Robust Analysis of Short Panels*

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Abstract

Many structural econometric models include latent variables on whose probability distributions one may wish to place minimal restrictions. Leading examples in panel data models are individual-specific variables sometimes treated as "fixed effects" and, in dynamic models, initial conditions. This paper presents a generally applicable method for characterizing sharp identified sets when models place no restrictions on the probability distribution of certain latent variables and no restrictions on their covariation with other variables. In our analysis latent variables on which restrictions are undesirable are removed, leading to econometric analysis robust to misspecification of restrictions on their distributions which are commonplace in the applied panel data literature. Endogenous explanatory variables are easily accommodated. Examples of application to some static and dynamic binary, ordered and multiple discrete choice and censored panel data models are presented.

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1 Introduction

This paper deals with models of processes delivering values of outcomes, Y, given values of exogenous variables, Z, and latent, that is unobserved, variables U and V. The models that are the focus of this paper all leave the distribution of V on its known support and its covariation with all other variables completely unrestricted. By contrast, latent variable U may be required to be, to some degree, independent of Z.

Leading examples of latent variables in structural econometric models employed in practice on whose distribution one may not want to impose restrictions are the individual-specific unobserved variables included in many panel data models, sometimes called "fixed effects" and the historic values of outcomes dynamically determined by a process, commonly called "initial conditions".

The following example has both elements, a "fixed effect", C, and an initial condition, Y_{10}.

Example 1 *A dynamic binary response model specifies that for all* $t \in [T] \equiv \{1, ..., T\}$,

$$Y_{1t} = \begin{cases} 1 & , \quad \alpha Y_{2t} + Z_t\beta + \gamma Y_{1t-1} + C + U_t \geq 0, \\ 0 & , \quad \alpha Y_{2t} + Z_t\beta + \gamma Y_{1t-1} + C + U_t \leq 0, \end{cases} \tag{1}$$

with $Y_{1t} = 0$ *or* $Y_{1t} = 1$ *permitted when both inequalities hold,*[1] *and* $U \perp\!\!\!\perp Z$ *where*

$$U \equiv (U_1, \ldots, U_T), \qquad Z \equiv (Z_1, \ldots, Z_T).$$

Realizations of (Y, Z) *are observed where*

$$Y = (Y_{11}, \ldots, Y_{1T}, Y_{21}, \ldots, Y_{2T}),$$

and Z_t *and* Y_{2t} *may be vectors. If the value of* Y_{10} *is observed then unrestricted* $V = C$, *otherwise* $V = (C, Y_{10})$. *If* α *is not restricted equal to zero there are endogenous explanatory variables.*

In many cases found in practice in which T is large, the value V takes for each observational unit is identified. In this case econometric analysis can proceed placing no restrictions at all on the distribution of V and treating it as a parameter to be

[1]This is equivalent to the representation $Y_{1t} = 1[\alpha Y_{2t} + Z_t\beta + \gamma Y_{1t-1} + C + U_t > 0]$ when the indicator function $1[a > b]$ takes the value 1 if $a > b$, 0 if $a < b$, and either value if $a = b$.

estimated. When T is not large this is unattractive because there may be intolerable inaccuracy in the estimation of V which may contaminate estimates of other parameters. Additionally there is the incidental parameters problem set out in Neyman and Scott (1948) and reviewed in Lancaster (2000).

Faced with this problem, for small T, most papers proceed to obtain information on structural features by placing distributional restrictions on V. Section 2 lists many examples. It is good to know what knowledge of structural features can be obtained absent such restrictions. That allows the force of distributional restrictions on V to be assessed and offers the possibility of detecting misspecification. This paper shows how that knowledge can be obtained. The results given here open the way to a relatively robust analysis of models like panel models with fixed effects in which there are latent variables on which one desires to place no distributional restrictions.

This paper presents characterizations of identified sets of structures and structural features in models admitting unobserved variables such as V whose distribution is unrestricted. There can be endogenous explanatory variables as in Example 1 when $\alpha \neq 0$. A model may be *incomplete* in the sense that, given values of all observed and all unobserved variables and a specification of parameter values and functional forms, the model can deliver a nonsingleton set of values of outcomes. Identified sets are characterized by systems of moment inequalities. Estimation and inference can proceed using established econometric methods.

The strategy employed here removes unrestricted latent variables, V, by projection.[2] We derive, for each value of the observed variables, the set of values of unobserved U compatible with that value. Values of U in such a set are associated with alternative values of V. Typically a value of U in such a set can deliver more than one value of Y. So, on removing latent variables V, there remains an incomplete model.[3]

Identification analysis is conducted in the context of the Generalized Instrumental Variable (GIV) framework introduced in Chesher and Rosen (2017) in which probability distributions of such sets of values of U induced by the observed distributions of outcomes are essential elements.

Section 2 considers the relationship of this work to some other results in the literature. Section 3 presents characterizations of identified sets of structures. Sections 4 to 9 set out applications to linear panel models and to models of binary response

[2] Our eschewal of restrictions on the distribution of V accords with the approach in Neyman and Scott (1948) in which the elements of V are treated as parameters, subject to no restrictions.

[3] If the model is incomplete before projection then different values of V can deliver different sets of values of Y.

panels, ordered choice panels, multiple discrete choice panels, simultaneous binary outcome panels, and models of panels with censored continuous outcomes.

2 Related literature

Rasch (1960), Rasch (1961), Andersen (1970), and Chamberlain (2010) study point identifying static panel models (i.e. $\gamma = 0$ in (1)) with restrictions requiring $U_1, ..., U_T$ to be independent over time and distributed independently of Z and independently of the fixed effect and each with logistic marginal distributions. Like all the papers referred to in this section, except one paper which is noted, these models do not admit endogenous explanatory variables.

In the linear panel data model with fixed effects, differencing across time periods removes the fixed effect, delivering events whose probability of occurrence can be known and is invariant with respect to changes in the value of the fixed effect. Under suitable support restrictions this leads to point identification. In nonlinear panel data models with fixed effects, this simple differencing strategy does not apply. Nonetheless, in the Rasch-Andersen-Chamberlain set up, events whose probabilities of occurrence are invariant to changes in the value of the fixed effect are found. Under particular distributional restrictions point identification results. More recent papers on nonlinear panel data models have taken a similar approach.

Honoré and Kyriazidou (2000) study a dynamic model as in (1) but with no endogenous explanatory variable ($\alpha = 0$) with the U_t's independent of the fixed effect, independent over time, distributed independently of Z and with logistic distributions. That paper also studies a case in which the logistic distribution restriction is dropped and a case with multinomial logit panels with latent variables U independent of the fixed effects and independent of Z. Honoré and Kyriazidou (2019) extends this work, studying multivariate dynamic panel data logit models with fixed effects. Many papers, like these, invoke restrictions requiring independence between U_t's and the fixed effect conditional on some of the other observable variables including Honoré and Tamer (2006), Honoré and De Paula (2021),[4] Dobronyi, Gu, and Kim (2021), Honoré and Weidner (2022), Davezies, D'Haultfœuille, and Laage (2022), Kitazawa (2022), Bonhomme, Dano, and Graham (2023), Dano (2023), Davezies, D'Haultfœuille, and Mugnier (2023), and Honoré, Muris, and Weidner (2023).

[4] This paper considers models in which there are simultaneous equations in binary outcomes and so, endogenous explanatory variables.

Such independence restrictions are not imposed here.[5] This permits for example the U_t's to exhibit heteroskedastic variation with observational-unit-specific fixed effects.

There are many papers studying panel models of binary outcomes and multiple discrete choice under conditional *stationarity* restrictions on the distribution of the time varying latent variables introduced in Manski (1987). These papers include Chernozhukov, Fernandez-Val, Hahn, and Newey (2013), Shi, Shum, and Song (2018), Gao and Li (2020), Khan, Ouyang, and Tamer (2021), Pakes, Porter, Shepard, and Calder-Wang (2021), Pakes and Porter (2022), Dobronyi, Ouyang, and Yang (2023), Khan, Ponomareva, and Tamer (2023), and Mbakop (2023).

In all of these cases the stationarity restriction placed on time-varying unobservable heterogeneity is required to hold conditional on the value of the fixed effect and the observable exogenous variables, which restricts the covariation of the fixed effect and U.[6] In contrast, the models considered in this paper impose no restrictions on the covariation of the fixed effect with any variable.

The only previous paper of which we are aware that provides partial identification analysis for discrete outcome panel data models absent restrictions on the covariation of the fixed effect with any other variables is Aristodemou (2021). That paper provides set-identifying moment inequalities in panel data models of binary response and ordered choice when the covariation of the fixed effects with other variables is unrestricted. The results developed in this paper provide a rule-directed procedure for enumerating all events whose probability is invariant with respect to the value of unrestricted latent variables thereby delivering sharp set identification for these and other nonlinear panel data models.

Application of sharp set identification analysis to panel data models with censored outcomes is demonstrated in Section 9. Observable implications in the form of moment equalities are derived for such models with Tobit-type censoring at zero in both static and dynamic contexts in Honoré (1992, 1993), Honoré and Hu (2002), and Hu (2002), all in models in which the U_t's satisfy the conditional stationarity assumption that has also been used in discrete outcome panel models. The only previous paper of

[5] One approach in such settings, demonstrated by e.g. Honoré and Weidner (2022) and Honoré, Muris, and Weidner (2023), is the functional differencing approach developed in Bonhomme (2012). This however requires knowledge of the distribution of $F_{Y|Z,C}$, which one does not have in models such as that of Example 1 without knowledge of the joint distribution of U and C.

[6] In the binary response specification (1) conditional stationarity implies that for all z, $F_{U_1|Z=z,C=c} = F_{U_2|Z=z,C=c}$ and $F_{U_1|Z=z,C=c'} = F_{U_2|Z=z,C=c'}$ for any c, c', which restricts how the conditional distribution of U can change with values of the fixed effect C. As pointed out by Chernozhukov, Fernandez-Val, Hahn, and Newey (2013) the stationarity restriction $U_t|C, Z \overset{d}{=} U_1|C, Z$ for all t is equivalent to $(U_t, C)|Z \overset{d}{=} (U_1, C)|Z$ for all t.

which we are aware that provides identification analysis for censored outcome panel models without restricting the covariation of the fixed effect with other variables is Khan, Ponomareva, and Tamer (2016) (KPT), which provides the sharp identified set for slope coefficient β in a static two-period model in which $U_2 - U_1$ and Z are independent. Extensions are provided to some specialized dynamic models with two periods of observations with an observed initial condition and inequality restrictions on parameters. The analysis here additionally accommodates more periods, unobserved initial conditions, and endogenous explanatory variables. Like KPT we allow the censoring value to vary and to be endogenous, nesting the classical case of fixed censoring found in Tobit models.

This paper presents a generally applicable approach to identification analysis in a wide class of nonlinear panel data models in which there are distributionally unrestricted latent variables and gives examples of the results it produces. Most of our examples feature discrete outcomes, but the application to the censored outcome model in Section 9 demonstrates that the analysis applies more broadly.

3 Identified sets

First the notation employed in this paper is introduced.

