A Theory of Stable Market Segmentations

Nima Haghpanah and Ron Siegel*

This draft: July 12, 2023
First draft: February 10, 2022

Abstract

We study market segmentation as the outcome of a cooperative game between consumers who interact with a monopolistic seller in groups. We introduce two new solution concepts, the weakened core and stability, that coincide with the core whenever it is nonempty. We show that these concepts are equivalent and characterized by efficiency and saturation. A segmentation is saturated if shifting consumers from a segment with a higher price to a segment with a lower price leads the seller to optimally increase the lower price. We show that stable segmentations that maximizes average consumer surplus (across all segmentations) always exist.

*Department of Economics, the Pennsylvania State University, University Park, PA 16802 (e-mail: nuh47@psu.edu and rus41@psu.edu). We thank Rohan Dutta, Matt Mitchell, Debraj Ray, Daniel Rappoport, Doron Ravid, Joe Root, Jakub Steiner, Roland Strausz, Rakesh Vohra, Asher Wolinsky, and participants at various seminars and conferences for comments.

1 Introduction

Consumers often interact with sellers as a group. Workers' unions, employers representing their employees, homeowner associations, and student groups are some examples of consumer groups. Recent technological advances greatly facilitate and expand the ability of consumers to form groups. Social media and other platforms make coordination among consumers easier. Novel enterprises such as data cooperatives and mediators of individual data make it possible for consumers to coordinate the information that sellers can access, thus controlling the degree to which sellers can distinguish between consumers and effectively interacting with the seller in groups.[1]

This raises the question of which consumer groups will form and what will be the resulting welfare consequences. We investigate this question in a market setting with a monopolistic seller of a single product, assuming that all partitions of the consumers into groups are possible. Each partition of the consumers induces a segmentation of the market into consumer groups, and we assume that the seller sets a profit-maximizing price for each group, that is, for each market segment.[2] In other words, consumers control how they are segmented into groups, and the seller controls the prices consumers face given the segmentation.[3]

The fundamental tension between consumers is that different consumers may prefer different segmentations. For example, suppose that a measure 0.45 of consumers have value 1 for the product, a measure 0.35 have value 2, and the remaining measure 0.2 have value 3. One possible segmentation of the market is into a group that consists of the consumers with values 1 and 2 and a group that consists of the consumers with value 3. Given this segmentation, the seller optimally sets a price of 1 for the first group and a price of 3 for the second group. Another segmentation of the market groups the consumers with values 1 and 3 together and the consumers with value 2 together, which leads to a price of 1 for the first group and price of 2 for the second group. Consumers with value 2 prefer the first segmentation to the second one, whereas

[1]For example, MiData and Salus Coop enable their members to pool and share health data. Swash pools web surfing data from its members. Mediators of individual data were recently introduced by Lanier and Weyl (2018).

[2]Group formation based on consumer unions, home owner associations, etc. can be thought of as leading to an "access-based" market segmentation, whereas control of the consumer data the seller can access leads to a "data-based" market segmentation.

[3]In practice, some groups may also increase consumers' bargaining power. We abstract from this effect in our investigation of market segmentation by consumers and maintain the seller's ability to unilaterally set prices for each consumer group.

consumers with value 3 have the opposite preference. Of course, even in this example many other segmentations are possible that do not group all the consumers with the same value in the same group. And, in general, even consumers with the same value may rank segmentations differently.[4]

To study this tension we model the interaction between consumers as a cooperative game with non-transferable utility. Given a segmentation and a seller-optimal price for each segment, each consumer decides whether to purchase the product at the price she faces in her segment. This determines every consumer's utility given the segmentation. A segmentation is in the the core of the game if no new group of consumers can form that *objects* to the segmentation, that is, every consumer in the new group weakly prefers the price set by the seller for the new group to the price she faces in the segmentation, with a strict preference for some members of the group.

A key determinant of the core is whether the market is efficient, that is, whether in the unsegmented market the seller optimally sets the lowest possible price, which is equal to the lowest consumer value in the market. We observe that all consumers agree on the best segmentations if and only if the market is efficient, and in this case the core consists of the market segmentations in which every consumer faces the lowest possible price. This makes intuitive sense: if all consumers agree on the best segmentations, then such a segmentation would be a reasonable prediction. However, whenever the market is inefficient, that is, whenever consumers do not all agree on the best segmentations, so there is some conflict of interest among consumers, the core is empty. The reason is that if there is any conflict of interest, then starting from any segmentation, some consumers who are not facing the lowest possible price can form a new group together with the lowest-value consumers and obtain the lowest possible price. This disrupts the existing segmentation.

To study what happens in the economically interesting case in which not all consumers agree on the best segmentations, we develop two alternative solution concepts. The first is the weakened core, which is similar to the core but rules out certain deviations. One motivation for the weakened core is that "breaking apart" an existing segment involves a small cost (perhaps an administrative fee), which must be paid by

[4]For example, suppose that a measure 0.6 of consumers have value 1 and the remaining consumers have value 2. One segmentation groups the consumers with value 1 and half of the consumers with value 2 together, with the remaining value 2 consumers forming a second group. The other segmentation groups the consumers with value 1 and the other half of the consumers with value 2 together, with the remaining value 2 consumers forming a second group. Half of the value 2 consumers prefer the first segmentation to the second one, and the other half have the opposite preference.

some of the deviating consumers from every segment that is broken apart in order to form the deviation. The second solution concept is stability, which in a sense is the "opposite" of the core: a stable segmentation is one that for any alternative, non-equivalent segmentation contains a segment that objects to the alternative segmentation.[5] This notion of stability captures a kind of "coalitional individual rationality (IR):" once a segment forms, its members cannot be regrouped into a different segment or segments if they all oppose this change (at least some strictly). Thus, whereas the core can be thought of as prioritizing deviations, stability prioritizes the prevailing segmentation by allowing it to prevent deviations to other segmentations. We show that both solution concepts coincide with the core whenever it is not empty, that is, whenever the market is efficient.[6]

Our main result shows that both solution concepts always coincide and characterizes them. Because a segmentation is in the weakened core if and only if it is stable, we refer to such segmentations as stable segmentations. Our characterization shows that a segmentation is stable if and only if it is *efficient* and *saturated*. Efficiency means that every consumer buys the product, and saturation means that consumers in each segment are not willing to accept additional consumers from segments with higher prices because doing so increases the price in their own segment. We also show that stable segmentations are Pareto undominated, that is, there is no other segmentation that makes all consumers better off.

Our characterization helps address the following question. Suppose that consumers can choose a segmentation before they learn their value for the product but cannot commit to the segmentation, so that some consumers may want to deviate by forming a new group after they learn their value. Which segmentation would arise? We formalize this as asking which stable segmentations maximize expected consumer surplus among all stable segmentations. We answer this question by constructively showing that a stable segmentation exists that maximizes average consumer surplus among all segmentations (stable or not). This stable segmentation is the *maximal equal-revenue segmentation*, identified by Bergemann, Brooks, and Morris (2015).[7] Our result that the maximal equal-revenue segmentation is stable shows that stability does not reduce

[5]Two segmentations are equivalent if each consumer faces the same price in both.

[6]Section 6 and Appendix C discuss the relationship between our solution concepts and existing cooperative solution concepts other than the core.

[7]Bergemann, Brooks, and Morris (2015) characterized the set of average consumer-producer surplus pairs achievable across all segmentations, and did not consider stability or the question of which segmentations will arise.

the surplus consumers can achieve even in the absence of commitment or a central planner that enforces the segmentation. The result also implies that stable segmentations always exist. We then show that multiple stable segmentations may exist and that maximizing average consumer surplus neither implies nor is implied by stability.

Our analysis of stable segmentations may be relevant to policy discussions regarding monopolies, price discrimination, data sharing, and data intermediaries. Monopolies lead to inefficiencies, and these inefficiencies may be reduced with regulation or increased competition. Market segmentation arising from the monopolist's access to consumer data can also reduce inefficiency, but may harm consumers, as first-degree price discrimination demonstrates. Our results show that as long as consumers can form groups, market segmentation leads to efficiency and Pareto undominated outcomes for consumers. If consumers can choose a segmentation before they know their value, they can achieve the highest possible expected surplus without the help of a social planner even if they cannot commit to maintaining the segmentation after they learn their value. This indicates that policies or information intermediaries, such as data cooperatives, that facilitate consumer group formation while allowing the seller to price discriminate across groups may offer an alternative or complimentary tool to addressing the inefficiencies associated with monopolistic markets.

The rest of the paper is organized as follows. Section 1.1 describes the relationship of our work to the existing literature. Section 2 describes the model. Section 3 introduces the core and our solution concepts. Section 4 shows the equivalence of our solution concepts and characterizes them. Section 5 shows that segmentations that satisfy these solution concepts exist and relates them to consumer-optimal segmentations. Section 6 concludes.

1.1 Related literature

Peivandi and Vohra (2021) consider stability of centralized markets against deviations by coalitions of agents. They show that fragmentation of such markets is unavoidable, despite its efficiency costs, except in special circumstances. They study a bilateral trade setting, whereas we study a setting with a population of consumers and a seller. When centralized markets are fragmented, Peivandi and Vohra (2021) do not predict what the resulting segmentation looks like, whereas characterizing stable segmentations is a main focus of our paper. Another important difference is that in their setting a coalition chooses the trading mechanism, whereas in our setting each coalition of consumers faces a profit-maximizing price set by the seller.

A recent literature on third-degree price discrimination studies consumer and producer surplus across all possible segmentations of a given market. Bergemann, Brooks, and Morris (2015) identify the set of average producer and consumer surplus pairs that result from all segmentations of a given market. Their results also identify segmentations that maximize average consumer surplus. Cummings et al. (2020) study an extension in which only certain segmentations may be chosen. Glode, Opp, and Zhang (2018) study optimal disclosure by an informed agent in a bilateral trade setting, and show that the optimal disclosure policy leads to socially efficient trade, even though information is revealed only partially. Haghpanah and Siegel (2023) study when a market served by a multi-product seller can be segmented in a way that is Pareto improving. Yang (2022) studies how a profit-maximizing data broker sells market segmentations to a monopolist. Ichihashi (2020), Hidir and Vellodi (2021), Braghieri (2017), and Haghpanah and Siegel (2022) consider maximum average consumer surplus when a multi-product seller offers different products in each market segment. These papers can be seen as identifying segmentations that are chosen *ex ante* by a consumer who does not know her type, because such a consumer chooses the segmentation that maximizes her expected payoff. In contrast, in this paper we study market segmentation when consumers know their type.

The related papers that study disclosure decisions by consumers who know their type model these interactions as non-cooperative games. Ali, Lewis, and Vasserman (2023) consider voluntary disclosure of data by a single consumer, and analyze the welfare implications of various disclosure policies in both a monopolistic and a competitive environment. Sher and Vohra (2015) study a disclosure setting in which the seller can commit to the mechanism that he will use after receiving information. In our setting, the seller cannot commit and chooses a profit-maximizing price for each segment. Acemoglu et al. (2019), Bergemann, Bonatti, and Gan (2022), Baumann and Dutta (2022), and Galperti and Perego (2023) also study the consequences of consumers' disclosure decisions on prices and other market outcomes. Kuang et al. (2022) study the formation of market segmentations as the equilibrium of a non-cooperative game in which each consumer chooses which segment to join. In their setting consumers can unilaterally move between segments, whereas in our setting, once a segmentation arises consumers cannot join an existing segment if all consumers in that segment oppose this change. These differences lead to different results: in the setting of Kuang et al. (2022), the seller's surplus in any equilibrium segmentation is equal to the seller's surplus in the unsegmented market and consumers are better off, but the outcome need not be efficient. In contrast, our stable segmentations are Pareto undominated, efficient, and increase the seller's surplus.

2 Model

A monopolistic seller faces a unit mass of consumers uniformly distributed on the unit interval $[0, 1]$. Consumers have unit demand for the monopolist's product. The value of the product for consumer $c \in [0, 1]$ is $v(c) \in V = \{v_1, \ldots, v_n\} \subseteq R_{>0}$, where v is a measurable function and v_i increases in i.[8] Let μ be the Borel measure on the unit interval. The measure of consumers with value v_i is $f(v_i) = \mu(\{c : c \in [0, 1], v(c) = v_i\})$. We assume without loss of generality that $f(v_i) > 0$ for every $v_i \in V$, and normalize the seller's production cost to zero.

A *coalition* is a measurable subset $C \subseteq [0, 1]$ of consumers. Let $f^C(v_i) = \mu(\{c : c \in C, v(c) = v_i\})$ denote the measure of consumers with value v_i in coalition C. We say that *consumers with value v_i are in C* (or that C contains consumers with value v_i) if $f^C(v_i) > 0$. A price $p \in V$ is *optimal for coalition C* if it maximizes the revenue from selling the product to consumers in C, that is, for any other price $p' \in V$,

$$p \sum_{i:v_i \geq p} f^C(v_i) \geq p' \sum_{i:v_i \geq p'} f^C(v_i).$$

We restrict attention to prices in V because for any other price there exists a price in V with a weakly higher revenue.

A *segment* is a pair (C, p), where C is a coalition and p is an optimal price for that coalition.[9] A *segmentation* S is a finite set of segments $\{(C_j, p_j)\}_{j=1,\ldots,k}$ such that $C_1, \ldots, C_k$ partitions the set of consumers $[0, 1]$. That is, a segmentation partitions the set of consumers into coalitions and assigns an optimal price for each coalition.

Denote by $CS(c, p) = \max\{v(c) - p, 0\}$ the surplus of consumer c who is offered the product at price p. If consumer c belongs to segment (C, p), then her surplus is $CS(c, p)$. Given a segmentation S, denote by $p_S(c)$ the price in the unique segment that includes consumer c. Let $CS(c, S) = CS(c, p_S(c))$ denote the surplus of consumer

[8] Appendix D discusses variants of our model with a continuum of possible values.

[9] That is, the seller cannot price discriminate between the different consumers in C, and sets a price for these consumers that maximizes his revenue.

c in segmentation S.

Consumers' preferences over segmentations may differ because the prices different consumers face vary within and across segmentations. We investigate how consumers reconcile these differences by modeling the interaction between consumers as a cooperative game with non-transferable utility (NTU) that determines the market segmentation.[10] That is, the consumers cooperatively determine how they are partitioned into coalitions, and the seller sets an optimal price for each coalition. As described in the introduction, we have in mind both access-driven settings, in which groups of consumers such as home-owner associations and employees of a university interact with the seller as a bloc, and data-driven settings, in which data cooperatives, mediators of individual data, and other online platforms facilitate coordination among consumers regarding the consumer data the seller can access.

3 Solution Concepts

3.1 The Core and Why it is Unsatisfactory

We start by studying the core of the cooperative game. We show that the core is unsatisfactory because it is empty if and only if consumers disagree about which segmentations are most preferred. Because our main goal is to understand which segmentations arise when consumers disagree, we develop two other solution concepts, which are less demanding than the core but coincide with it whenever consumers agree on the best segmentations, that is, whenever the core is not empty.

To define the core, we first formalize what it means for a segment to object to a segmentation.

Definition 1 (Objection) *A segment (C, p) objects to a segmentation S if $CS(c, p) \geq CS(c, S)$ for all consumers c in C, with a strict inequality for a positive measure of consumers c in C.*

A segment (C, p) objects to a segmentation S if all the consumers in C are weakly better off and some consumers in C are strictly better off in the segment compared to the segmentation. In particular, an objection (C, p) *is not* a segment in S (otherwise

[10] Formally, for each coalition C of consumers, the set of utility vectors feasible for C comprises the payoff profiles of the consumers in C across all segmentations of C (when C, instead of $[0, 1]$, is taken to be the set of consumers).

all consumers in C would be indifferent between the segment and the segmentation). Notice that the definition would be vacuous if we required the preference to be strict for all (or almost all) consumers in C, because the optimality of p for C requires that the surplus of the consumers with the lowest value in C is zero.

