identified.

Corollary 1' (Identification with time-invariant variables in x_t when $T = 3$)**.** *When $T = 3$, the parameters are identified if $\mathbb{E}(X^*_{3,1}X^{*\prime}_{3,1})$, or equivalently the second moment matrix of $(x_1', \Delta\bar{x}_1^{*2\prime}, \Delta\bar{x}_{1,1}^{2\prime}, \Delta\bar{q}_1^{K\prime})'$, has full rank.*

Corollary 2' (Identification with time-invariant variables in x_t when $T = 3$)**.** *When $T = 3$, the parameters are identified if: (i) the second moment matrix of $(x_2', \Delta\bar{q}_2^{K\prime})'$ has full rank (i.e., d_x+K), and (ii) the second moment matrix of $(x_1', \Delta\bar{x}_1^{*2\prime})'$ has full rank (i.e, $d_x + d_{x^*}$).*

We define

$$\underbrace{X^*_{T,T-1}}_{(d_x+K)\times 1} \equiv X_{T,T-1}, \qquad \underbrace{X^*_{T,t}}_{((T-t)d_x+(T-t-1)d_{x^*}+K)\times 1} \equiv \begin{bmatrix} x_t \\ \beta\Delta\bar{x}_t^{*t+1} \\ \beta\Delta\bar{x}_{1,t}^{t+1} \\ \beta^2\Delta\bar{x}_t^{*t+2} \\ \beta^2\Delta\bar{x}_{1,t}^{t+2} \\ \vdots \\ \beta^{T-1-t}\Delta\bar{x}_t^{*T-1} \\ \beta^{T-1-t}\Delta\bar{x}_{1,t}^{T-1} \\ \beta^{T-t}\Delta\bar{q}_t^K \end{bmatrix}$$

for $T \geq 2$ and $t = 1, 2, \dots, T-2$, and let

$$\underbrace{\tilde{L}^*_{T,t}}_{((T-t)d_x+(T-t-1)d_{x^*}+K)\times((T-t)d_x+(T-t-1)d_{x^*}+K)} \equiv \mathbb{E}(X^*_{T,t}X^{*\prime}_{T,t}) \text{ and}$$

$$\underbrace{L^*_{T,t}}_{((T-t)d_x+(T-t-1)d_{x^*}+K)\times((T-1)d_x+(T-2)d_{x^*}+K)} \equiv \left[\begin{array}{cc} 0_{((T-t)d_x+(T-t-1)d_{x^*}+K)\times((t-1)d_x+(t-1)d_{x^*})} & \tilde{L}^*_{T,t} \end{array}\right]$$

for $T \geq 2$ and $t = 1,\ldots,T-1$. Again, it is obvious from the definition that the $(T(T-1)d_x/2 + (T-1)(T-2)d_{x^*}/2 + (T-1)K) \times ((T-1)d_x + (T-2)d_{x^*} + K)$ matrix L^*_T has a block-triangular structure as follows:

$$L^*_T = \left[\begin{array}{cccc} 0_{(d_x+K)\times(d_x+d_{x^*})} & 0_{(d_x+K)\times((T-4)d_x+(T-4)d_{x^*})} & 0_{(d_x+K)\times(d_x+d_{x^*})} & \mathbb{E}(X^*_{T,T-1}X^{*\prime}_{T,T-1}) \\ \hdashline 0_{(2d_x+d_{x^*}+K)\times(d_x+d_{x^*})} & 0_{(2d_x+d_{x^*}+K)\times((T-4)d_x+(T-4)d_{x^*})} & \mathbb{E}(X^*_{T,T-2}X^{*\prime}_{T,T-2}) & \\ \hdashline \vdots & & \iddots & \\ \hdashline 0_{((T-2)d_x+(T-3)d_{x^*}+K)\times(d_x+d_{x^*})} & \mathbb{E}(X^*_{T,2}X^{*\prime}_{T,2}) & & \\ \hdashline \mathbb{E}(X^*_{T,1}X^{*\prime}_{T,1}) & & & \end{array}\right].$$

Theorem 2’ (Identification and over-identification for general T)**.** *The parameters of interest are* $(\delta^{*\prime}, \gamma^{K\prime})'$.