Notation. *Generically $\mathcal{R}_A$ denotes the support of random variable A and $L_{A|Z=z}$ denotes a conditional probability distribution of random variable A given $Z = z$. $L_{A|Z=z}(\mathcal{S})$ is the conditional probability A takes a value in set $\mathcal{S}$ given $Z = z$. $\mathcal{L}_{A|Z} \equiv \{L_{A|A=z}; z \in R_Z\}$ is the collection of conditional distributions delivered by a joint distribution L_{AZ} when the support of Z is $\mathcal{R}_Z$. $A \perp\!\!\!\perp B$ denotes A and B are independently distributed. Sets and set-valued random variables are expressed using calligraphic font. Collections of sets are expressed using sans serif font. $\mathbb{R}$ denotes the real line. The empty set is denoted $\emptyset$. For $T > 1$, notation $[T]$ denotes $\{1, ...T\}$. For any random vectors $X_1, ..., X_T$ notation $\Delta_{ts}X \equiv X_t - X_s$ is used throughout.*

Variables Y are endogenous outcomes, variables Z are exogenous[7] and variables U and V are latent variables. Random vectors (Y, Z, U, V) are defined on a probability space $(\Omega, \mathsf{L}, \mathbb{P})$ endowed with the Borel sets on Ω. The support of (Y, Z, U, V) is a subset of a finite dimensional Euclidean space. The sampling process identifies F_{YZ}, equivalently the collection of conditional distributions $\mathcal{F}_{Y|Z}$ and F_Z, as occurs for example under random sampling of observational units. It is assumed throughout for

[7] In the sense that their values are not affected by the evolution of the process.

ease of exposition that each observational unit delivers the same number of observations, but unbalanced panels are easily accommodated with some added notation.

Models place restrictions on a structural function $h : \mathcal{R}_{YZUV} \to \mathbb{R}$ which specifies the combinations of these variables that can occur *via* the following restriction.[8]

$$\mathbb{P}[h(Y, Z, U, V) = 0] = 1$$

Models place restrictions on the conditional probability distributions of U given Z which are elements of a collection $\mathcal{G}_{U|Z}$. Coupled pairs $(h, \mathcal{G}_{U|Z})$ are called *structures*. A model $\mathcal{M}$ is a collection of structures that obey the restrictions imposed a priori on the data generation process. This paper provides sharp identification analysis of structures $(h, \mathcal{G}_{U|Z}) \in \mathcal{M}$ and functionals thereof given knowledge of $\mathcal{F}_{Y|Z}$.

The essential element of the models considered here is that they place *no* restrictions on the marginal distribution of V and *no* restrictions on the covariation of V with (Z, U).

This paper shows how the framework set out in Chesher and Rosen (2017) (CR) can be used to study cases with unobserved variables whose distribution and covariation with other variables is not subject to restrictions. The support of any initial condition components of V is assumed known and the support of all "fixed effect" components of V is assumed to be the entirety of the Euclidean space in which it resides. It is straightforward to generalize the analysis to cases in which the support of the fixed effect is restricted.

For all characterizations of identified sets of values of the pair $(h, \mathcal{G}_{U|Z}) \in \mathcal{M}$ it is assumed that a priori restrictions on $\mathcal{G}_{U|Z}$ are such that $U|Z$ is restricted absolutely continuous with respect to Lebesgue measure almost surely. This renders the boundary of sets $\mathcal{U}^*(y, z; h)$ to be measure zero with respect to any distribution $G_{U|Z=z}$. It is convenient to define the structural function h such that sets $\mathcal{U}^*(Y, Z; h)$ are closed almost surely in the usual Euclidean topology, and we do so here, but this is of no substantive consequence and can be relaxed.[9]

[8]In the case of (1) a suitable h function would be

$$h(Y, Z, U, V) = \sum_{t=1}^{T} \max\{0, (1 - 2Y_{1t}) \cdot (\alpha Y_{2t} + Z_t\beta + \gamma Y_{1t-1} + C + U_t)\}.$$

with $V = (C, Y_0)$.

[9]With some care equivalent results could be obtained allowing for random open sets and random closed sets, or by working with an alternative topology in which the sets under consideration are closed, such as the discrete topology when $\mathcal{R}_Y$ is discrete. One could also allow sets of values of unobservables that deliver "ties" in the optimal choice of discrete outcome with positive probability,

Taken together the restrictions set out above ensure that Restrictions A1 - A6 of CR hold in the models considered, suitably modified to accommodate unobservable variables (U, V) with the distribution of V unrestricted.[10]

Theorem 1 provides a characterization of the identified set of structures, denoted $\mathcal{I}(\mathcal{M}, \mathcal{F}_{Y|Z})$, delivered by a model $\mathcal{M}$ and a collection of distributions, $\mathcal{F}_{Y|Z}$. This is the collection of distributions *marginal* with respect to V obtained from some collection $\mathcal{F}_{YV|Z}$.

Theorem 1 *Let $\mathcal{R}_V$ denote the support of V. Define $\mathcal{U}^*(y, z; h)$ as follows.*

$$\mathcal{U}^*(y, z; h) \equiv \{u : \exists v \in \mathcal{R}_V \quad \textit{such that} \quad h(y, z, u, v) = 0\} \tag{2}$$

Let $\mathcal{F}_{Y|Z}$ be a collection of distributions whose members are marginal distributions of the members of some collection of distributions $\mathcal{F}_{YV|Z}$. The set of structures $(h, \mathcal{G}_{U|Z})$ identified by model $\mathcal{M}$ and the collection of distributions $\mathcal{F}_{Y|Z}$ comprises all structures admitted by the model $\mathcal{M}$ such that for all $z \in \mathcal{R}_Z$, the probability distribution $G_{U|Z=z} \in \mathcal{G}_{U|Z}$ is selectionable with respect to the conditional distribution of the random set $\mathcal{U}^*(Y, Z; h)$ delivered by the probability distribution $F_{Y|Z=z} \in \mathcal{F}_{Y|Z}$.

Formally the Theorem defines the identified set of structures $(h, \mathcal{G}_{U|Z})$ as

$$\begin{aligned}\mathcal{I}(\mathcal{M}, \mathcal{F}_{Y|Z}) \equiv \big\{ & (h, \mathcal{G}_{U|Z}) \in \mathcal{M} : G_{U|Z=z} \preceq \mathcal{U}^*(Y, Z; h) \\ & \text{conditional on } Z = z \quad \text{a.e.} \quad z \in \mathcal{R}_Z \big\} ,\end{aligned} \tag{3}$$

where, as in Chesher and Rosen (2020), for any random variable A with distribution F_A and random set $\mathcal{A}$, $F_A \preceq \mathcal{A}$ denotes that F_A is selectionable with respect to the distribution of $\mathcal{A}$.[11]

The proof relies on the following Lemma.

Lemma 1 *Define*

$$\mathcal{Y}^*(z, u; h) \equiv \{y : \exists v \in \mathcal{R}_V \quad \textit{such that} \quad h(y, z, u, v) = 0\}. \tag{4}$$

and apply results of CR, with suitable care.

[10] The latent variables U in restrictions A1-A6 of CR should be taken to include both the variables U and V of this paper. For completeness, these restrictions, adapted to the present context, are collected in Appendix A.

[11] The probability distribution of random variable A is selectionable with respect to the probabilty distribution of random set $\mathcal{A}$ when there exists (i) $\tilde{A}$ having the same distribution as A, and (ii) $\widetilde{\mathcal{A}}$ having the same distribution as $\mathcal{A}$, both defined on the same probability space such that $\mathbb{P}[\tilde{A} \in \widetilde{\mathcal{A}}] = 1$. See Definition 2 of Chesher and Rosen (2020).

The sets $\mathcal{Y}^*(z.u;h)$ and $\mathcal{U}^*(y,z;h)$ possess the duality property

$$\forall z, y^+, u^+ \quad y^+ \in \mathcal{Y}^*(z,u^+;h) \iff u^+ \in \mathcal{U}^*(y^+,z;h).$$

Proof. The result follows because

$$y^+ \in \mathcal{Y}^*(z,u^+;h) \iff \exists v \in \mathcal{R}_V \quad \text{such that} \quad h(y^+,z,u^+,v) = 0$$

$$u^+ \in \mathcal{U}^*(y^+,z;h) \iff \exists v \in \mathcal{R}_V \quad \text{such that} \quad h(y^+,z,u^+,v) = 0.$$

■

The proof of Theorem 1 above proceeds as the proof of Theorem 2 in CR, replacing U sets with U^* sets.

The identified set of structures can be characterized as shown in Corollary 1 using the characterization of selectionability given in Artstein (1983), as in Corollary 2 of CR.

Corollary 1 *Let $\mathsf{F}(\mathcal{R}_U)$ denote the collection of closed sets on the support of U. The set of structures identified by model $\mathcal{M}$ and the collection of distributions $\mathcal{F}_{Y|Z}$ is as follows.*

$$\begin{aligned}\mathcal{I}(\mathcal{M},\mathcal{F}_{Y|Z}) \equiv \big\{ &(h,\mathcal{G}_{U|Z}) \in \mathcal{M} : \forall \mathcal{S} \in \mathsf{F}(\mathcal{R}_U) \\ &F_{Y|Z=z}(\{y : \mathcal{U}^*(y,z;h) \subseteq \mathcal{S}\}) \leq G_{U|Z=z}(\mathcal{S}) \ \textit{a.e.}\ z \in \mathcal{R}_Z \big\}. \end{aligned} \tag{5}$$

Remarks

1. The probability $F_{Y|Z=z}(\{y : \mathcal{U}^*(y,z;h) \subseteq \mathcal{S}\})$ is the probability conditional on $Z = z$ of the occurrence of a value of Y that can *only* occur when $U \in \mathcal{S}$. We will refer to such a probability as a *containment probability* and employ the notation $\mathcal{A}(\mathcal{S},z;h) \equiv \{y : \mathcal{U}^*(y,z;h) \subseteq \mathcal{S}\}$.

2. Because the inequalities defining $\mathcal{I}(\mathcal{M},\mathcal{F}_{Y|Z})$ only involve probabilities of events under which U^* sets are subsets of test sets, $\mathcal{S}$, the collection of test sets $\mathsf{F}(\mathcal{R}_U)$ in the definition of $\mathcal{I}(\mathcal{M},\mathcal{F}_{Y|Z})$ can, for each $z \in \mathcal{R}_Z$, be replaced by the collection of all unions of U^* sets,

$$\mathsf{U}^*(z;h) \equiv \left\{ \bigcup_{y \in \mathcal{Y}} \mathcal{U}^*(y,z;h) : \mathcal{Y} \subseteq \mathcal{R}_Y \right\}. \tag{6}$$

3. Let $\mathsf{Q}(z;h)$ be a core determining collection (CDC) of sets such that if the inequality

$$F_{Y|Z=z}(\{y : \mathcal{U}^*(y,z;h) \subseteq \mathcal{S}\}) \leq G_{U|Z=z}(\mathcal{S})$$

holds for all $\mathcal{S} \in \mathsf{Q}(z;h)$ then it holds for all $\mathcal{S} \in \mathsf{F}(\mathcal{R}_U)$. The collection $\mathsf{U}^*(z;h)$ defined in (6) is such a CDC for the specified value of z.

   (a) If disjoint $\mathcal{S}_1$ and $\mathcal{S}_2$ are members of a CDC and $\mathcal{A}(\mathcal{S}_1, z; h) \cap \mathcal{A}(\mathcal{S}_2, z; h) = \emptyset$, which occurs for example when all U^* sets are connected sets, then $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ can be excluded from the CDC. Theorem 3 of CR applies and gives further refinements.

   (b) If $\mathcal{S}_1$ and $\mathcal{S}_2$ are members of a CDC with $\mathcal{S} \equiv \mathcal{S}_1 \cap \mathcal{S}_2$ and $\mathcal{S}_1 \cup \mathcal{S}_2 = \mathcal{R}_U$, and

$$\mathcal{A}(\mathcal{S}_1, z; h) \cup \mathcal{A}(\mathcal{S}_2, z; h) = \mathcal{R}_Y$$

$$\mathcal{A}(\mathcal{S}_1, z; h) \cap \mathcal{A}(\mathcal{S}_2, z; h) = \mathcal{A}(\mathcal{S}, z; h)$$

   then $\mathcal{S}$ can be excluded from the CDC by results in Luo and Wang (2018) and Ponomarev (2022).