We define the core to be the set of segmentations to which there is no objection.[11]

Definition 2 (Core) *The core is the set of segmentations S to which no segment objects.*

According to the core, a possible objection can be *any* segment and is not constrained by the existing coalitions in the prevailing segmentation. An objecting coalition is unconcerned with what happens to the rest of the players, and only considers their own payoffs.

The core is not a useful solution concept in our setting. This is because, as the proposition below shows, the core is empty if and only if the market is inefficient, that is, price v_1 is not optimal for the set $[0, 1]$ of all consumers. If price v_1 is optimal for the set of all consumers, then the segmentations in which every consumer faces price v_1 (including the "trivial" segmentation $\{([0, 1], v_1)\}$) are most preferred by every consumer because v_1 is the lowest price the seller would ever offer. In this case, as expected, the core consists of the segmentations in which every consumer faces price v_1. On the other hand, if price v_1 is *not* optimal for the set of all consumers, then there is no segmentation in which every consumer faces price v_1. Any consumer who faces a price higher than v_1 ranks segmentations in which she faces price v_1 higher (the proof shows that such segmentations exist). Thus, the core is empty precisely when consumers disagree on their top-ranked segmentations.

Proposition 1 *If the market is inefficient, then the core is empty. Otherwise, the core consists of the segmentation $\{([0, 1], v_1)\}$ and all the other segmentations in which every consumer faces price v_1.*

Proof. For the first claim, suppose that the market is inefficient, so v_1 is not optimal for the set $[0, 1]$ of all consumers. Then any segmentation includes a segment with a price strictly higher than v_1. Otherwise, the price in every segment is equal to v_1 (because

[11] For our purposes it is more convenient to refer to a set of segmentations instead of the payoff vectors they induce.

it cannot be less than v_1), so price v_1 remains optimal for the single coalition $[0,1]$, which contradicts the assumption that v_1 is not optimal for the set of all consumers.[12]

Now, take a segmentation S and a segment (C,p) in it with $p > v_1$. Because p is optimal for C, C contains a positive measure of consumers with value p. Consider a coalition C' that consists of a positive measure of consumers from C with value p and a positive measure of consumers with value v_1 (from any segment). If $f^{C'}(p)$ is small enough relative to $f^{C'}(v_1)$, then price v_1 is optimal for C', so (C', v_1) is a segment. The surplus of value p consumers in C' is $p - v_1 > 0$, whereas their surplus in S is zero. The surplus of consumers with value v_1 in any segment is zero. Therefore, segment (C', v_1) objects to S, so S is not in the core. Since S was a arbitrary segmentation, the core is empty.

For the second claim, suppose that the market is efficient. Then $([0,1], v_1)$ is a segment and there is no objection to the segmentation $\{([0,1], v_1)\}$ because in any segment the price is at least v_1. For the same reason, any segmentation in which every consumer faces price v_1 is also in the core. Now consider a segmentation S in which the price p in some segment exceeds v_1. Because p is optimal for the coalition in the segment, the coalition contains a positive measure of consumers with value p, who strictly prefer price v_1. And because the price in any segment is at least v_1, the segment $([0,1], v_1)$ objects to S, so S is not in the core. ■

The following example illustrates how disagreement between consumers about which segmentations are the best leads to an empty core.

Example 1 (The core might be empty) *There are two values,* $1, 2$. *Consumers from* 0 *to* 0.4 *have value 1, and consumers from* 0.4 *to 1 have value 2, as shown in Figure 1.*

Consider segmentation $S = \{(C_1, 1), (C_2, 2)\}$, where $C_1 = [0, 0.6)$ and $C_2 = [0.6, 1]$. This segmentation is not in the core. One segment that objects to S is $(C_1', 1)$, where $C_1' = [0, 0.8)$. This objection is obtained by adding some consumers from the higher priced segment in S to the lower priced segment in S in a way that leaves the low price 1 optimal for the seller. By doing so, the surplus of consumers in C_1 does not change, and the surplus of the added consumers strictly increases.

[12]Suppose that price p is optimal for disjoint coalitions $C^1, \dots, C^k$, that is, $p \sum_{i:v_i \geq p} f^{C^j}(v_i) \geq p' \sum_{i:v_i \geq p'} f^{C^j}(v_i)$ for all C^j and p'. Summing over all j, and letting $C = \bigcup_j C^j$, we have $p \sum_{i:v_i \geq p} f^{C}(v_i) \geq p' \sum_{i:v_i \geq p'} f^{C}(v_i)$ for all p', so price p is optimal for coalition C.

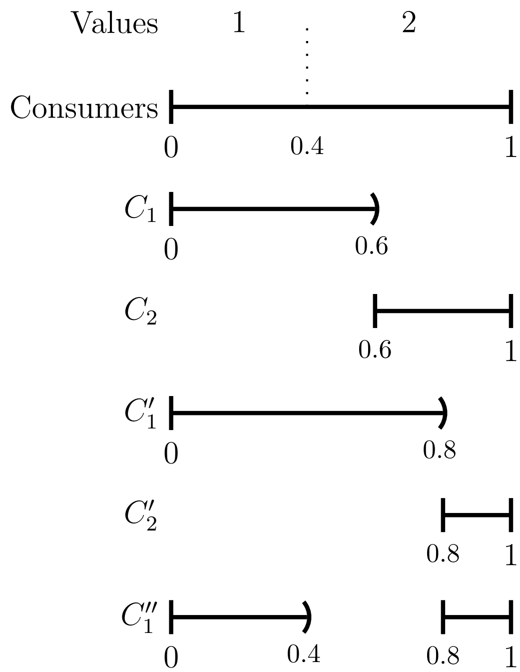

Figure 1: Example 1.

Now consider segmentation $S' = \{(C_1', 1), (C_2', 2)\}$, where C_1' is as defined above and $C_2' = [0.8, 1]$. Adding any positive measure of consumers from C_2' to C_1' necessarily increases the optimal price in the first segment, so such a coalition would not be an objection because the value 2 consumers in C_1' would be strictly harmed. However, segmentation S' is still not in the core because segment $(C_1'', 1)$, where $C_1'' = [0, 0.4] \cup [0.8, 1]$, objects to it: value 1 consumers in C_1' are indifferent, and value 2 consumers in C_1'' strictly prefer $(C_1'', 1)$ to S', where they face price 2.

This example clarifies why the core is empty. Because there is only a measure 0.4 of value 1 consumers, in any segmentation at most a measure 0.4 of value 2 consumers may be offered price 1. So in any segmentation, at least a measure 0.2 of value 2 consumers are offered price 2. By combining these value 2 consumers with all the value 1 consumers, we obtain a coalition to which the seller optimally offers price 1, which results in an objection.

Our main goal in studying the cooperative game is to understand how the conflict of interest between consumers is resolved when they do not all agree on a top-ranked segmentation. If the market is efficient, there is no conflict of interest, and core gives us the "correct" prediction. But in this case we do not really need to write down a game. And whenever there is a conflict of interest, that is, when the market is inefficient, the core does not help.

To obtain useful predictions, in the next two subsections we introduce two solution concepts. The first one is a slight relaxation of the core; the second, stability, is based on a different motivation and is in some sense the "opposite" of the core.[13]

3.2 The Weakened Core

Our first solution concept is a less demanding version of the core. The idea is that the core allows for "too many" objections, especially if forming an objection involves some friction. Our solution concept, which we call the weakened core, rules out certain objections by slightly strengthening the requirements from an objection. Even though the strengthening is slight, we later show that this change guarantees non-emptiness. After defining the weakened core below, we provide a cost-based motivation for the definition.

To introduce the weakened core, consider Example 1 and the segmentation $S' = \{(C'_1, 1), (C'_2, 2)\}$, where $C'_1 = [0, 0.8)$ and $C'_2 = [0.8, 1]$. As discussed earlier, segment $(C''_1, 1)$, where $C''_1 = [0, 0.4] \cup [0.8, 1]$, objects to S'. But notice that this objection requires "breaking apart" the existing segment C'_1, taking consumers with value 1 from C'_1 and adding them to the objection, even though none of these consumers strictly benefits from joining the objection. The weakened core excludes these kind of objections by requiring that if a deviating coalition breaks apart an existing coalition, then at least some members of the existing coalition who form part of the deviation strictly benefit.

To formalize this, we first define strict objections. A strict objection is an objection with the additional requirement that if the objecting coalition breaks apart an existing coalition, then at least some members of the existing coalition who are in the objecting coalition strictly benefit from this rearrangement. Formally, for the definition we say that C' *breaks apart* C if coalition C' contains some but not all of the consumers in coalition C (so $C \cap C'$ and $C \backslash C'$ both have positive measures).

Definition 3 *A segment (C', p') strictly objects to a segmentation S if*

1. *(C', p') objects to S, and*

[13] Appendix C compares our solution concepts to existing solution concepts for cooperative games other than the core, and shows that our solution concepts refine them.

2. for any segment (C,p) in S, if C' breaks apart C, then $CS(c,p') > CS(c,S)$ for a positive measure of consumers c in $C \cap C'$.

Definition 4 *The weakened core is the set of segmentations S to which no segment strictly objects.*

Because strict objections are objections, the weakened core is a superset of the core.[14]

One cost-based justification for the weakened core is as follows. Once a segmentation forms, a small fee must be paid to break apart any of its existing coalitions. This fee must be paid by some members of the coalition who want to leave the coalition and join a deviation. More precisely, suppose that the prevailing segmentation is S and a new segment (C',p') objects to S. If C' breaks apart an existing segment (C,p) in S, then some of the consumers in $C \cap C'$ must cover the small breakup fee in order dissolve C, which allows the consumers in $C \cap C'$ to leave and form C'. Consumers in $C \cap C'$ who strictly benefit from the deviation are willing to pay this fee. So if for any existing coalition that is broken apart by C', some consumers in the intersection of the two coalitions strictly benefit from the deviation, all the breakup fees are paid and the objection forms. But if for some existing coalition C that is broken apart by C' all consumers in $C \cap C'$ are indifferent between S and (C',p'), then none of these consumers is willing to pay the fee and the objection fails.

3.3 Stability

Our second solution concept is stability. Stability is conceptually different from the core and the weakened core. The core (and the weakened core) give priority to deviations, in that an objection (or a strict objection) to a segmentation prevents that segmentation from being realized. Stability, on the other hand, gives priority to the candidate segmentation by requiring it to *prevent* deviations to any other segmentation: for any other segmentation, a stable segmentation contains a segment that objects to the other segmentation. This motivation is closely related to various existing stable set notions in cooperative game theory, which we discuss in Appendix C.

To motivate stability, consider again Example 1 and the segmentation $S' = \{(C_1',1),(C_2',2)\}$, where $C_1' = [0,0.8)$ and $C_2' = [0.8,1]$. As discussed earlier, segment $(C_1'',1)$ objects to

[14] However, not every objection is a strict objection. Going back to Example 1, $(C_1'',1)$ is not a strict objection because it breaks apart the coalition C_1' in S' but none of the consumers in $C_1' \cap C_1''$ strictly prefer the segment to the segmentation.

this segmentation, where $C''_1 = [0, 0.4] \cup [0.8, 1]$, so S' is not in the core (and the core is in fact empty). But notice that segment $(C'_1, 1)$ in S' objects to any segmentation S'' that contains $(C''_1, 1)$ (in any such segmentation, any consumer not in C''_1 has value 2 and faces price 2). The formation of $(C''_1, 1)$ requires rearranging the consumers in coalition C'_1 even though *all* the consumers in C'_1 oppose this rearrangement (some weakly and some strictly). Stability says that rearrangements that are unanimously opposed by an existing segment do not occur.[15]

To define stability, we introduce the notion of a blocking segmentation. For the definition, it is helpful to think of segmentation S as the prevailing segmentation and segmentation S' as a proposed deviation.

Definition 5 (Blocking) *A segmentation S blocks a segmentation S' if there exists a segment (C, p) in S that objects to S'.*

For the definition of stability, we say that segmentations S and S' are *equivalent* if almost all consumers face the same price in the two segmentations, that is, for almost all consumers c in $[0, 1]$, c is in a segment with price p in segmentation S if and only if c is in a segment with price p in segmentation S'.[16]

Definition 6 (Stability) *A segmentation is stable if it blocks any non-equivalent segmentation.*

One way to think of stability is as follows. Suppose that S is the prevailing segmentation and a deviation is proposed (say by some consumer who prefers the deviation to S). This deviation requires rearranging the consumers of some segments in S but may leave other segments unaffected. For every affected segment, the proposal must specify how the consumers in the affected segment are rearranged. Once this proposal is made, every affected segment in S is asked if it is willing to participate in this rearrangement. If an affected segment objects to this rearrangement (in the sense that all its consumers prefer S to the rearrangement, some weakly and some strictly), then the proposal fails. In a sense, if a deviation requires rearranging consumers in an existing

[15]If a rearrangement can occur without modifying an existing segment, then this segment does not oppose the rearrangement.

[16]Notice that two equivalent segmentations need not be identical when viewed as two sets of segments. For example, given a segmentation, if we replace a segment (C, p) with two segments $(C \backslash C', p)$ and (C', p) for some subset C' of C (so that price p is optimal for both coalitions C' and $C \backslash C'$), then we obtain a new segmentation that is not identical to the original segmentation but is equivalent to it.

segment, these consumers as a group "have the right" to prevent being rearranged. Notice that preventing the rearrangement requires *unanimous* agreement by the members of an existing segment. If some consumers in an existing segment strictly prefer the rearrangement, then this segment does not object - these consumers cannot be prevented from leaving (and dissolving the segment) by the other consumers in the segment.

Thus, stability captures a kind of "coalitional individual rationality (IR)," in that no segment can be forced to regroup into one or more different segments if all its members oppose this change. A segmentation is stable if moving to any other non-equivalent segmentation violates coalitional IR.

Stability may appear to demand more than just coalitional IR because it requires that a segmentation block *every* non-equivalent segmentation, even those that are not attractive alternatives (because they do not block the original segmentation), and even those that may be "difficult to reach" from the original segmentation. We later use our characterization of stable segmentations from the next section to address these issues and show that stability is in fact the right notion for capturing coalitional IR.

4 Main Result: Equivalence and Characterization

Before characterizing our two solution concepts, we relate them to the core. We show that even though the weakened core is a superset of the core, and the idea underlying stability is the "opposite" of that underlying the core, all three solution concepts coincide whenever the core is not empty, that is, when the market is efficient.

Proposition 2 *If the core is not empty, then it is equal to the weakened core and the set of stable segmentations.*

Proof. Suppose that the core is not empty. By Proposition 1, the core consists of the segmentation $\{([0,1],v_1)\}$ and all its equivalent segmentations. These segmentations are clearly stable because any such segmentation S only contains segments of the form (C,v_1), so in any non-equivalent segmentation a positive measure of consumers are offered a price higher than v_1, and then there is a segment in S that objects to the non-equivalent segmentation. Any segmentation that is not equivalent to the segmentation $\{([0,1],v_1)\}$ is not stable because it does not block $\{([0,1],v_1)\}$ (since v_1 is the lowest price that any consumer faces in any segmentation).

The weakened core is a superset of the core, so any segmentation that is equivalent to $\{([0,1], v_1)\}$ is in the weakened core. For the other direction, any segmentation that is not equivalent to $\{([0,1], v_1)\}$ is not in the weakened core because segment $([0,1], v_1)$ strictly objects to it ($([0,1], v_1)$ is an objection that does not break apart any segment, so it is a strict objection). ■

We now show that stability and the weakened core are in fact equivalent, and we characterize them with two properties. These properties are easy to check and only refer to the segmentation under consideration, without involving objections or blocking, which relate to other segments or segmentation. While it is fairly straightforward to show that these properties are necessary and that any stable segmentation is in the weakened core (so stability is more demanding than the weakened core), sufficiency of the properties and the reverse inclusion are less obvious.