(i) The parameters are identified if and only if the L^*_T matrix defined above has full column rank.

(ii) If more than one matrix that consists of $(T-1)d_x + (T-2)d_{x^*} + K$ distinct rows from L^*_T has full rank, then the parameters are over-identified.

Corollary 3’ (Identification with time-invariant variables in x_t for general T)**.** $(\delta^{*\prime}, \gamma^{K\prime})'$ *is identified if the second moment matrix of* $(x_1', \Delta\bar{x}_1^{*2\prime}, \Delta\bar{x}_{1,1}^{2\prime}, \ldots, \Delta\bar{x}_1^{*\tau\prime}, \Delta\bar{x}_{1,1}^{\tau\prime}, \ldots, \Delta\bar{x}_1^{*T-1\prime}, \Delta\bar{x}_{1,1}^{T-1\prime}, \Delta\bar{q}_1^{K\prime})'$ *has full rank.*

Corollary 4’ (Identification with time-invariant variables in x_t for general T)**.** $(\delta^{*\prime}, \gamma^{K\prime})'$ *is identified if: (i) the second moment matrix of* $(x_{T-1}', \Delta\bar{q}_{T-1}^{K\prime})'$ *has full rank (i.e., $d_x + K$); and (ii) the second moment matrix of* $(x_t', \Delta\bar{x}_t^{*t+1\prime})'$ *has full rank (i.e, $d_x + d_{x^*}$) for all $t = 1,\ldots,T-2$.*

D.2 Estimation

We use $\Delta\bar{x}^*$ to collectively denote all $\Delta\bar{x}^\tau_{1,t}$ and $\Delta\bar{x}^{*\tau}_t$ ($t = 1,\ldots,T-2$, $t < \tau \leq T-1$). Redefine the moment functions as We need to redefine the moment functions using the above notation:

$$\begin{aligned} m^*_{T-1}(D,\delta^*,\gamma^K,p,\Delta\bar{\eta},\Delta\bar{x}^*,\Delta\bar{q}^K) &\equiv -v^*_{T-1}(D,\delta^*,\gamma^K,p,\Delta\bar{\eta},\Delta\bar{x}^*,\Delta\bar{q}^K)X^*_{T,T-1}, \text{ and} \\ m^*_t(D,\delta^*,\gamma^K,p,\Delta\bar{\eta},\Delta\bar{x}^*,\Delta\bar{q}^K) &\equiv -v^*_t(D,\delta^*,\gamma^K,p,\Delta\bar{\eta},\Delta\bar{x}^*,\Delta\bar{q}^K)X^*_{T,t}, \end{aligned}$$

where $v^*_{T-1}(D,\delta^*,\gamma^K,p,\Delta\bar{\eta},\Delta\bar{x}^*,\Delta\bar{q}^K) \equiv y_{T-1} - x_{T-1}'\Delta_{T-1} - \beta\Delta\bar{q}^{K\prime}_{T-1}\gamma^K$ and $v^*_t(D,\delta^*,\gamma^K,p,\Delta\bar{\eta},\Delta\bar{x}^*,\Delta\bar{q}^K) \equiv y_t - x_t'\Delta_t - \sum_{\tau=t+1}^{T-1}\beta^{\tau-t}\Delta\bar{x}^{*\tau\prime}_t\delta^*_{0,\tau} - \sum_{\tau=t+1}^{T-1}\beta^{\tau-t}\Delta\bar{x}^{\tau\prime}_{1,t}\Delta_\tau - \beta^{T-t}\Delta\bar{q}^{K\prime}_t\gamma^K$ for $t = 1,\ldots,T-2$. Let $m^*(D,\delta^*,\gamma^K,p,\Delta\bar{\eta},\Delta\bar{x}^*,\Delta\bar{q}^K)$ be the stack of the moment functions $m^*_t(D,\delta^*,\gamma^K,p,\Delta\bar{\eta},\Delta\bar{x}^*,\Delta\bar{q}^K)$ for $t = T-1,\ldots,1$ (in that order). δ^* and γ^K can still be estimated through the steps in

