Local Regression Distribution Estimators

Matias D. Cattaneo† Michael Jansson‡ Xinwei Ma§

January 29, 2021

Abstract

This paper investigates the large sample properties of local regression distribution estimators, which include a class of boundary adaptive density estimators as a prime example. First, we establish a pointwise Gaussian large sample distributional approximation in a unified way, allowing for both boundary and interior evaluation points simultaneously. Using this result, we study the asymptotic efficiency of the estimators, and show that a carefully crafted minimum distance implementation based on “redundant” regressors can lead to efficiency gains. Second, we establish uniform linearizations and strong approximations for the estimators, and employ these results to construct valid confidence bands. Third, we develop extensions to weighted distributions with estimated weights and to local $L^2$ least squares estimation. Finally, we illustrate our methods with two applications in program evaluation: counterfactual density testing, and IV specification and heterogeneity density analysis. Companion software packages in Stata and R are available.

Keywords: distribution and density estimation, local polynomial methods, uniform approximation, efficiency, optimal kernel, program evaluation.

---

*Prepared for “Celebrating Whitney Newey’s Contributions to Econometrics” Conference at MIT, May 17-18, 2019. We thank the conference participants for comments, and Guido Imbens and Yingjie Feng for very useful discussions. We are also thankful to the handling co-Editor, Xiaohong Chen, an Associate Editor and two reviewers for their input. Cattaneo gratefully acknowledges financial support from the National Science Foundation through grant SES-1947805, and Jansson gratefully acknowledges financial support from the National Science Foundation through grant SES-1947662 and the research support of CREATES.

†Department of Operations Research and Financial Engineering, Princeton University.
‡Department of Economics, UC Berkeley and CREATES.
§Department of Economics, UC San Diego.
1 Introduction

Kernel-based nonparametric estimation of distribution and density functions, as well as higher-
order derivatives thereof, play an important role in econometrics. These nonparametric estimators
often feature both as the main object of interest and as preliminary ingredients in multi-step semi-
parametric procedures (Newey and McFadden, 1994; Ichimura and Todd, 2007). Whitney Newey’s
path-breaking contributions to non/semiparametric econometrics employing kernel smoothing are
numerous.\footnote{See, for example, Newey and Stoker (1993), Newey (1994a), Newey (1994b), Hausman and Newey (1995), Robins,
Hsieh, and Newey (1995), Newey, Hsieh, and Robins (2004), Newey and Ruud (2005), Ichimura and Newey (2020), and
Chernozhukov, Escanciano, Ichimura, Newey, and Robins (2020).} This paper hopes to honor his influentia work in this area by studying the main large
sample properties of a new class of local regression distribution estimators, which can be used for
non/semiparametric estimation and inference.

The class of local regression distribution estimators is constructed using a local least squares
approximation to the empirical distribution function of a random variable $x \in X \subseteq \mathbb{R}$, where the
localization at the evaluation point $x \in X$ is implemented via a kernel function and a bandwidth
parameter. The local functional form approximation is done using a finite-dimension basis function.
When the basis function contains polynomials up to order $p \in \mathbb{N}$, the associated least squares coef-
ficients give estimators of the distribution function, density function, and higher-order derivatives
(up to order $p - 1$), all evaluated at $x \in X$. If only a polynomial basis is used, then the estimator
reduces to the one recently proposed in Cattaneo, Jansson, and Ma (2020).

We present two main large sample distributional results for the local regression distribution
estimators. First, in Section 3, we establish a pointwise (in $x \in X$) Gaussian distributional ap-
proximation with consistent standard errors. Because these estimators have a U-statistic structure
with an $n$-varying kernel, where $n$ denotes the sample size, we construct a fully automatic Student-
ization given a choice of basis, kernel, and bandwidth. Furthermore, we show that when the basis
function includes polynomials, the associated density and its higher-order derivatives estimators are
boundary adaptive without further modifications. This result generalizes Cattaneo, Jansson, and
Ma (2020) by allowing for arbitrary local basis functions, which is particularly useful for efficiency
considerations.

To be more precise, for the special case of local polynomial density estimation, Cattaneo,
Jansson, and Ma (2020) showed that the asymptotic variance of the estimator is of the “sandwich”
form, which does not reduce to a single matrix (up to a proportional factor) by a choice of kernel function. This finding indicates that more efficient estimators can be constructed via a minimum distance approach based on “redundant” regressors, following well-known results in econometrics (Newey and McFadden, 1994). In Section 3.3, we present a novel minimum distance construction for estimation of the density and its derivatives, and obtain an efficiency bound for the new minimum distance density estimator. Furthermore, we show that the efficiency bound coincides with the well-known asymptotic variance lower bound for kernel-based density estimation (Granovsky and Müller, 1991; Cheng, Fan, and Marron, 1997). We also show that this efficiency bound is tight: we construct a feasible minimum distance procedure exploiting carefully chosen redundant regressors, which leads to an estimator with asymptotic variance arbitrarily close to the theoretical efficiency bound. These results offer not only a novel theoretical perspective on efficiency of classical nonparametric kernel-based density estimation, but also a new class of more efficient boundary adaptive density estimators for practice. We also discuss how these results generalize to other local regression distribution estimators in the supplemental appendix.

Our second main large sample distributional result, in Section 4, concerns uniform estimation and inference over a region \( I \subseteq \mathcal{X} \), based on either the basic local regression distribution estimators or the associated more efficient estimators obtained via our proposed minimum distance procedure. More precisely, we establish a strong approximation to the boundary adaptive Studentized statistic, uniformly over \( x \in I \), relying on a “coupling” result in Giné, Koltchinskii, and Sakhanenko (2004); see also Rio (1994) and Giné and Nickl (2010) for closely related results, and Zaitsev (2013) for a review on strong approximation methods. This approach allows us to deduce a distributional approximation for many functionals of the Studentized statistic, including its supremum, following ideas in Chernozhukov, Chetverikov, and Kato (2014b). For further discussion and references on strong approximations and their applications to non/semiparametric econometrics see Chernozhukov, Chetverikov, and Kato (2014a), Belloni, Chernozhukov, Chetverikov, and Kato (2015), Belloni, Chernozhukov, Chetverikov, and Fernandez-Val (2019), and Cattaneo, Farrell, and Feng (2020), Cattaneo, Crump, Farrell, and Feng (2021), and references therein.

We employ our strong approximation results for local regression distribution estimators to construct asymptotically valid confidence bands for the density function and derivatives thereof. Other applications of our results, not discussed here to conserve space, include specification and
shape restriction testing. As a by-product, we also establish a linear approximation to the boundary adaptive Studentized statistic, uniformly over $x \in \mathcal{I}$, which gives uniform convergence rates and can be used for further theoretical developments. See the supplemental appendix for more details.

In addition to our main large sample results for local regression distribution and related estimators, we briefly discuss several extensions in Section 5. First, we allow for a weighted empirical distribution function entering our estimators, where the weights themselves may be estimated. Our results continue to hold in this more general case, which is practically relevant as illustrated in our empirical applications. Second, we present and study an alternative class of estimators that employ a non-random $L^2$ loss function, instead of the more standard least squares approximation underlying our local regression distribution estimators. These alternative estimators enjoy certain theoretical advantages, but require ex-ante knowledge of the boundary location of $X$. In particular, we show in the supplemental appendix how these alternative estimators can be implemented to achieve maximum asymptotic efficiency in estimating the density function and its derivatives. Third, we also discuss incorporating shape restrictions using the general local basis function entering the local regression distribution estimators.

Finally, in Section 6, we illustrate our methods with two applications in program evaluation (for a review see Abadie and Cattaneo, 2018). First, we discuss counterfactual density analysis following DiNardo, Fortin, and Lemieux (1996); see also Chernozhukov, Fernandez-Val, and Melly (2013) for related discussion based on distribution functions. Second, we discuss specification testing and heterogeneity analysis in the context of instrumental variables following Kitagawa (2015) and Abadie (2003), respectively; see also Imbens and Rubin (2015) for background and other applications of nonparametric density estimation to causal inference and program evaluation. In all these applications, we develop formal estimation and inference methods based on nonparametric density estimation using local regression distribution estimators implemented with weighted distribution functions. We showcase our new methods using a subsample of the data in Abadie, Angrist, and Imbens (2002), corresponding to the Job Training Partnership Act (JTPA).

From both methodological and technical perspectives, our proposed class of local regression distribution estimators is different from, and exhibits demonstrable advantages over, other related estimators available in the literature. For the special case of density estimation (i.e., when the basis function is taken to be polynomial), our resulting kernel-based density estimator enjoys boundary
carpentry over the possibly unknown boundary of $X$, does not require preliminary smoothing of the data and hence avoids preliminary tuning parameter choices, and is easy to implement and interpret. Cattaneo, Jansson, and Ma (2020) gave a detailed discussion of that density estimator and related approaches in the literature, which include the influential local polynomial estimator of Cheng, Fan, and Marron (1997) and related estimators (Zhang and Karunamuni, 1998; Karunamuni and Zhang, 2008, and references therein). The class of estimators we consider here can be more efficient by employing minimum distance estimation ideas (Section 3), easily delivers intuitive estimators of density-weighted averages (Section 5.1), and allows for incorporating shape and other restrictions (Section 5.3), among other features that we discuss below. Last but not least, some of the technical results presented herein for the general class of estimators, such as asymptotic efficiency (Section 3.3) and uniform inference (Section 4) are new, even for the special case of density estimation in Cattaneo, Jansson, and Ma (2020).

The rest of the paper proceeds as follows. Section 2 introduces the class of local regression distribution estimators. Section 3 establishes a pointwise distributional approximation, along with a consistent standard error estimator, and discusses efficiency focusing in particular on the leading special case of density estimation. Section 4 establishes uniform results, including valid linearizations and strong approximations, which are then used to construct confidence bands. Section 5 discusses extensions of our methodology, while Section 6 illustrates our new methods with two distinct program evaluation applications. Section 7 concludes. The supplemental appendix (SA) includes all proofs of our theoretical results as well as other technical, methodological and numerical results that may be of independent interest. Software packages for Stata and R implementing the main results in this paper are discussed in Cattaneo, Jansson, and Ma (2021).

2 Setup

Suppose $x_1, x_2, \ldots, x_n$ is a random sample from a univariate random variable $x$ with absolute continuous cumulative distribution function $F(\cdot)$, and associated Lebesgue density $f(\cdot)$, over its support $X \subseteq \mathbb{R}$, which may be compact and not necessarily known. We propose, and study the large sample properties of a new class of nonparametric estimators of $F(\cdot)$, $f(\cdot)$, and derivatives thereof, both pointwise at $x \in X$ and uniformly over some region $I \subseteq X$.