We start by introducing the notion of a *canonical* segmentation. A segmentation is canonical if no two segments in it have the same price.[17] Each segmentation S is equivalent to a unique canonical segmentation, which we call the *induced canonical segmentation* of S. The induced canonical segmentation is obtained by merging segments that have the same price into a single segment with that price (see footnote 12).

Our characterization says that our two solution concepts are equivalent, and a segmentation satisfies the concepts if and only if its induced canonical segmentation satisfies two properties: efficiency and saturation. A segmentation is *efficient* if all consumers buy the product, that is, for any segment (C, p) in the segmentation, the price p is equal to the lowest value $\underline{v}(C) := \min\{v : f^C(v) > 0\}$ of consumers in C. A segmentation is *saturated* if for any segment (C, p) in the segmentation, whenever we add consumers to coalition C from a segment with a price strictly higher than p, price p is sub-optimally low for this larger coalition. That is, for any two segments (C, p) and (C', p') in the segmentation with $p < p'$, and any positive-measure set $C'' \subseteq C'$ of consumers, any optimal price for coalition $C \cup C''$ is strictly higher than p.

The following lemma shows that saturation can be expressed more succinctly by looking at the set of prices that are optimal for different segments.

Lemma 1 *A segmentation is saturated if and only if for any two segments (C, p) and (C', p') in the segmentation with $p < p'$, there exists a price $\hat{p}$ that is optimal for C such that $p < \hat{p} \leq \underline{v}(C')$.*

[17] Bergemann, Brooks, and Morris (2015) refer to these as *direct* segmentations.

Proof. If such a $\hat{p}$ exists, then by adding consumers from C' to C, all of whose values are at least $\underline{v}(C')$, and therefore at least $\hat{p}$, the revenue from price $\hat{p}$ increases more than the revenue from price p. And because both p and $\hat{p}$ are optimal for C, p (and any price lower than p) is not optimal when we add these consumers. Conversely, if no such $\hat{p}$ exists, then p is the highest optimal price for C that does not exceed $\underline{v}(C')$, so if we add a small measure of consumers with value $\underline{v}(C')$ from C' to C, price p remains optimal for C. ■

We now state and prove our main result. In addition to showing that our two solution concepts are equivalent and characterizing them, the result also shows that the weakened core and stability are equivalent to a third, intermediate solution concept. This intermediate solution concept only requires a segmentation to block "potentially attractive segmentations," that is, segmentations that blocks it, instead of blocking all non-equivalent segmentations, which is required by stability. We will motivate and discuss this intermediate solution concept after establishing the equivalence.

Theorem 1 *For any segmentation S, the following statements are equivalent*

1. *S is in the weakened core.*
2. *S blocks any segmentation that blocks S.*
3. *S is stable.*
4. *The canonical segmentation of S is efficient and saturated.*

Proof. The three solution concepts can be ranked: statement (3) in the theorem implies statement (2) and statement (2) implies statement (1). Indeed, if a segmentation S is stable and S' blocks S, then S and S' are not equivalent, so S blocks S' because S is stable. If a segmentation S blocks any segmentation that blocks it, then it is in the weakened core. To see this, suppose that S is not in the weakened core, so there is a strict objection (C', p') to S. We can then construct a segmentation S' that blocks S but is not blocked by S as follows. Start with S' being the empty set. Add (C', p') to S'. For any segment (C, p) in S such that $C \cap C'$ is empty (has zero measure), add (C, p) to S'. For any segment (C, p) that is broken apart by (C', p'), add segment $(C \backslash C', p'')$ to S', where p'' is any optimal price for $C \backslash C'$. Segmentation S' blocks S because it contains segment (C', p'), which objects to S. To see that no segment (C, p) in S objects to S', consider three cases. If $C \subseteq C'$, then because (C', p') objects to S, the consumers in C weakly prefer S' to S so (C, p) does not object to S'. If $C \cap C'$ is empty, then (C, p) is a segment in S' so the consumers in C are indifferent between S and S' and (C, p) does not object to S'. If (C, p) is broken apart by C', then because (C', p') is a strict objection to S, some consumers in $C \cap C'$ strictly prefer S' to S so (C, p) does not object to S' regardless of how the consumers in $C \backslash C'$ are rearranged.

We have shown that statement (3) in the theorem implies statement (2), and statement (2) implies statement (1). We will show that statement (1) implies statement (4), and statement (4) implies statement (3). This last implication is the least straightforward part of the proof.

We first show that statement (1) implies statement (4). To see that any segmentation in the weakened core has an efficient canonical segmentation, consider a segmentation S' with an induced canonical segmentation S. Suppose that S is inefficient. We will construct a strict objection to S'. Because S is inefficient, so is S', so there is a segment (C', p') in S' with $p' > \underline{v}(C')$. Consider a coalition $\bar{C} \subseteq C'$ that consists of all the consumers in C' with values strictly lower than p', in addition to a positive measure of the highest value consumers in C' that is small enough that any optimal price for $\bar{C}$ is strictly lower than p'. Denote by $p < p'$ an optimal price for $\bar{C}$, so $(\bar{C}, p)$ is a segment. Observe that p' remains optimal for $C' \backslash \bar{C}$. Indeed, removing from C' consumers with values strictly lower than p', who do not purchase the product, does not change the revenue from p'; and removing from C' some consumers with the highest value in C' can only lower the optimal price, but p' is already the lowest value of consumers in C after removing the consumers with values lower than p', so p' remains optimal. We argue that $(\bar{C}, p)$ is a strict objection to S'. To see this, first notice that $(\bar{C}, p)$ is an objection to S': because $p < p'$, the consumers in $\bar{C}$ weakly prefer segment $(\bar{C}, p)$ to S', and the preference is strict for the consumers with the highest value in C' that are in $\bar{C}$. Further, notice that (C', p') is the only segment in S' that is broken apart by $(\bar{C}, p)$, and the consumers in $\bar{C} \cap C'$ who have the highest value in C' strictly prefer $(\bar{C}, p)$ to S', so $(\bar{C}, p)$ is a strict objection to S'.

To see that any segmentation in the weakened core has a saturated canonical segmentation, consider a segmentation S' with an induced canonical segmentation S. Suppose that S is not saturated. If S is inefficient, then the argument above implies that S' is not in the weakened core. Suppose that S is efficient, which together with non-saturation implies (by Lemma 1) that there are two segments $(C, \underline{v}(C))$ and $(C', \underline{v}(C'))$ in S with $\underline{v}(C) < \underline{v}(C')$ such that no $\hat{p}$ with $\underline{v}(C) < \hat{p} \leq \underline{v}(C')$ is optimal for C. In particular, if we add a positive-measure set $C'' \subset C'$ of consumers with value $\underline{v}(C')$ to C, price $\underline{v}(C)$ remains optimal provided that the measure of C'' is sufficiently small. Let C'' be such a set that contains a positive measure of consumers with value $\underline{v}(C')$ from *every* segment in S' in which the price is $\underline{v}(C')$ (recall that S' need not be canonical). Let $\bar{C} = C \cup C''$ and consider the segment $(\bar{C}, \underline{v}(C))$.

We argue that $(\bar{C}, \underline{v}(C))$ is a strict objection to S'. Segment $(\bar{C}, \underline{v}(C))$ objects to S' because the consumers in C'' strictly prefer it to the segmentation, and the consumers in $C = \bar{C} \backslash C''$ are indifferent between $(\bar{C}, \underline{v}(C))$ and S'. Now consider which coalitions in S' are broken apart by $\bar{C}$. Coalitions in segments with a price different from $\underline{v}(C)$ or $\underline{v}(C')$ do not intersect with $\bar{C}$. The coalition of a segment with price $\underline{v}(C)$ in S' is completely contained in $\bar{C}$, so these coalitions are not broken apart by $\bar{C}$. The only coalitions that *are* broken apart by $\bar{C}$ are those in which the price is $\underline{v}(C')$. By construction, from any such coalition there are some consumers in $C'' \subseteq \bar{C}$, who strictly prefer $(\bar{C}, \underline{v}(C))$ to S'. So $(\bar{C}, \underline{v}(C))$ is a strict objection to S'.

We now show that statement (4) implies statement (3). Consider a segmentation S' with an induced canonical segmentation S that is efficient and saturated. Let $\bar{S}$ be a segmentation that is not blocked by S'. We will show that $\bar{S}$ is equivalent to S'. Since the canonical representation of $\bar{S}$ is also not blocked by S' and is equivalent to S' if and only if $\bar{S}$ is equivalent to S', we suppose without loss of generality that $\bar{S}$ is canonical. Write the two canonical segmentations as $S = \{(C_1, v_1), \ldots, (C_n, v_n)\}$ and $\bar{S} = \{(\bar{C}_1, v_1), \ldots, (\bar{C}_n, v_n\}$, where for each i either C_i is empty or $\underline{v}(C_i) = v_i$ (because S is efficient), and each $\bar{C}_i$ may be empty. We will show by induction that $C_i = \bar{C}_i$ for all i, which will prove that $\bar{S}$ is equivalent to S'. (For the rest of this proof, $C_i = \bar{C}_i$ is in the "almost all" sense, that is, the measure of consumers in C_i but not in $\bar{C}_i$ is zero, and the measure of consumers in $\bar{C}_i$ but not in C_i is zero).

Suppose that $C_j = \bar{C}_j$ for all $j < i$ (the basis of the induction is $i = 1$). We show that $C_i = \bar{C}_i$. If $i = n$, then we are done because S and $\bar{S}$ partition the same set $[0, 1]$ of consumers. Suppose that $i < n$. Since $C_j = \bar{C}_j$ for all $j < i$, a consumer faces a price $p \geq v_i$ in S if and only if she faces a price $p' \geq v_i$ in $\bar{S}$ (up to a measure zero of consumers). In particular, consumers in C_i face a price of at least v_i in $\bar{S}$. Consumers in C_i with values higher than v_i must be in $\bar{C}_i$, otherwise these consumers face a price strictly higher than v_i in $\bar{S}$, so any segment in S' that contains some of these consumers objects to $\bar{S}$ (and thus S' blocks $\bar{S}$): the consumers in this segment are in C_i so face price v_i in S' (because S is the canonical representation of S'), and in $\bar{S}$ the consumers in this segment face prices no lower than v_i (by the claim earlier in the paragraph). So C_i and $\bar{C}_i$ are identical (up to a measure zero of consumers), except that $\bar{C}_i$ possibly does not contain some consumers of value v_i from C_i and may contain some consumers from coalitions $C_{i+1}, \ldots, C_n$, all of whom have values strictly higher than v_i (because S is efficient). But, as we now argue, if C_i and $\bar{C}_i$ are not identical, then the fact that S is saturated contradicts the fact that v_i is optimal for $\bar{C}_i$ (so $(\bar{C}_i, v_i)$ is not a segment). To see this, suppose first that $\bar{C}_i$ does not contain some consumers of value v_i from C_i. Since S is saturated and $i < n$, by Lemma 1 some price $p > v_i$ is optimal for C_i and p is lower than the value of all consumers in coalitions $C_{i+1}, \ldots, C_n$. Removing from C_i some consumers with value v_i reduces the revenue from price v_i but not the revenue from price p, which makes price v_i sub-optimal. Then, if needed, adding to C_i consumers from coalitions $C_{i+1}, \ldots, C_n$, all of whose values are at least p, to obtain $\bar{C}_i$ makes price v_i even worse (weakly) relative to price p, so v_i is not optimal for $\bar{C}_i$. Now suppose that $\bar{C}_i$ differs from C_i only because $\bar{C}_i$ contains some consumers from coalitions $C_{i+1}, \ldots, C_n$, all of whom have value strictly higher than v_i. Adding these consumers to C_i makes price v_i sub-optimal because S is saturated, so v_i is not optimal for $\bar{C}_i$. ■

The proof of Theorem 1 in fact shows that if a segmentation is stable, then it is efficient and saturated, regardless of whether it is canonical. But the other direction of the proof, that efficiency and saturation imply stability, relies on the segmentation being canonical. Example 5 in Appendix A describes a non-canonical segmentation that is efficient and saturated but not stable (and therefore, by Theorem 1, is also not in the weakened core and does not satisfy the intermediate notion of stability).

4.1 Interpretation and Implications of the Main Result

Let us examine Theorem 1 through the lens of "coalitional IR," which states that a segment cannot be rearranged if all its coalition members are weakly harmed by the rearrangement, and some are strictly harmed. Formally, given a prevailing segmentation S, we say that segmentation S' respects coalitional IR if no segment in S objects to S', that is, if S does not block S'.[18]

If a segment (C', p') strictly objects to the prevailing segmentation S, then *regardless* of how we rearrange the remaining consumers from the coalitions in S that are

[18] If S blocks S', the interpretation is that this prevents a deviation (rearrangement) to S'.

broken apart by C', coalitional IR will be respected.[19] A weaker way to demand that coalitional IR be respected would be to say that, given a prevailing segmentation S and an objection to it, there is *some way* to complete the objection into a segmentation S' so that S' respects coalitional IR. If we refer to objections that satisfy this as "intermediate objections," then it is immediate that a segmentation S blocks any segmentation that blocks S (which is what the intermediate solution concept in statement (2) of Theorem 1 says) if and only if it has no intermediate objections.[20] Clearly, any strict objection is an intermediate objection but the reverse does not hold.[21] Theorem 1 shows that a segmentation has an intermediate objection if and only if it has a strict objection.

The equivalence between stability and the weakened core implies that if a segmentation is not stable, then it has a strict objection. This supports the idea that stability is not an unreasonably strong demand.[22] Moreover, if a segmentation is not stable, there is a finite sequence of strict objections that takes us from the original segmentation to one that is stable.[23]

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[19]Each coalition that is broken apart includes some consumers who strictly benefit from leaving their segment in S to form (C', p'), so their segment in S does not object to the resulting segmentation. Segments whose coalitions do not intersect with C' are not rearranged.

[20]Appendix B interprets and formalizes intermediate objections as driving a contagion process among consumers.

[21]Consider a setting with two values, 1 and 2, with measure 1/2 each. Consider a segmentation in which all value 1 consumers are in one segment with price 1 and all value 2 consumers are in another segment with price 2. Consider an objection that consists of measure 1/3 of value 1 consumers and measure 1/4 of value 2 consumers with price 1. This is not a strict objection, because it breaks apart the segment of type 1 consumers, who are all indifferent. But if we add two segments to this objection, each containing the remaining consumers with a single value, then we obtain a segmentation that respects coalitional IR, that is, a segmentation to which S does not object.

[22]A prevailing segmentation S may fail to be stable because another non-equivalent segmentation S' respects coalitional IR, so S does not block S', even if no segment in S' objects to S. Theorem 1 clarifies that this cannot be the only reason S is not stable: in this case there is another segment that strictly objects to S.

[23]This process follows the construction in the proof of Theorem 1 and consists of two phases. In the first phase, we convert the segmentation to an efficient one, and in the second phase, to one whose canonical representation is also saturated. In the first phase, for each segment (C, p) such that $p > \underline{v}(C)$, we proceed exactly as in the proof of Theorem 1 and construct a strict objection that contains consumers in C with value lower than p and a small measure of consumers with the highest value in C. Because we do this once for each inefficient segment, the first phase ends after a finite number of steps. In the second phase, as long as there is a segment in the canonical representation of the segmentation that can accommodate some consumers form a segment with a higher price, take such a segment (C, p) with the lowest price, and let (C', p') be the segment with the next lowest price. Merge all the segments with price p in the segmentation (recall that the segmentation might not be canonical). Starting with $\epsilon = 0$, add a subset C'' of C' with proportional measures, $f^{C''}(v) = \epsilon f^{C'}(v)$ for all v, to C and increase ϵ continuously until either increasing ϵ increases the optimal price for $C \cup C''$ or C' becomes empty (in which case we keep adding consumers from the segment with the

Before turning to the issue of existence, we comment on the role that weak and strict improvements in the definition of an objecting segment play in the definition of the core. Weak improvements lead to many objecting segments, which excludes many segmentations from being in the core. But, unlike for stability, where it is natural for consumers to stay in their current segment if they do not strictly benefit from deviating, for the core it may be natural to require strict improvements for all consumers in an objecting segment, since forming this segment requires these consumers to actively deviate from their current segment. In our setting with a finite number of values, it is never optimal for the seller to set a price that gives all consumers in a segment positive surplus, so no objections with strict improvements for all its members exist. As we discuss in Appendix D, modifying the model to accommodate a continuum of values and requiring strict improvements does not solve the issue of the core being uninteresting; rather, it makes the core "too large" instead of "too small" (and often empty) as in our baseline model.