   (c) In particular applications some members of a CDC need not be considered because they deliver inequalities that are dominated by others.

4. If there is additionally the restriction $U \perp\!\!\!\perp Z$ then $\mathcal{G}_{U|Z} = \{G_U\}$ and there is the following simplification.

$$\mathcal{I}(\mathcal{M}, \mathcal{F}_{Y|Z}) \equiv \left\{ \quad (h, G_U) \in \mathcal{M} : \quad \forall \mathcal{S} \in \mathsf{F}(\mathcal{R}_U) \right. \\ \left. \sup_{z \in \mathcal{R}_Z} F_{Y|Z=z}(\{y : \mathcal{U}^*(y,z;h) \subseteq \mathcal{S}\}) \leq G_U(\mathcal{S}) \quad \right\}. \quad (7)$$

Quantile and mean independence restrictions can be accommodated.

5. Sets $\mathcal{U}^*(y,z;h)$ compatible with realizations (y,z) of observable variables can be obtained in a variety of ways. In many models $h(y,z,u,v) = 0$ if and only if there exist functions $d_t(\cdot; h) : \mathcal{R}_{YZUV} \to \mathbb{R}$ such that for all $t \in [T]$, $d_t(y,z,u,v) = 0$. The generalized inverse of this mapping with respect to v is

$$\mathcal{D}_t(y,z,u;h) \equiv \{v : d_t(y,z,u,v) = 0\},$$

point-valued when $d_t(y,z,u,v)$ is strictly monotone in scalar v and more generally set-valued.

The U^* sets can then be written as

$$\mathcal{U}^*(y,z;h) = \left\{ u : \bigcap_{t\in[T]} \mathcal{D}_t(y,z,u;h) \neq \emptyset \right\}.$$

In models in which outcomes Y_{1t} are determined by a weakly monotone transformation of an index function that is linear in v, sets $\mathcal{D}_t(y,z,u)$ are characterized by linear equalities and inequalities. Variables v can then be analytically removed from these linear systems, for example by way of Fourier-Motzkin elimination, to obtain linear inequalities characterizing $\mathcal{U}^*(y,z;h)$.

This can be used both in the examples studied in this paper, in which scalar fixed effects enter additively in an index function, as well as in more general models that allow individual-specific coefficients in such index functions.

6. Many of our illustrative examples will employ the restriction that U and Z are fully independent, but the characterizations afforded by Theorem 1 and Corollary 1 allow for a much wider variety of restrictions on the collection of conditional distributions $\mathcal{G}_{U|Z}$. For example, restrictions could require that $U_t \perp\!\!\!\perp (Z_1, ..., Z_t)$ for all t, while permitting dependence between U_t and Z_s for $s > t$, hence allowing models that impose only weak exogeneity.

7. Identified sets of values of a structural feature, defined as a functional, $\theta\left((h, \mathcal{G}_{U|Z})\right)$, are obtained by projection.

$$\mathcal{I}_\theta(\mathcal{M}, \mathcal{F}_{Y|Z}) = \{\theta(\left(h, \mathcal{G}_{U|Z}\right)) : \left(h, \mathcal{G}_{U|Z}\right) \in \mathcal{I}(\mathcal{M}, \mathcal{F}_{Y|Z})\}.$$

An example of such a structural feature is a vector of coefficients multiplying included exogenous variables in models in which h is parametrically specified with a linear index restriction.

8. Outer sets for the projection of the identified set of structures onto the space of structural functions can be obtained. Impose the restriction $U \perp\!\!\!\perp Z$ and let there be no further restrictions on G_U. All structures in $\mathcal{I}(\mathcal{M}, \mathcal{F}_{Y|Z})$ satisfy the inequality

$$\sup_{z\in\mathcal{R}_Z} F_{Y|Z=z}(\{y : \mathcal{U}^*(y,z;h) \subseteq \mathcal{S}\}) \leq G_U(\mathcal{S})$$

and applying this with $\mathcal{S}$ replaced by its complement delivers

$$G_U(\mathcal{S}) \leq \inf_{z \in \mathcal{R}_Z} F_{Y|Z=z}(\{y : \mathcal{U}^*(y,z;h) \cap \mathcal{S} \neq \emptyset\}).$$

Let $\mathcal{H}(\mathcal{M})$ denote the set of structural functions admitted by model $\mathcal{M}$. There is the following outer identified set on the space of structural functions.

$$\mathcal{I}_h(\mathcal{M}, \mathcal{F}_{Y|Z}) \equiv \left\{ \begin{array}{l} h \in \mathcal{H}(\mathcal{M}) : \quad \forall \mathcal{S} \in \mathsf{F}(\mathcal{R}_U) \\ \quad \sup_{z \in \mathcal{R}_Z} F_{Y|Z=z}(\{y : \mathcal{U}^*(y,z;h) \subseteq \mathcal{S}\}) \leq \\ \qquad\qquad \inf_{z \in \mathcal{R}_Z} F_{Y|Z=z}(\{y : \mathcal{U}^*(y,z;h) \cap \mathcal{S} \neq \emptyset\}) \end{array} \right\} \tag{8}$$

9. There is an alternative characterization of the identified set of structures

$$\mathcal{I}(\mathcal{M}, \mathcal{F}_{Y|Z}) \equiv \big\{(h, \mathcal{G}_{U|Z}) \in \mathcal{M} : F_{Y|Z=z} \preceq \mathcal{Y}^*(z,U;h) \\ \text{conditional on } Z = z \quad \text{a.e.} \quad z \in \mathcal{R}_Z, \big\} \tag{9}$$

where the set $\mathcal{Y}^*(\cdot,\cdot;\cdot)$ is defined in (4).[12] Using the Artstein characterization of selectionability this leads to the following representation

$$\mathcal{I}(\mathcal{M}, \mathcal{F}_{Y|Z}) \equiv \big\{(h, \mathcal{G}_{U|Z}) \in \mathcal{M} : \forall \mathcal{K} \in \mathsf{K}(\mathcal{R}_Y) \\ G_{U|Z=z}(\{u : \mathcal{Y}^*(z,u;h) \subseteq \mathcal{K}\}) \leq F_{Y|Z=z}(\mathcal{K}) \quad \text{a.e. } z \in \mathcal{R}_Z\big\} \tag{10}$$

where $\mathsf{K}(\mathcal{R}_Y)$ is the collection of closed sets on the support of Y. In many cases arising in econometrics in which distributional restrictions are put on U this is less convenient to work with than the characterization (5). Y^* sets for a dynamic binary response two period panel model are shown in Table 1 for the case in which the initial value Y_0 is observed and for the case in which it is not.

Some examples of the application of these results are now presented.[13]

[12] See for example Beresteanu, Molchanov, and Molinari (2011) and Molinari (2020) and further references therein for such characterizations. Theorem 1 of CR implies equivalence of characterizations of the form (3) and (9).

[13] The development of some of these results was done by exploiting the symbolic computational power of Mathematica, Wolfram Research, Inc. (2023).

4 Linear panel data model

The approach set out in this paper delivers classical results when taken to the simple linear panel data model. Consider the simplest case with two periods of observation and the following model incorporating a conditional mean independence restriction

$$Y_t = \beta_0 + \beta_1 Z_t + V + U_t, \quad \mathbb{E}[U_t|Z] = 0, \quad t \in \{1, 2\},$$

where Z_1 and Z_2 are scalar, $Z \equiv (Z_1, Z_2)$, and $z \equiv (z_1, z_2)$.[14]

The Y^* and U^* sets are as follows.

$$\mathcal{Y}^*(u, z; \beta) = \{(y_1, y_2) : y_2 - y_1 = \beta_1 (z_2 - z_1) + u_2 - u_1\}$$

$$\mathcal{U}^*(y, z; \beta) = \{(u_1, u_2) : u_2 - u_1 = y_2 - y_1 - \beta_1 (z_2 - z_1)\}$$

Theorem 5 of CR delivers the result that the values of β_1, say β_1^+ in the identified set are all values such that zero is an element of the Aumann expectation of the set $\mathcal{U}^*(Y, Z; \beta_1^+)$ conditional on $Z = z$ for all $z \in \mathcal{R}_Z$. The set $\mathcal{U}^*(Y, Z; \beta_1)$ is singleton in this example, so the Aumann expectation is simply the classical expectation of point-valued random variables and there is

$$\mathbb{E}[\mathcal{U}^*(Y, Z; \beta_1)|Z = z] = \mathbb{E}[Y_2 - Y_1|Z = z] - \beta_1 (z_2 - z_1)$$

which, set equal to zero, delivers the correspondence

$$\beta_1 = \frac{\mathbb{E}[Y_2 - Y_1|Z = z]}{(z_2 - z_1)}$$

which is point identifying as long as $z_2 \neq z_1$.

Extension to $T > 2$ and dynamic models is straightforward and need not be rehearsed here. The point is that the general approach proposed here delivers classical results.

However the approach will *not* deliver the well-known point identification result in binary response panel data models with logistic independently distributed time-varying latent variables because those models further impose $U \perp\!\!\!\perp V$.[15] In this paper

[14] The function

$$\sum_{t=1}^{T} (Y_t - (\beta_0 + Z_t\beta_1 + V + U_t))^2$$

can serve as the function $h(Y, Z, U, V)$.

[15] See Chamberlain (2010).

the covariation of V with all other variables is unrestricted.

5 Binary response panel models

This section studies the dynamic binary response model of Example 1 under a variety of restrictions. Only in the final Section 5.3 are models admitting endogenous explanatory variables considered. Section 2 lists many papers that study binary response panel models with fixed effects. In all but one previous paper known to us there is a restriction on the joint distribution of the fixed effect and other variables such that the conditional distribution of other variables given the fixed effect is subject to restrictions. No such restrictions are imposed here. The one exception of which we are aware is Aristodemou (2021), in which bounds are provided for binary response panel data models with an observed initial condition.

Section 5.1 gives results for the two period dynamic binary response model when the initial condition (Y_0) is observed. This model is studied in Aristodemou (2021). Three period dynamic models with unobserved initial condition are studied in Section 5.2. Section 5.3 gives results for a general case in which there may be endogenous explanatory variables. Extension to models with multiple lagged dependent variables is straightforward.

Define $Y = (Y_1, \ldots, Y_T)$ and Z and U similarly.

5.1 Two period dynamic binary response model, initial condition observed

In the case considered in this section, $T = 2$ and Y_0 is observed. Define $\Delta u \equiv u_2 - u_1$, $\Delta z \equiv z_2 - z_1$, and $\theta = (\beta', \gamma)'$.

The U^* sets are as follows.

$$\mathcal{U}^*(y, z, y_0; \theta) = \begin{cases} \mathcal{R}_U & , \quad y = (0,0) \\ \{u : \Delta u \geq -\Delta z\beta + y_0\gamma\} & , \quad y = (0,1) \\ \{u : \Delta u \leq -\Delta z\beta + (y_0 - 1)\gamma\} & , \quad y = (1,0) \\ \mathcal{R}_U & , \quad y = (1,1) \end{cases}$$

Unions of these U^* sets do not deliver additional informative inequalities.[16]

Under the independence restriction $U \perp\!\!\!\perp Z | Y_0$ the identified set of values of $(\theta, G_{U|Y_0})$ comprises those values such that the following inequalities hold for $y_0 \in \{0, 1\}$ and

[16] Unions are either disjoint or equal to the support of U depending on the sign of γ.

Table 1: Y^* sets in the binary response two period panel.