Our proposed estimators are applicable whenever $F(\cdot)$ is suitably smooth near $x$ and admits a
sufficiently accurate linear-in-parameters local approximation of the form:

\[ \varrho(h, x) = \sup_{|x - x'| \leq h} |F(x) - R(x - x)' \theta(x)| \quad \text{is small for } h \text{ small,} \tag{1} \]

where \( R(\cdot) \) is a known local basis function and \( \theta(x) \) is a parameter vector to be estimated. As an estimator of \( \theta(x) \) in (1), we consider the local regression estimator

\[ \hat{\theta}(x) = \arg\min_{\theta} \sum_{i=1}^{n} W_i (\hat{F}_i - R_i' \theta)^2, \tag{2} \]

where \( W_i = K((x_i - x)/h)/h \) for some kernel \( K(\cdot) \) and some bandwidth \( h \), \( R_i = R(x_i - x) \), and

\[ \hat{F}_i = \frac{1}{n} \sum_{j=1}^{n} \mathbb{I}(x_j \leq x_i) \tag{3} \]

is the empirical distribution function evaluated at \( x_i \), with \( \mathbb{I}(\cdot) \) denoting the indicator function.

The generic formulation (1) is motivated in part by the important special case where \( F(\cdot) \) is sufficiently smooth, in which case

\[ F(x) \approx F(x) + f(x)(x - x) + \ldots + f^{(p-1)}(x) \frac{1}{p!}(x - x)^p \quad \text{for } x \approx x, \tag{4} \]

and \( f^{(s)}(x) = \frac{d^s f(x)}{dx^s}|_{x=x} \) are higher-order density derivatives. Of course, the approximation (4) is of the form (1) with \( R(u) = (1, u, \ldots, u^{p}/p!)' \), and hence \( \theta(x) = (F(x), f(x), \ldots, f^{(p-1)}(x))' \). In such special case, the estimator \( \hat{\theta}(x) \) corresponds to one of the estimators introduced in Cattaneo, Jansson, and Ma (2020). But, as further discussed below, other choices of \( R(\cdot) \) and/or \( \theta(\cdot) \) can be attractive, and as a consequence we take (1) as the starting point for our analysis. Section 5 discusses other extensions and generalization of the basic local regression distribution estimator \( \hat{\theta}(x) \) in (2).

The class of estimators defined in (2) is motivated by standard local polynomial regression methods (Fan and Gijbels, 1996). However, well-known results for local polynomial regression are not applicable to the local regression distribution estimator, \( \hat{\theta}(x) \), because the empirical distribution function estimator, \( \hat{F}_i \), which plays the role of the “dependent” variable in the construction, depends on not only \( x_i \) but also all of the “independent” observations \( x_1, x_2, \ldots, x_n \). This implies
that, unlike the case of standard local polynomial regression, $\hat{\theta}(x)$ cannot be studied by conditioning on the “covariates” $x_1, x_2, \ldots, x_n$. Instead, we employ U-statistic methods for analyzing the statistical properties of $\hat{\theta}(x)$. This observation explains the quite different asymptotic variance of our estimator: see Section 3.3 for details. Furthermore, as discussed in Section 5.1, when a weighted distribution function is used in place of $\hat{F}_i$ in (2), the resulting (weighted) local regression distribution estimators are consistent for a density-weighted regression function, as opposed to being consistent for the regression function itself (as it is the case for standard local polynomial regression methods). Finally, the SA highlights other technical differences between the two types of local regression estimators.

3 Pointwise Distribution Theory

This section discusses the large sample properties of the estimator $\hat{\theta}(x)$, pointwise in $x \in X$. We first establish asymptotic normality, and then discuss asymptotic efficiency. Other results are reported in the SA to conserve space. We drop the dependence on the evaluation point $x$ whenever possible.

3.1 Assumptions

We impose the following assumption throughout this section. We do not restrict the support of $X$, which can be a compact set or unbounded, because our estimator automatically adapts to boundary evaluation points.

**Assumption 1** $x_1, \ldots, x_n$ is a random sample from a distribution $F(\cdot)$ supported on $X \subseteq \mathbb{R}$, and $x \in X$.

(i) For some $\delta > 0$, $F(\cdot)$ is absolutely continuous on $[x - \delta, x + \delta]$ with a density $f(\cdot)$ admitting constants $f(x-), \dot{f}(x-), f(x+), \dot{f}(x+)$ such that

$$\sup_{u \in [-\delta, 0)} \frac{|f(x + u) - f(x-) - \dot{f}(x-)u|}{|u|^2} + \sup_{u \in (0, \delta]} \frac{|f(x + u) - f(x+) - \dot{f}(x+)u|}{|u|^2} < \infty.$$  

(ii) $K(\cdot)$ is nonnegative, symmetric, and continuous on its support $[-1, 1]$, and integrates to 1.

(iii) $R(\cdot)$ is locally bounded, and there exists a positive-definite diagonal matrix $\Upsilon_h$ for each $h > 0$, such that $\Upsilon_h R(u) = R(u/h)$.
(iv) Let $X_{h,x} = \frac{x-x}{h}$. For all $h$ sufficiently small, the minimum eigenvalues of $\Gamma_{h,x}$ and $h^{-1}\Sigma_{h,x}$ are bounded away from zero, where

$$
\Gamma_{h,x} = \int_{X_{h,x}} R(u)R(u)'^{\top}K(u)f(x+hu)du,
$$

$$
\Sigma_{h,x} = \int_{X_{h,x}} \int_{X_{h,x}} R(u)R(v)'^{\top}\left[F(x+h\min\{u,v\}) - F(x+hu)F(x+hv)\right] \cdot K(u)K(v)f(x+hu)f(x+hv)dudv.
$$

Part (i) imposes smoothness conditions on the distribution function $F(\cdot)$, separately for the two regions on the left and on the right of the evaluation point $x$. In most applications, the distribution function will also be smooth at the evaluation point, in which case $f(x-) = f(x+)$ and $f'(x-) = f'(x+)$. However, there are important situations where $F(\cdot)$ only has one-sided derivatives, such as at boundary or kink evaluation points. Part (ii) imposes standard restrictions on the kernel function, which allows for all commonly used (compactly supported) second-order kernel functions. Part (iii) requires that the local basis $R(\cdot)$ can be stabilized by a suitable normalization. Parts (iv) give assumptions on two (non-random) matrices which will feature in the asymptotic distribution.

The error of the approximation in (1) depends on the choice of $R(\cdot)$ and $\theta$, and is quantified by $\varrho(h)$, where we suppress the dependence on the evaluation point $x$ to save notation. The approximation error will be required to be “small” in the sense that $n\varrho(h)^2/h \to 0$. In the cases of main interest (i.e., when $R(\cdot)$ is polynomial), we have either $\varrho(h) = O(h^{p+1})$ or $\varrho(h) = o(h^p)$ for some $p$. The condition can therefore be stated as $nh^{2p+1} \to 0$ and $nh^{2p-1} = O(1)$, respectively, in those cases.

We do not discuss how to choose the bandwidth $h$, or the order $p$ if $R(\cdot)$ contains polynomials, as both choices can be developed following standard ideas in the local polynomial literature. We focus instead on distributional approximation (Section 3.2) and asymptotic variance minimization (Section 3.3), given a choice of bandwidth sequence and polynomial order. Bandwidth selection can be developed by extending the results in Cattaneo, Jansson, and Ma (2020) and polynomial order selection can be developed following Fan and Gijbels (1996, Section 3.3). In particular, a larger $p$ can lead to more bias reduction whenever the target population function is smooth enough at the expense of a larger asymptotic variance. We discuss this trade-off explicitly in our efficiency calculations (Section 3.3).
3.2 Asymptotic Normality

We show that, under regularity conditions and if $h$ vanishes at a suitable rate as $n \to \infty$, then

$$\hat{\Omega}^{-1/2}(\hat{\theta} - \theta) \overset{d}{\sim} \mathcal{N}(0, I), \quad \hat{\Omega} = \hat{\Gamma}^{-1}\hat{\Sigma}\hat{\Gamma}^{-1},$$  \hspace{1cm} (5)

where

$$\hat{\Gamma} = \frac{1}{n} \sum_{i=1}^{n} W_i R_i R_i', \quad \hat{\Sigma} = \frac{1}{n^2} \sum_{i=1}^{n} \hat{\psi}_i \hat{\psi}_i', \quad \hat{\psi}_i = \frac{1}{n} \sum_{j=1}^{n} W_j R_j (\mathbb{1}(x_i \leq x_j) - \hat{F}_j).$$

It follows from this result that inference on $\theta$ can be based on $\hat{\theta}$ by employing the (pointwise) distributional approximation $\hat{\theta} \overset{d}{\sim} \mathcal{N}(\theta, \hat{\Omega})$. The three matrices, $\hat{\Gamma}$, $\hat{\Sigma}$ and $\hat{\Omega}$, depend on the evaluation point $x$, but such dependence is again suppressed for simplicity. This distributional result will rely on the “small” bias condition $n \varrho(h)^2 / h \to 0$ mentioned above, which makes the asymptotic approximation (or smoothing) bias of $\hat{\theta}$ negligible relative to the standard error. From an inference perspective, such bias condition can be achieved by employing undersmoothing or robust bias correction: see Calonico, Cattaneo, and Farrell (2018, 2020) for discussion and background references. The SA includes more details on the bias of the estimator.

To provide some insight into the distributional approximation (5), and to see why it cannot be established using standard results for local polynomial regression, first observe that

$$\hat{\theta} - \theta = \hat{\Gamma}^{-1}S, \quad S = \frac{1}{n} \sum_{i=1}^{n} W_i R_i (\hat{F}_i - R_i'\theta),$$

assuming $\hat{\Gamma}$ is invertible with probability approaching one. The statistic $S$ can be written as

$$S = U + B, \quad U = \frac{1}{n(n-1)} \sum_{i,j=1,i\neq j}^{n} W_j R_j \left( \mathbb{1}(x_i \leq x_j) - F(x_j) \right),$$  \hspace{1cm} (6)

where $B$ consists of a leave-in bias term and a smoothing bias term. Since $S$ is approximately a second-order $U$-statistic, result (5) should follow from a central limit theorem for ($n$-varying) $U$-statistics under suitable regularity conditions, including conditions ensuring that the approximation errors are negligible. More specifically, result (5) follows if $U$ is asymptotically mean-zero Gaussian $\mathbb{V}[U]^{-1/2}U \overset{d}{\sim} \mathcal{N}(0, I)$, where $\mathbb{V}[U]$ denotes the variance of $U$, $\mathbb{V}[U]^{-1/2}B \to_{\mathbb{P}} 0$, and if the variance
estimator $\hat{\Sigma}$ is consistent in the sense that $\mathbb{V}[U]^{-1}(\hat{\Sigma} - \mathbb{V}[U]) \rightarrow_p 0$. Moreover, the projection theorem for $U$-statistics implies that, under appropriate regularity conditions,

$$\mathbb{V}[U] \approx \frac{1}{n} \mathbb{E}[\psi_i \psi_i'], \quad \psi_i = \mathbb{E}[W_j R_j \mathbbm{1}(x_i \leq x_j) - F(x_j|x_i)],$$

which motivates the functional form of the variance estimator $\hat{\Sigma}$ used to form $\hat{\Omega}$.

The following theorem formalizes the above intuition with precise sufficient conditions.

**Theorem 1 (Pointwise Asymptotic Normality)** Suppose Assumption 1 holds. If $ng(h^2)/h \rightarrow 0$ and $nh^2 \rightarrow \infty$, then (5) holds.