5 Existence and Consumer-optimal Stable Segmentations

Given Theorem 1, throughout the rest of the paper we refer to a segmentation that satisfies any of the four equivalent conditions in Theorem 1 as a stable segmentation.

We show that stable segmentations always exist by using the characterization from Theorem 1 to construct a stable segmentation. We start with an example that demonstrates the construction.

Example 2 (Maximal equal-revenue segmentation) *There are three values,* $1, 2, 3$. *Consumers from* 0 *to* $\frac{1}{3}$ *have value 1, those from* $\frac{1}{3}$ *to* $\frac{1}{2}$ *have value 2, and those from* $\frac{1}{2}$ *to 1 have value 3, with measures* $\frac{1}{3}, \frac{1}{6}, \frac{1}{2}$, *respectively, as shown in Figure 1.*

Consider the segmentation $S = \{(C_1, 1), (C_2, 2), (C_3, 3)\}$, where $C_1 = [0, \frac{4}{9}) \cup [\frac{7}{9}, 1]$, $C_2 = [\frac{4}{9}, \frac{11}{18})$, and $C_3 = [\frac{11}{18}, \frac{7}{9})$, shown in Figure 2. Coalition C_1 is the largest "equal-revenue" coalition that includes all values. That is, the measures $\frac{1}{3}, \frac{1}{9}, \frac{2}{9}$ of the three values in coalition C_1 are such that prices 1, 2, and 3 are all optimal, and C_1 is the

next lowest price). Because we are adding consumers proportionally, the optimal price for $C' \backslash C''$ does not change and efficiency is preserved. The second phase also ends in a finite number of steps because there are at most as many steps as there are values.

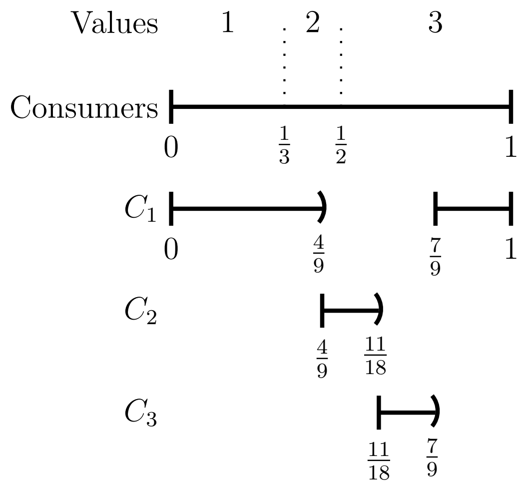

Figure 2: Example 2.

largest such coalition because it contains all the consumers with value 1.[24] Segment $(C_1, 1)$ is efficient. Consumers not in C_1 have value either 2 or 3, so adding them to C_1 makes price 1 no longer optimal.

Having put all the consumers with value 1 in segment C_1, we define the rest of the segmentation recursively to guarantee efficiency and saturation. The values of the remaining consumers are 2 and 3, and the measures of these consumers are $\frac{1}{18}$ and $\frac{5}{18}$, respectively. Coalition C_2, in which values 2 and 3 have measures $\frac{1}{18}$ and $\frac{2}{18}$, is the largest coalition for which prices 2 and 3 are both optimal. The segment $(C_2, 2)$ is efficient, and adding any of the remaining consumers, all of whom have value 3, increases the optimal price. The last segment is $(C_3, 3)$, which is efficient. Thus, segmentation S is efficient and saturated. Because it is also canonical, it is stable by Theorem 1.

We now formally define the maximal equal-revenue segmentation (MERS). Let $\bar{F}^C(v_i)$ be the cumulative measure of consumers with values v_i or higher in coalition C. If $v_i\bar{F}^C(v_i) = v_j\bar{F}^C(v_j)$, then prices v_i and v_j generate the same revenue for coalition C. Coalition C is an equal-revenue coalition if all consumer values in the coalition generate the same revenue, that is, $v_i\bar{F}^C(v_i)$ is the same for all v_i with $f^C(v_i) > 0$. The MERS is defined recursively. The first coalition, C_1, is the largest equal-revenue coalition that includes all the values. To construct C_1, let λ_1 be the eventual revenue in coalition C_1 from each of the values, that is, $\lambda_1 = v_i\bar{F}^{C_1}(v_i)$ for all v_i in V. Recalling

[24] Any two coalitions C and C' for which all three prices are optimal are proportional, that is, $f^C(v) = \alpha f^{C'}(v)$ for some $\alpha > 0$ and every value v, so the largest such coalition is well-defined.

that $f^C(v_i)$ is the measure of consumers with value v_i in coalition C, and $f(v_i)$ is the overall measure of consumers with value v_i, we have that $f(v_i) \geq f^{C_1}(v_i) = \bar{F}^{C_1}(v_i) - \bar{F}^{C_1}(v_{i+1}) = \lambda_1(\frac{1}{v_i} - \frac{1}{v_{i+1}})$ for all v_i, where $\frac{1}{v_{n+1}} \equiv 0$. Therefore, the highest value that λ_1 can take is such that $f^{C_1}(v_i) = f(v_i)$ for some i. That is, λ_1 is the smallest value such that $\lambda_1(\frac{1}{v_i} - \frac{1}{v_{i+1}}) = f(v_i)$ for some i. Denote the index of this value by i_1, so $\lambda_1(\frac{1}{v_{i_1}} - \frac{1}{v_{i_1+1}}) = f(v_{i_1})$. Then, more succinctly, we define C_1 by letting

$$\lambda_1 = \min_{v_i \in V} \frac{f(v_i)}{\frac{1}{v_i} - \frac{1}{v_{i+1}}} = \frac{f(v_{i_1})}{\frac{1}{v_{i_1}} - \frac{1}{v_{i_1+1}}}, \tag{1}$$

and letting $\bar{F}^{C_1}(v_i) = \lambda_1/v_i$ for all v_i.

Coalition C_1 contains all the consumers with value v_{i_1}, and adding a positive measure of consumers with other values to C_1 makes price v_{i_1} sub-optimal. Therefore coalition C_1 cannot be any larger and still be an equal-revenue coalition. The first segment in the MERS is (C_1, v_1), and the rest of the segmentation is defined recursively, where coalition C_j in the j'th segment is the largest equal-revenue coalition that includes all the values that remain after removing the consumers in $C_1, \ldots, C_{j-1}$, that is, $\{v_i : f^{C_j}(v_i) > 0\} = \{v_i : f^{[0,1] \setminus \cup_{j'<j} C_{j'}}(v_i) > 0\}$, and the price in the j'th segment is $\min\{v_i : f^{C_j}(v_i) > 0\}$. This process ends because in each step the number of remaining values decreases by at least 1.

The MERS is not necessarily canonical. For example, if the first equal-revenue coalition C_1 exhausts some value other than v_1, then the second coalition, C_2, will also include consumers with value v_1, so the MERS will contain two segments, (C_1, v_1) and (C_2, v_1), that have the same price. By Theorem 1, to establish that the MERS is stable, we need show that its induced canonical segmentation is efficient and saturated.

Proposition 3 *The maximal equal-revenue segmentation (MERS) is stable.*

Proof. The price in each segment of the MERS is equal to the lowest consumer value in the segment, so the segmentation is efficient and the same is true for its induced canonical segmentation. It remains to show that the induced canonical segmentation is saturated.

By construction of the MERS, for any two segments (C_i, v_i) and (C_j, v_j) with $i < j$, the set of consumer values in C_j is a subset of that in C_i. Since coalition C_j contains consumers with value v_j ($f^{C_j}(v_j) > 0$), so does coalition C_i. Because C_i is an equal-revenue segmentation, price v_j is optimal for coalition C_i. Consider the segment with price v_i in the induced canonical segmentation. By definition of the induced canonical segmentation, the coalition in this segment is the union of all the coalitions with price v_i in the MERS. Price v_j is optimal for each of these coalitions, as argued above, and is therefore optimal for the union of these coalitions (see footnote 12). Thus, the induced canonical segmentation is saturated by Lemma 1. ■

The MERS is not the unique stable segmentation. Here is an informal description of another construction of a stable segmentation. Put all consumers with value v_1 in the first coalition, and continually add consumers with the lowest remaining value to the first coalition until some price v_i other than v_1 also becomes optimal. This forms the first coalition, C_1. The first segment is (C, v_1). Repeat this process with the remaining consumers (the last segment may have only one optimal price). The resulting segmentation is canonical, efficient, and saturated. Saturation follows because given a segment (C, v_j) so constructed, a value $v_k > v_j$ becomes optimal for coalition C only when we have already added all the available consumers with values lower than v_k to C, so the value of any consumer in a segment with a higher price is at least v_k, and adding such consumers to C makes price v_j sub-optimal. This segmentation is also typically different from the MERS because the first segment does not generally include all values.[25]

5.1 Consumer-Optimal Stable Segmentation

Proposition 3 helps address the following question. Suppose that consumers choose a segmentation before they learn their value for the product (by, for example, coordinating their data disclosure decisions or interacting with the seller through a third party) but can deviate from the segmentation by forming new groups after they learn their value if the segmentation is not stable. Which segmentation will they choose?

Before they learn their value, all consumers rank segmentations by the average consumer surplus they generate. Thus, a segmentation that maximizes average consumer

[25]This alternative stable segmentation is related to the greedy algorithm of Ali, Lewis, and Vasserman (2023), adapted to the case of a finite number of values. However, whereas our construction starts from the lowest type, theirs starts from the highest type. As a result, their segmentation may not be stable. For example, suppose that there are three possible values, $1, 2, 4$, each with measure $\frac{1}{3}$. The greedy algorithm from Ali, Lewis, and Vasserman (2023) puts consumers with values 2 and 4 in one segment with price 2, and those with value 1 in another segment with price 1. This segmentation is not saturated because we can add consumers from the segment with price 2 to the one with price 1 without increasing the price in the latter segment. Our construction puts consumers with values 1 and 2 in one segment with price 1, and those with value 4 in another segment with price 4. This segmentation is stable.

surplus across all segmentations (stable or not) is preferred by all consumers. And if such a segmentation is stable, then consumers will not deviate from it after they learn their value. Bergemann, Brooks, and Morris (2015), who first introduced the MERS, showed that the MERS maximizes average consumer surplus across all segmentations. Proposition 3 shows that the MERS is stable, so consumers can achieve the maximal average surplus by choosing the MERS (or another equivalent stable segmentation) before they learn their value, even in the absence of commitment or a central planner that enforces the segmentation.

As an illustrative example, consider a genetically determined disease whose probability of impacting a given individual varies across individuals. Suppose that this probability can be verifiably determined by genetic testing, and suppose for simplicity that such testing is costless. Also for simplicity, suppose that the disutility in monetary terms of having the disease is the same across individuals. A monopolistic drug maker produces a drug at zero marginal cost that has no side effects and is guaranteed to prevent the disease if taken. Before learning their individual probability of having the disease, all individuals agree that grouping individuals by their probability in a way that implements the MERS achieves the highest ex-ante utility for them. Thus, they will agree to form "risk pools" that will be populated by the individuals based on their test results.[26] After the individuals learn their test result, any group of individuals is free to deviate and form its own risk pool. But since the MERS is stable, no such deviations will succeed, so the individuals can implement the MERS and achieve the maximal expected surplus across all segmentations.

Bergemann, Brooks, and Morris (2015) also showed that segmentations other than the MERS maximize average consumer surplus. This raises the question of the relationship between stability and maximizing average consumer surplus. The following two examples show that stability is neither necessary nor sufficient for maximizing average consumer surplus.

Example 3 (A segmentation that maximizes average consumer surplus and is not stable) *Consider again the setting from Example 2, with three values,* $1, 2, 3$*, where consumers from* 0 *to* $\frac{1}{3}$ *have value 1, those from* $\frac{1}{3}$ *to* $\frac{1}{2}$ *have value 2, and those from* $\frac{1}{2}$ *to 1 have value 3, with measures* $\frac{1}{3}, \frac{1}{6}, \frac{1}{2}$*, respectively. Consider segmentation* $S = \{(C_1, 1), (C_2, 2)\}$ *with coalitions* $C_1 = [0, \frac{1}{3}] \cup [\frac{5}{6}, 1]$ *and* $C_2 = (\frac{1}{3}, \frac{5}{6})$*.*

[26] A lottery, perhaps in the form of a priority list, can be used if several pools include individuals with the same test results.

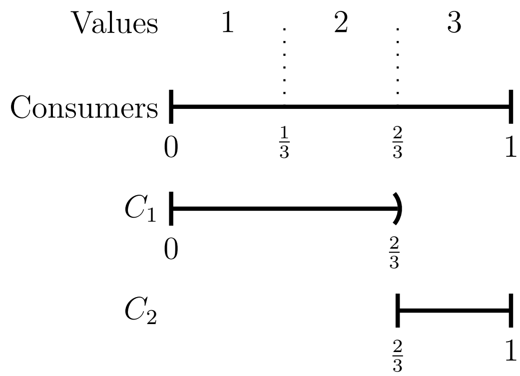

Figure 3: Example 4.

Coalition C_1 contains all the value 1 consumers and some value 3 consumers in a proportion that makes prices 1 and 3 optimal. Coalition C_2 contains the remaining consumers, whose proportions are such that prices 2 and 3 are optimal. Segmentation S maximizes average consumer surplus across all segmentations.[27]

But the segmentation is not stable. Since it is canonical and efficient, to show that it is not stable, we show that it is not saturated. Indeed, adding a small measure $\epsilon > 0$ of value 2 consumers to C_1 does not make price 1 sub-optimal: price 2 is not optimal for C_1, so if ϵ is small enough, price 2 remains sub-optimal, and the addition increases the revenue from price 1 but does not change the revenue from price 3.

Example 4 (A stable segmentation that does not maximize average consumer surplus) *There are three values,* $1, 2, 3$. *Consumers from* 0 *to* $\frac{1}{3}$ *have value 1, those from* $\frac{1}{3}$ *to* $\frac{2}{3}$ *have value 2, and those from* $\frac{2}{3}$ *to 1 have value 3, with measures* $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$, *respectively, as shown in Figure 3.*

Consider the segmentation $S = \{(C_1, 1), (C_2, 3)\}$ where $C_1 = [0, \frac{2}{3})$ and $C_2 = [\frac{2}{3}, 1]$ shown in Figure 3. Coalition C_1 consists of the consumers with values 1 and 2. Coalition C_3 consists of the consumers with value 3. This segmentation is canonical, efficient, and saturated. It is therefore stable.

The average consumer surplus of this segmentation is $\frac{1}{3}$, whereas the maximum average consumer surplus is $\frac{2}{3}$.[28] For some intuition, it is illuminating to study the marginal improvement in the average consumer surplus of S obtained by swapping the

[27] The segmentation is efficient so it maximizes total surplus. It also minimizes the seller's revenue across all segmentations. This is because price 3, which is optimal for the set of all consumers, is also optimal for each coalition.