Y_0	$\mathcal{Y}$	$\{u : \mathcal{Y}^*(z, u; \theta) = \mathcal{Y}\}$
observed	$\{(0,0),(1,1)\}$	$\{u : -\Delta z\beta + (y_0 - 1)\gamma < \Delta u < -\Delta z\beta + y_0\gamma\}$
	$\{(0,0),(0,1),(1,1)\}$	$\{u : \Delta u \geq -\Delta z\beta + y_0\gamma \wedge \Delta u > -\Delta z\beta + (y_0 - 1)\gamma\}$
	$\{(0,0),(1,0),(1,1)\}$	$\{u : \Delta u \leq -\Delta z\beta + (y_0 - 1)\gamma \wedge \Delta u < -\Delta z\beta + y_0\gamma\}$
	$\mathcal{R}_Y$	$\{-\Delta z\beta + y_0\gamma \leq \Delta u \leq -\Delta z\beta + (y_0 - 1)\gamma\}$
not observed	$\{(0,0),(0,1),(1,1)\}$	$\{u : \Delta u > \max(-\Delta z\beta - \gamma, -\Delta z\beta)\}$
	$\{(0,0),(1,0),(1,1)\}$	$\{u : \Delta u < \min(-\Delta z\beta + \gamma, -\Delta z\beta)\}$
	$\mathcal{R}_Y$	$\{u : \min(-\Delta z\beta + \gamma, -\Delta z\beta) \leq \Delta u \leq \max(-\Delta z\beta - \gamma, -\Delta z\beta)\}$

a.e. $z \in \mathcal{R}_Z$.

$$\mathbb{P}[Y = (0,1)|Z = z, Y_0 = y_0] \leq G_{U|Y_0=y_0}(\{u : \Delta u \geq -\Delta z\beta + y_0\gamma\})$$

$$\mathbb{P}[Y = (1,0)|Z = z, Y_0 = y_0] \leq G_{U|Y_0=y_0}(\{u : \Delta u \leq -\Delta z\beta + (y_0 - 1)\gamma\})$$

These are the inequalities of Theorem 1 of Aristodemou (2021). Setting $\gamma = 0$ with $U \perp\!\!\!\perp Z$, dropping conditioning on Y_0, delivers the inequalities defining the identified set in the two period static binary response panel model.

Table 1 shows the Y^* sets for the two period dynamic binary response panel model. The top half of the table shows the sets for the case in which Y_0 is observed. The bottom part shows the sets obtained when Y_0 is not observed.

Appendix C derives sharp bounds on θ absent any specification of the distribution of U using the method set out in Remark 8 of Section 3.

5.2 A three period dynamic binary response model with the initial condition not observed

For any $s, t \in [T]$ define $\Delta_{st}u \equiv u_s - u_t$, and $\Delta_{st}z \equiv z_s - z_t$. With $T = 3$, and treating both V and Y_0 as unobserved latent variables with unrestricted distributions the U^* sets are as shown in Table 2.

For sets of values of Y, $\mathcal{T} \subset \mathcal{R}_Y$, define functions

$$\mathcal{S}(\mathcal{T}, z; \theta) \equiv \bigcup_{y \in \mathcal{T}} \mathcal{U}^*(y, z; \theta) \tag{11}$$

Table 2: U^* sets in the dynamic binary response panel data model with 3 periods and Y_0 not observed.

	y	$\mathcal{U}^*(y,z;\theta)$
1	$(0,0,0)$	$\mathcal{R}_U$
2	$(0,0,1)$	$\{u : (\Delta_{31}u \geq -\Delta_{31}z\beta + \min(\gamma,0)) \wedge (\Delta_{32}u \geq -\Delta_{32}z\beta)\}$
3	$(0,1,0)$	$\{u : (\Delta_{21}u \geq -\Delta_{21}z\beta + \min(\gamma,0)) \wedge (\Delta_{32}u \leq -\Delta_{32}z\beta - \gamma)\}$
4	$(0,1,1)$	$\{u : (\Delta_{21}u \geq -\Delta_{21}z\beta + \min(\gamma,0)) \wedge (\Delta_{31}u \geq -\Delta_{31}z\beta - \max(\gamma,0))\}$
5	$(1,0,0)$	$\{u : (\Delta_{21}u \leq -\Delta_{21}z\beta - \min(\gamma,0)) \wedge (\Delta_{31}u \leq -\Delta_{31}z\beta + \max(\gamma,0))\}$
6	$(1,0,1)$	$\{u : (\Delta_{21}u \leq -\Delta_{21}z\beta - \min(\gamma,0)) \wedge (\Delta_{32}u \geq -\Delta_{32}z\beta + \gamma)\}$
7	$(1,1,0)$	$\{u : (\Delta_{31}u \leq -\Delta_{31}z\beta - \min(\gamma,0)) \wedge (\Delta_{32}u \leq -\Delta_{32}z\beta)\}$
8	$(1,1,1)$	$\mathcal{R}_U$

and[17]

$$\mathcal{Y}(\mathcal{T},z;\theta) \equiv \{y : \mathcal{U}^*(y,z;\theta) \subseteq \mathcal{S}(\mathcal{T},z;\theta)\}. \tag{12}$$

The identified set of values of $\left(\beta,\gamma,\mathcal{G}_{U|Z=z}\right)$ comprises the values satisfying inequalities of the form

$$\mathbb{P}[Y \in \mathcal{Y}(\mathcal{T},z;\theta)|Z=z] \leq G_{U|Z=z}(\mathcal{S}(\mathcal{T},z;\theta)), \text{ a.e. } z \in \mathcal{R}_Z,$$

where the sets $\mathcal{Y}(\mathcal{T},z;\theta)$ and $\mathcal{T}$ are shown in the first and second columns of Tables 7, 8, and 9, covering the cases in which $\gamma = 0$, $\gamma > 0$, and $\gamma < 0$, respectively.

5.3 General dynamic binary response panel models

Consider now the general specification of a dynamic panel data model from Example 1, allowing for endogeneity admitting $\alpha \neq 0$. This section illustrates application of our identification analysis to such cases, also allowing for arbitrary finite T.[18]

Define

$$\mathcal{T}_0 \equiv \{t \in [T] : Y_{1t} = 0\}, \qquad \mathcal{T}_1 \equiv \{t \in [T] : Y_{1t} = 1\}, \tag{13}$$

denoting the sets of periods in which $Y_{1t} = 0$ and $Y_{1t} = 1$, respectively. Let $\mathcal{Y}_0$ denote the set of values in which the initial condition Y_{10} is known to lie, with $\mathcal{Y}_0 = \{Y_{10}\}$ if the initial condition is observed and $\mathcal{Y}_0 = \{0,1\}$ if the initial condition is not observed.

[17] The set $\mathcal{T}$ can be a strict subset of $\mathcal{Y}(\mathcal{T},z;\theta)$. For example, this is the case when $\mathcal{T}$ contains two values of Y and there is a third value of Y such that its U^* set is a subset of $\mathcal{S}(\mathcal{T},z;\theta)$ as in row 7 of Table 7.

[18] Here we impose $\mathcal{R}_C = \mathbb{R}$, as typically done in the literature. Extension to cases in which $\mathcal{R}_C$ is a subset of $\mathbb{R}$ is straightforward.

The set $\mathcal{U}^*(Y, Z; h)$ defined in (2) in this model can be written

$$\mathcal{U}^*(Y,Z;h) = \Big\{u \in \mathcal{R}_U : \exists Y_{10} \in \mathcal{Y}_0 \text{ such that} \\ \max_{t\in\mathcal{T}_0}\{Y_{2t}\alpha + Z_t\beta + Y_{1t-1}\gamma + u_t\} \leq \min_{t\in\mathcal{T}_1}\{Y_{2t}\alpha + Z_t\beta + Y_{1t-1}\gamma + u_t\}\Big\}. \quad (14)$$

This is so because the constituent inequalities may be equivalently expressed as

$$\underline{C} \leq \overline{C}$$

where

$$\underline{C} \equiv \max_{t\in\mathcal{T}_1}\big\{-(Y_{2t}\alpha + Z_t\beta + Y_{1t-1}\gamma + u_t)\big\},$$
$$\overline{C} \equiv \min_{t\in\mathcal{T}_0}\big\{-(Y_{2t}\alpha + Z_t\beta + Y_{1t-1}\gamma + u_t)\big\}.$$

That $\underline{C} \leq \overline{C}$ for some $Y_{10} \in \mathcal{Y}_0$ guarantees there exist values $C \in \big[\underline{C}, \overline{C}\big]$ and $Y_{10} \in \mathcal{Y}_0$ such that (1) holds.[19]

Define $\theta = (\alpha', \beta', \gamma)'$. For any panel data model for a binary outcome as in (1) with $U \sim G_U$ independent of Z, the identified set of values of (θ, G_U) are those pairs satisfying, for an appropriately chosen collection[20] of sets $\mathcal{T}$, the inequalities

$$\mathbb{P}[Y \in \mathcal{Y}(\mathcal{T}, z; \theta)|Z = z] \leq G_U(\mathcal{S}(\mathcal{T}, z; \theta)), \text{ a.e. } z \in \mathcal{R}_Z,$$

where the sets $\mathcal{S}(\mathcal{T}, z; \theta)$ and $\mathcal{Y}(\mathcal{T}, z; \theta)$ are as defined in (11) and (12).

This characterization applies for dynamic models and static models (for which $\gamma = 0$ is imposed), models allowing endogenous explanatory variables (for which $\alpha \neq 0$ is permitted), and for arbitrary T.

6 Static multiple discrete choice panel models

In this section multiple discrete choice panel models are considered. The presence of the fixed effect renders the model incomplete as in the multiple discrete choice analysis of Chesher, Rosen, and Smolinski (2013). In that analysis incompleteness arose due to the inclusion of potentially endogenous explanatory variables. Analysis

[19]The max and min operators applied to the empty set are defined to be $-\infty$ and ∞, respectively.

[20]The collection of all unions of U^* sets, $\mathsf{U}^*(z; h)$, defined in (6), will suffice. In practice there may be unions in this collection which need not be considered because they deliver redundant inequalities.

Table 3: U^* sets in the multiple discrete three choice two period panel model.

	y	$\mathcal{U}^*(y,z;\theta)$
1	$(1,1)$	$\mathcal{R}_U$
2	$(1,2)$	$\{u : \Delta u_2 - \Delta u_1 \geq \Delta z\beta_1 - \Delta z\beta_2\}$
3	$(1,3)$	$\{u : \Delta u_1 - \Delta u_3 \leq -\Delta z\beta_1\}$
4	$(2,1)$	$\{u : \Delta u_2 - \Delta u_1 \leq \Delta z\beta_1 - \Delta z\beta_2\}$
5	$(2,2)$	$\mathcal{R}_U$
6	$(2,3)$	$\{u : \Delta u_2 - \Delta u_3 \leq -\Delta z\beta_2\}$
7	$(3,1)$	$\{u : \Delta u_1 - \Delta u_3 \geq -\Delta z\beta_1\}$
8	$(3,2)$	$\{u : \Delta u_2 - \Delta u_3 \geq -\Delta z\beta_2\}$
9	$(3,3)$	$\mathcal{R}_U$

of a static panel model with $T = 2$ periods is considered.

In a three-choice model with two periods there is

$$Y_t = \underset{d}{\text{argmax}}\{J_{dt} : d \in \{1,2,3\}\}, \quad t \in \{1,2\}$$

where the J_{dt} terms are random utilities with parameters $\theta \equiv (\beta_1', \beta_2')'$ as follows.

$$J_{1t} \equiv Z_t\beta_1 + V_1 + U_{1t}, \quad t \in \{1,2\},$$

$$J_{2t} \equiv Z_t\beta_2 + V_2 + U_{2t}, \quad t \in \{1,2\},$$

$$J_{3t} \equiv U_{3t}, \quad t \in \{1,2\}.$$

The terms V_1 and V_2 are "fixed effects" whose distribution and covariation with other variables is unrestricted.