This theorem establishes a (pointwise) Gaussian distributional approximation for the Studentized statistic $\hat{\Omega}^{-1/2}(\hat{\theta} - \theta)$, which is valid for each evaluation point $x \in \mathcal{X}$. For example, letting $c$ be a vector of conformable dimension and $\alpha \in (0, 1)$, this result justifies the standard $100(1 - \alpha)$% confidence interval

$$\text{CI}_\alpha(x) = \left[ c'\hat{\theta}(x) - q_{1 - \alpha/2} \sqrt{c'\hat{\Omega}(x)c} \right. \left. , \quad c'\hat{\theta}(x) - q_{\alpha/2} \sqrt{c'\hat{\Omega}(x)c} \right],$$

where $q_a = \inf \{ u \in \mathbb{R} : \mathbb{P}[\mathcal{N}(0, 1) \leq u] \geq a \}$. The above confidence interval is asymptotically valid for each evaluation point $x$, which is reflected by the notation $\text{CI}_\alpha(x)$. That is,

$$\lim_{n \rightarrow \infty} \mathbb{P}\left[ c'\theta(x) \in \text{CI}_\alpha(x) \right] = 1 - \alpha, \quad \text{for all } x \in \mathcal{X}.$$

Section 4 develops asymptotically valid confidence bands, which will be denoted by $\text{CI}_\alpha(I)$ for some region $I \subseteq \mathcal{X}$.

**3.3 Efficiency**

As it is well known in the literature (Fan and Gijbels, 1996), the standard local polynomial regression estimator of $\mathbb{E}[y|x = x]$, for dependent variable $y$ and independent variable $x$, has a limiting asymptotic variance of the “sandwich form” $e_0' \Gamma^{-1} A \Gamma^{-1} e_0$, where $e_\ell$ denotes the $(\ell + 1)$th standard
basis vector, and
\[ \Gamma = f(x) \int_{-1}^{1} R(u)R(u)'K(u)du, \quad A = V[y|x= x]f(x) \int_{-1}^{1} R(u)R(u)'K(u)^2du. \]

This variance structure implies that setting \( K(\cdot) \) to be the uniform kernel makes \( \Gamma \) proportional to \( A \) (i.e., \( K(u) = K(u)^2 \) whenever \( K(u) = 1(\|u\| \leq 1) \)), and hence minimizes the above asymptotic variance, at least in the sense that \( \Gamma^{-1}A\Gamma^{-1} \geq A^{-1} \). See also Granovsky and Müller (1991) for a more general discussion on the optimality of the uniform kernel for kernel-based estimation.

Unlike the case of the asymptotic variance of local polynomial regression, however, our local regression distribution estimators exhibit a more complex and uneven asymptotic variance formula due to their construction. As a result, employing the uniform kernel may not exhaust the potential efficiency gains. For example, in the case of local polynomial density estimation (Cattaneo, Jansson, and Ma, 2020), \( R(u) \) is polynomial of order \( p \geq 1 \) and the asymptotic variance of the density estimator \( \hat{f}(x) = e_1'\hat{\theta}(x) \) takes the form \( e_1'\Gamma^{-1}\Sigma\Gamma^{-1}e_1 \) with
\[ \Sigma = f(x)^3 \int_{-1}^{1} \int_{-1}^{1} \min\{u, v\}R(u)R(v)'K(u)K(v)dudv, \]
which implies that \( \Gamma \) is no longer proportional to \( \Sigma \) even when the kernel function is uniform. (To show this result, one first recognizes that the asymptotic variance of \( \hat{f}(x) \) is \( h^{-1}e_1'\Gamma_h^{-1}\Sigma_h\Gamma_h^{-1}e_1 \), where the matrices are defined in Assumption 1. Then the expression reduces to \( e_1'\Gamma^{-1}\Sigma\Gamma^{-1}e_1 \) after taking the limit \( h \to 0 \), provided that \( x \) is an interior evaluation point. See the SA for omitted details.) This observation applies to the general case where the local basis function \( R(\cdot) \) needs not to be of polynomial form, or when higher-order derivatives are of interest. See the SA for further discussion and detailed formulas.

In this section we employ a minimum distance approach to develop a lower bound on the asymptotic variance of the local regression distribution estimators, and also propose more efficient estimators based on the observation that their asymptotic variance is of the sandwich form \( \Gamma^{-1}\Sigma\Gamma^{-1} \) but with \( \Gamma \) not proportional to \( \Sigma \) even when the uniform kernel is used.

To motivate our approach, notice that in many cases it is possible to specify \( R(\cdot) \) in such a way that \( \theta \) can be partitioned as \( \theta = (\theta_1', \theta_2')', \) where \( \theta_2 = 0 \). In such cases several distinct estimators of \( \theta_1 \) are available. To describe some leading candidates and their salient properties, partition \( \hat{\theta}, \hat{\Gamma}, \)
Σ, and \( \hat{\Omega} \) conformable with \( \theta \) as \( \hat{\theta} = (\hat{\theta}_1', \hat{\theta}_2')' \) and

\[
\hat{\Gamma} = \begin{pmatrix}
\hat{\Gamma}_{11} & \hat{\Gamma}_{12} \\
\hat{\Gamma}_{21} & \hat{\Gamma}_{22}
\end{pmatrix}, \quad \hat{\Sigma} = \begin{pmatrix}
\hat{\Sigma}_{11} & \hat{\Sigma}_{12} \\
\hat{\Sigma}_{21} & \hat{\Sigma}_{22}
\end{pmatrix}, \quad \hat{\Omega} = \begin{pmatrix}
\hat{\Omega}_{11} & \hat{\Omega}_{12} \\
\hat{\Omega}_{21} & \hat{\Omega}_{22}
\end{pmatrix}.
\]

The “short” regression counterpart of \( \hat{\theta}_1 \) obtained by dropping \( R_2(\cdot) \) from \( R(\cdot) = (R_1(\cdot)', R_2(\cdot)')' \) is given by

\[
\hat{\theta}_{R,1} = \hat{\theta}_1 + \hat{\Gamma}^{-1}_{11} \hat{\Gamma}_{12} \hat{\theta}_2,
\]

while an optimal minimum distance estimator of \( \theta_1 \) is given by

\[
\hat{\theta}_{MD,1} = \arg\min_{\theta_1} \left( \frac{1}{2} \Omega_{11}^{-1} \left( \hat{\theta}_1 - \theta_1 \right)' \Omega_{22}^{-1} \left( \hat{\theta}_1 - \theta_1 \right) \right) = \hat{\theta}_1 - \Omega_{12} \Omega_{22}^{-1} \hat{\theta}_2.
\]  

(7)

As a by-product of results obtained when establishing (5), it follows that

\[
\hat{\Omega}_{11}^{-1/2} (\hat{\theta}_1 - \theta_1) \sim N(0, I),
\]

\[
\hat{\Omega}_{R,11}^{-1/2} (\hat{\theta}_{R,1} - \theta_1) \sim N(0, I), \quad \hat{\Omega}_{R,11} = \hat{\Gamma}_{11}^{-1} \hat{\Gamma}_{11} \hat{\Gamma}_{11}^{-1},
\]

and

\[
\hat{\Omega}_{MD,11}^{-1/2} (\hat{\theta}_{MD,1} - \theta_1) \sim N(0, I), \quad \hat{\Omega}_{MD,11} = \hat{\Omega}_{11} - \hat{\Omega}_{12} \hat{\Omega}_{22}^{-1} \hat{\Omega}_{21},
\]

under regularity conditions. Since \( \hat{\Omega} \) is of “sandwich” form, the estimators \( \hat{\theta}_1 \) and \( \hat{\theta}_{R,1} \) cannot be ranked in terms of (asymptotic) efficiency in general. On the other hand, \( \hat{\theta}_{MD,1} \) will always be (weakly) superior to both \( \hat{\theta}_1 \) and \( \hat{\theta}_{R,1} \). In fact, because

\[
\hat{\theta}_1 = \arg\min_{\theta_1} \left( \frac{1}{2} \Omega_{11}^{-1} \left( \hat{\theta}_1 - \theta_1 \right)' \Omega_{22}^{-1} \left( \hat{\theta}_1 - \theta_1 \right) \right),
\]

and

\[
\hat{\theta}_{R,1} = \arg\min_{\theta_1} \left( \frac{1}{2} \Gamma (\hat{\theta}_1 - \theta_1) \right),
\]

each estimator admits a minimum distance interpretation, but only \( \hat{\theta}_{MD,1} \) can be interpreted as an optimal minimum distance estimator based on \( \hat{\theta} \). See Newey and McFadden (1994) for more discussion on minimum distance estimation.

As a consequence, we investigate whether an appropriately implemented \( \hat{\theta}_{MD,1} \) can lead to asymp-
totic efficiency gains relative to $\hat{\theta}_1$ and $\hat{\theta}_{R,1}$. More generally, as a by-product, we obtain an efficiency bound among minimum distance estimators and show that this bound coincides with those known in the literature for kernel-based density estimation at interior points (Granovsky and Müller, 1991; Cheng, Fan, and Marron, 1997).

In the remaining of this section we focus on the case of local polynomial density estimation at an interior point for concreteness, but the SA presents more general results. Consequently, we assume that $F(\cdot)$ is $p$-times continuously differentiable in a neighborhood of $x$. Then, (4) is satisfied and a natural choice of $R(\cdot)$ is

$$R(u) = \left(R_1(u)', R_2(u)'ight)' = \left(1, P(u)', Q(u)'ight)', \quad (8)$$

where $P(u) = (u, u^2/2, \cdots, u^p/p!')$ is a polynomial basis, and $Q(\cdot)$ represent redundant regressors. Therefore, in our minimum distance construction, the parameters are

$$\theta = \left(\begin{array}{c} F(x) \\ \text{intercept} \\ f(x), \cdots, f^{(p-1)}(x) \\ \text{slope, } P(\cdot) \\ 0, \cdots, 0 \\ \text{redundant, } Q(\cdot) \end{array}\right)', \quad (9)$$

with smoothing error of order $\varrho(h) = o(h^p)$.

With (8) and (9), we define the minimum distance density estimator as $\hat{f}_{MD}(x) = e_1'\hat{\theta}_{MD,1}$. Similarly, we have $\hat{f}(x) = e_1'\hat{\theta}_1$ and $\hat{f}_R(x) = e_1'\hat{\theta}_{R,1}$. Of course, if it is known a priori that the distribution function is $p+q$ times continuously differentiable, then one can specify $Q(\cdot)$ to include higher order polynomials: $Q(u) = (u^{p+1}/(p+1)!, \cdots, u^{p+q}/(p+q)!)'$. By redefining the parameters as $\theta = (F(x), f(x), \cdots, f^{(p+q-1)}(x))'$, the smoothing error will be of order $\varrho(h) = o(h^{p+q})$. Notice that, in this case, $\hat{f}(x)$ and $\hat{f}_R(x)$ correspond to the density estimator introduced in Cattaneo, Jansson, and Ma (2020) implemented with $R(u) = (1, u, \cdots, u^{p+q}/(p+q)!)'$ and $R(u) = (1, u, \cdots, u^p/p!)'$, respectively. Since the purpose of this section is to investigate the efficiency gains of incorporating additional redundant regressors, we do not exploit the extra smoothness condition, and we will treat $Q(\cdot)$ as redundant regressors even if $Q(\cdot)$ contains higher order polynomials.