Section 3.2, provided that we replace $\Delta\bar{x}_t^{\tau}$ with $\Delta\bar{x}_t^{*\tau}$ in the collection of "dependent variables" $\Delta\bar{x}$ in step (II) (now denoted as $\Delta\bar{x}^*$) and let

$$(\hat{\delta}^{*\prime}, \hat{\gamma}^{K\prime})' \equiv \underset{\delta^* \in \mathbb{R}^{(T-1)d_x + (T-2)d_{x^*}}, \gamma^K \in \mathbb{R}^K}{\arg\min} \bar{m}_N(\delta^*, \gamma^K)' W_N^* \bar{m}_N(\delta^*, \gamma^K), \tag{D.1}$$

where $\bar{m}_N^*(\delta^*, \gamma^K) \equiv \frac{1}{N}\sum_{i=1}^N \hat{m}^*(D_i, \delta^*, \gamma^K)$, $\hat{m}^*(D, \delta^*, \gamma^K) \equiv m^*(D, \delta^*, \gamma^K, \hat{p}, \widehat{\Delta\bar{\eta}}, \widehat{\Delta\bar{x}}^*, \widehat{\Delta\bar{q}}^K)$ and W_N^* is a symmetric weighting matrix of conformable dimensions that converges in probability to a positive definite matrix W^* as $N \to \infty$. Note that the "regressors" of the nonparametric regressions in step (II) are still s_t, including time-invariant variables in x_t.

Proposition 3' (Asymptotic distribution of $\hat{\delta}^*$ with time-invariant variables in x_t)**.** *Under Assumptions 1 to 5 and the regularity conditions in Section 5 of Newey (1994a), we have*

$$\sqrt{N}\left(\hat{\delta}^* - \delta^*\right) \xrightarrow{d.} \mathcal{N}\left(0, V^*\right),$$

where $V^* \equiv \mathbb{E}(\psi_{\delta^*}(D_i)\psi_{\delta^*}'(D_i))$, and $\psi_{\delta^*}(\cdot)$ is the first $(T-1)d_x + (T-2)d_{x^*}$ coordinates of the following influence function $\psi^*(\cdot)$:

$$\psi^*(D) \equiv -(L_T^{*\prime} W^* L_T^*)^{-1} L_T^{*\prime} W^* \alpha^*(D),$$