Section 2 lists many papers that study multiple discrete panel models with fixed effects. In all studies of multiple discrete choice panel data models known to us there are conditions imposed on the joint distribution of fixed effects and other variables such that the conditional distribution of other variables given the fixed effect is subject to restriction. No such restrictions are imposed here.

The U^* sets are shown in Table 3 using notation $\Delta U_d \equiv U_{d2} - U_{d1}$.

The identified set of values of (θ, G_U) are those pairs satisfying, for all $z \in \mathcal{R}_Z$ the inequalities

$$\mathbb{P}[Y \in \mathcal{Y}(\mathcal{T}, z;\theta)|Z = z] \leq G_{U|Z=z}(\mathcal{S}(\mathcal{T}, z;\theta)), \quad \text{a.e. } z \in \mathcal{R}_Z,$$

where the sets $\mathcal{S}(\mathcal{T}, z;\theta)$ and $\mathcal{Y}(\mathcal{T}, z;\theta)$ are as defined in (11) and (12) and the sets $\mathcal{T}$ and $\mathcal{Y}(\mathcal{T}, z; \theta)$ are shown in Table 10. As we show for ordered choice panels in the following section, this characterization can be generalized to allow arbitrary periods T and alternatives $\{1, ..., K\}$, and can allow for dependence on lagged choices. Endogenous covariates can be permitted as is done for cross sectional multiple discrete choice in Chesher, Rosen, and Smolinski (2013).

7 Ordered response panel models

This section generalizes the binary response models of Section 5 to models in which the outcome is an ordered response variable. Section 7.1 gives results for a static two period model with three ordered outcomes. Section 7.2 then gives results for a general ordered outcome model allowing an arbitrary finite number of ordered outcomes, arbitrary periods, and dynamics.

7.1 Two period ordered response panel models with three categories

There are structural equations as follows.

$$Y_t = \begin{cases} 0 & , \quad Z_t\beta + V + U_t \leq c_1 \\ 1 & , \quad c_1 \leq Z_t\beta + V + U_t \leq c_2 \\ 2 & , \quad c_2 \leq Z_t\beta + V + U_t \end{cases} , \quad t \in \{1, 2\}$$

Let $Y = (Y_1, Y_2)$, $Z = (Z_1, Z_2)$, $U = (U_1, U_2)$. Let $\theta = (\beta', c_1, c_2)'$.[21] There is the restriction $U \perp\!\!\!\perp Z$. This model is studied in Aristodemou (2021) where, as here, no restrictions are placed on the distribution of V or on its covariation with other variables.

Define $\Delta u \equiv u_2 - u_1$ and $\Delta z \equiv z_2 - z_1$. The U^* sets are shown in Table 4. The identified set of values of (θ, G_U) comprises the values satisfying, for $z \in \mathcal{R}_Z$, 7 inequalities of the form

$$\mathbb{P}[Y \in \mathcal{Y} | Z = z] \leq G_U(\mathcal{S})$$

where $\mathcal{Y}$ and $\mathcal{S}$ are given in Table 5.

Theorem 5 of Aristodemou (2021) delivers an outer set using the inequalities 1, 2

[21] In some applications c_1 and c_2 can have known values.

Table 4: U^* sets in the ordered response three category two period panel model.

	y	$\mathcal{U}^*(y,z;\theta)$
1	$(0,0)$	$\mathcal{R}_U$
2	$(0,1)$	$\{u : \Delta u \geq -\Delta z\beta\}$
3	$(0,2)$	$\{u : \Delta u \geq c_2 - c_1 - \Delta z\beta\}$
4	$(1,0)$	$\{u : \Delta u \leq -\Delta z\beta\}$
5	$(1,1)$	$\{u : (\Delta u \leq c_2 - c_1 - \Delta z\beta) \wedge (\Delta u \geq c_1 - c_2 - \Delta z\beta)\}$
6	$(1,2)$	$\{u : \Delta u \geq -\Delta z\beta\}$
7	$(2,0)$	$\{u : \Delta u \leq c_1 - c_2 - \Delta z\beta\}$
8	$(2,1)$	$\{u : \Delta u \leq -\Delta z\beta\}$
9	$(2,2)$	$\mathcal{R}_U$

Table 5: Sets $\mathcal{Y}$ and $\mathcal{S}$ in the inequalities defining the identified set of values of β and G_U in the two period ordered response panel model with three categories.

	$\mathcal{Y}$	$\mathcal{S}$
1	$\{(0,2)\}$	$\{u : \Delta u \geq c_2 - c_1 - \Delta z\beta\}$
2	$\{(1,1)\}$	$\{u : (\Delta u \leq c_2 - c_1 - \Delta z\beta) \wedge (\Delta u \geq c_1 - c_2 - \Delta z\beta)\}$
3	$\{(2,0)\}$	$\{u : \Delta u \leq c_1 - c_2 - \Delta z\beta\}$
4	$\{(0,1),(0,2),(1,2)\}$	$\{u : \Delta u \geq -\Delta z\beta\}$
5	$\{(1,0),(2,0),(2,1)\}$	$\{u : \Delta u \leq -\Delta z\beta\}$
6	$\{(0,1),(0,2),(1,1),(1,2)\}$	$\{u : \Delta u \geq c_1 - c_2 - \Delta z\beta\}$
7	$\{(1,0),(1,1),(2,0),(2,1)\}$	$\{u : \Delta u \leq c_2 - c_1 - \Delta z\beta\}$

and 3 in Table 5 and the inequalities:

$$\mathbb{P}(Y = (0,1)|Z = z] \leq G_U(\{u : \Delta u > -\Delta z\beta\})$$

and

$$\mathbb{P}(Y = (1,2)|Z = z] \leq G_U(\{u : \Delta u > -\Delta z\beta\})$$

which are implied by inequality 4, and

$$\mathbb{P}(Y = (1,0)|Z = z] \leq G_U(\{u : \Delta u < -\Delta z\beta\})$$

and

$$\mathbb{P}(Y = (2,1)|Z = z] \leq G_U(\{u : \Delta u < -\Delta z\beta\})$$

which are implied by inequality 5.

7.2 General ordered response panel models

Consider now a general specification of an ordered response panel data model with $\mathcal{R}_Y = \{0, ..., J\}$ and allowing for dynamics as in e.g. Honoré, Muris, and Weidner (2023) in which for all $j \in \mathcal{R}_Y$:

$$Y_t = j \implies c_j \leq Z_t\beta + \imath_t\gamma + V + U_t \leq c_{j+1}, \tag{15}$$

where $c_0 \equiv -\infty$, $c_{J+1} \equiv \infty$, and $\imath_t \equiv (1[Y_{t-1} = 0], \ldots, 1[Y_{t-1} = J])$ with each component of γ encoding the impact of lagged Y on Y_t.[22] Let $\tilde{Z}_t \equiv (Z_t, \imath_t)$, $\tilde{\beta} \equiv (\beta', \gamma')'$, $Y \equiv (Y_1, ..., Y_T)$, $Z \equiv (Z_1, ..., Z_T)$, $U \equiv (U_1, ..., U_T)$. Let $\theta \equiv (\beta', \gamma', c_1, ..., c_J)'$ denote parameters of the structural function, restricted such that $c_1 < \cdots < c_J$. The initial condition Y_0 is assumed observed, but it is straightforward to accommodate an unobserved initial condition as for the binary panel studied in Section 5.2.

Sets $\mathcal{U}^*(Y, Z; h)$ are given by

$$\mathcal{U}^*(Y, Z; h) = \{u \in \mathcal{R}_U : \forall s, t \in [T], \\ u_t - u_s \leq c_{Y_t+1} - c_{Y_s} - (\tilde{Z}_t - \tilde{Z}_s)\tilde{\beta}\},$$

This is verified by noting that for all $u \in \mathcal{U}^*(Y, Z; h)$ we have that

$$\forall s, t \in [T], \qquad c_{Y_s} - \tilde{Z}_s\tilde{\beta} - u_s \leq c_{Y_t+1} - \tilde{Z}_t\tilde{\beta} - u_t,$$

[22]It is straightforward to accommodate multiple lags.

in turn implying the existence of v such that

$$\forall s,t \in [T], \qquad c_{Y_s} - \tilde{Z}_s\tilde{\beta} - u_s \leq v \leq c_{Y_t+1} - \tilde{Z}_t\tilde{\beta} - u_t.$$

For all such u, v it follows that (15) holds for all t with $U = u$ and $V = v$.

When the independence restriction $U \perp\!\!\!\perp Z$ is imposed, the identified set for (θ, G_U) are those pairs satisfying

$$\mathbb{P}[Y \in \mathcal{Y}(\mathcal{T}, z; \theta) | Z = z] \leq G_U(\mathcal{S}(\mathcal{T}, z; \theta)), \text{ a.e. } z \in \mathcal{R}_Z$$

for an appropriately chosen collection of sets $\mathcal{T}$ where the sets $\mathcal{S}(\mathcal{T}, z; \theta)$ and $\mathcal{Y}(\mathcal{T}, z; \theta)$ are as defined in (11) and (12).[23] This characterization can be generalized to allow for endogenous variables on the right hand side of (15) as done for cross section analysis of ordered choice models in Chesher and Smolinski (2012) and Chesher, Rosen, and Siddique (2023). It is straightforward to allow $G_{U|Z=z}$ to vary with z by replacing G_U with $G_{U|Z=z}$ in the inequality above, which then delivers an identified set for pairs $\left(\theta, \mathcal{G}_{U|Z}\right)$.

8 Simultaneous binary response panel models

There is the model

$$\begin{aligned} Y_{1t} &= 1[\alpha_1 Y_{2t} + Z_t\beta_1 + V_1 + U_{1t} \geq 0] \\ Y_{2t} &= 1[\alpha_2 Y_{1t} + Z_t\beta_2 + V_2 + U_{2t} \geq 0] \end{aligned}$$

with $t \in [T]$ and the independence restriction $(U_1, U_2) \perp\!\!\!\perp Z \equiv (Z_1, \ldots, Z_T)$ where for each $j \in \{1, 2\}$, $U_j \equiv (U_{j1}, \ldots, U_{jT})$.[24]

This is a simultaneous equations model with binary outcomes such as is found in simultaneous firm entry applications[25] and models of social interactions, put into a panel context with "fixed effects", constant through time, one for each outcome.

Honoré and De Paula (2021) study a restricted version of this model with $\beta_1 = \beta_2$, $\alpha_1 = \alpha_2$ and U and V restricted to be independently distributed. No such restrictions are imposed here.

[23] Once again the collection of all unions of U^* sets, $\mathsf{U}^*(z; h)$, defined in (6), will suffice, but in practice some of these unions may not be necessary.

[24] This strong exogeneity restriction can be relaxed.

[25] See for example Tamer (2003).