As both $\hat{f}(x)$ and $\hat{f}_R(x)$ are (weakly) asymptotically inefficient relative to $\hat{f}_{MD}(x)$ for any choice of $Q(\cdot)$, we consider the asymptotic variance of the minimum distance estimator, which can be
obtained by establishing asymptotic counterparts of $\hat{\Gamma}$ and $\hat{\Sigma}$ after suitable scaling. Under regularity conditions (e.g., lack of perfect collinearity between $P$ and $Q$), the asymptotic variance of the minimum distance $\ell$-th derivative density estimator, $\hat{f}_{\text{MD}}^{(\ell)}(x) = e_{\ell+1}' \hat{\theta}_{\text{MD},1}$ with $0 \leq \ell \leq p - 1$, is

$$\text{AsyVar}[\hat{f}_{\text{MD}}^{(\ell)}(x)] = e_{\ell}' \left[ \Omega_{PP} - \Omega_{PQ} \Omega_{QQ}^{-1} \Omega_{QP} \right] e_{\ell},$$

where

$$\begin{pmatrix} \Omega_{11} & \Omega_{1P} & \Omega_{1Q} \\ \Omega_{P1} & \Omega_{PP} & \Omega_{PQ} \\ \Omega_{Q1} & \Omega_{QP} & \Omega_{QQ} \end{pmatrix} = \Gamma^{-1} \Sigma^{-1}.$$

Therefore, the objective is to find a function $Q(\cdot)$ that minimizes the asymptotic variance $\text{AsyVar}[\hat{f}_{\text{MD}}^{(\ell)}(x)]$. Taking $Q(\cdot)$ scalar and properly orthogonalized, without loss of generality, we have

$$\int_{-1}^{1} P(u)K(u)du = 0 \quad \text{and} \quad \int_{-1}^{1} (1, P(u)')Q(u)K(u)du = 0.$$

It follows that the problem of selecting an optimal $Q(\cdot)$ to minimize $\text{AsyVar}[\hat{f}_{\text{MD}}^{(\ell)}(x)]$ is equivalent to the following variational problem:

$$\sup_{Q \in \mathcal{Q}} \left[ \int_{-1}^{1} \int_{-1}^{1} P_\ell(u)Q(v) \min\{u, v\} K(u)K(v)du dv \right]^2 \int_{-1}^{1} \int_{-1}^{1} Q(u)Q(v) \min\{u, v\} K(u)K(v)du dv$$

(10)

where

$$\mathcal{Q} = \left\{ Q(\cdot) : \int_{-1}^{1} Q(u)K(u)du = 0, \quad \int_{-1}^{1} P(u)Q(u)K(u)du = 0 \right\},$$

with $P_\ell(u) = e_{\ell}' \left( \int_{-1}^{1} P(u)P(u)'K(u)du \right)^{-1} P(u)$ and $\ell = 1, 2, \ldots, p - 1$. The objective function is obtained from the fact that, after proper orthogonalization, the matrix $\Gamma$ becomes block diagonal. See the SA for all other omitted details.

The following theorem characterizes a lower bound for the asymptotic variance of the minimum distance density estimator among all possible choices of redundant regressors.

**Theorem 2 (Efficiency: Local Polynomial Density Estimator at Interior Points)** Suppose the conditions of Theorem 1 hold. If $x \in X$ is an interior point, then

$$\inf_{Q \in \mathcal{Q}} \text{AsyVar}[\hat{f}_{\text{MD}}^{(\ell)}(x)] \geq \nu_\ell, \quad \nu_\ell = f(x)e_{\ell}' \left( \int_{-1}^{1} \hat{P}(u)\hat{P}(u)'du \right)^{-1} e_{\ell}, \quad 0 \leq \ell \leq p - 1,$$

where $\hat{P}(u) = (1, u, \ldots, u^{p-1}/(p-1)!)'$ is the derivative of $P(u)$. 

13
This theorem establishes a lower bound among minimum distance estimators. Importantly, it is shown in the SA that this bound coincides with the variance bound of all kernel-type density (and derivatives thereof) estimators employing the same order of the (induced) kernel function (Granovsky and Müller, 1991). Therefore, our minimum distance approach sheds new light on minimum variance results for nonparametric kernel-based estimators of the density function and its derivatives.

This lower bound can be (approximately) achieved by setting the redundant regressor \( Q(\cdot) \) to include a certain higher order polynomial function. By direct calculation for each \( p = 1, 2, \ldots, 10 \), it is also shown in the SA that \( \lim_{j \to \infty} \text{AsyVar}[\hat{f}^{(\ell)}_{MD,j}(x)] = \nu_\ell \), where the minimum distance estimator \( \hat{f}^{(\ell)}_{MD,j}(x) = e'\hat{\theta}_{MD,j} \) is constructed with

\[
Q(u) = u^{2j+1} - P(u)' \left( \int_{-1}^{1} P(u)P(u)' \, du \right)^{-1} \int_{-1}^{1} P(u)u^{2j+1} \, du, \quad \text{for } \ell = 0, 2, 4, \cdots
\]

or

\[
Q(u) = u^{2j+2} - P(u)' \left( \int_{-1}^{1} P(u)P(u)' \, du \right)^{-1} \int_{-1}^{1} P(u)u^{2j+2} \, du, \quad \text{for } \ell = 1, 3, 5, \cdots
\]

and \( K(\cdot) \) being the uniform kernel. While we found that other kernel shapes can also be used, we chose the uniform kernel in this construction for three reasons. First, this choice is intuitive and coincides with the optimal choice in standard local polynomial regression settings. Second, when \( p \geq 3 \) the other allowed kernel shapes overweight observations near the boundary of the kernel’s support. Third, the uniform kernel makes the asymptotic variance calculation more tractable. See the SA for further details.

The resulting recipe for implementation is simple: it proposes a specific choice of \( Q(\cdot) \) so that the corresponding minimum distance estimator approximately achieves the variance bound for \( j \) large enough. Interestingly, \( Q(\cdot) \) is scalar and known, but the larger \( j \) the closer the asymptotic variance of the minimum distance density estimator will be to the efficiency bound. We assume \( Q(\cdot) \) is orthogonal to \( P(\cdot) \) for theoretical convenience. To implement this estimator, one only needs to run a local polynomial regression of the empirical distribution function on a constant, the polynomial basis \( P(\cdot) \), and one additional regressor, either \( u^{2j+1} \) or \( u^{2j+2} \) (depending on the choice of \( \ell \)), and then apply (7) with the corresponding estimated variance-covariance matrix.
Figure 1: Equivalent Kernels of the Minimum Distance Density Estimators.

Notes. We set $P(u) = u$ or $P(u) = (u, u^2/2)'$, and $K$ uniform. The redundant regressor is $Q(u) = u^{2j+1}$ for $j = 1, 2, \ldots, 30$. The initial equivalent kernel is quadratic (black solid line), and the minimum variance kernel is uniform (red solid line).

In Figure 1, we consider the local linear/quadratic density estimator ($\ell = 0$) with the redundant regressor being a higher order polynomial (i.e., $P(u) = u$ or $P(u) = (u, u^2/2)'$, and $Q(u) = u^{2j+1}$), and plot the corresponding equivalent kernel of our minimum distance density estimator for $j = 1, 2, \ldots, 30$. As $j$ increases, the equivalent kernel converges to the uniform kernel, which is well-known to minimize the (asymptotic) variance among all density estimators employing second order kernels (Granovsky and Müller, 1991). The asymptotic variance of our proposed minimum distance density estimator converges to the optimal asymptotic variance as $j \to \infty$.

Finally, in this paper we focus on minimizing the asymptotic variance of the estimator $\hat{\theta}$ and its variants because our main goal is inference. However, our results could be modified and extended to optimize the asymptotic mean square error (MSE). We do not pursue point estimation optimality further for brevity, but we do note that in the case of local polynomial density estimation (Cattaneo, Jansson, and Ma, 2020), the resulting estimator is automatically MSE-optimal at interior points when $p \leq 2$, because the induced equivalent kernel coincides with the Epanechnikov kernel.
The distributional result presented in Theorem 1 is valid pointwise for $x \in \mathcal{X}$. We now develop an uniform distributional approximation for the Studentized process

$$T(x) = \frac{c'\hat{\theta}(x) - c'\theta(x)}{\sqrt{c'\hat{\Omega}(x)c}} : x \in \mathcal{I},$$

using the notation in (5), where $c$ is a conformable vector and $\mathcal{I} \subseteq \mathcal{X}$ is some prespecified region. This stochastic process is not asymptotically tight, and hence does not converge in distribution. Our approximation proceeds in two steps. First, for an positive (vanishing) sequence, $r_{L,n}$, we establish a uniform “linearization” of the process $T(\cdot)$ of the form:

$$\sup_{x \in \mathcal{I}} |T(x) - \hat{T}(x)| = O_P(r_{L,n}),$$

where

$$\hat{T}(x) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} K_{h,x}(x_i) : x \in \mathcal{I}$$

with

$$K_{h,x}(x_i) = \frac{c'\hat{\Upsilon}_h \Gamma_{h,x}^{-1} \int_{\mathcal{X}_n} R(u) \left[ \mathbf{1}(x_i \leq x + hu) - F(x + hu) \right] K(u) f(x + hu) du}{\sqrt{c'\hat{\Upsilon}_h \Omega_{h,x} \Gamma_{h,c}}},$$

and $\Omega_{h,x} = \Gamma_{h,x}^{-1} \Sigma_{h,x} \Gamma_{h,x}^{-1}$. In words, we show that the Studentized process $T(\cdot)$, which involves various pre-asymptotic estimated quantities, is uniformly close to the linearized process $\hat{T}(\cdot)$, which is a sample average of independent observations. To obtain (11), we develop new uniform approximations with precise convergence rates, which may be of independent interest in semiparametric estimation and inference settings. See the SA for more details.

Second, in a possibly enlarged probability space, we show that there exists a copy of $\hat{T}(\cdot)$, denoted by $\tilde{T}(\cdot)$, and a centered Gaussian process $\{\mathcal{B}(x) : x \in \mathcal{I}\}$, with a suitable variance-covariance structure, such that

$$\sup_{x \in \mathcal{I}} |\tilde{T}(x) - \mathcal{B}(x)| = O_P(r_{G,n}),$$

where $r_{G,n}$ is a suitable sequence.
where \( r_{G,n} \) is another positive (vanishing) sequence. This type of strong approximation result, when established with suitably fast rate \( r_{G,n} \to 0 \), can be used to deduce distributional approximations for statistics such as \( \sup_{x \in I} |T(x)| \), which are useful for constructing confidence bands or for conducting hypothesis tests about shape or other restrictions on the function of interest. To obtain (12), we employ a result established by Rio (1994), and later extended in Giné, Koltchinskii, and Sakhanenko (2004); see also Giné and Nickl (2010, Proof of their Proposition 5).