in which

$$\begin{aligned}
\alpha^*(D) &\equiv \alpha_\eta(D) + \alpha_{x^*}(D) + \alpha_q(D) + \alpha_{p,direct}(D) + \alpha_{p,indirect}(D),\\
\alpha_{x^*}(D) &\equiv \begin{bmatrix} 0_{(d_x+K)\times 1} \\ \beta X_{T,T-2,0}^* \delta_{0,T-2,0}^{*\prime}\left[\left(\frac{a_{T-2}}{p_{T-2,0}} - \frac{1-a_{T-2}}{1-p_{T-2,0}}\right) x_{T-1}^* - \Delta\bar{x}_{T-2,0}^{*T-1}\right] \\ \vdots \\ \sum_{\tau=t+1}^{T-1} \beta^{\tau-t} X_{T,t,0}^* \delta_{0,\tau,0}^{*\prime}\left[\left(\frac{a_t}{p_{t,0}} - \frac{1-a_t}{1-p_{t,0}}\right) x_\tau^* - \Delta\bar{x}_{t,0}^{*\tau}\right] \\ \vdots \\ \sum_{\tau=2}^{T-1} \beta^{\tau-1} X_{T,1,0}^* \delta_{0,\tau,0}^{*\prime}\left[\left(\frac{a_1}{p_{1,0}} - \frac{1-a_1}{1-p_{1,0}}\right) x_\tau^* - \Delta\bar{x}_{1,0}^{*\tau}\right] \end{bmatrix} \\
&\quad + \begin{bmatrix} 0_{(d_x+K)\times 1} \\ \beta X_{T,T-2,0}^* \Delta_{T-2,0}'\left[\left(\frac{a_{T-2}}{p_{T-2,0}} - \frac{1-a_{T-2}}{1-p_{T-2,0}}\right) p_{T-1,0} x_{T-1} - \Delta\bar{x}_{1,T-2,0}^{T-1}\right] \\ \vdots \\ \sum_{\tau=t+1}^{T-1} \beta^{\tau-t} X_{T,t,0}^* \Delta_{\tau,0}'\left[\left(\frac{a_t}{p_{t,0}} - \frac{1-a_t}{1-p_{t,0}}\right) p_{\tau,0} x_\tau - \Delta\bar{x}_{1,t,0}^{\tau}\right] \\ \vdots \\ \sum_{\tau=2}^{T-1} \beta^{\tau-1} X_{T,1,0}^* \Delta_{\tau,0}'\left[\left(\frac{a_1}{p_{1,0}} - \frac{1-a_1}{1-p_{1,0}}\right) p_{\tau,0} x_\tau - \Delta\bar{x}_{1,1,0}^{\tau}\right] \end{bmatrix},
\end{aligned}$$

and $\alpha_\eta(D)$, $\alpha_q(D)$, $\alpha_{p,direct}(D)$ and $\alpha_{p,indirect}(D)$ are the same as those defined above in eq. (C.2), eq. (C.4), eq. (C.5) and eq. (C.6), respectively, except the only difference that $X_{T,t,0}$ is replaced by $X_{T,t,0}^*$ for $t = 1, \ldots, T-1$ in every occurrence. Note that again, the influence function consists of only the "adjustment terms" since $m_0^*(D) \equiv m_0(D, \delta_0^*, \gamma_0^K, p_0, \Delta\bar{\eta}_0, \Delta\bar{x}_0^*, \Delta\bar{q}_0^K) = 0$.

A consistent estimator of V^* is provided in eq. (38), with relevant objects replaced by their "*" counterparts whenever applicable. Again, its consistency can be justified by a proposition that is essentially the same as Proposition 4, which we omit here for conciseness.

E Proofs of the Results in Section 4.1

Proof of Lemma 4. By the definition of $\Delta\bar{r}_t^K$ and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\mathbb{E}\big((\Delta\bar{r}_t^K)^2\big) &= \mathbb{E}\big([\Delta\mathbb{E}(r^K(x_T,z_T)\mid x_t,z_t)]^2\big)\\ &= \mathbb{E}\big([\mathbb{E}(r^K(x_T,z_T)\mid x_t,z_t,a_t=1)-\mathbb{E}(r^K(x_T,z_T)\mid x_t,z_t,a_t=0)]^2\big)\\ &\le 2\mathbb{E}\big([\mathbb{E}(r^K(x_T,z_T)\mid x_t,z_t,a_t=1)]^2\big)\\ &\quad + 2\mathbb{E}\big([\mathbb{E}(r^K(x_T,z_T)\mid x_t,z_t,a_t=0)]^2\big).\end{aligned} \tag{E.1}$$