[28] To find maximum average consumer surplus, we can consider the MERS. The MERS is

same measure of value 2 and 3 consumers. For this, consider a coalition C_1' obtained from C_1 by removing measure ϵ of value 2 consumers and adding a measure ϵ of value 3 consumers, and a coalition C_2' that contains the remaining consumers. If $\epsilon > 0$ is small enough, price 1 is optimal for C_1' and price 3 is optimal for C_3', so $S' = \{(C_1', 1), (C_3', 3)\}$ is a segmentation. To compare the average consumer surplus of S and S', it suffices to consider the swapped consumers. Each value 2 consumer loses 1 unit of surplus: their surplus is 1 in S and 0 in S'. Each value 3 consumer gains 2 units of surplus: their surplus is 0 in S and 2 in S'.[29]

Even though stable segmentations might not maximize average consumer surplus, they satisfy another desirable welfare property. We say that segmentation S' *Pareto dominates* segmentation S if $CS(c, S') \geq CS(c, S)$ for all consumers c in $[0, 1]$, with a strict inequality for a positive measure of consumers. A segmentation S is *Pareto undominated* if no segmentation Pareto dominates S.

Lemma 2 *Stable segmentations are Pareto undominated. Pareto undominated segmentations are efficient.*

Proof. If S' Pareto dominates S, then no segment in S objects to S', and S' is not equivalent to S. Therefore, S is not stable.

For the second statement, suppose that S is inefficient, so there is a segment (C, p) in S with $p > \underline{v}(C)$. Consider a coalition $\bar{C} \subseteq C$ that consists of the consumers in C with values strictly lower than p and a positive measure of the highest value consumers in C that is small enough that any optimal price for $\bar{C}$ is strictly lower than p. Denote by $p' < p$ an optimal price for $\bar{C}$, so $(\bar{C}, p')$ is a segment. Observe that p remains optimal for $C \backslash \bar{C}$. Indeed, removing from C consumers with values strictly lower than p, who do not purchase the product, does not change the revenue from p; and removing from C some consumers with the highest value in C can only lower the optimal price, but p is already the lowest value of consumers in C after removing the consumers with values lower than p, so p remains optimal. Now consider segmentation $\bar{S}$ obtained

$\{(C_1'', 1), (C_2'', 2)\}$, where $(f^{C_1''}(1), f^{C_1''}(2), f^{C_1''}(3)) = (\frac{1}{3}, \frac{1}{9}, \frac{2}{9})$ and $(f^{C_2''}(1), f^{C_2''}(2), f^{C_2''}(3)) = (0, \frac{2}{9}, \frac{1}{9})$. The surplus of value 2 and 3 consumers is 1 and 2 in the first segment, and the surplus of value 3 consumers is 1 in the second segment. The average consumer surplus is therefore $\frac{1}{9} \cdot 1 + \frac{2}{9} \cdot 2 + \frac{1}{9} \cdot 1 = \frac{2}{3}$.

[29] Notice that the change in the offered price is 2 for the value 2 consumers (from 1 to 3) and -2 for the value 3 consumers (from 3 to 1). But even though this change has the same absolute value, the surplus change for the value 3 consumers is higher than for the value 2 consumers because value 2 consumers do not buy the product at a price higher than 2 (so increasing the price they face from 2 to 3 does not change their surplus.)

from segmentation S by replacing segment (C, p) with the two segments $(C \backslash \bar{C}, p)$ and $(\bar{C}, p')$. The consumers in $\bar{C}$ with the highest value in C have a strictly higher surplus in $\bar{S}$ than in S, and all the other consumers in C have a weakly higher surplus in $\bar{S}$ than in S. Thus, $\bar{S}$ Pareto dominates S. ■

Any segmentation that maximizes average consumer surplus must be Pareto undominated. And because we have already seen examples of segmentations that maximize average consumer surplus but are not stable, such as Example 3, there are Pareto undominated segmentations that are not stable.

6 Conclusions

We study settings in which consumers form groups in their interaction with a monopolistic seller. These groups correspond to market segments, which together form a segmentation of the market. Because the seller offers an optimal price in each market segment, different consumers may rank the possible segmentations differently, so it is not clear which market segmentation would arise. We model the interaction between consumers that determines the segmentation as a cooperative game. If the market is efficient, then all consumers agree on the best segmentations and the core of the game consists of these segmentations. But whenever the market is inefficient, consumers disagree and the core is empty. To investigate how consumers resolve their differences, we introduce two new solution concepts, the weakened core and stability. The weakened core is a slight relaxation of the core, which rules out certain objections and can be motivated by a small cost of breaking up existing segments. A stable segmentation is one that, for each segmentation considered as a possible deviation, contains a segment of consumers that object to the deviation. This captures a kind of "coalitional individuals rationality (IR)." The weakened core and the set of stable segmentations coincide with the core whenever the core is not empty.

Our main result shows that a segmentation is in the weakened core if and only if it is stable, and characterizes these segmentations as those that are efficient and saturated, in that enlarging any segment by adding consumers who face higher prices necessarily increases the profit-maximizing price for the segment. We use this characterization to show that stable segmentations always exist by showing that a particular segmentation that maximizes average consumer surplus (the MERS), identified by Bergemann, Brooks, and Morris (2015), is stable. We also show that efficiency and maximizing

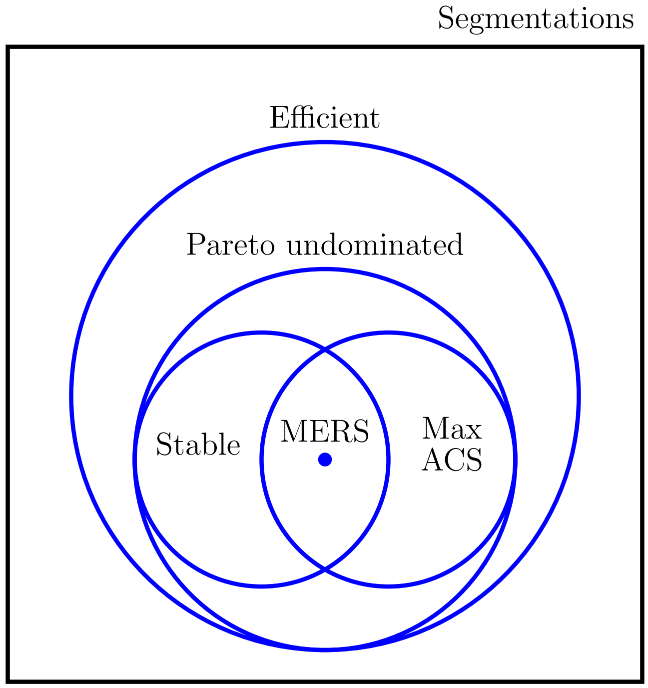

Figure 4: Summary of the relation between different notions.

consumer surplus are neither necessary nor sufficient for a segmentation to be stable. We show that stable segmentations are Pareto undominated, which are in turn efficient. The relationship between the various concepts is illustrated in Figure 4.

The main result also implies that stability and the weakened core are not overly demanding. If a segmentation is not stable, then the characterization implies that the segmentation is either not efficient or not saturated. In both cases it is easy to construct an alternative segmentation that is a relatively minor modification of the original one, is intuitively easy to obtain from the original segmentation, that contains a segment that objects to the original segmentation, and such that no segment in the original segmentation objects to the alternative segmentation.

Our notion of stability refines several solution concepts for NTU games (other than the core) applied to our setting. In Appendix C, we show that a stable segmentation together with its equivalent segmentations form a Morgenstern and Von Neumann (1953) stable set, a Harsanyi (1974) farsighted stable set, a Ray and Vohra (2015) farsighted stable set, and a Ray and Vohra (2019) maximal farsighted stable set.[30]

[30]Our notion of "coalitional IR" is closely related to the notion of "coalitional sovereignty" in Ray and Vohra (2015) applied to our setting.

We point out that stable sets and farsighted stable sets do not always exist in NTU games. And when they do exist they may necessarily include multiple, non-equivalent utility vectors or coalitions of players. Our results show that in our market game, "singleton" (up to equivalent segmentations) stable and farsighted stable sets always exist. In particular, for any deviation from a stable segmentation S to a non-equivalent segmentation S', there exists a path of "credible" and "maximal" (in the sense of Ray and Vohra, 2019) segmentations that leads back from S' to S. This provides another justification for our notion of stability.

Appendix D discusses a variant of our model with a continuum of values, and shows that our results extend to this setting. We also discuss the effect of requiring strict improvements for all members of an objecting coalition, and show that it does not make the core a more interesting solution concept because the core becomes too large and includes all efficient segmentations; strict improvements also make stability too demanding and less interesting by not allowing objections that include indifferent consumers who "go along" with members of their existing segment who strictly prefer the existing segment.

Our results indicate that when consumers interact with the seller in groups, the loss of efficiency associated with monopoly pricing may be overcome. While efficiency is also achieved with first-degree price discrimination, we show that when consumers can form groups, the resulting efficient segmentation is Pareto undominated and may increase consumer surplus up to the highest amount possible in the "surplus triangle" of Bergemann, Brooks, and Morris (2015). If consumers can choose a segmentation before they learn their value, then even if they can deviate by forming new groups after they learn their value, they can obtain the highest consumer surplus in the surplus triangle by choosing the MERS (or another suitable stable segmentation). Thus, monopolistic price discrimination when consumers can form groups can be viewed as a possible alternative or addition to standard anti-trust regulation. Consumer blocs, employee unions, online platforms, and data cooperatives, which serve as intermediaries that collect data from consumers and negotiate with companies on consumers' behalf, may be ways to achieve this.

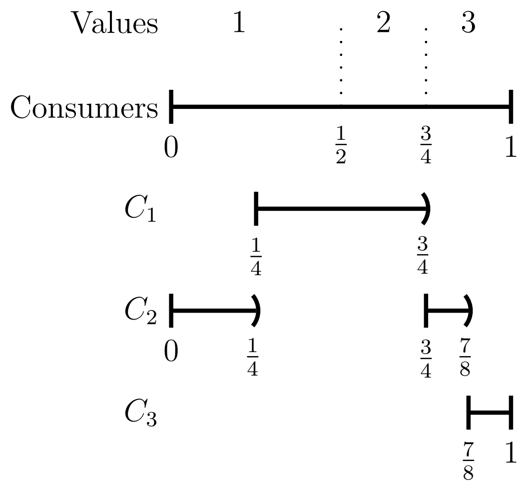

Figure 5: Example 5.

Appendix

A An Example from Section 4

The following example describes a non-canonical segmentation that is efficient and saturated but not stable. Thus, to verify the stability of a segmentation, efficiency and saturation must be checked for its induced canonical segmentation.

Example 5 *There are three values,* $1, 2, 3$*, with measures* $0.5, 0.25, 0.25$*, respectively, as shown in Figure 5.*

Consider the segmentation $S = \{(C_1, 1), (C_2, 1), (C_3, 3)\}$ with coalitions C_1, C_2, C_3, shown in Figure 5.

For coalition C_1, prices 1 and 2 are optimal. For coalition C_2, prices 1 and 3 are optimal. Adding consumers from segment $(C_3, 3)$ to either segment $(C_1, 1)$ or $(C_2, 1)$ necessarily increases the optimal price in the latter segments. Thus, the segmentation is saturated. The segmentation is also clearly efficient.

For coalition $C_1 \cup C_2 = [0, \frac{7}{8})$, however, price 1 is the unique optimal price, so the segmentation $\{(C_1 \cup C_2, 1), (C_3, 3)\}$ is efficient but not saturated: we can add some consumers from C_3, all of whom have value 3, to $C_1 \cup C_2$ without changing the optimal price. To see that S is not stable, note that segmentation $S' = \{([0, \frac{7}{8} + \epsilon), 1), ([\frac{7}{8} + \epsilon, 1], 3)\}$ for some small $\epsilon > 0$ Pareto dominates S, so S is not stable by Lemma 2.

B A Procedural Way to Describe Stability

The intermediate solution concept in statement (2) of Theorem 1 can be thought of in a procedural way that links the different segments of a blocking segmentation. To see this, consider a prevailing segmentation S and a proposed objecting segment (C', p'). Each consumer in C' weakly prefers (C', p') to S, and for some consumers the preference is strict. Consumers in C' for whom the preference is not strict can be thought of as consulting their coalition partners in S, saying "I was asked to join a deviating segment (C', p'); if I leave our current segment in S to do so, would you be able to form new segments, perhaps with consumers from other segments in S, so that not all of you are hurt (some weakly and some strictly)? I will only join the deviating segment if your answer is affirmative." If these coalition partners can form new segments so that not all of the coalition partners are hurt, the process continues; if these new segments include consumers from additional segments in S, then some consumers in those segments may consult their coalition partners in S, and so on. In this way, the formation of a blocking segmentation can be thought of as a "contagion" process that starts from the initial objecting segment (C', p'). If all consumers are able to successfully rearrange into new segments, then a blocking segmentation forms and S is not stable. But if S is stable, then for any initial objecting segment, this process will get "stuck:" at some point in the process, some indifferent consumers who are considering deviating from S belong to a segment that objects to *any* rearrangement of the remaining consumers.

We now formalize the procedural description of stability. Given a prevailing segmentation S and an objecting segment (C', p'), the following iterative procedure attempts to construct a segmentation that includes (C', p') and which S does not block.

Let $C_d^0 = C'$ (consumers already assigned to a deviating segment) and set $k = 0$.

(*) Let $C_l^k = [0,1] \backslash C_d^k$ (consumers not yet assigned to a deviating segment), $S_d^k = \{(C, p) \in S : C \cap C_d^k \neq \emptyset\}$ (original segments that include consumers already assigned to a deviating segment), $F^k = \cup_{(C,p) \in S_d^k} C$ (the consumers in those original segments), and $D^k = F^k \backslash C_d^k$ (the consumers in those original segments not yet assigned to a deviating segment).

If $D^k = \emptyset$, then stop and output the segmentation that consists of the deviating segments so far constructed, whose coalitions contain precisely the consumers in $C_d^k = F^k$, and the segments in $S \backslash S_d^k$.

If $D^k \neq \emptyset$, continue the construction by, for some set of consumers $E^k \subset C_l^k$, assigning all the consumers in $D^k \cup E^k$ to new deviating segments, in a way that no segment in S_d^k objects to the deviating segments.[31] If there is no way to do this for any $E^k \subset C_l^k$, then stop and output 'stuck.' Otherwise, set $C_d^{k+1} = C_d^k \cup D^k \cup E^k$, increase k by 1, and repeat from (*).

This procedure ends after finite number of iterations because for any k, if $D^{k+1} \neq \emptyset$ then $S_d^{k+1} \backslash S_d^k$ contains at least one segment from S, and S consists of a finite number of segments.

Claim 1 *A segmentation S is stable if and only if for any segment (C', p') that objects to S the procedure always outputs 'stuck.'*

Proof. Suppose that for some segment (C', p') that objects to S the procedure does not output 'stuck,' that is, it outputs a segmentation S'. By construction, S' contains (C', p'), so S' is not equivalent to S, and no segment in S objects to S'. Thus, S is not stable.

For the other direction, suppose that for any segment (C', p') that objects to S the procedure outputs 'stuck.' By Theorem 1, it is enough to show that S blocks every segmentation S' that blocks S. Suppose that this is not the case, that is, there exists a segmentation S' that blocks S but S does not block S'. Consider a segment (C', p') in S' that objects to S and run the procedure. We will show that for appropriate sets $E^0, E^1, \ldots$, the procedure will output a segmentation and not get stuck. Let $E^k = \cup_{\{(C'', p'') \in S' : C'' \cap F^k\}} C''$, where $F^k = \cup_{(C,p) \in S_d^k} C$ as defined in the procedure. By running the procedure with $E^0, E^1, \ldots$ and, if the procedure has not stopped by step k, constructing in step k the set of deviating segments $\{(C'', p'') \in S' : C'' \cap F^k\}$, we see that at each step the procedure adds at least one more segment from S' (starting with the objecting segment (C', p')). Since S does not object to S', the procedure will not get stuck. Indeed, when it stops, the procedure outputs a segmentation S'' that includes (C', p'), possibly additional segments from S', and possibly segments from S (if the procedure stops at step k with $D^k = \emptyset$ and $C_d^k \neq [0, 1]$). This completes the proof. ■

C Relation to Existing Cooperative Concepts

We here relate stability to several existing solution concepts for NTU games, including various stable set notions and the bargaining set. The cooperative solution concepts

[31] This assignment implies in particular that all the consumers in F^k have been assigned to deviating segments, so we can determine whether any segment in S_d^k objects to the deviating segments.

we discuss, which are typically defined for games with a finite number of players, require minor adjustments because we have a continuum of consumers. With these adjustments, we show that stability is a strict refinement of all these concepts. Stability therefore inherits the justifications for these concepts that often rely on farsighted behavior of agents, but has the additional (myopic) justification based on coalitional IR.