In this section we consider a fixed linear combination \( c \) for ease of exposition, but in the SA we discuss the more general case where \( c \) can depend on both the evaluation point \( x \) and the tuning parameter \( h \), which is necessary to establish uniform distribution approximations for the minimum distance estimator introduced in Section 3.3. All the results reported in this section apply to the latter class of estimators as well.

4.1 Assumptions

In addition to Assumption 1, we impose the following conditions on the data generating process. In the sequel, continuity and differentiability conditions at boundary points should be interpreted as one-sided statements (i.e., as in part (i) of Assumption 1).

**Assumption 2** Let \( I \subseteq \mathcal{X} \) be a compact interval.

(i) The density function \( f(x) \) is twice continuously differentiable and bounded away from zero on \( I \).

(ii) There exists some \( \delta > 0 \) and compactly supported kernel functions \( K^\dagger(\cdot) \) and \( \{K^{\dagger,d}(\cdot)\}_{d \leq \delta} \), such that (ii.1) \( \sup_{u \in \mathbb{R}} |K^\dagger(u)| + \sup_{d \leq \delta, u \in \mathbb{R}} |K^{\dagger,d}(u)| < \infty \), (ii.2) the support of \( K^{\dagger,d}(\cdot) \) has Lebesgue measure bounded by \( Cd \), where \( C \) is independent of \( d \); and (ii.3) for all \( u \) and \( v \) such that \( |u - v| \leq \delta \),

\[
|K(u) - K(v)| \leq |u - v| \cdot K^\dagger(u) + K^{\dagger,|u-v|}(u).
\]

(iii) The basis function \( R(\cdot) \) is Lipschitz continuous in \([-1,1]\).

(iv) For all \( h \) sufficiently small, the minimum eigenvalues of \( \Gamma_{h,x} \) and \( h^{-1}\Sigma_{h,x} \) are bounded away from zero uniformly for \( x \in I \).

The above strengthens and expands Assumption 1. Part (i) requires the density function to be reasonably smooth uniformly in \( I \). Part (ii) imposes additional requirements on the kernel
function. Although seemingly technical, it permits a decomposition of the difference $|K(u) - K(v)|$ into two parts. The first part, $|u - v| \cdot K'(u)$, is a kernel function which vanishes uniformly as $|u - v|$ becomes small. Note that this will be the case for all piecewise smooth kernel functions, such as the triangular or the Epanechnikov kernel. However, difference of discontinuous kernels, such as the uniform kernel, cannot be made uniformly close to zero. This motivates the second term in the above decomposition. Part (iii) requires the basis function to be reasonably smooth. Together, parts (i)–(iii) imply that the estimator $\hat{\theta}(x)$ will be “smooth” in $x$, which is important to control the complexity of the process $T(\cdot)$. Finally, part (iv) implies that the matrices $\Gamma_{h,x}$ and $\Sigma_{h,x}$ are well-behaved uniformly for $x \in I$.

4.2 Strong Approximation

We first discuss the covariance of the process $\Xi(\cdot)$. It is straightforward to show that

$$\text{Cov}[\Xi(x), \Xi(y)] = \frac{c' \Omega_h \Omega_{h,xy} \Gamma_h c}{\sqrt{c' \Omega_h \Omega_{h,x} \Gamma_h c \cdot c' \Omega_h \Omega_{h,y} \Gamma_h c}}, \quad \Omega_{h,xy} = \Gamma_{h,x}^{-1} \Sigma_{h,xy} \Gamma_{h,y}^{-1},$$

where

$$\Sigma_{h,x,y} = \int_{X_{h,y}} \int_{X_{h,x}} R(u) R(v) \left[ F(\min\{x + hu, y + hv\}) - F(x + hu)F(y + hv) \right]$$

$$\cdot K(u)K(v)f(x + hu)f(y + hv) du dv,$$

and $\Sigma_{h,x,x} = \Sigma_{h,x}$.

Now we state the second main distributional result of this paper in the following theorem.

**Theorem 3 (Strong Approximation)** Suppose Assumptions 1 and 2 hold, and that $h \to 0$ and $nh^2/\log(n) \to \infty$.

1. (11) holds with

$$r_{L,n} = \sqrt{\frac{n}{h} \sup_{x \in I} \rho(h, x) + \frac{\log(n)}{\sqrt{nh^2}}}$$

2. On a possibly enlarged probability space, there exists a copy $\tilde{\Xi}(\cdot)$ of $\Xi(\cdot)$, and a centered Gaussian process, $\{\mathcal{B}(x), x \in I\}$, defined with the same covariance as $\Xi(\cdot)$, such that (12)
The first part of this theorem gives conditions such that the feasible Studentized process $T(\cdot)$ is well approximated by the infeasible (linear) process $\mathfrak{T}(\cdot)$, uniformly for $x \in I$. The latter process is mean zero, and takes a kernel-based form. However, standard strong approximation results for kernel-type estimators do not apply directly to the process $\mathfrak{T}(\cdot)$, as the implied (equivalent, Studentized) kernel $K_{h,x}(\cdot)$ depends not only on the bandwidth but also on the evaluation point in a non-standard way. That is, due to the boundary adaptive feature of the local regression distribution estimators, the shape of the implied kernel automatically changes for different evaluation points depending on whether they are interior or boundary points.

Putting the two results together, it follows that the distribution of $T(\cdot)$ is approximated by that of $\mathfrak{B}(\cdot)$, provided the following condition holds:

$$\sqrt{n} h \sup_{x \in I} \varrho(h,x) + \frac{\log(n)}{\sqrt{nh}} \to 0.$$ 

To facilitate understanding of this rate restriction, we consider the local polynomial density estimation setting of Cattaneo, Jansson, and Ma (2020), where the basis function takes the form $R(u) = (1, u, u^2/2, \cdots, u^p/p!)'$ for some $p \geq 1$, and the second element of $\hat{\theta}(x)$ estimates the density $f(x)$. That is, $e_1' \hat{\theta}(x) = \hat{f}(x) \to_P f(x)$ under Assumption 1, where $c = e_1$. By a Taylor expansion argument, it is easy to see that the smoothing bias has order $h^{p+1}$ as long as the distribution function $F(\cdot)$ is suitably smooth. Then, the above rate restriction reduces to $\sqrt{nh^{2p+1}} + \frac{\log(n)}{\sqrt{nh^2}} \to 0$.

Finally, if the goal is to approximate the distribution of $\sup_{x \in I} |T(x)|$, then an extra $\sqrt{\log(n)}$ factor is needed in the rate restriction, as discussed in Chernozhukov, Chetverikov, and Kato (2014a). A formal statement of such result is given below, after we discuss how we can further approximate the infeasible Gaussian process $\mathfrak{B}(\cdot)$.

### 4.3 Confidence Bands

Feasible inference cannot be based on the Gaussian process $\mathfrak{B}(\cdot)$, as its covariance structure is unknown and has to be estimated in practice. For estimation, first recall from Sections 2 and 3 that $W_i(x) = K((x_i - x)/h)/h$, $R_i(x) = R(x_i - x)$, and $\hat{\Gamma}(x) = \frac{1}{n} \sum_{i=1}^{n} W_i(x) R_i(x) R_i(x)'$. Then, we
construct the plug-in estimator of $\Omega_{h,x,y}$ as follows:

$$\hat{\Omega}_{h,x,y} = n \Upsilon_h^{-1} \hat{\Gamma}(x)^{-1} \hat{\Sigma}(x,y) \hat{\Gamma}(y)^{-1} \Upsilon_h^{-1}, \quad \hat{\Sigma}(x,y) = \frac{1}{n^2} \sum_{i=1}^n \hat{\psi}_i(x) \hat{\psi}_i(y)'$$

where

$$\hat{\psi}_i(x) = \frac{1}{n} \sum_{j=1}^n W_j(x) R_j(x) \left( \mathbf{1}(x_i \leq x_j) - \hat{F}_j \right).$$

The following theorem combines previous results, and justifies the uniform confidence band constructed using critical values from $\sup_{x \in I} |\hat{B}(x)|$. Let $X_n = (x_1, x_2, \ldots, x_n)'$.

**Theorem 4 (Kolmogorov-Smirnov Distance)** Suppose Assumptions 1 and 2 hold, and that $n \sup_{x \in I} \rho(h,x)^2 \log(n)/h + \log(n)^5/(nh^2) \to 0$. Then, conditional on $X_n$, there exists a centered Gaussian process $\{\hat{B}(x), x \in I\}$ with covariance

$$\text{Cov} \left[ \hat{B}(x), \hat{B}(y) \right| X_n] = \frac{c' \Upsilon_h \hat{\Omega}_{h,x,y} \Upsilon_h c}{\sqrt{c' \Upsilon_h \hat{\Omega}_{h,x,x} \Upsilon_h c} \sqrt{c' \Upsilon_h \hat{\Omega}_{h,y,y} \Upsilon_h c}},$$

such that

$$\sup_{u \in \mathbb{R}} \left| \mathbb{P} \left[ \sup_{x \in I} |T(x)| \leq u \right] - \mathbb{P} \left[ \sup_{x \in I} |\hat{B}(x)| \leq u \right| X_n \right] = o_p(1).$$

From Theorem 4, an asymptotically valid $100(1 - \alpha)%$ confidence band for $\{c'\theta(x) : x \in I\}$ is given by

$$\text{CI}_\alpha(I) = \left\{ c' \hat{\theta}(x) - q_{1-\alpha} \sqrt{c' \hat{\Omega}(x)c}, \quad c' \hat{\theta}(x) + q_{1-\alpha} \sqrt{c' \hat{\Omega}(x)c}, \quad x \in I \right\},$$

where $q_{1-\alpha}$ is the $1 - \alpha$ quantile of $\sup_{x \in I} |\hat{B}(x)|$, conditional on the data. That is,

$$q_a = \inf \left\{ u \in \mathbb{R} : \mathbb{P} \left[ \sup_{x \in I} |\hat{B}(x)| \leq u \right| X_n \right] \geq a \right\},$$

which can be obtained by simulating the process $\hat{B}(\cdot)$ on a dense grid.

As an alternative to analytic estimation of the covariance kernel, it is possible to consider resampling methods as in Chernozhukov, Chetverikov, and Kato (2014a), Cheng and Chen (2019), Cattaneo, Farrell, and Feng (2020), and references therein. We relegate resampling-based inference for future research.
5 Extensions and Other Applications

We briefly outline some extensions of our main results. First, we introduce a re-weighted version of \( \hat{\theta} \), which is useful in applications as illustrated in Section 6. Second, we discuss a new class of local regression estimators based on a non-random least-squares loss function, which has some interesting theoretical properties and may be of interest in some semiparametric settings. Finally, we discuss how to incorporate restrictions in the estimation procedure, employing the generic structure of the local basis \( R(\cdot) \).

5.1 Re-weighted Distribution Estimator

Suppose \((x_1, w_1), (x_2, w_2), \ldots, (x_n, w_n)\) is a random sample, where \(x_i\) is a continuous random variable with a smooth cumulative distribution function, but now \(w_i\) is an additional “weighting” variable, possibly random and involving unknown parameters. We consider the generic weighted distribution parameter

\[
H(x) = \mathbb{E}[w_i \mathbb{1}(x_i \leq x)],
\]

whose practical interpretation depends on the specific choice of \(w_i\).