Under the conditions (i) to (iv) of this lemma and by Theorem 8 (p. 90) in Lorentz (1966), we have

$$\sup_{s\in\mathcal{S}}|r^K(s)|\le C_1K^{-\frac{m}{d_s}},$$

for some constant C_1. By this uniform bound of $|r^K(s)|$ and the Jensen's inequality, we have

$$\begin{aligned}[\mathbb{E}(r^K(x_T,z_T)\mid x_t,z_t,a_t=a)]^2 &\le \mathbb{E}(|r^K(x_T,z_T)|^2\mid x_t,z_t,a_t=a)\\ &\le C_1^2K^{-\frac{2m}{d_s}} \text{ for } a=0,1.\end{aligned} \tag{E.2}$$

Combining eq. (E.1) and eq. (E.2), we get

$$\mathbb{E}\big((\Delta\bar{r}_t^K)^2\big)\le 4C_1^2K^{-\frac{2m}{d_s}},$$

and the result of the lemma follows. ■

Proof of Theorem 3. It is easy to see that the probability limit of the linear MD estimators $\big(\hat{\delta},\hat{\gamma}^K\big)'$ is $\big(\delta_{pseudo}^{K\prime},\gamma_{pseudo}^{K\prime}\big)'=(L'WL)^{-1}L'WR$. Recalling the definitions of R and $\Delta\bar{r}_t^K$ $(t=1,\ldots,T-1)$, we get

$$\begin{aligned}\left\|\begin{bmatrix}\delta_{pseudo}^K-\delta\\ \gamma_{pseudo}^K-\gamma^K\end{bmatrix}\right\| &\le \lambda_{max}\left(\left(L'WL\right)^{-1}\right)\sqrt{\lambda_{max}(LL')}\lambda_{max}(W)\left\|\begin{bmatrix}\mathbb{E}(X_{T,T-1}\Delta\bar{r}_{T-1}^K)\\ \vdots\\ \mathbb{E}(X_{T,1}\Delta\bar{r}_1^K)\end{bmatrix}\right\|\\ &= \frac{\sqrt{\lambda_{max}(LL')}\lambda_{max}(W)}{\lambda_{min}\left(L'WL\right)}\sqrt{\sum_{t=1}^{T-1}\left\|\mathbb{E}(X_{T,t}\Delta\bar{r}_t^K)\right\|^2}\\ &\le C_2\sqrt{\sum_{t=1}^{T-1}\left\|\mathbb{E}(X_{T,t}\Delta\bar{r}_t^K)\right\|^2}\\ &\le C_3K^{-\frac{m}{d_s}}\sqrt{\sum_{t=1}^{T-1}\|\sqrt{\mathbb{E}(X_{T,t}^2)}\|}\end{aligned}$$

$$\leq C_4 K^{-\frac{m}{d_s}},$$

where C_2 is some constant. In the above, the first inequality holds by the Cauchy-Schwarz inequality and the properties of eigenvalues, the first equality holds by the property of eigenvalues and the definition of the Frobenius norm, the second inequality holds by the conditions on the eigenvalues of $L'L$ and W, the third inequality holds by the Cauchy-Schwarz inequality and Lemma 4, and the last inequality holds by finite second moments of $q^K(x_T, z_T)$ and x_t for $t = 1, \ldots, T-1$. Then, the result of the proposition follows. ■

F Generating Simulation Sample

To generate a simulation sample from the model specified in Section 5.1 is not a trivial task, because all the state variables in our parameterization are continuous. In this appendix, we provide details on the three steps we took to solve the dynamic programming (DP) problem to generate a simulation sample. Section F.1 describes how to discretize the choice-specific VAR(1) state transition processes, so the DP problem with continuous state variables can be approximated by a DP problem with discrete states. Section F.2 explains how to solve the DP problem with discrete states backwardly. Section F.3 describes how to draw a simulation sample forwardly.

F.1 Efficient Discretization of States and State Transition Distributions

In general, randomly drawing state variables s_t from their stationary distribution is not efficient, especially when d_s is large, because most of the draws will end up in regions with low probabilities. In addition, the VAR(1) processes of s_t in our model depend on whether $a_t = 0$ or $a_t = 1$ is chosen.