C.1 Stable Set

Our notion of stability is related to the notion of a *stable set* from Morgenstern and Von Neumann (1953). The stable set is defined for any cooperative game; we present its application to our game. Notice that whereas the stability notion of Morgenstern and Von Neumann (1953), stated below, is a property of a *set* of segmentations, our notion of stability is a property of a single segmentation.

Definition 7 (Stable set, Morgenstern and Von Neumann, 1953) *A* set *of segmentations $\mathcal{S}$ is a stable set if it satisfies the following two properties:*

1. *Internal Stability: For any $S \in \mathcal{S}$, no $S' \in \mathcal{S}$ blocks S.*
2. *External Stability: For any $S \notin \mathcal{S}$, some $S' \in \mathcal{S}$ blocks S.*

If a segmentation S is stable, then the set of all segmentations that are equivalent to S is a stable set. This is easy to see: internal stability is trivially satisfied because a segmentation does not block an equivalent segmentation, and external stability is satisfied by definition of stability. Because stable segmentations always exist, stable sets exist in our setting. This is noteworthy because stable sets do not exist for some cooperative games. Moreover, even when stable sets exist, they may necessarily contain multiple elements. In contrast, our characterization of stable sets in the Online Appendix (Section E.1) shows that any stable set in our setting contains an essentially unique element in the sense that it consists of all the segmentations that are equivalent to some segmentation S.

Perhaps surprisingly, our characterization of stable sets shows that the set of segmentations that are equivalent to a segmentation S may be a stable set even if S is not stable. This is because stability requires that a *single* segmentation block any other non-equivalent segmentation; for the set of segmentations that are equivalent to S to be a stable set, on the other hand, requires that any segmentation that is not equivalent to S be blocked by *some* segmentation that is equivalent to S. It may be that S does not block S' but a segmentation that is equivalent to S does. To see this, suppose that S is canonical but not stable, and consider another segmentation S' that is not blocked by S. Take a segment (C, p) in S. Since (C, p) does not object to S', C may contain some consumers who prefer S to S' and some consumers who prefer S' to S. If coalition $C' \subseteq C$ is such that (C', p) and $(C \backslash C', p)$ are segments and we replace (C, p) with (C', p) and $(C \backslash C', p)$, it could be that (C', p) objects to S', yielding a segmentation that is equivalent to S and blocks S'. We provide an example of this in the Online Appendix (Section E.1).

C.2 Farsighted Stable Set

Our stability notion is also related to two other notions motivated by farsighted stability: the Harsanyi stable set and the Ray and Vohra farsighted stable set (henceforth RV stable set).

Both notions define a stable set as one that satisfies internal and external stability, just like the stable set of Morgenstern and Von Neumann (1953). But the notion of blocking used to define internal and external stability is "farsighted." A segmentation blocks another segmentation if there is a sequence of segmentations that begins with the segmentation to be blocked and ends with the blocking segmentation such that each intermediate segmentation contains a coalition that prefers the blocking segmentation to the one that preceded the intermediate segmentation. These objecting coalitions allow the blocking segmentation to be "reached" starting from the original segmentation. The two notions differ in what is assumed about the segments along the sequence other than the objecting segments, with the RV stable set assuming a kind of "coalitional autonomy" similar to the "coalitional IR" that motivates our definition of stability.

Our notion of stability satisfies these two notions which, although differing in general, coincide in our setting. Moreover, although these are set notions, in our setting they are satisfied only by singleton sets. Importantly, however, these notions are not particularly useful in our setting because they are too permissive. More precisely, for each notion we have a weak and a strong version; any segmentation that does not eliminate all consumer surplus satisfies the weak versions, and any Pareto undominated segmentation satisfied the strong versions. We provide the details in the Online Appendix (Section E.2).

C.3 Bargaining Set

Our notion of stability is closely related to the bargaining set. Roughly speaking, a segmentation S is in a bargaining set if for any objection to S, there is another objection to S that would in a sense "cancel" the original objection. In contrast, stability requires that S contain an objection to any other non-equivalent segmentation. Section C.3 contains an example with two values in which all segmentations are in the bargaining set.

There are many ways to define a bargaining set depending on whether or not we require strict or weak preferences. We only state one possibility here.

Definition 8 *The bargaining set is a set of segmentations S with the property that for any objection (C, p) to S, there exist a segment (C', p') such that, $C \not\subseteq C'$, $C' \not\subseteq C$, $CS(c, p') \geq CS(c, p)$ for almost all $c \in C \cap C'$, and $CS(c, p') \geq CS(c, S)$ for almost all $c \in C' \backslash C$.*

For simplicity, suppose that there are only two values, v_1 and v_2. Suppose that v_1 is not optimal for the set of all consumers $[0, 1]$, that is, $v_1 < v_2 f(v_2)$. We show that all segmentations are in the bargaining set.

Consider any segmentation S and an objection (C, p) to S. We must have $p = v_1$ because otherwise the surplus of all consumers in (C, p) is zero. The coalition C contains a positive measure, but not all, of value v_2 consumers. Construct a coalition C' by removing an ϵ measure of value v_2 consumers from C and replace them with the same measure of value v_2 consumers that are not in C. Notice that (C', v_1) is a segment. The consumers in $C \cap C'$ have the same surplus in (C, v_1) and in (C', v_1). Consumers in $C' \backslash C$ have a weakly higher surplus in (C', v_1) than in S. Therefore, S is in the bargaining set.

D Continuum of Values

One modeling assumption we make is that the number of possible consumer values is finite. We consider a variant of our setting with a continuum of values and show that our main results extend: the core is empty unless the unsegmented market is efficient, and stability is characterized by efficiency and saturation.

In this setting with a continuum of values, we also study what happens if we strengthen the definition of an objection to require that almost all consumers in a segment strictly benefit from deviating to the segment. This reduces the number of deviations relative to the setting that allows for objections with indifferences, which makes the core a less demanding solution concept. With this stronger form of objections, the core becomes too permissive in the sense that it contains all finite efficient segmentations. We also study what stability looks like with this stronger form of objections. Because we require a given segmentation to veto any deviation in a stronger sense, we should expect stability to become more demanding. In fact, stability becomes too demanding in the sense that no stable segmentations exist. Our view is that indifferences are consistent with an intuitive notion of coalitional IR and stability: a consumer who is indifferent could reasonably agree to passively go along with the objecting votes of her current coalition members who are strictly harmed by a proposed deviation. In contrast, when defining the core, it may be reasonable to require strict improvements: to be willing to actively deviate from a segmentation, consumers in the coalition should strictly benefit.

In the variant of our setting we consider there is a unit mass of consumers whose values are distributed on an interval $[\underline{v}, \bar{v}]$, $0 < \underline{v} < \bar{v} < \infty$ according to a distribution F with a derivative that is bounded from above and away from 0. Each consumer $c \in [\underline{v}, \bar{v}]$ is identified by her unique value for the product. A coalition C consists of a finite union of sub-intervals of $[\underline{v}, \bar{v}]$. A segment (C, p) consists of a coalition C and a price p that is optimal (revenue-maximizing) when the values are drawn according to F conditional on being in C. The assumption that the derivative of F is bounded from above and away from 0 means that there exists some $\delta > 0$ such that for any consumer c, price $p = c$ is uniquely optimal for the set of consumers $[c, c + \epsilon]$ for any $\epsilon \leq \delta$.

We focus on finite segmentations. A (finite) segmentation is a finite set of segments $\{(C_j, p_j)\}_{j=1,\ldots,k}$ such that $C_1, \ldots, C_k$ partition the set of all consumers $[\underline{v}, \bar{v}]$.[32] Let $CS(c, p)$ denote the surplus of consumer c from being offered price p, and $CS(c, S)$ the surplus of this consumer in segmentation S.

We first study the core and stability according to the notion of objection used

[32]In this formulation each consumer has a unique value. An alternative formulation, inspired by the information design literature, would be as follows. An unsegmented market is a distribution F over values $[\underline{v}, \bar{v}]$. A segment is a pair (G, p), where p is an optimal price when values are distributed according to G. A segmentation is a finite set of segments $\{(G_j, p_j)\}_{j=1,\ldots,k}$ and a distribution over the segments given by probabilities $\alpha_1, \ldots, \alpha_k$ satisfying Bayes-plausibility, $F = \sum_j \alpha_j G_j$. In this formulation, it is possible for multiple segments to contain consumers of some value. The issue now is that it is ambiguous to talk about a segment (G, p) objecting to a segmentation S, because there is no way to keep track of *which* segments in the original segmentation the consumers in the segment are coming from. We leave an appropriate formalization of such a model for future work.

throughout the paper, requiring that all consumers in the segment weakly, and some of them strictly, prefer the segment to the segmentation. Recall that stability is defined using a notion of equivalence. Here we say that two segmentations are equivalent if, for any segment (C, p) in one segmentation, there exists a segment (C', p) in the other segmentations such that C and C' are almost identical, that is, both $C \backslash C'$ and $C' \backslash C$ have zero measure. Our characterization is unchanged: the core is empty unless the unsegmented market is efficient, and stability is characterized by efficiency and saturation.[33]

Proposition 4 *If the unsegmented market is efficient, that is, price $\underline{v}$ is optimal for the set $[\underline{v}, \bar{v}]$ of all consumers, then the core consists of the segmentation $\{([\underline{v}, \bar{v}], \underline{v})\}$. Otherwise, the core is empty. A segmentation is stable if and only if it is efficient and saturated.*

Proof. If the unsegmented market is efficient, then in the segmentation $\{([\underline{v}, \bar{v}], \underline{v})\}$ all consumers are offered the lowest price $\underline{v}$ and so there is no objection to the segmentation. Further, in any segmentation in the core, all consumers must be offered price $\underline{v}$, otherwise the segment that contains all consumers together with price $\underline{v}$ is an objection, so the segmentation $\{([\underline{v}, \bar{v}], \underline{v})\}$ is the unique segmentation in the core.

Suppose the unsegmented market is inefficient. Then, in any segmentation, there must exist a segment with price p strictly higher than $\underline{v}$. Take a small measure δ of consumers with value at least p from that segment, and add them to coalition $[\underline{v}, \underline{v} + \epsilon)$ to form a new coalition C. For small enough ϵ, price $\underline{v}$ is uniquely optimal for $[\underline{v}, \underline{v} + \epsilon)$, so if δ is small enough, price $\underline{v}$ is also optimal for coalition C so $(C, \underline{v})$ is an objection.

We now turn to stability. If a segmentation S is inefficient, then it contains some segment (C, p) in which the price p is higher than the lowest value. Replace (C, p) with two segments: a new segment (C', p') with consumers whose values are lower than p, and another segment containing the remaining consumers $(C \backslash C', p)$. We must have $p' < p$ because all consumers in C' have value at most p. Also C' must contain a positive measure of consumers with value strictly higher than p' (otherwise the revenue is zero). Call the resulting segmentation S'. Notice that (C, p) does not object to S' because some of the consumers, those in (C', p') with values above p', strictly prefer S' to S. So S is not stable.

[33] In this setting with a continuum of values, any segmentation is canonical because any two segments in any segmentation have different prices. As a result, any two segmentations are non-equivalent.

Suppose a segmentation S is efficient but not saturated. Saturation means there are two segments $(C, \underline{v}(C))$ and $(C', \underline{v}(C'))$ with prices $\underline{v}(C) < \underline{v}(C')$ in S such that we can add some consumers from C' to C without increasing the price in the first segment. Call the resulting segmentation S'. The segment $(C, \underline{v}(C))$ does not object to S' because the consumers in it are indifferent. The segment $(C', \underline{v}(C'))$ does not object to S' because some of its consumers, those added to the segment with the lower price, strictly prefer S' to S. So S is not stable.

Suppose that a segmentation $S = \{(C_1, p_1), \ldots, (C_k, p_k)\}$, $p_1 < \ldots < p_k$ is efficient and saturated. Suppose there is some segmentation S' that is not blocked by S. Let $C'_1, \ldots, C'_k$ be the coalitions in S' belonging to segments with prices $p_1, \ldots, p_k$, respectively (these coalitions might be empty). Because S is efficient, p_1 is equal to the lowest possible value $\underline{v}$. Therefore, if C_1 contains a positive measure of consumers that are not in C'_1, then (C_1, p_1) objects to S'. By saturation, C'_1 cannot be a superset of C_1, therefore we must have $C'_1 = C_1$ (in the almost all sense). Now again because S is efficient, p_2 is equal to the lowest value among consumers that are not in C_1. A similar argument implies that we must therefore have $C'_2 = C_2$. A recursive argument implies that S and S' must be equivalent. ■

We now consider a stronger notion of an objection in which almost all consumers are required to strictly prefer the segment.

Definition 9 (Strong objection) *A segment (C, p) strongly objects to a segmentation S if $CS(c, p) > CS(c, S)$ for almost all consumers in C.*

Definition 10 (Strong core) *The strong core is the set of segmentations S to which no segment strongly objects.*[34]

Definition 11 (Strong stability) *A segmentation is strongly stable if it contains a strong objection to any non-equivalent segmentation.*

We first demonstrate the main ideas with an example. Suppose values are uniformly distributed on $[1, 3]$. For any $\delta \leq 1$, price v is optimal for a coalition $[v, v + \delta)$.[35] So $S = \{([1, 1 + \delta), 1), ([1 + \delta, 1 + 2\delta), 1 + \delta), \ldots, ([1 + k\delta, 3], 1 + k\delta)\}$ is a segmentation and is efficient.

[34] This notion is in fact weaker than the corresponding notion with weak objections. We call it "strong" to clarify that it is defined based on strong objections.

[35] The revenue from price p is proportional to $p(v + \delta - p)$, and its derivative is $v + \delta - 2p \leq \delta - v$ because $p \leq v$, and $\delta - v \leq 0$ because $\delta \leq 1 \leq v$.

We argue that S is in the core for any $\delta \leq 1$. So suppose $\delta \leq 1$ and there exists some strong objection (C, p) to S. Because consumers in segment $([1, 1+\delta), 1)$ are offered the lowest possible price, they cannot be a part of any strong objection, so $[1, 1+\delta) \cap C$ has measure zero. This in turn implies that $p \geq 1+\delta$. But then consumers in segment $([1+\delta, 1+2\delta), 1+\delta)$ cannot be a part of a strong objection either. An inductive argument shows that a strong objection does not exist.

We show below that the strong core is the set of all efficient segmentations. Our interpretation is that the strong core is a weak solution concept because there are many efficient segmentations. Recall that because the derivative of F is bounded above and away from 0, there exists $\delta > 0$ such that for any consumer c, price c is revenue-maximizing for the set of consumers $[c, c+\epsilon]$ for any $\epsilon \leq \delta$. Now we construct a class of efficient segmentations. Starting from $c = \underline{v}$, choose an arbitrary $\epsilon \leq c + \delta$ and let $C = [c, c+\epsilon]$ and add segment (C, c) to the segmentation. Repeat until all consumers are in the segmentation.

Proposition 5 *A segmentation is in the strong core if and only if it is efficient.*

Proof. Suppose first that a segmentation S is inefficient. Therefore, there exists a segment (C, p) in S such that p is strictly higher than $\underline{v}(C) = \inf\{v(c)|c \in C\}$. Let p' be the optimal price for the set $C \cap \{v|v \leq p\}$ (which is a positive-measure set because $p > \underline{v}(C)$). Notice that p' is also optimal for the set $C' := C \cap \{v|p' \leq v \leq p\}$. The segment (C', p') is a strong objection to S because all these consumers get zero surplus in S but almost all of them get a positive surplus in (C', p').