We discuss some examples. If \(w_i = 1\), \(H(\cdot)\) becomes the distribution function \(F(\cdot)\), and hence the results above apply. If \(w_i\) is set to be a certain ratio of propensity scores for subpopulation membership, then the derivative \(dH(x)/dx\) becomes a counterfactual density function, as in Dinoardo, Fortin, and Lemieux (1996); see Section 6.1 below. If \(w_i\) is set to be a combination of the treatment assignment and treatment status variables, then the resulting derivative can be used to conduct specification testing in IV models, or if \(w_i\) is set to be a certain ratio of propensity scores for a binary instrument, then the derivative can be used to identify distributions of compliers, as in Imbens and Rubin (1997), Abadie (2003), and Kitagawa (2015); see Section 6.2 below. Other examples of applicability of this extension include bunching, missing data, measurement error, data combination, and treatment effect settings.

More generally, when weights are allowed for, there is another potentially interesting connection between the estimand \(dH(x)/dx\) and classical weighted averages featuring prominently in econometrics because \(dH(x)/dx = \mathbb{E}[w_i | x_i = x]f(x)\), which is useful in the context of partial means and related problems as in Newey (1994b).
Our main results extend immediately to allow for $\sqrt{n}$-consistent estimated weights $w_i$ or, more generally, to estimated weights that converge sufficiently fast. Specifically, we let $\hat{F}_{w,i}(x) = \frac{1}{n} \sum_{j=1}^{n} w_j \mathbb{1}(x_j \leq x)$ in place of $\hat{F}_i$, and investigate the large sample properties of our proposed estimator in (2) when $w_i$ is replaced by $\hat{w}_i = w_i(\hat{\beta})$ with $\hat{\beta}$ an $a_n$-consistent estimator, for some $a_n \to \infty$, and $w_i(\cdot)$ a known function of the data. That is, when estimated weights are used to construct the weighted empirical distribution function $\hat{F}_{w,i}(x)$. Provided that $a_n^{-1} \to 0$ sufficiently fast, this extra estimation step will not affect the asymptotic properties of our estimator of the density function or its derivatives (which will be true, for example, in parametric estimation cases, where $a_n = \sqrt{n}$ under regularity conditions). All the results reported in the previous sections apply to this extension, which we illustrate empirically below.

5.2 Local $L^2$ Distribution Estimators

The local regression distribution estimator is obtained from a least squares projection of the empirical distribution function onto a local basis, where the projection puts equal weights at all observations. That is, (2) employs an $L^2(\hat{F})$-projection

$$\hat{\theta}(x) = \arg\min_{\theta} \int \left( \hat{F}(u) - R(u-x) \theta \right)^2 K \left( \frac{u-x}{h} \right) d\hat{F}(u).$$

This representation motivates a general class of local $L^2$ distribution estimators given by

$$\hat{\theta}_G(x) = \arg\min_{\theta} \int \left( \hat{F}(u) - R(u-x) \theta \right)^2 K \left( \frac{u-x}{h} \right) dG(u)$$

for some measure $G$. We show in the SA that all our theoretical results continue to hold for $\hat{\theta}_G$, provided that $G$ is absolutely continuous with respect to the Lebesgue measure and the Radon-Nikodym derivative is reasonably smooth. (Note that $G$ does not need to be a proper distribution function.)

The estimator $\hat{\theta}_G$ involves only one average, while the local regression estimator $\hat{\theta}$ has two layers of averages (one from the construction of the empirical distribution function, and the other from the $L^2(\hat{F})$-projection/regression). As a result, with suitable centering and scaling, the local $L^2$ distribution estimator, $\hat{\theta}_G$, can be written as the sum of a mean-zero influence function and a smoothing bias term. Since $\hat{\theta}_G$ no longer involves a second order U-statistic (c.f. (6)), or a leave-in
bias, pointwise asymptotic normality can be established under weaker conditions: it is no longer needed to assume \( nh^2 \to \infty \) (Theorem 1), and \( nh \to \infty \) will suffice. Similarly, for the strong approximation results we only need to restrict \( \log(n)/\sqrt{nh} \) as opposed to \( \log(n)/\sqrt{n h^2} \) (part 1 of Theorem 3).

In addition, the local \( L^2 \) distribution estimator \( \hat{\theta}_G \) is robust to “low” density. To see this, recall that the local regression estimator \( \hat{\theta} \) involves regressing the empirical distribution on a local basis, which means that this estimator can be numerically unstable if there are only a few observations near the evaluation point. More precisely, the matrix \( \hat{\Gamma} \) will be close to singular if the effective sample size is small.

Although the local \( L^2 \) distribution estimator \( \hat{\theta}_G \) takes a simpler form, is robust to low density, and its large sample properties can be established under weaker bandwidth conditions, it does have one drawback: it requires knowledge of the support \( \mathcal{X} \). To be more precise, let \( G \) be the Lebesgue measure, then the local \( L^2 \) distribution estimator may be biased at or near boundaries of \( \mathcal{X} \) if it is compact. In contrast, the local regression distribution estimator is fully boundary adaptive, even in cases where the location of the boundary is unknown. See Cattaneo, Jansson, and Ma (2020) for further discussion for the case of density estimation.

### 5.3 Incorporating Restrictions

The formulation (2) is general enough to allow for some interesting extensions in the definition of the local regression distribution estimator. The key observation is that the estimator has a weighted least squares representation with a generic local basis function \( R(\cdot) \), which allows for deploying well-know results from linear regression models. We briefly illustrate this idea with three examples.

First, consider the case where the local basis \( R(u) \) incorporates specific restrictions, such as continuity or lack thereof, on the distribution function, density function or higher-order derivatives at the evaluation point \( x \). To give a concrete example, suppose that \( F(x) \) and \( f(x) \) are known to be continuous at some interior point \( x \in \mathcal{X} \), while no information is available for the higher-order derivatives. Then, these restriction can be effortlessly incorporated to the local regression
distribution estimator by considering the local basis function

\[ R(u) = \left(1, u, \frac{u^2}{2}1(u < x), \frac{u^2}{2}1(u \geq x), \frac{u^3}{6}1(u < x), \frac{u^3}{6}1(u \geq x), \cdots, \frac{u^p}{p!}1(u < x), \frac{u^p}{p!}1(u \geq x)\right)'. \]

It follows that \( \hat{f}(x) = e'_1\hat{\theta}(x) \) consistently estimates the density \( f(x) \) at the kink point \( x \), while \( e'_2\hat{\theta}(x) \) and \( e'_3\hat{\theta}(x) \) are consistent estimators of the left and the right derivatives of the density function, respectively (and similarly for other higher-order one-sided derivatives). In this example, the generalized formulation not only reduces the bias of \( \hat{f}(x) = e'_1\hat{\theta}(x) \) even in the absence of continuity of higher-order derivatives, but also provides the basis for testing procedures for continuity of higher-order derivatives; e.g., by considering an statistic based on \( (e'_2 - e'_3)'\hat{\theta}(x) \). This provides a concrete illustration of the advantages of allowing for generic local basis. A distinct example was developed in Cattaneo, Jansson, and Ma (2018, 2020) for density discontinuity testing in regression discontinuity designs.

As a second example, consider imposing shape constraints, such as positivity or monotonicity, in the construction of the local regression distribution estimator. Such constraints amount to specific restrictions on the parameter space of \( \theta \), which naturally leads to restricted weighted least squares estimation in the context of our estimator. To be concrete, consider constructing a local polynomial density estimator which is non-negative, in which case \( R(u) \) is a polynomial basis of order \( p \geq 1 \) and (2) is extended to:

\[
\hat{\theta}(x) = \arg\min_{\theta} \sum_{i=1}^{n} W_i(\hat{F}_i - R'_i\theta)^2 \quad \text{subject to } T\theta \geq 0,
\]

where \( T \) denotes a matrix of restrictions; in this example, \( T = e'_1 \) to ensure that \( \hat{f}(x) = e'_1\hat{\theta}(x) \geq 0 \). This example showcases the advantages of the weighted least squares formulation of our estimator. Local monotonicity constraints, for instance, could also be easily incorporated in a similar fashion.

The final example of extensions of our basic local regression distribution estimation approach concerns non-identity link functions, leading to a non-linear least squares formulation. Specifically, (2) can be generalized to \( \hat{\theta}(x) = \arg\min_{\theta} \sum_{i=1}^{n} W_i(\hat{F}_i - \Lambda(R'_i\theta))^2 \) for some known link function \( \Lambda(\cdot) \). For instance, such extension may be useful to model distributions with large support or to impose specific local shape constraints.

All of the examples above, as well as many others, can be analyzed using the large sample
results developed in this paper and proper extensions thereof. We plan to investigate these and other extensions in future research.

6 Applications

We discuss two applications of our main results in the context of program evaluation (see Abadie and Cattaneo, 2018, and references therein).

6.1 Counterfactual Densities

In this first application, the objects of interest are density functions over their entire support, including boundaries and near-boundary regions, which are estimated using estimated weighting schemes, as this is a key feature needed for counterfactual analysis (and many other applications). Our general estimation strategy is specialized to the counterfactual density approach originally proposed by DiNardo, Fortin, and Lemieux (1996). We focus on density estimation, and we refer readers to Chernozhukov, Fernandez-Val, and Melly (2013) for related methods based on distribution functions as well as for an overview of the literature on counterfactual analysis.

To construct a counterfactual density or, more generally, re-weighted density estimators, we simply need to set the weights \( (w_1, w_2, \ldots, w_n) \) appropriately. In most applications, this also requires constructing preliminary consistent estimators of these weights, as we illustrate in this section. Suppose the observed data is \((x_i, t_i, z_i')\), \( i = 1, 2, \ldots, n \), where \( x_i \) continues to be the main outcome variable, \( z_i \) collects other covariates, and \( t_i \) is a binary variable indicating to which group unit \( i \) belongs. For concreteness, we call these two groups control and treatment, though our discussion does not need to bear any causal interpretation.

The marginal distribution of the outcome variable \( x_i \) for the full sample can be easily estimated without weights (that is, \( w_i = 1 \)). In addition, two conditional densities, one for each group, can be estimated using \( w_i^1 = t_i / P[t_i = 1] \) for the treatment group and \( w_i^0 = (1 - t_i) / P[t_i = 0] \) for the control group, and are denoted by \( \hat{f}_1(x) \) and \( \hat{f}_0(x) \), respectively. For example, in the context of randomized controlled trials, these density estimators can be useful to depict the distribution of the outcome variables for control and treatment units.

A more challenging question is: what would the outcome distribution have been, had the treated units had the same covariates distribution as the control units? The resulting density is called the
counterfactual density for the treated, which is denoted by \( f_{1>0}(x) \). Knowledge about this distribution is important for understanding differences between \( f_1(x) \) and \( f_0(x) \), as the outcome distribution is affected by both group status and covariates distribution. Furthermore, the counterfactual distribution has another useful interpretation: Assume the outcome variable is generated from potential outcomes, \( x_i = t_i x_i(1) + (1 - t_i) x_i(0) \), then under unconfoundedness, that is, assuming \( t_i \) is independent of \((x_i(0), x_i(1))'\) conditional on the covariates \( z_i \), \( f_{1>0}(x) \) is the counterfactual distribution for the control group: it is the density function associated with the distribution of \( x_i(1) \) conditional on \( t_i = 0 \).