Table 6: U^* sets in the simultaneous binary response two period panel.

	y	$\mathcal{U}^*(y,z;\theta)$
1	$(0,0,0,0)$	$\mathcal{R}_U$
2	$(0,0,0,1)$	$\{u : \Delta u_2 \geq -\Delta z\beta_2\}$
3	$(0,0,1,0)$	$\{u : \Delta u_2 \leq -\Delta z\beta_2\}$
4	$(0,0,1,1)$	$\mathcal{R}_U$
5	$(0,1,0,0)$	$\{u : \Delta u_1 \geq -\Delta z\beta_1\}$
6	$(0,1,0,1)$	$\{u : (\Delta u_1 \geq -\Delta z\beta_1 - \alpha_1) \wedge (\Delta u_2 \geq -\Delta z\beta_2 - \alpha_2)\}$
7	$(0,1,1,0)$	$\{u : (\Delta u_1 \geq -\Delta z\beta_1 + \alpha_1) \wedge (\Delta u_2 \leq -\Delta z\beta_2 - \alpha_2)\}$
8	$(0,1,1,1)$	$\{u : \Delta u_1 \geq -\Delta z\beta_1\}$
9	$(1,0,0,0)$	$\{u : \Delta u_1 \leq -\Delta z\beta_1\}$
10	$(1,0,0,1)$	$\{u : (\Delta u_1 \leq -\Delta z\beta_1 - \alpha_1) \wedge (\Delta u_2 \geq -\Delta z\beta_2 + \alpha_2)\}$
11	$(1,0,1,0)$	$\{u : (\Delta u_1 \leq -\Delta z\beta_1 + \alpha_1) \wedge (\Delta u_2 \leq -\Delta z\beta_2 + \alpha_2)\}$
12	$(1,0,1,1)$	$\{u : \Delta u_1 \leq -\Delta z\beta_1\}$
13	$(1,1,0,0)$	$\mathcal{R}_U$
14	$(1,1,0,1)$	$\{u : \Delta u_2 \geq -\Delta z\beta_2\}$
15	$(1,1,1,0)$	$\{u : \Delta u_2 \leq -\Delta z\beta_2\}$
16	$(1,1,1,1)$	$\mathcal{R}_U$

Define $\theta \equiv (\alpha_1, \alpha_2, \beta_1', \beta_2')'$. The distribution of $V \equiv (V_1, V_2)$ and the covariation of V with other variables is unrestricted.

Consider the case with $T = 2$ when $Y = (Y_{11}, Y_{12}, Y_{21}, Y_{22})$. Extension to more time periods and outcomes is straightforward.

Define $\Delta u_1 \equiv u_{12} - u_{11}$, $\Delta u_2 \equiv u_{22} - u_{21}$, $\Delta z \equiv z_2 - z_1$. The U^* sets, $\mathcal{U}^*(y,z;\theta)$, are as shown in Table 6.

There are 12 U^* sets that are not equal to $\mathcal{R}_U$ and 4 pairs of these U^* sets are identical - for example $\mathcal{U}^*(0,0,0,1),z;\theta) = \mathcal{U}^*(1,1,0,1),z;\theta)$, so there are unions of 8 U^* sets to be considered when calculating the identified set, that is 254 unions in total. Only 24 of these deliver inequalities that characterize the identified set of parameter values, the remaining unions delivering redundant inequalities.

The configuration of the unions of these U^* sets depends on the signs of α_1 and α_2 and in practice there are likely to be restrictions on these. For example in a simultaneous firm entry application $\alpha_1 \leq 0$ and $\alpha_2 \leq 0$ would likely be imposed and in a model of couple's choices of activity (e.g. cinema attendance) $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$.

Only the case with $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$ is presented here. In this case, among the U^* sets only the sets $\mathcal{U}^*((0,1,0,1),z;\theta)$ and $\mathcal{U}^*((1,0,1,0),z;\theta)$ have a non-empty intersection.

The identified set of values of (θ, G_U) are those pairs satisfying, for all $z \in \mathcal{R}_Z$ the inequalities

$$\mathbb{P}[Y \in \mathcal{Y}(\mathcal{T}, z; \theta)|Z = z] \leq G_U(\mathcal{S}(\mathcal{T}, z; \theta))$$

where the sets $\mathcal{S}(\mathcal{T}, z; \theta)$ and $\mathcal{Y}(\mathcal{T}, z; \theta)$ are as defined in (11) and (12) and the sets $\mathcal{T}$ and $\mathcal{Y}(\mathcal{T}, z; \theta)$ are shown in Table 11.

9 Censored outcome panels

In a panel model with a censored outcome there is the following.

$$Y_{1t} = \max(\alpha Y_{2t} + Z_t\beta + \gamma Y_{1t-1} + C + U_t, Y_{3t}), \quad t \in [T], \quad C \in \mathbb{R}, \quad Y_{10} \in \mathcal{Y}_{10},$$

with $V \equiv (C, Y_{10})$ and Y_{3t} denoting a censoring threshold, such that the outcome variable Y_{1t} takes the value of the index $\alpha Y_{2t} + Z_t\beta + \gamma Y_{1t-1} + C + U_t$ when it exceeds the censoring threshold, and otherwise takes the value Y_{3t}. The censoring indicator $W_t \equiv 1[Y_{1t} = Y_{3t}]$ is observed.

As in the models studied in KPT, the censoring threshold Y_{3t} can be endogenous, and it may be correlated with elements of U and V.[26] As before, endogenous Y_{2t} is permitted in models with $\alpha \neq 0$, as in cross-sectional Tobit models studied in Chesher, Kim, and Rosen (2023). The set $\mathcal{Y}_{10}$ denotes the feasible set of values for the initial condition Y_{10} given the observed variables.[27]

Define

$$\mathcal{T}_0 \equiv \{t \in [T] : W_t = 0\}, \qquad \mathcal{T}_1 \equiv \{t \in [T] : W_t = 1\}.$$

Adopting the strategy for obtaining U^* sets described in remark 5 of Section 3 we can define

$$\mathcal{D}_t(y, z, u; h) = \begin{cases} \{(c, y_{10}) \in \mathbb{R} \times \mathcal{Y}_{10} : c = y_{1t} - \alpha y_{2t} - z_t\beta - \gamma y_{1t-1} - u_t\} & , \quad t \in \mathcal{T}_0 \\ \{(c, y_{10}) \in \mathbb{R} \times \mathcal{Y}_{10} : \alpha y_{2t} + z_t\beta + \gamma y_{1t-1} + c + u_t \leq y_{1t}\} & , \quad t \in \mathcal{T}_1 \end{cases}$$

The U^* sets are then as follows.

$$\begin{aligned} \mathcal{U}^*(Y, Z; h) = \{u : \exists Y_{10} \in \mathcal{Y}_{10} \text{ s.t. } \forall s, t \in \mathcal{T}_0,\ \Delta_{ts}u = \Delta_{ts}Y_1 - \alpha\Delta_{ts}Y_2 - \Delta_{ts}Z\beta - \gamma\Delta_{t-1,s-1}Y_1 \\ \wedge\ \forall(s,t) \in \mathcal{T}_1 \times \mathcal{T}_0,\ \Delta_{ts}u \geq \Delta_{ts}Y_1 - \alpha\Delta_{ts}Y_2 - \Delta_{ts}Z\beta - \gamma\Delta_{t-1,s-1}Y_1\}, \end{aligned}$$

where $\Delta_{ts}U \equiv U_t - U_s$, $\Delta_{ts}Z \equiv Z_t - Z_s$, $\Delta_{ts}Y_1 \equiv Y_{1t} - Y_{1s}$ and $\Delta_{ts}Y_2 \equiv Y_{2t} - Y_{2s}$.

[26] It is not necessary for Y_{3t} to be observed in periods without censoring.

[27] So $\mathcal{Y}_{10}$ is the singleton $\{Y_{10}\}$ if the initial condition is observed, and would typically be its entire support if it is not observed. In a static model the analysis applies with $\gamma = 0$ and Y_{10} absent.

Following the approach set out in Theorem 1, the identified set of values of $\left(\theta, \mathcal{G}_{U|Z}\right)$, where $\theta \equiv (\alpha, \beta, \gamma)$ are those satisfying the inequalities

$$\mathbb{P}[Y \in \mathcal{Y}(\mathcal{T}, z; \theta) | Z = z] \leq G_{U|Z=z}(\mathcal{S}(\mathcal{T}, z; \theta)) \text{ a.e. } z \in \mathcal{R}_Z,$$

for an appropriate selection of sets $\mathcal{T}$, where the sets $\mathcal{S}(\mathcal{T}, z; \theta)$ and $\mathcal{Y}(\mathcal{T}, z; \theta)$ are defined in (11) and (12). The required selection can be characterized following the same steps taken in the models studied in prior sections.

10 Concluding remarks

This paper delivers methods for producing identified sets when models admit unobserved, latent, variables on which no distributional restrictions are placed, opening the way to robust analysis of short panels. Examples found in econometric practice include models incorporating so-called fixed effects and initial conditions. Endogenous explanatory variables are easily accommodated.

The identified sets delivered by the models in this paper that place no restriction on the distribution of latent V will contain the structures identified by more restrictive models if the restrictions of those models are satisfied by the process under study. The analysis set out here will show how sensitive the findings obtained using that more restrictive model are to those additional restrictions. In some cases it may be found that estimation employing a point-identifying model delivers a structure outside an estimator of the identified set obtained using a less restrictive model of the type studied in this paper. Such a finding would suggest the more restrictive model is misspecified. Formal development of such specification tests may be of interest for future research.

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Appendix A CR Restrictions A1-A6

This section collects restrictions from Chesher and Rosen (2017) adapted to the present setting with unobservable variables (U, V), which are imposed throughout the paper.

Restriction A1: (Y, Z, U, V) are random vectors defined on a probability space $(\Omega, \mathsf{L}, \mathbb{P})$, endowed with the Borel sets on Ω. The support of (Y, Z, U, V) is a subset of Euclidean space. $\square$

Restriction A2: A collection of conditional distributions

$$\mathcal{F}_{Y|Z} \equiv \left\{F_{Y|Z}\left(\cdot|z\right) : z \in \mathcal{R}_Z\right\},$$

is identified by the sampling process, where for all $\mathcal{T} \subseteq \mathcal{R}_{Y|z}$, $F_{Y|Z}\left(\mathcal{T}|z\right) \equiv \mathbb{P}\left[Y \in \mathcal{T}|z\right]$. $\square$

Restriction A3: There is an L-measurable function $h\left(\cdot, \cdot, \cdot, \cdot\right) : \mathcal{R}_{YZUV} \to \mathbb{R}$ such that

$$\mathbb{P}\left[h\left(Y, Z, U\right) = 0\right] = 1,$$

and there is a collection of conditional distributions

$$\mathcal{G}_{U|Z} \equiv \left\{G_{U|Z}\left(\cdot|z\right) : z \in \mathcal{R}_Z\right\},$$

where for all $\mathcal{S} \subseteq \mathcal{R}_{U|z}$, $G_{U|Z}\left(\mathcal{S}|z\right) \equiv \mathbb{P}\left[U \in \mathcal{S}|z\right]$. $\square$

Restriction A4: The pair $\left(h, \mathcal{G}_{U|Z}\right)$ belongs to a known set of admissible structures $\mathcal{M}$. $\square$

Restriction A5: $\mathcal{U}^*\left(Y, Z; h\right)$ is closed almost surely $\mathbb{P}\left[\cdot|z\right]$, each $z \in \mathcal{R}_Z$. $\square$

Restriction A6: $\mathcal{Y}^*\left(Z, U; h\right)$ is closed almost surely $\mathbb{P}\left[\cdot|z\right]$, each $z \in \mathcal{R}_Z$. $\square$

Appendix B Sets $\mathcal{T}$ for sharp identified sets

This section collects tables of $\mathcal{T}$ and $\mathcal{Y}(\mathcal{T}, z; \theta)$ defined in (12) as

$$\mathcal{Y}(\mathcal{T}, z; \theta) \equiv \{y : \mathcal{U}^*(y, z; \theta) \subseteq \mathcal{S}(\mathcal{T}, z; \theta)\},$$

such that inequalities of the form

$$\mathbb{P}[Y \in \mathcal{Y}(\mathcal{T}, z; \theta) | Z = z] \leq G_{U|Z=z}(\mathcal{S}(\mathcal{T}, z; \theta))$$

for all $\mathcal{T}$ listed characterize the identified set for $\left(\theta, \mathcal{G}_{U|Z}\right)$ in all examples covered in Sections 5–8. Recall from (11) the definition of $\mathcal{S}(\mathcal{T}, z; \theta)$:

$$\mathcal{S}(\mathcal{T}, z; \theta) \equiv \bigcup_{y \in \mathcal{T}} \mathcal{U}^*(y, z; \theta).$$

Table 7: Sets $\mathcal{Y}(\mathcal{T}, z; \theta)$ and $\mathcal{T}$ in the inequalities defining the identified set of structures in the static binary response 3 period panel data model ($\gamma = 0$).