Now suppose that a segmentation S is not in the core, so it has an objection (C, p). Because S is finite and the coalition in each segment in S consists of a finite union of intervals, there is an $\epsilon > 0$ such that all consumers in $(p, p+\epsilon)$ belong to a single segment in S, say (C', p'). Because price p is optimal for C, a positive measure of consumers in $(p, p+\epsilon)$ must be in C (otherwise we can increase the price without decreasing revenue). So because (C, p) is a strong objection to S, the consumers in $(p, p+\epsilon) \cap C$ must strictly prefer (C, p) to the segmentation, and it must be that $p < p'$. But this implies that p' is less than the value of a positive measure of consumers in C', so (C', p') is inefficient and therefore S is also inefficient.[36] ■

With strong objections, stable segmentations do not exist.

[36]Notice that this argument does not require the objecting coalition C to consist of a finite union of intervals.

Proposition 6 *There exists no strongly stable segmentation.*

Proof. Consider any segment (C, p) in any segmentation S. Let C' be a subset of C consisting of consumers with value at least $c' > p$ (with c' chosen so that C' has positive measure). Any optimal price p' for C' is strictly higher than p. Because we removed consumers with the highest values from C, any optimal price p'' for $C \backslash C'$ is at most p. Now replace (C, p) with (C', p') and $(C \backslash C', p'')$, and call the resulting segmentation S'. The two segmentations are not equivalent because $p' \neq p$. But (C, p) is not a strong objection to S' because the consumers in $C \backslash C'$ weakly prefer S' to S. ■

References

Acemoglu, Daron, Ali Makhdoumi, Azarakhsh Malekian, and Asuman Ozdaglar. 2019. "Too much data: Prices and inefficiencies in data markets." Tech. rep., National Bureau of Economic Research.

Ali, S Nageeb, Greg Lewis, and Shoshana Vasserman. 2023. "Voluntary disclosure and personalized pricing." *The Review of Economic Studies,* forthcoming .

Baumann, Leonie and Rohan Dutta. 2022. "Strategic Evidence Disclosure in Networks and Equilibrium Discrimination." *Available at SSRN 4305083* .

Bergemann, Dirk, Alessandro Bonatti, and Tan Gan. 2022. "The economics of social data." *The RAND Journal of Economics* .

Bergemann, Dirk, Benjamin Brooks, and Stephen Morris. 2015. "The limits of price discrimination." *American Economic Review* 105 (3):921–57.

Braghieri, Luca. 2017. "Targeted Advertising and Price Discrimination in Intermediated Online Markets." *Working paper* .

Cummings, Rachel, Nikhil R Devanur, Zhiyi Huang, and Xiangning Wang. 2020. "Algorithmic price discrimination." In *Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms*. SIAM, 2432–2451.

Galperti, Simone and Jacopo Perego. 2023. "Competitive Markets for Personal Data." *Working paper* .

Glode, Vincent, Christian C Opp, and Xingtan Zhang. 2018. "Voluntary disclosure in bilateral transactions." *Journal of Economic Theory* 175:652–688.

Haghpanah, Nima and Ron Siegel. 2022. "The Limits of Multiproduct Price Discrimination." *American Economic Review: Insights* 4 (4):443–58.

———. 2023. "Pareto improving segmentation of multi-product markets." *Journal of Political Economy,* forthcoming .

Harsanyi, John C. 1974. "An equilibrium-point interpretation of stable sets and a proposed alternative definition." *Management science* 20 (11):1472–1495.

Hidir, Sinem and Nikhil Vellodi. 2021. "Privacy, personalization, and price discrimination." *Journal of the European Economic Association* 19 (2):1342–1363.

Ichihashi, Shota. 2020. "Online Privacy and Information Disclosure by Consumers." *American Economic Review* 110 (2):569–95.

Kuang, Zhonghong, Sanxi Li, Yi Liu, and Yang Yu. 2022. "Stable Market Segmentation against Price Discrimination." *Working Paper* .

Lanier, Jaron and E Glen Weyl. 2018. "A blueprint for a better digital society." *Harvard Business Review* 26:2–18.

Morgenstern, Oskar and John Von Neumann. 1953. *Theory of games and economic behavior*. Princeton university press.

Peivandi, Ahmad and Rakesh V Vohra. 2021. "Instability of Centralized Markets." *Econometrica* 89 (1):163–179.

Ray, Debraj and Rajiv Vohra. 2015. "The farsighted stable set." *Econometrica* 83 (3):977–1011.

———. 2019. "Maximality in the farsighted stable set." *Econometrica* 87 (5):1763–1779.

Sher, Itai and Rakesh Vohra. 2015. "Price discrimination through communication." *Theoretical Economics* 10 (2):597–648.

Yang, Kai Hao. 2022. "Selling consumer data for profit: Optimal market-segmentation design and its consequences." *American Economic Review* 112 (4):1364–1393.

Online Appendix

E Additional details for Appendix C

E.1 Proofs for Section C.1

We first provide an example where a segmentation S does not block S' but a segmentation that is equivalent to S does.

Example 6 *There are three values,* $1, 2, 3$*, with measures* $\frac{6}{21}, \frac{4}{21}, \frac{11}{21}$*, respectively, as shown in Figure 6, and a segmentation* $S = \{(C_1, 1), (C_2, 2)\}$ *with* $C_1 = [0, \frac{6}{21}) \cup [\frac{18}{21}, 1]$ *and* $C_2 = [\frac{6}{21}, \frac{18}{21})$*.*

Segmentation S is not stable because it is not saturated. This is because we can add some consumers with value 2 from C_2 to C_1 without increasing the optimal price $p = 1$ in the first segment. It is also easy to see directly that S is not stable. For example, segmentation $S' = \{(C'_1, 1), (C'_2, 3)\}$ with coalitions $C'_1 = [0, \frac{7}{21}) \cup [\frac{18}{21}, 1]$ and $C'_2 = [\frac{7}{21}, \frac{18}{21})$ shown in Figure 6 is not blocked by S. Segment $(C_1, 1)$ in S does not object to S' because all consumers in C_1 are indifferent between the two segmentations. Segment $(C_2, 2)$ does not object to S' because the value 2 consumers who join the first segment in S' strictly prefer S' to S. However, segmentation S' is blocked by segmentation $S'' = \{(C''_1, 1), (C''_2, 2), (C'''_2, 2)\}$ with coalitions $C''_1 = [0, \frac{6}{21}) \cup [\frac{18}{21}, 1]$, $C''_2 = [\frac{6}{21}, \frac{7}{21}) \cup [\frac{16}{21}, \frac{18}{21})$, and $C'''_2 = [\frac{7}{21}, \frac{16}{21})$, which is equivalent to S. In particular, segment $(C'''_2, 2)$ objects to S' because the consumers in C''' face price 2 in S'' and price 3 in S'.

The proposition below characterizes stable sets and shows that in this example the set of all segmentations that are equivalent to S is in fact a stable set. To state the proposition, we first define two weak notions of objection and blocking.

Definition 12 (Weak Objections) *A segment (C, p) weakly objects to a segmentation S if $CS(c, p) > CS(c, S)$ for a positive measure of consumers c in C, and, for every price p' that is optimal for C, $CS(c, p) \geq CS(c, S)$ for a positive measure of consumers c in C whose value is p'.*

Any objection is also a weak objection. To see this, observe that both objections and weak objections require that some consumers strictly prefer the segment to the segmentation. But objections also require that all consumers in the segment weakly

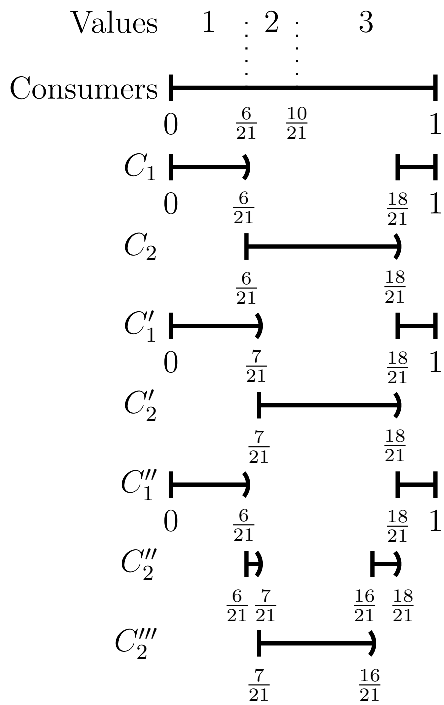

Figure 6: Example 6

prefer the segment. Weak objections do not require this for consumers whose value is not an optimal price for the segment. And for values that are optimal prices for the segment, only some consumers with such values are required to prefer the segment. We now define the corresponding notion of weak blocking.

Definition 13 (Weak Blocking) *A segmentation S weakly blocks a segmentation S' if there exists a segment (C, p) in S that weakly objects to S'.*

Segmentation S in Example 6 weakly blocks (but does not block) segmentation S' because segment $(C_2, 2)$ weakly objects to S': consumers with value 3 in C_2 strictly prefer the segment to S', consumers with value 2 in $C_2 \cap C'_2$ weakly prefer the segment to S', and 2 is an optimal price for C_2.[37]

The following proposition characterizes the stable sets.

Proposition 7 *A set of segmentations $\mathcal{S}$ is a stable set if and only if it comprises all the segmentations that are equivalent to some canonical segmentation S that weakly blocks any segmentation S' that is not equivalent to S.*

[37] $(C_2, 2)$ does not object to S' because consumers in $C_2 \backslash C'_2$ face a price of 2 in $(C_2, 2)$ and a price of 1 in S'.

Proof. To see the necessity of these conditions, consider a stable set $\mathcal{S}$ of segmentations. We first show that any segmentation in $\mathcal{S}$ is Pareto undominated. Suppose for contradiction that a segmentation S in $\mathcal{S}$ is Pareto dominated by another segmentation S'. If S' is in $\mathcal{S}$, then internal stability is violated because S' blocks S. If S' is not in $\mathcal{S}$, then, by external stability, there is a segmentation S'' in $\mathcal{S}$ that blocks S'. But then S'' also blocks S, which violates internal stability. Pareto undominance implies that any segmentation in $\mathcal{S}$ is efficient.

We now show that any two segmentations in $\mathcal{S}$ are equivalent. Suppose for contradiction that segmentations S_1 and S_2 in $\mathcal{S}$ are not equivalent. Their induced canonical segmentations S_1' and S_2' are also not equivalent, so there is a price p and segments (C_1, p) in S_1' and (C_2, p) in S_2' with $C_1 \neq C_2$ (in the "almost all" sense), where C_1 or C_2 may be empty. Suppose without loss of generality that p is the lowest such price, so any consumer in C_1 is either in C_2 or in a segment of S_2' with a higher price, and similarly any consumer in C_2 is either in C_1 or in a segment of S_1' with a higher price (up to a set of consumers of measure 0). Because $C_1 \neq C_2$, either $C_1 \backslash C_2$ or $C_2 \backslash C_1$ has positive measure. Suppose without loss of generality that $C_1 \backslash C_2$ has positive measure.

First observe that $C_1 \backslash C_2$ cannot contain a positive measure of consumers with value p. Indeed, such consumers would be in segments of S_2' with prices strictly higher than p, so S_2' would not be efficient, contradicting the efficiency of S_2. Therefore, $C_1 \backslash C_2$ contains a positive measure of consumers with values higher than p.

Consider any segment (C'', p) in S_1 that contains some such consumers, that is, $C'' \cap (C_1 \backslash C_2)$ has positive measure. Because $C'' \subseteq C_1$, the consumers in C'' face prices no lower than p in S_2, and the consumers in $C'' \cap (C_1 \backslash C_2)$ face prices strictly higher than p in S_2. So S_1 blocks S_2, which contradicts internal stability.

We have established that $\mathcal{S}$ may only contain segmentations that are equivalent to a Pareto undominated segmentation S. If some S' that is equivalent to S is not in $\mathcal{S}$, then no segmentation in $\mathcal{S}$ blocks S' so external stability is violated. So $\mathcal{S}$ must contain *all* segmentations that are equivalent to a Pareto undominated segmentation S, which we can assume to be canonical without loss of generality. To complete the necessity direction, it remains to show that the canonical segmentation S weakly blocks any non-equivalent segmentation.

Suppose for contradiction that S does not weakly block some non-equivalent segmentation S'. Because $\mathcal{S}$ is a stable set and contains all segmentations that are equivalent to S, there is a segmentation S'' that blocks S' and is equivalent to S. Consider a segment (C'', p) in S'' that objects to S', and the unique segment (C, p) in S in which the price is p. Because (C'', p) objects to S' and $C'' \subseteq C$, there is a positive measure of consumers in (C, p) that strictly prefer S to S'. Because S does not weakly block S', there exists some optimal price v for C such that all consumers with value v in C strictly prefer S' to (C, p). We claim that v is also optimal for any segment in S'' with price p, and therefore for C''. To see this, consider all the segments $(C''_1, p), \ldots, (C''_k, p)$ in S'' with price is p, so C is the union of all these coalitions, one of which is C''. Because p is optimal for C''_j, $j = 1, \ldots, k$, we have $vF^{C''_j}(v) \leq pF^{C''_j}(p)$. If price v is not optimal for some C''_j, then $vF^{C''_j}(v) < pF^{C''_j}(p)$. In this case, summing up over all j, we have $vF^C(v) < pF^C(p)$, which contradicts the optimality of price p for coalition C. So v must be optimal for C''. Therefore, v is optimal for C'', which means C'' contains a positive measure of consumers with value v. And because all consumers with value v in C strictly prefer S' to S, and S'' is equivalent to S', all these consumers strictly prefer S' to (C'', p) so (C'', p) cannot object to S', a contradiction.

To establish sufficiency, consider any canonical segmentation $S = \{(C_1, v_1), \ldots, (C_n, v_n)\}$ that weakly blocks any non-equivalent segmentation. The set of segmentations that are equivalent to S satisfies internal stability because no segmentation blocks an equivalent segmentation. For external stability, we show that for any segmentation S' that is not equivalent to S, there is a segmentation S'' that is equivalent to S and blocks S'.