Regardless of the interpretation taken, \( f_{1>0}(x) \) is of interest and can be estimated using our generic density estimator \( \hat{f}(x) \) with the following weights:

\[
w_i^{1>0} = t_i \cdot \frac{P[t_i = 0|z_i] P[t_i = 1]}{P[t_i = 1|z_i] P[t_i = 0]}.
\]

In practice, this choice of weighting scheme is unknown because the conditional probability \( P[t_i = 1|z_i] \), a.k.a. the propensity score, is not observed. Thus, researchers estimate this quantity using a flexible parametric model, such as Probit or Logit. Our technical results allow for these estimated weights to form counterfactual density estimators after replacing the theoretical weights by their estimated counterparts, provided the estimated weights converge sufficiently fast to their population counterparts.

To be more precise, we can model
\[
P[t_i = 1|z_i] = G(b(z_i)'\beta)
\]
for some known link function \( G(\cdot) \), such as Logit or Probit, and \( K \)-dimensional basis expansion \( b(z_i) \), such as power series or B-splines. If the model is correctly specified for some fixed \( K \) and basis function \( b(\cdot) \), then
\[
\max_{1 \leq i \leq n} |w_i - \hat{w}_i| = O_p(a_n^{-1})
\]
with \( a_n = \sqrt{n} \) under mild regularity conditions, and all our results carry over to the setting with estimated weights mentioned in Section 5.1. Alternatively, from a nonparametric perspective, if \( K \to \infty \) as \( n \to \infty \), and for appropriate basis function \( b(\cdot) \) and regularity conditions, \( \max_{1 \leq i \leq n} |w_i - \hat{w}_i| = O_p(a_n^{-1}) \) with \( a_n \) depending on both \( K \) and \( n \). Then, as in the parametric case, our main results carry over if \( a_n^{-1} \to 0 \) fast enough. The exact rate requirements can be deduced from the main theorems above.
Table 1: Summary Statistics for the JTPA data.

<table>
<thead>
<tr>
<th></th>
<th>Full</th>
<th>JTPA Offer</th>
<th>JTPA Enrollment</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>N</td>
<td>Y</td>
<td>N</td>
</tr>
<tr>
<td>Income</td>
<td>17949.20</td>
<td>17191.13</td>
<td>18321.59</td>
</tr>
<tr>
<td>HS or GED</td>
<td>0.72</td>
<td>0.71</td>
<td>0.72</td>
</tr>
<tr>
<td>Male</td>
<td>0.46</td>
<td>0.47</td>
<td>0.46</td>
</tr>
<tr>
<td>Nonwhite</td>
<td>0.36</td>
<td>0.36</td>
<td>0.36</td>
</tr>
<tr>
<td>Married</td>
<td>0.28</td>
<td>0.27</td>
<td>0.29</td>
</tr>
<tr>
<td>Work ≤ 12</td>
<td>0.44</td>
<td>0.43</td>
<td>0.44</td>
</tr>
<tr>
<td>AFDC</td>
<td>0.17</td>
<td>0.17</td>
<td>0.17</td>
</tr>
<tr>
<td>Age</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>22-25</td>
<td>0.24</td>
<td>0.25</td>
<td>0.24</td>
</tr>
<tr>
<td>26-29</td>
<td>0.21</td>
<td>0.20</td>
<td>0.21</td>
</tr>
<tr>
<td>30-35</td>
<td>0.24</td>
<td>0.25</td>
<td>0.24</td>
</tr>
<tr>
<td>36-44</td>
<td>0.19</td>
<td>0.19</td>
<td>0.19</td>
</tr>
<tr>
<td>45-54</td>
<td>0.08</td>
<td>0.08</td>
<td>0.08</td>
</tr>
<tr>
<td>Sample Size</td>
<td>9872</td>
<td>3252</td>
<td>6620</td>
</tr>
</tbody>
</table>

Columns: (i) Full: full sample; (ii) JTPA Offer: whether offered JTPA services; (iii) JTPA Enrollment: whether enrolled in JTPA. Rows: (i) Income: cumulative income over 30-month period post random selection; (ii) HS or GED: whether has high school degree or GED; (iii) Male: gender being male; (iv) Nonwhite: black or Hispanic; (v) Married: whether married; (vi) Work ≤ 12: worked less than 12 weeks during one year period prior to random assignment; (vii) Age: age groups.

6.1.1 Empirical Illustration

We demonstrate empirically how marginal, conditional, and counterfactual densities can be estimated with our proposed method. We consider the effect of education on earnings using a subsample of the data in Abadie, Angrist, and Imbens (2002). The data consists of individuals who did not enroll in the Job Training Partnership Act (JTPA). The main outcome variable is the sum of earnings in a 30-month period, and individuals are split into two groups according to their education attainment: \( t_i = 1 \) for those with high school degree or GED, and \( t_i = 0 \) otherwise. Also available are demographic characteristics, including gender, ethnicity, age, marital status, AFDC receipt (for women), and a dummy indicating whether the individual worked at least 12 weeks during a one-year period. The sample size is 5,447, with 3,927 being either high school graduates or GED. Summary statistics are available as the fourth column in Table 1. We leave further details on the JTPA program to Section 6.2, where we utilize a larger sample and conduct distribution estimation in a randomized controlled (intention-to-treat) and instrumental variables (imperfect compliance) setting.
It is well-known that education has significant impact on labor income, and we first plot earning distributions separately for subsamples with and without high school degree or GED. The two estimates, \( \hat{f}_1(x) \) and \( \hat{f}_0(x) \), are plotted in panel (a) of Figure 2. There, it is apparent that the earning distribution for high school graduates is very different compared to those without high school degree. More specifically, both the mean and median of \( \hat{f}_1(x) \) are higher than \( \hat{f}_0(x) \), and \( \hat{f}_1(x) \) seems to have much thinner left tail and thicker right tail.

As mentioned earlier, direct comparison between \( \hat{f}_1(x) \) and \( \hat{f}_0(x) \) does not reveal the impact of having high school degree on earning, since the difference is confounded by the fact that individuals with high school degree can have very different characteristics (measured by covariates) compared to those without. We employ covariates adjustments, and ask the following question: what would the earning distribution have been for high school graduates, had they had the same characteristics as those without such degree?
We estimate the counterfactual distribution \( f_{1>0}(x) \) by our proposed method, and is shown in panel (b) of Figure 2. The difference between \( \hat{f}_{1>0}(x) \) and \( \hat{f}_1(x) \) is not very profound, although it seems \( \hat{f}_{1>0}(x) \) has smaller mean and median. On the other hand, difference between \( \hat{f}_0(x) \) and \( \hat{f}_{1>0}(x) \) remains highly nontrivial. Our empirical finding is compatible with existing literature on return to education: it is generally believed that education leads to significant accumulation of human capital, hence increase in labor income. As a result, educational attainment is usually one of the most important “explanatory variables” for differences in income.

6.2 IV Specification and Heterogeneity

Self-selection and treatment effect heterogeneity are important concerns in causal inference and studies of socioeconomic programs. It is now well understood that classical treatment parameters, such as the average treatment effect or the treatment effect on the treated, are not identifiable even when treatment assignment is fully randomized due to imperfect compliance. Indeed, what can be recovered is either an intention-to-treat parameter or, using the instrumental variables method, some other more local treatment effect, specific to a subpopulation: the “compliers.” See Imbens and Rubin (2015) and references therein for further discussion. Practically, this poses two issues for empirical work employing instrumental variables methods focusing on local average treatment effects. First, since compliers are usually not identified, it is crucial to understand how different their characteristics are compared to the population as a whole. Second, it is often desirable to have a thorough estimate of the distribution of potential outcomes, which provides information not only on the mean or median, but also its dispersion, overall shape, or local curvatures.

Motivated by these observations, and to illustrate the applicability of our density estimation methods, we now consider two related problems. First, we investigate specification testing in the context of local average treatment effects based on comparison of two (rescaled) densities as discussed by Kitagawa (2015). This method requires estimating two densities nonparametrically. Second, we consider estimating the density of potential outcomes for compliers in the IV setting of Abadie (2003), which allows for conditioning on covariates. The resulting density plots not only provide visual guides on treatment effects, but also can be used for further analysis to construct a rich set of summary statistics or as inputs for semiparametric procedures. Both methods require estimated weights.
We first introduce the notation and the potential outcomes framework. For each individual there is a binary indicator of treatment assignment (a.k.a. the instrument), denoted by $d_i$. The actual treatment (takeup), however, can be different, due to imperfect compliance. More specifically, let $t_i(0)$ and $t_i(1)$ be the two potential treatments, corresponding to $d_i = 0$ and $1$, then the observed binary treatment indicator is $t_i = d_i t_i(1) + (1 - d_i) t_i(0)$. We also have a pair of potential outcomes, $x_i(0)$ and $x_i(1)$, associated with $t_i = 0$ and $1$, and what is observed is $x_i = t_i x_i(1) + (1 - t_i) x_i(0)$. Finally, also available are some covariates, collected in $z_i$. We assume that the observed data is a random sample $\{(x_i, t_i, d_i, z'_i) : 1 \leq i \leq n\}$.

There are three important assumptions for identification. First, the instrument has to be exogenous, meaning that conditional on covariates, it is independent of the potential treatments and outcomes. Second, the instrument has to be relevant, meaning that conditional on covariates, the instrument should be able to induce changes in treatment takeups. Third, there are no defiers (a.k.a. the monotonicity assumption). We do not reproduce the exact details of those assumptions and other technical requirements for identification; see the references given for more details.

Building on Imbens and Rubin (1997), Kitagawa (2015) discusses interesting testable implications in this IV setting, which can be easily adapted to test instrument validity using our density estimator. In the current context, the testable implications take the following form: for any (measurable) set $I \subset \mathbb{R}$,

$$\mathbb{P}[x_i \in I, \ t_i = 1|d_i = 1] \geq \mathbb{P}[x_i \in I, \ t_i = 1|d_i = 0],$$

and

$$\mathbb{P}[x_i \in I, \ t_i = 0|d_i = 0] \geq \mathbb{P}[x_i \in I, \ t_i = 0|d_i = 1].$$

The first requirement holds trivially in the JTPA context, since the program does not allow enrollment without being offered (that is, $\mathbb{P}[t_i = 1|d_i = 0] = 0$). Therefore we demonstrate the second with our density estimator. Let $f_{d=0,t=0}(x)$ be the earning density for the subsample $d_i = 0$ and $t_i = 0$, that is, for individuals without JTPA offer and not enrolled. Similarly let $f_{d=1,t=0}(x)$ be the earning density for individuals offered JTPA but not enrolled. Then the second inequality in the above display is equivalent to, for all $x \in \mathbb{R}$,

$$\mathbb{P}[t_i = 0|d_i = 0] \cdot f_{d=0,t=0}(x) \geq \mathbb{P}[t_i = 0|d_i = 1] \cdot f_{d=1,t=0}(x).$$
Thus, our density estimator can be used directly, where \( f_{d=0,t=0}(x) \) is consistently estimated with weights \( w_{d=0,t=0}^i = (1 - d_i)(1 - t_i)/\mathbb{P}[d_i = 0, t_i = 0] \), and \( f_{d=1,t=0}(x) \) is consistently estimated with \( w_{d=1,t=0}^i = d_i(1 - t_i)/\mathbb{P}[d_i = 1, t_i = 0] \).