	$\mathcal{Y}(\mathcal{T}, z; \theta)$	$\mathcal{T}$
1	$\{(0,0,1)\}$	$\{(0,0,1)\}$
2	$\{(0,1,0)\}$	$\{(0,1,0)\}$
3	$\{(0,1,1)\}$	$\{(0,1,1)\}$
4	$\{(1,0,0)\}$	$\{(1,0,0)\}$
5	$\{(1,0,1)\}$	$\{(1,0,1)\}$
6	$\{(1,1,0)\}$	$\{(1,1,0)\}$
7	$\{(0,0,1),(0,1,0),(0,1,1)\}$	$\{(0,0,1),(0,1,0)\}$
8	$\{(0,0,1),(0,1,1)\}$	$\{(0,0,1),(0,1,1)\}$
9	$\{(0,0,1),(1,0,0),(1,0,1)\}$	$\{(0,0,1),(1,0,0)\}$
10	$\{(0,0,1),(1,0,1)\}$	$\{(0,0,1),(1,0,1)\}$
11	$\{(0,1,0),(0,1,1)\}$	$\{(0,1,0),(0,1,1)\}$
12	$\{(0,1,0),(1,0,0),(1,1,0)\}$	$\{(0,1,0),(1,0,0)\}$
13	$\{(0,1,0),(1,1,0)\}$	$\{(0,1,0),(1,1,0)\}$
14	$\{(0,0,1),(0,1,1),(1,0,1)\}$	$\{(0,1,1),(1,0,1)\}$
15	$\{(0,1,0),(0,1,1),(1,1,0)\}$	$\{(0,1,1),(1,1,0)\}$
16	$\{(1,0,0),(1,0,1)\}$	$\{(1,0,0),(1,0,1)\}$
17	$\{(1,0,0),(1,1,0)\}$	$\{(1,0,0),(1,1,0)\}$
18	$\{(1,0,0),(1,0,1),(1,1,0)\}$	$\{(1,0,1),(1,1,0)\}$
19	$\{(0,0,1),(0,1,0),(0,1,1),(1,0,1)\}$	$\{(0,0,1),(0,1,0),(1,0,1)\}$
20	$\{(0,0,1),(0,1,0),(0,1,1),(1,1,0)\}$	$\{(0,0,1),(0,1,0),(1,1,0)\}$
21	$\{(0,0,1),(0,1,1),(1,0,0),(1,0,1)\}$	$\{(0,0,1),(0,1,1),(1,0,0)\}$
22	$\{(0,0,1),(1,0,0),(1,0,1),(1,1,0)\}$	$\{(0,0,1),(1,0,0),(1,1,0)\}$
23	$\{(0,1,0),(0,1,1),(1,0,0),(1,1,0)\}$	$\{(0,1,0),(0,1,1),(1,0,0)\}$
24	$\{(0,1,0),(1,0,0),(1,0,1),(1,1,0)\}$	$\{(0,1,0),(1,0,0),(1,0,1)\}$

Table 8: Sets $\mathcal{Y}(\mathcal{T}, z; \theta)$ and $\mathcal{T}$ in the core determining inequalities defining the identified set of structures in the dynamic binary response 3 period panel with Y_0 not observed and $\gamma > 0$.

	$\mathcal{Y}(\mathcal{T}, z; \theta)$	$\mathcal{T}$
1	$\{(0,0,1)\}$	$\{(0,0,1)\}$
2	$\{(0,1,0)\}$	$\{(0,1,0)\}$
3	$\{(0,1,1)\}$	$\{(0,1,1)\}$
4	$\{(1,0,0)\}$	$\{(1,0,0)\}$
5	$\{(1,0,1)\}$	$\{(1,0,1)\}$
6	$\{(1,1,0)\}$	$\{(1,1,0)\}$
7	$\{(0,0,1),(0,1,1)\}$	$\{(0,0,1),(0,1,1)\}$
8	$\{(0,0,1),(1,0,0),(1,0,1)\}$	$\{(0,0,1),(1,0,0)\}$
9	$\{(0,0,1),(1,0,1)\}$	$\{(0,0,1),(1,0,1)\}$
10	$\{(0,1,0),(0,1,1)\}$	$\{(0,1,0),(0,1,1)\}$
11	$\{(0,1,0),(1,1,0)\}$	$\{(0,1,0),(1,1,0)\}$
12	$\{(0,1,0),(0,1,1),(1,1,0)\}$	$\{(0,1,1),(1,1,0)\}$
13	$\{(1,0,0),(1,0,1)\}$	$\{(1,0,0),(1,0,1)\}$
14	$\{(1,0,0),(1,1,0)\}$	$\{(1,0,0),(1,1,0)\}$
15	$\{(0,0,1),(0,1,0),(0,1,1)\}$	$\{(0,0,1),(0,1,0),(0,1,1)\}$
16	$\{(0,0,1),(0,1,1),(1,0,0),(1,0,1)\}$	$\{(0,0,1),(0,1,1),(1,0,0)\}$
17	$\{(0,0,1),(0,1,1),(1,0,1)\}$	$\{(0,0,1),(0,1,1),(1,0,1)\}$
18	$\{(0,0,1),(0,1,0),(0,1,1),(1,1,0)\}$	$\{(0,0,1),(0,1,1),(1,1,0)\}$
19	$\{(0,0,1),(1,0,0),(1,0,1),(1,1,0)\}$	$\{(0,0,1),(1,0,0),(1,1,0)\}$
20	$\{(0,1,0),(0,1,1),(1,0,0),(1,1,0)\}$	$\{(0,1,0),(0,1,1),(1,0,0)\}$
21	$\{(0,1,0),(1,0,0),(1,1,0)\}$	$\{(0,1,0),(1,0,0),(1,1,0)\}$
22	$\{(1,0,0),(1,0,1),(1,1,0)\}$	$\{(1,0,0),(1,0,1),(1,1,0)\}$
23	$\{(0,0,1),(0,1,0),(0,1,1),(1,0,1)\}$	$\{(0,0,1),(0,1,0),(0,1,1),(1,0,1)\}$
24	$\{(0,0,1),(0,1,0),(1,0,0),(1,0,1)(1,1,0)\}$	$\{(0,0,1),(0,1,0),(1,0,0),(1,1,0)\}$
25	$\{(0,0,1),(0,1,0),(0,1,1),(1,0,1),(1,1,0)\}$	$\{(0,0,1),(0,1,1),(1,0,1),(1,1,0)\}$
26	$\{(0,1,0),(1,0,0),(1,0,1),(1,1,0)\}$	$\{(0,1,0),(1,0,0),(1,0,1),(1,1,0)\}$

Table 9: Sets $\mathcal{Y}(\mathcal{T}, z; \theta)$ and $\mathcal{T}$ in the inequalities defining the identified set of structures in the dynamic binary response 3 period panel with Y_0 not observed and $\gamma < 0$.

	$\mathcal{Y}(\mathcal{T}, z; \theta)$	$\mathcal{T}$
1	$\{(0,0,1)\}$	$\{(0,0,1)\}$
2	$\{(0,1,0)\}$	$\{(0,1,0)\}$
3	$\{(0,1,1)\}$	$\{(0,1,1)\}$
4	$\{(1,0,0)\}$	$\{(1,0,0)\}$
5	$\{(1,0,1)\}$	$\{(1,0,1)\}$
6	$\{(1,1,0)\}$	$\{(1,1,0)\}$
7	$\{(0,0,1),(0,1,0),(0,1,1\}$	$\{(0,0,1),(0,1,0)\}$
8	$\{(0,0,1),(0,1,1)\}$	$\{(0,0,1),(0,1,1)\}$
9	$\{(0,0,1),(1,0,0)\}$	$\{(0,0,1),(1,0,0)\}$
10	$\{(0,0,1),(1,0,1)\}$	$\{(0,0,1),(1,0,1)\}$
11	$\{(0,1,0),(0,1,1)\}$	$\{(0,1,0),(0,1,1)\}$
12	$\{(0,1,0),(1,0,0),(1,1,0)\}$	$\{(0,1,0),(1,0,0)\}$
13	$\{(0,1,0),(1,0,1)\}$	$\{(0,1,0),(1,0,1)\}$
14	$\{(0,1,0),(1,1,0)\}$	$\{(0,1,0),(1,1,0)\}$
15	$\{(0,0,1),(0,1,1),(1,0,1)\}$	$\{(0,1,1),(1,0,1)\}$
16	$\{(0,1,1),(1,1,0)\}$	$\{(0,1,1),(1,1,0)\}$
17	$\{(1,0,0),(1,0,1)\}$	$\{(1,0,0),(1,0,1)\}$
18	$\{(1,0,0),(1,1,0)\}$	$\{(1,0,0),(1,1,0)\}$
19	$\{(1,0,0),(1,0,1),(1,1,0)\}$	$\{(1,0,1),(1,1,0)\}$
20	$\{(0,0,1),(0,1,0),(0,1,1),(1,0,1)\}$	$\{(0,0,1),(0,1,0),(1,0,1)\}$
21	$\{(0,0,1),(0,1,0),(0,1,1),(1,1,0)\}$	$\{(0,0,1),(0,1,0),(1,1,0)\}$
22	$\{(0,0,1),(0,1,1),(1,0,0),(1,0,1)\}$	$\{(0,0,1),(0,1,1),(1,0,0)\}$
23	$\{(0,0,1),(1,0,0),(1,0,1)\}$	$\{(0,0,1),(1,0,0),(1,0,1)\}$
24	$\{(0,0,1),(1,0,0),(1,0,1),(1,1,0)$	$\{(0,0,1),(1,0,0),(1,1,0)\}$
25	$\{(0,1,0),(0,1,1),(1,0,0),(1,1,0)\}$	$\{(0,1,0),(0,1,1),(1,0,0)\}$
26	$\{(0,1,0),(0,1,1),(1,1,0)\}$	$\{(0,1,0),(0,1,1),(1,1,0)\}$
27	$\{(0,1,0),(1,0,0),(1,0,1),(1,1,0)\}$	$\{(0,1,0),(1,0,0),(1,0,1)\}$

Table 10: Sets $\mathcal{Y}(\mathcal{T}, z; \theta)$ and $\mathcal{T}$ in the inequalities defining the identified set of structures in the 3 choice multiple discrete choice 2 period panel data model.

	$\mathcal{Y}(\mathcal{T}, z; \theta)$	$\mathcal{T}$
1	$\{(1,2)\}$	$\{(1,2)\}$
2	$\{(1,3)\}$	$\{(1,3)\}$
3	$\{(2,1)\}$	$\{(2,1)\}$
4	$\{(2,3)\}$	$\{(2,3)\}$
5	$\{(3,1)\}$	$\{(3,1)\}$
6	$\{(3,2)\}$	$\{(3,2)\}$
7	$\{(1,2),(1,3)\}$	$\{(1,2),(1,3)\}$
8	$\{(1,2),(3,2)\}$	$\{(1,2),(3,2)\}$
9	$\{(1,3),(2,3)\}$	$\{(1,3),(2,3)\}$
10	$\{(2,1),(2,3)\}$	$\{(2,1),(2,3)\}$
11	$\{(2,1),(3,1)\}$	$\{(2,1),(3,1)\}$
12	$\{(3,1),(3,2)\}$	$\{(3,1),(3,2)\}$
13	$\{(1,2),(1,3),(2,3)\}$	$\{(1,2),(2,3)\}$
14	$\{(1,2),(1,3),(3,2)\}$	$\{(1,3),(3,2)\}$
15	$\{(1,2),(3,1),(3,2)\}$	$\{(1,2),(3,1)\}$
16	$\{(1,3),(2,1),(2,3)\}$	$\{(1,3),(2,1)\}$
17	$\{(2,1),(2,3),(3,1)\}$	$\{(2,3),(3,1)\}$
18	$\{(2,1),(3,1),(3,2)\}$	$\{(2,1),(3,2)\}$

Table 11: Sets $\mathcal{Y}(\mathcal{T},z;\theta)$ and $\mathcal{T}$ in the inequalities defining the identified set of values of θ in the simultaneous binary response 2 period panel.