The original DP problem in Section 5.1 involves a 3×1 vector of continuous state variables $s_{i,t}$, whose transition is governed by two choice-specific VAR(1) processes.[34] Let $f(s_{t+1} \mid a_t = a)$ for $a = 0, 1$ denote the density function of the stationary distribution of s_{t+1} when $a_t = a$. The choice $a_t = 1$ is a "reset" choice such that the distribution of the next-period state variables s_{t+1} does not depend on the current state variables s_t, that is, $f(s_{t+1} \mid s_t, a_t = 1) = f(s_{t+1} \mid a_t = 1)$. It is tempting to separately discretize the two choice-specific VAR(1) processes, but this will result in complicated choice-specific discretized state transition matrices.

To make the discretized state transition matrices simple, we impose the restriction that the stationary distributions of the state variables to be the same across both choices: $f(s_{t+1} \mid a_t = 1) = f(s_{t+1} \mid a_{i,t} = 0)$. We first discretize the VAR(1) process when $a_t = 0$ by using the "EDS" method, proposed by Maliar and Maliar (2015) and specialized for a VAR(1) process by Gordon (2021) to obtain a discrete grid of s_t values that consists of only 1000 points but approximates the stochastic

[34] Our method can handle larger d_s, but we chose $d_s = 3$ because simulating the sequences of state variables and choices to implement the HM estimator puts a rapidly increasing strain on memory as d_s increases.

behavior of the original VAR(1) process well.[35] Let $\vec{s} = (s^1, \ldots, s^{1000})'$ denote the 1000 "EDS" grid points, and each point s^j ($j = 1, \ldots, 1000$) is to a vector of $(x_1, x_2, z)'$ values. The "EDS" method also gives us the discretized state transition matrix $f^{dis}(s_{t+1} = s^j \,|\, s_t = s^{j'}, a_t = 0)$ and the stationary probability mass function $f^{dis}(s_{t+1} = s^j \mid a_t = 0)$, for $\forall j, j' = 1, \ldots, 1000$. This probability mass function equals to each row of the state transition matrix $f^{dis}(s_{t+1} = s^j \,|\, s_t = s^{j'}, a_t = 1)$ when $a_t = 1$, because $f(s_{t+1} \,|\, s_t, a_t = 1) = f(s_{t+1} \,|\, a_t = 0)$.

F.2 Solving the Model Backwardly with the Discrete States

Having obtained the discrete states and discrete state transition matrices, we can solve the discretized DP problem backwardly.

F.2.1 $t = 3$ (Decision Terminal period)

We start from the terminal period. First, the choice-specific expected payoff in period $t = 3$ is

$$v_3(a, s^j) = u_3(a, x^j) = \delta_{a,3,0} + \delta_{a,3,1} x_1^j + \delta_{a,3,2} x_2^j, \text{ for } j = 1, \ldots, 1000, \tag{F.1}$$

where $x^j = (x_1^j, x_2^j)'$ is the first 2×1 subvector of s^j, and we get a vector $\vec{v}_3(a) \equiv (v_3(a, s^1), \ldots, v_3(a, s^{1000}))'$ using this formula. Then, the CCPs in period t is

$$p_t(s^j) = \frac{\exp v_t(1, s^j)}{\exp v_t(0, s^j) + \exp v_t(1, s^j)}, \text{ for } j = 1, \ldots, 1000. \tag{F.2}$$

Let $t = 3$ in eq. (F.2) and we get a vector $\vec{p}_3 \equiv (p_3(s^1), \ldots, p_3(s^{1000}))'$. Lastly, the expected optimal payoff in period t (i.e., the value of the integrated value function) is (see Arcidiacono and Miller, 2011, for example)

$$\bar{V}_t(s^j) = v_t(0, s^j)) - \ln(1 - p_t(s^j)), \text{ for } j = 1, \ldots, 1000. \tag{F.3}$$

Let $t = 3$ in eq. (F.3) and we get a vector $\vec{\bar{V}}_3 \equiv (\bar{V}_3(s^1), \ldots, \bar{V}_3(s^{1000}))'$.