Consider the segment (C, p) in S that weakly objects to S'. We will construct a coalition $C'' \subseteq C$ and show that the segmentation S'' that is the same as S except that (C, p) is replaced with $(C \backslash C'', p)$ and (C'', p), and is therefore equivalent to S, objects to S'. The construction of C'' has two steps. First, let C''_1 be a small coalition that comprises consumers with all the values that are optimal prices for C in proportions that make these values optimal prices for C''_1. That is, $\epsilon = vF^{C''_1}(v)$ for some small ϵ and all v that are optimal prices for C. For the second step, let v' be such that a positive measure of consumers in C with value v' strictly prefer (C, p) to S'. For some $\delta > 0$, add to C''_1 a measure δ of consumers in C with value v' that strictly prefer (C, p) to S', and remove from C''_1 the same measure δ of consumers with the highest value in C''_1 that is at most v' (a positive measure of these consumers exists because some consumers in C''_1 have value p and $p < v'$, otherwise consumers with value v' have zero surplus in S so do not strictly prefer (C, p) to S'). The resulting coalition is C'', which, if δ is small relative to ϵ, satisfies that $\epsilon = vF^{C''}(v)$ for all prices v that are optimal for C. So if δ is small relative to ϵ, then C'' has the same set of optimal prices as C''_1, and

(C'', p) is a segment. Similarly, if ϵ and δ are small enough, then $C \backslash C''$ has the same set of optimal prices as C, so $(C \backslash C'', p)$ is a segment. By construction, (C'', p) objects to S', so S'', which is equivalent to S, blocks S'. ■

The canonical segmentation S in Example 6 weakly blocks any non-equivalent segmentation, so the set of segmentations that are equivalent to S is a stable set. To see that S weakly blocks any non-equivalent segmentation, consider some segmentation S' that is not weakly blocked by S. Suppose without loss of generality that S' is canonical, so $S' = \{(C'_1, 1), (C'_2, 2), (C'_3, 3)\}$. Because all consumers with value 1 have zero surplus, if some consumers with value 3 in C_1 strictly preferred S to S', then $(C_1, 1)$ would weakly object to S'. Thus, the consumers with value 3 in C_1 are in C'_1. It is impossible for all value 2 consumers in C_2 to be in C'_1, because then the revenue from price 2, $2 \cdot (\frac{4}{21} + \frac{11}{21})$, would be strictly higher than the revenue from price 1, which is at most $\frac{6}{21} + \frac{4}{21} + \frac{11}{21}$. Thus, some consumers with value 2 in C_2 weakly prefer S to S'. This implies that the consumers with value 3 in C_2 must be in $C'_1 \cup C'_2$, otherwise some such consumers, those in C'_3, would strictly prefer S to S' and then $(C_2, 2)$ would weakly object to S'. Therefore C'_3 is empty (C'_3 cannot only contain consumers with value other than 3 because then price 3 cannot be optimal). Let $\delta_1 \geq 0$ be the measure of value 1 consumers in C'_2, and let $\delta_2, \delta_3 \geq 0$ be the measures of value 2 and value 3 consumers from C_2 that are in C'_1. For price 1 to be optimal for C'_1, the revenue from this price, $\frac{6}{21} - \delta_1 + \delta_2 + \frac{3}{21} + \delta_3$, must be no lower than the revenue from price 3, $3 \cdot (\frac{3}{21} + \delta_3)$, which means that $-\delta_1 + \delta_2 + \delta_3 \geq 3\delta_3$. Similarly, for price 2 to be optimal for C'_2 we must have $2 \cdot (\frac{4}{21} - \delta_2 + \frac{8}{21} - \delta_3) \geq 3 \cdot (\frac{8}{21} - \delta_3)$, which means that $3\delta_3 \geq 2(\delta_2 + \delta_3)$. These two inequalities hold if and only if $\delta_1 = \delta_2 = \delta_3 = 0$. We therefore have that $C_1 = C'_1$ and $C_2 = C'_2$, so S' is equivalent to S.

E.2 Proofs for Section C.2

To apply the Harsanyi and RV stable sets to our setting, we need to address two technical issues. First, these notions are defined for a finite number of players. Second, they involve a definition of objection that requires a strict improvement for all members of the objecting coalition. In our setting, consumers with the lowest value in a coalition have zero surplus, so these notions become trivial (every segmentation satisfies them) if we require a strict improvement for every consumer. We define modified versions of these notions below, allowing for a continuum of players and weak improvements. Because farsighted stability considers sequences of deviations, there are two ways to allow for weak improvements. We therefore define two versions of each solution concept.

Definition 14 *A segmentation S Harsanyi blocks a segmentation S' if there is a sequence $S^0 = S', S^1, \ldots, S^n = S$ of segmentations and a sequence $(C^1, p^1), \ldots, (C^n, p^n)$ of segments such that for $i = 1 \ldots n$, $(C^i, p^i) \in S^i$ and $CS(c, S^{i-1}) \leq CS(c, S)$ for all consumers $c \in C^i$, with a strict inequality for a positive measure of consumers $c \in C^i$ for some i. If, in addition, $CS(c, S^{i-1}) < CS(c, S)$ for a positive measure of consumers $c \in C^i$ for* all *$i = 1 \ldots n$, we say that S strongly Harsanyi blocks S'.*

Definition 15 *A* set *of segmentations $\mathcal{S}$ is a (strong) Harsanyi stable set if it satisfies the following two properties:*

1. *Internal Stability: For all $S \in \mathcal{S}$, there exists no $S' \in \mathcal{S}$ that (strong) Harsanyi blocks S.*
2. *External Stability: For all $S \notin \mathcal{S}$, there exists $S' \in \mathcal{S}$ that (strong) Harsanyi blocks S.*

Definition 16 *A segmentation S RV blocks a segmentation S' if there is a sequence $S^0 = S', S^1, \ldots, S^n = S$ of segmentations and a sequence $(C^1, p^1), \ldots, (C^n, p^n)$ of segments such that for $i = 1 \ldots n$, $(C^i, p^i) \in S^i$ and $(C, p) \in S^i$ whenever $(C, p) \in S^{i-1}$ and $C \cap C^i = \emptyset$, and $CS(c, S^{i-1}) \leq CS(c, S)$ for all consumers $c \in C^i$, with a strict inequality for a positive measure of consumers $c \in C^i$ for some i. If, in addition, $CS(c, S^{i-1}) < CS(c, S)$ for a positive measure of consumers $c \in C^i$ for* all *$i = 1 \ldots n$, we say that S strongly RV blocks S'.*

Definition 17 *A* set *of segmentations $\mathcal{S}$ is a (strong) RV stable set if it satisfies the following two properties:*

1. *Internal Stability: For all $S \in \mathcal{S}$, there exists no $S' \in \mathcal{S}$ that (strong) RV blocks S.*
2. *External Stability: For all $S \notin \mathcal{S}$, there exists $S' \in \mathcal{S}$ that (strong) RV blocks S.*

For the following characterization of Harsanyi and RV stable sets we denote by $ACS(S)$ the average consumer surplus in segmentation S.

Proposition 8 *The following are equivalent for any set of segmentations $\mathcal{S}$:*

- *$\mathcal{S}$ is a Harsanyi stable set*

- *$\mathcal{S}$ is a RV stable set*

- *$\mathcal{S} = \{S\}$ for some S with $ACS(S) > 0$.*

The proof of Proposition 8 uses the following lemma.

Lemma 3 *For any two segmentations S and S', the following are equivalent:*

- *S Harsanyi blocks S'.*

- *S RV blocks S'.*

- *$ACS(S) > 0$.*

Proof. If $ACS(S) = 0$, then $CS(c, S) = 0$ for all consumers. Therefore, S cannot Harsanyi block or RV block any segmentation.

Suppose that $ACS(S) > 0$. We show that S RV blocks any segmentation S', which also implies that S Harsanyi blocks S'. We do so by constructing a sequence of segmentations in several steps that gradually transform S' to an elementary segmentation in which each segment includes consumers with a single value. We then proceed from the elementary segmentation to S.

In step 0 we set $S^0 = S'$. In each following step $i > 0$, we take the segmentation S^{i-1} and a segment (C, p) in S^{i-1} that contains consumers of at least two types. For each value v_j, we let $C^i_{v_j}$ be the set of all consumers with value v_j in C. S^i is constructed from S^{i-1} by replacing (C, p) with the segments $(C^i_{v_j}, v_j)$ for all j such that $f^C(v_j) > 0$. Let $C^i = C^i_p$. The first phase ends with a segmentation in which every segment contains consumers of only a single type, so the surplus of all consumers is zero. The second phase has one step per segment in S. In particular, for each (C, p) in S, given S^{i-1}, we remove the consumers in C from the segments in S^{i-1} with single values and the new segment (C, p) to construct S^i with $C^i = C$. The second phase ends with $S^n = S$.

To see that S RV blocks S', notice that in each step $i = 1, \ldots, n$, consumers in C^i have zero surplus in S^{i-1}. Therefore, they weakly prefer S to S^{i-1}. Additionally, because $ACS(S) > 0$, there is a segment (C, p) in S in which a positive measure of consumers obtain positive surplus. As a result, a positive measure of consumers strictly prefer $S = S^n$ to S^{n-1}. ■

Proof of Proposition 8. Suppose that $\mathcal{S} = \{S\}$ for some S with $ACS(S) > 0$. Then, by Lemma 3, S RV blocks and Harsanyi blocks any $S' \neq S$, so $\mathcal{S}$ is a RV stable set and a Harsanyi stable set.

Consider any Harsanyi (respectively RV) stable set $\mathcal{S}$. The set $\mathcal{S}$ must contain at least one segmentation S with $ACS(S) > 0$, otherwise a segmentation $S' \notin \mathcal{S}$ is not Harsanyi (RV) blocked by any segmentation in $\mathcal{S}$ by Lemma 3. If the set contains more than one segmentation, then, by Lemma 3, the segmentation S that satisfies $ACS(S) > 0$ Harsanyi (RV) blocks the other segmentations in the set. Therefore, $\mathcal{S}$ contains a single segmentation S, and $ACS(S) > 0$. ■

Proposition 9 *The following are equivalent for any set of segmentations $\mathcal{S}$:*

- *$\mathcal{S}$ is a strong Harsanyi stable set*
- *$\mathcal{S}$ is a strong RV stable set*
- *$\mathcal{S}$ is the set of all segmentations that are equivalent to some segmentation S that is Pareto undominated.*

The proof uses the following lemma. The lemma uses a weak notion of equivalence of segmentations. Namely, we say that two segmentations S and S' are surplus-equivalent if almost all consumers have the same surplus in the two segmentations, that is, for almost all $c \in [0, 1]$, $CS(c, S) = CS(c, S')$. Any two equivalent segmentations are surplus-equivalent.

Lemma 4 *For any two segmentations S and S', the following are equivalent:*

- *Some surplus-equivalent segmentation to S strong Harsanyi blocks S'.*
- *Some surplus-equivalent segmentation to S strong RV blocks S'.*
- *There exist a positive measure of consumers c such that $CS(c, S) > CS(c, S')$.*

Proof. If some segmentation S'' that is surplus-equivalent to S strong Harsanyi (RV) blocks S', then, by definition, a positive measure of consumers strictly prefer S'', and therefore S, to S'.

Suppose that a positive measure of consumers strictly prefer S to S'. We show that some segmentation S'' that is surplus-equivalent to S strong RV blocks segmentation S', which also implies that S'' strong Harsanyi blocks S'.

We do so by constructing a sequence of segmentations in two phases. The first phase consists of two steps. In the first step, consider some segment (C, p) in S that contains a positive measure of consumers that strictly prefer S to S'. Let coalition C^1 contain a positive measure of consumers with value p from C, a positive measure of (but not all the) consumers from C that strictly prefer S to S', and, for *every* segment in S', a positive measure of consumers with the lowest value in that segment, where the proportions of consumers in C^1 are such that (C^1, p) is a segment. Consider a segmentation S^1 that consists of (C^1, p) and, for each consumer value, a segment that contains only the consumers in $[0, 1]\backslash C^1$ with that value, so their surplus is zero. In the second step, replace (C^1, p) with (C, p) and, for each consumer value, put the consumers in $C^1\backslash C$ with that value in a separate segment (all other segments remain intact). Denote the resulting segmentation by S^0.

The second phase consists of (potentially) several steps. In each step $i > 0$, take segmentation S^{i-1} and, for some segment (C', p') in S that is not already in S^{i-1} and contains a positive measure of consumers with positive surplus, let $C^i = C'$. S^i is constructed from S^{i-1} by taking all segments (C'', p'') that contain a positive measure of consumers from C' (so C'' contains only consumers with value p'') and replacing them with $(C''\backslash C', p'')$, and finally adding segment (C', p') to S^i. This process ends with a final segmentation S^n that may differ from S but is surplus-equivalent to it because for any segment in S that is not in S^n, all consumers in that segment obtain zero surplus in both segmentations. So, for the remainder of the proof, suppose without loss of generality that $S = S^n$.

To see that S RV blocks S', notice that in the first step of the first phase, coalition C^1 contains some consumers that strictly prefer S to S', and all other consumers in C^1 weakly prefer S to S' because they have surplus zero in S'. In the second step of the first phase, by definition, some consumers in C strictly prefer S to S^1 and all the consumers in $C\backslash C^1$ because they have zero surplus in S^1. Similarly, in each step i of the second phase, consumers in C^i have surplus zero in S^{i-1}, and some consumers in C^i strictly prefer S to S^{i-1} because they have a positive surplus in S. ■

Proof of Proposition 9. We first show for any Pareto undominated S, the set of segmentations that are equivalent to S is the same as the set of segmentations that are surplus-equivalent to S. For this, we show that a segmentation S' is equivalent to Pareto undominated S if and only if it is surplus-equivalent to S. If S' is equivalent to it, then almost all consumers have the same surplus in the two segmentations, and therefore they are surplus-equivalent. Suppose S and S' are surplus equivalent. Because S is Pareto undominated, so is S'. By Lemma 2, both segmentations must be efficient. Let $\{(C_1, v_1), \ldots, (C_n, v_n)\}$ and $\{(C'_1, v_1), \ldots, (C'_n, v_n)\}$ be the canonical representations of S and S', respectively. Because the two segmentations are efficient, all consumers in value v_1 are in both C_1 and C'_1. If some consumer with value higher than v_1 is in C_1 but not C'_1, the the consumer's surplus is different across the two segmentations, violating surplus-equivalence. A similar argument apples to consumers in C'_1 but not C_1. So we must have $C_1 = C'_1$. An inductive argument implies that the two segmentations are equivalent.

Suppose that $\mathcal{S}$ is the set of all segmentations that are equivalent, and hence surplus-equivalent, to some Pareto undominated segmentation S. Because S is Pareto undominated, for any segmentation that is not in $\mathcal{S}$, and hence is not surplus-equivalent to S, there are some consumers that strictly prefer S to S'. By Lemma 4, S strong RV blocks and strong Harsanyi blocks any such S', so external stability is satisfied. Any two segmentations in $\mathcal{S}$ are surplus-equivalent and therefore by Lemma 4, neither strong RV blocks nor strong Harsanyi blocks the other, and therefore internal stability is satisfied. So $\mathcal{S}$ is a strong RV stable set and a strong Harsanyi stable set.

Consider a strong Harsanyi (RV) stable set $\mathcal{S}$. If the set contains two segmentations S and S' in $\mathcal{S}$ that are not surplus-equivalent, then there is a positive measure of consumers that either prefer S to S' or S' to S. Then, by Lemma 4, one of the two segmentations strong Harsanyi (RV) blocks the other one, violating internal stability. Therefore, $\mathcal{S}$ contains only surplus-equivalent segmentations. Further, if S and S' are surplus-equivalent and one of them is in $\mathcal{S}$, then other other one must be too, because otherwise again by Lemma 4 the segmentation that is not in $\mathcal{S}$ is not Harsanyi (RV) blocked by any segmentation in $\mathcal{S}$, violating external stability. A segmentation S in $\mathcal{S}$ cannot be Pareto dominated by any segmentation S' not in $\mathcal{S}$ because otherwise, by Lemma 4, S would not strong Harsanyi (RV) block S', violating external stability. So $\mathcal{S}$ must be the set of all surplus-equivalent segmentations, and therefore equivalent segmentations, to the Pareto undominated segmentation S. ■

The stable set notions we study in this section allow for weak inequalities along a sequence of segmentations. If we require strict improvement in every step, then no segmentation can block any other segmentation because in any segmentation some consumers get zero surplus. So in that case, the unique RV stable set (and also Harsanyi and also maximal RV stable set) is the set of all segmentations.

Ray and Vohra (2019) define a notion of *maximality* of a stable set and show that any single-payoff RV stable set is also a maximal RV stable set. Because both Proposition 8 and Proposition 9 characterize stable sets as ones that contain a single segmentation (or surplus-equivalent ones), those stable sets are also maximal.[38] Roughly speaking, maximality requires that in a chain of segmentations defined in Definition 16 that ends in S, at each step the move specified by the chain is "optimal" in the sense that no coalition C has another move that would lead to another segmentation in the stable set that the coalition C prefers to S. If the stable set is a singleton, then all chains necessarily end in the same segmentation, and therefore maximality is trivially satisfied.

[38] Theorem 1 together with Remark 1 in Ray and Vohra (2019) show that in their setting with a finite number of players and objections that are defined to require strict improvements, any singleton RV stable set is also a maximal RV stable set. A similar argument shows that this is also the case in our setting.