	$\mathcal{Y}(\mathcal{T},z;\theta)$	$\mathcal{T}$
1	$\{(0,0,0,1),(1,0,0,1),(1,1,0,1)\}$	$\{(0,0,0,1)\}$
2	$\{(0,0,1,0),(0,1,1,0),(1,1,1,0)\}$	$\{(0,0,1,0)\}$
3	$\{(0,1,0,0),(0,1,1,0),(0,1,1,1)\}$	$\{(0,1,0,0)\}$
4	$\{(1,0,0,0),(1,0,0,1),(1,0,1,1)\}$	$\{(1,0,0,0)\}$
5	$\{(0,1,0,1)\}$	$\{(0,1,0,1)\}$
6	$\{(0,1,1,0)\}$	$\{(0,1,1,0)\}$
7	$\{(1,0,0,1)\}$	$\{(1,0,0,1)\}$
8	$\{(1,0,1,0)\}$	$\{(1,0,1,0)\}$
9	$\left\{\begin{array}{c}(0,0,0,1),(0,1,0,0),(0,1,1,0),\\(0,1,1,1),(1,0,0,1),(1,1,0,1)\end{array}\right\}$	$\{(0,0,0,1),(0,1,0,0)\}$
10	$\left\{\begin{array}{c}(0,0,0,1),(1,0,0,0),(1,0,0,1),\\(1,0,1,1),(1,1,0,1)\end{array}\right\}$	$\{(0,0,0,1),(1,0,0,0)\}$
11	$\{(0,0,0,1),(0,1,0,1),(1,0,0,1),(1,1,0,1)\}$	$\{(0,0,0,1),(0,1,0,1)\}$
12	$\left\{\begin{array}{c}(0,0,0,1),(1,0,0,0),(1,0,0,1),\\(1,0,1,0),(1,0,1,1),(1,1,0,1)\end{array}\right\}$	$\{(0,0,0,1),(1,0,1,0)\}$
13	$\left\{\begin{array}{c}(0,0,1,0),(0,1,0,0),(0,1,1,0),\\(0,1,1,1),(1,1,1,0)\end{array}\right\}$	$\{(0,0,1,0),(0,1,0,0)\}$
14	$\left\{\begin{array}{c}(0,0,1,0),(0,1,1,0),(1,0,0,0),\\(1,0,0,1),(1,0,1,1),(1,1,1,0)\end{array}\right\}$	$\{(0,0,1,0),(1,0,0,0)\}$
15	$\left\{\begin{array}{c}(0,0,1,0),(0,1,0,0),(0,1,0,1),\\(0,1,1,0),(0,1,1,1),(1,1,1,0)\end{array}\right\}$	$\{(0,0,1,0),(0,1,0,1)\}$
16	$\{(0,0,1,0),(0,1,1,0),(1,0,1,0),(1,1,1,0)\}$	$\{(0,0,1,0),(1,0,1,0)\}$
17	$\{(0,1,0,0),(0,1,0,1),(0,1,1,0),(0,1,1,1)\}$	$\{(0,1,0,0),(0,1,0,1)\}$
18	$\left\{\begin{array}{c}(0,0,1,0),(0,1,0,0),(0,1,1,0),\\(0,1,1,1),(1,0,1,0),(1,1,1,0)\end{array}\right\}$	$\{(0,1,0,0),(1,0,1,0)\}$
19	$\left\{\begin{array}{c}(0,0,0,1),(0,1,0,1),(1,0,0,0),\\(1,0,0,1),(1,0,1,1),(1,1,0,1)\end{array}\right\}$	$\{(1,0,0,0),(0,1,0,1)\}$
20	$\{(1,0,0,0),(1,0,0,1),(1,0,1,0),(1,0,1,1)\}$	$\{(1,0,0,0),(1,0,1,0)\}$
21	$\{(0,1,0,1),(1,0,1,0)\}$	$\{(0,1,0,1),(1,0,1,0)\}$
22	$\left\{\begin{array}{c}(0,0,0,1),(0,1,0,0),(0,1,0,1),(0,1,1,0),\\(0,1,1,1),(1,0,0,1),(1,1,0,1)\end{array}\right\}$	$\{(0,0,0,1),(0,1,0,0),(0,1,0,1)\}$
23	$\left\{\begin{array}{c}(0,0,0,1),(0,1,0,1),(1,0,0,0),(1,0,0,1),\\(1,0,1,0),(1,0,1,1),(1,1,0,1)\end{array}\right\}$	$\{(0,0,0,1),(0,1,0,1),(1,0,1,0)\}$
24	$\left\{\begin{array}{c}(0,0,1,0),(0,1,1,0),(1,0,0,0),(1,0,0,1),\\(1,0,1,0),(1,0,1,1),(1,1,1,0)\end{array}\right\}$	$\{(0,0,1,0),(1,0,0,0),(1,0,1,0)\}$
25	$\left\{\begin{array}{c}(0,0,1,0),(0,1,0,0),(0,1,0,1),(0,1,1,0),\\(0,1,1,1),(1,0,1,0),(1,1,1,0)\end{array}\right\}$	$\{(0,0,1,0),(0,1,0,1),(1,0,1,0)\}$

Appendix C Bounds on structural parameters θ

Here calculation of the projection of the identified set of values of $(\theta, G_{U|Y_0})$ onto the space of θ as set out in Remark 8 of section 3 is demonstrated for the two-period dynamic binary reponse model studied in Section 5.1.

Define

$$p_{jk}(z, y_0) \equiv \mathbb{P}[Y = (j,k)|Z = z, Y_0 = y_0].$$

Consider sets $\{u : \Delta u \in (-\infty, w]\}$ and $\{u : \Delta u \in (w, \infty)\}$ for $w \in \mathbb{R}$. There is

$$\mathbb{P}[\mathcal{U}^*(Y,Z;\theta) \subseteq \{u : \Delta u \in (-\infty, w]\} | Z = z, Y_0 = y_0] = \\ p_{10}(z, y_0) \times 1[-\Delta z\beta + (y_0 - 1)\gamma \leq w],$$

and

$$\mathbb{P}[\mathcal{U}^*(Y,Z;\theta) \subseteq \{u : \Delta u \in (w, \infty)\} | Z = z, Y_0 = y_0] = \\ p_{01}(z, y_0) \times 1[-\Delta z\beta + y_0\gamma > w].$$

Thus, using $\mathcal{U}^*(Y,Z;\theta) \subseteq \mathcal{S} \implies U \in \mathcal{S}$ we have the inequalities

$$p_{10}(z, y_0) \times 1[-\Delta z\beta + (y_0 - 1)\gamma \leq w] \leq G_{U|Y_0=y_0}(\{u : \Delta u \leq w\})$$

and

$$p_{01}(z, y_0) \times 1[-\Delta z\beta + y_0\gamma > w] \leq G_{U|Y_0=y_0}(\{u : \Delta u > w\}).$$

Let $\tilde{G}_{\Delta U}(w; y_0) \equiv G_{U|Y_0=y_0}(\{u : \Delta u \leq w\})$. Since $G_{U|Y_0=y_0}(\{u : \Delta u > w\}) = 1 - G_{U|Y_0=y_0}(\{u : \Delta u \leq w\})$ it follows that

$$p_{10}(z, y_0) \times 1[-\Delta z\beta + (y_0 - 1)\gamma \leq w] \leq \tilde{G}_{\Delta U}(w; y_0) \leq 1 - p_{01}(z, y_0) \times 1[-\Delta z\beta + y_0\gamma > w]. \tag{16}$$

Bounds on θ are obtained as those values for which the above lower and upper inequalities never cross, as follows:

$$\Theta^* \equiv \left\{\theta \in \Theta : \forall (y_0, w) \in \{0,1\} \times \mathbb{R} \quad \tilde{G}^L_{\Delta U}(w; y_0) \leq \tilde{G}^U_{\Delta U}(w; y_0)\right\}, \tag{17}$$

where $\tilde{G}^L_{\Delta U}(w; y_0)$ and $\tilde{G}^U_{\Delta U}(w; y_0)$ correspond to the lower and upper envelopes for

$\tilde{G}_{\Delta U}(w; y_0)$ obtained from (16) upon taking intersections across z:

$$\tilde{G}^L_{\Delta U}(w; y_0) \equiv \begin{cases} \sup\limits_{z \in \mathcal{Z}_L(w,y_0)} p_{10}(z, y_0) & \text{if} \quad \mathcal{Z}_L(w, y_0) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

$$\tilde{G}^U_{\Delta U}(w; y_0) \equiv \begin{cases} \inf\limits_{z \in \mathcal{Z}_U(w,y_0)} 1 - p_{01}(z, y_0) & \text{if} \quad \mathcal{Z}_U(w, y_0) \neq \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

where

$$\mathcal{Z}_L(w, y_0) \equiv \{z \in \mathcal{R}_Z : -\Delta z\beta + (y_0 - 1)\gamma \leq w\},$$
$$\mathcal{Z}_U(w, y_0) \equiv \{z \in \mathcal{R}_Z : -\Delta z\beta + y_0\gamma > w\}.$$

The bounds Θ^* correspond to those of Theorem 2 of Aristodemou (2021), up to minor notational differences.

In a model in which U and Z are independent conditional on Y_0 with no restrictions placed on $G_{U|Y_0=y_0}$, the set Θ^* thus obtained is in fact the sharp identified set for θ because for every value of $\tilde{\theta} \in \Theta^*$ there is for each $y_0 \in \{0, 1\}$ a distribution of ΔU conditional on $Y_0 = y_0$, say $\tilde{G}_{\Delta U|Y_0=y_0}$, such that $(\tilde{\theta}, \tilde{G}_{\Delta U|Y_0=0}, \tilde{G}_{\Delta U|Y_0=1})$ is contained in the identified set of structures.

This is so because $\tilde{G}^L_{\Delta U}(w; y_0)$ and $\tilde{G}^U_{\Delta U}(w; y_0)$ are nondecreasing functions of w taking values on the unit interval. Accordingly, for each $\tilde{\theta} \in \Theta^*$ there exists for each value of y_0 a proper distribution $\tilde{G}_{\Delta U|Y_0=y_0}(\cdot, \tilde{\theta})$ such that for all $w \in \mathbb{R}$,

$$\tilde{G}^L_{\Delta U}(w; y_0) \leq \tilde{G}_{\Delta U|Y_0=y_0}((-\infty, w], \tilde{\theta}) \leq \tilde{G}^U_{\Delta U}(w; y_0),$$

for example

$$\tilde{G}_{\Delta U|Y_0=y_0}((-\infty, w], \theta) = \lambda\tilde{G}^L_{\Delta U}(w; y_0) + (1 - \lambda)\tilde{G}^U_{\Delta U}(w; y_0)$$

for any $\lambda \in (0, 1)$.[28] Thus any distributions $\tilde{G}_{U|Y_0=y_0}(\cdot)$ for U with corresponding distributions $\tilde{G}_{\Delta U|Y_0=y_0}(\cdot, \tilde{\theta})$ for each y_0 paired with $\tilde{\theta}$ can produce the observed distributions of Y given (Y_0, Z).

[28]This sharpness result relies on the identified set of structures being determined by the distribution of a scalar function of the unobserved variables, in this case ΔU. When $T > 2$ this will not be the case in this example but it is the case in the model of section 5.2.