F.2.2 $t = 2$

The choice-specific expected payoff in non-terminal period t is

$$v_t(a, s^j) = u_t(a, x^j) + \beta \mathbb{E}(\bar{V}_{t+1}(s^{j'}) \,|\, s_t = s^j, a_t = a), \tag{F.4}$$

where $u_t(a, x^j)$ is computed in a similar way as $u_3(a, x^j)$ in eq. (F.1). For $t = 2$, the second term in eq. (F.4) can be numerically computed using the vector $\vec{\bar{V}}_3$ obtained in Section F.2.1 and the choice-specific discrete state transition matrices $f^{dis}(s_{t+1} \,|\, s_t, a_t)$ obtained in Section F.1. Let $\vec{v}_2(a) \equiv (v_2(a, s^1), \ldots, v_2(a, s^{1000}))'$. Let $t = 2$ in eq. (F.2) eq. (F.3) and plug in $v_2(a, s^j)$ for $j = 1, \ldots, 1000$, then we get a vector $\vec{p}_2 = (p_2(s^1), \ldots, p_2(s^{1000}))'$ and another vector $\vec{\bar{V}}_2 \equiv (\bar{V}_2(s^1), \ldots, \bar{V}_2(s^{1000}))' = \vec{v}_2(0) - \ln(1 - \vec{p}_2)$.

[35] The number of points is up to the researchers' choice.

F.2.3 $t = 1$

Let $t = 1$ in eq. (F.4), in which again the second term can be numerically computed using the vector $\vec{\vec{V}}_2$ obtained in Section F.2.2 and the choice-specific discrete state transition matrices $f^{dis}(s_{t+1}|s_t, a_t)$ obtained in Section F.1. We let $\vec{v}_1(a) \equiv (v_1(a, s^1), \ldots, v_1(a, s^{1000}))'$. Let $t = 1$ in eq. (F.2) and we get $\vec{p}_1 = (p_1(s^1), \ldots, p_1(s^{1000}))'$. $\vec{\vec{V}}_1$ is unnecessary since $t = 1$ is the sample initial period.

F.3 Simulate A Sample Forwardly with the Original State Space

Section F.2 gives the solution of the DP problem on the discrete "EDS" grid points $s^1, \ldots, s^{1000}$ – the CCP vectors $\vec{p}_1, \vec{p}_2, \vec{p}_3$. Before simulating a sample, however, we need to interpolate CCPs for any continuous value of s using $\vec{p}_t$ as observations. In particular, for each $t = 1, 2, 3$, we regress $\ln(p_t(s^j)/(1 - p_t(s^j)))$ on a power series of x_1^j, x_2^j and z^j ($j = 1, \ldots, 1000$) to get the approximated CCP function $\tilde{p}_t(x_1, x_2, z)$.

Then we simulate a sample forwardly. For period $t = 1$, we first generate $N = 250$ random draws of $s_1 = (x_{1,1}, x_{1,2}, z_1)'$ from the stationary distribution of s_1, which is the same across the choices 0 and 1. We then evaluate the approximated CCP function $\tilde{p}_1(x_{1,1}, x_{1,2}, z_1)$ at the s_1 draws and randomly generate a choice a_1 for each draw. For period $t = 2$, we first generate random draws of s_2 from choice-specific VAR(1) processes based on the s_1 and a_1 values. We then evaluate the approximated CCP function $\tilde{p}_2(x_{2,1}, x_{2,2}, z_2)$ at the s_2 draws and randomly generate a choice a_2 for each draw. For period $t = 3$, repeat what we did for $t = 2$ with all time indices increased by one.