Abadie (2003) showed that the distributional characteristics of compliers are identified, and can be expressed as re-weighted marginal quantities. We focus on three distributional parameters here. The first one is the distribution of the observed outcome variable, \( x_i \), for compliers, which is denoted by \( f_c \). This parameter is important for understanding the overall characteristics of compliers, and how different it is from the populations. The other two parameters are distributions of the potential outcomes, \( x_i(0) \) and \( x_i(1) \), for compliers, since the difference thereof reveals the effect of treatment for this subgroup. They are denoted by \( f_{c,0} \) and \( f_{c,1} \), respectively. The three density functions can also be estimated using our proposed local polynomial density estimator \( \hat{f}(x) \) with, respectively, the following weights:

\[
\begin{align*}
  w_c^i &= \frac{1}{\mathbb{P}[t_i(1) > t_i(0)]} \cdot \left( 1 - t_i(1 - d_i) \right) \cdot \left( 1 - t_i(1 - d_i)/\mathbb{P}[d_i = 0|z_i] \right) - \left( 1 - t_i(1 - d_i)/\mathbb{P}[d_i = 1|z_i] \right), \\
  w_{c,0}^i &= \frac{1}{\mathbb{P}[t_i(1) > t_i(0)]} \cdot (1 - t_i) \cdot \left( 1 - d_i - \mathbb{P}[d_i = 0|z_i]/\mathbb{P}[d_i = 1|z_i] \right), \\
  w_{c,1}^i &= \frac{1}{\mathbb{P}[t_i(1) > t_i(0)]} \cdot t_i \cdot \frac{d_i - \mathbb{P}[d_i = 1|z_i]}{\mathbb{P}[d_i = 0|z_i]\mathbb{P}[d_i = 1|z_i]}.
\end{align*}
\]

Here, the weights need to be estimated in practice, unless precise knowledge about the treatment assignment mechanism is available. As mentioned previously, our results allow for estimated weights such as those obtained by fitting a flexible Logit or Probit model to approximate the propensity score \( \mathbb{P}[d_i = 1|z_i] \) so long as they converge sufficiently fast to their population counterparts.

### 6.2.1 Empirical Illustration

The JTPA is a large publicly funded job training program targeting at individuals who are economically disadvantaged and/or facing significant barriers to employment. Individuals were randomly offered JTPA training, the treatment take-up, however, was only about 67% among those who were offered. Therefore the JTPA offer provides valid instrument to study the impact of the job training program. We continue to use the same data as Abadie, Angrist, and Imbens (2002), who analyzed quantile treatment effects on earning distributions.

Besides the main outcome variable and covariates already introduced in Section 6.1, also avail-
Figure 3: Testing Validity of Instruments, JTPA.

Notes: (i) JTPA: Not Offered & Not Enrolled: the scaled density estimate \( \frac{\sum_i 1(t_i=0,d_i=0) \hat{f}_{d=0,t=0}(x)}{\sum_i 1(d_i=0)} \hat{f}_{d=0,t=0}(x) \); (ii) JTPA: Offered & Not Enrolled: the scaled density estimate \( \frac{\sum_i 1(t_i=0,d_i=1) \hat{f}_{d=1,t=0}(x)}{\sum_i 1(d_i=1)} \hat{f}_{d=1,t=0}(x) \). Point estimates are obtained by using local polynomial regression with order 2, and robust confidence bands are obtained with local polynomial of order 3. Bandwidths are chosen by minimizing integrated mean squared errors. All estimates are obtained using companion R (and Stata) package described in Cattaneo, Jansson, and Ma (2021).

able are the treatment take-up (JTPA enrollment) and the instrument (JTPA Offer). See Table 1 for summary statistics for the full sample and separately for subgroups. As the JTPA offers were randomly assigned, it is possible to estimate the intent-to-treat effect by mean comparison. Indeed, individuals who are offered JTPA services earned, on average, $1,130 more than those not offered. On the other hand, due to imperfect compliance, it is in general not possible to estimate the effect of job training (i.e. the effect of JTPA enrollment), unless one is willing to impose strong assumptions such as constant treatment effect.

We first implement the IV specification test, which is straightforward using our density estimator \( \hat{f}(x) \). We plot the two estimated (rescaled) densities in Figure 3. A simple eyeball test suggests no evidence against instrumental variable validity. A formal hypothesis test, justified using our theoretical results, confirms this finding.
Second, we estimate the density of the potential outcomes for compliers. In panel (a) of Figure 4, we plot earning distributions for the full sample and that for the compliers, where the second is estimated using the weights \( w_i^c \), introduced earlier. The two distributions seem quite similar, while compliers tend to have higher mean and thinner left tail in the earning distribution. Next we consider the intent-to-treat effect, as the difference in earning distributions for subgroups with and without JTPA offer (a.k.a. the reduced form estimate in the 2SLS context). This is given in panel (b) of Figure 4. The effect is significant, albeit not very large. We also plot earning distributions for individuals enrolled (and not) in JTPA in panel (c). Not surprisingly, the difference is much larger. Simple mean comparison implies that enrolling in JTPA is associated with $2,083 more income.

Unfortunately, neither panel (b) nor (c) reveals information on distribution of potential outcomes. To see the reason, note that in panel (b) earning distributions are estimated according to treatment assignment, but potential outcomes are defined according to treatment takeup. And panel (c) does not give potential outcome distributions since treatment takeup is not randomly assigned. In panel (d) of Figure 4, we use weighting schemes \( w_i^{c,0} \) and \( w_i^{c,1} \) to construct potential earning distributions for compliers, which estimates the identified distributional treatment effect in this IV setting. Indeed, treatment effect on compliers is larger than the intent-to-treat effect, but is smaller than that in panel (c). The result is compatible with the fact that JTPA has positive and nontrivial effect on earning. Moreover, it demonstrates the presence of self-selection: those who participated in JTPA on average would benefit the most, followed by compliers who are regarded as “on the margin of indifference.”

7 Conclusion

We introduced a new class of local regression distribution estimator, which can be used to construct distribution, density, and higher-order derivatives estimators. We established valid large sample distributional approximations, both pointwise and uniform over their support. Pointwise on the evaluation point, we characterized a minimum distance implementation based on redundant regressors leading to asymptotic efficiency improvements, and gave precise results in terms of (tight) lower bounds for interior points. Uniformly over the evaluation points, we obtained valid linearizations and strong approximations, and constructed confidence bands. Finally, we discussed several
Figure 4: Earning Distributions, JTPA.

Notes: (a) earning distributions in the full sample and for compliers; (b) earning distributions by JTPA offer; (c) earning distributions by JTPA enrollment; (d) distributions of potential outcomes for compliers. Point estimates are obtained by using local polynomial regression with order 2, and robust confidence bands are obtained with local polynomial of order 3. Bandwidths are chosen by minimizing integrated mean squared errors. All estimates are obtained using companion R (and Stata) package described in Cattaneo, Jansson, and Ma (2021).
extensions of our work.

Although beyond the scope of this paper, it would be useful to generalize our results to the case of multivariate regressors \( x_i \in \mathbb{R}^d \). Boundary adaptation is substantially more difficult in multiple dimensions, and hence our proposed methods are potentially very useful in such setting. In addition, multidimensional density estimation can be used to construct new conditional distribution, density and higher derivative estimators in a straightforward way. These new estimators would be useful in several areas of economics, including for instance estimation of auction models.

References


Local Regression Distribution Estimators*

Supplemental Appendix

Matias D. Cattaneo† Michael Jansson‡ Xinwei Ma§

January 29, 2021

Abstract

This Supplemental Appendix contains general theoretical results encompassing those discussed in the main paper, includes proofs of those general results, and discusses additional methodological and technical results.

---

* Cattaneo gratefully acknowledges financial support from the National Science Foundation through grant SES-1947805, and Jansson gratefully acknowledges financial support from the National Science Foundation through grant SES-1947662 and the research support of CREATES.
† Department of Operations Research and Financial Engineering, Princeton University.
‡ Department of Economics, UC Berkeley and CREATES.
§ Department of Economics, UC San Diego.
## Contents

1 Setup ..................................................................................................................................... 3

2 Pointwise Distribution Theory .............................................................................................. 5
   2.1 Local $L^2$ Distribution Estimation ................................................................................. 5
   2.2 Local Regression Distribution Estimation..................................................................... 7

3 Efficiency ............................................................................................................................... 9
   3.1 Effect of Orthogonalization........................................................................................... 10
   3.2 Optimal $Q$ ..................................................................................................................... 12

4 Uniform Distribution Theory ................................................................................................ 19
   4.1 Local $L^2$ Distribution Estimation ................................................................................. 22
   4.2 Local Regression Distribution Estimation..................................................................... 24

5 Proofs .................................................................................................................................... 27
   5.1 Proof of Theorem 1....................................................................................................... 27
   5.2 Proof of Theorem 2....................................................................................................... 28
   5.3 Proof of Theorem 3....................................................................................................... 28
   5.4 Proof of Theorem 4....................................................................................................... 30
   5.5 Proof of Corollary 5...................................................................................................... 32
   5.6 Proof of Corollary 6...................................................................................................... 33
   5.7 Proof of Lemma 7 ........................................................................................................ 34
   5.8 Proof of Theorem 8....................................................................................................... 35
   5.9 Additional Preliminary Lemmas ................................................................................... 36
   5.10 Proof of Theorem 9....................................................................................................... 38
   5.11 Proof of Lemma 10....................................................................................................... 39
   5.12 Proof of Lemma 11....................................................................................................... 41
   5.13 Proof of Lemma 12....................................................................................................... 41
   5.14 Proof of Theorem 13 ..................................................................................................... 42
   5.15 Proof of Theorem 14 ..................................................................................................... 42
   5.16 Proof of Lemma 15....................................................................................................... 43
   5.17 Proof of Lemma 16....................................................................................................... 43
   5.18 Proof of Lemma 17....................................................................................................... 46
   5.19 Proof of Lemma 18....................................................................................................... 47
   5.20 Proof of Theorem 19 ..................................................................................................... 47
   5.21 Proof of Theorem 20 ..................................................................................................... 48
1 Setup

Suppose \( x_1, x_2, \cdots, x_n \) is a random sample from a univariate distribution with cumulative distribution function \( F(\cdot) \). Also assume the distribution function admits a (sufficiently accurate) linear-in-parameters local approximation near an evaluation point \( x \):

\[
g(h, x) := \sup_{|x-x| \leq h} |F(x) - R(x-x)'\theta(x)| \text{ is small for } h \text{ small},
\]

where \( R(\cdot) \) is a known basis function. The parameter \( \theta(x) \) can be estimated by the following local \( L^2 \) method:

\[
\hat{\theta}_G = \arg\min_{\theta} \int_{\mathcal{X}} \left( \hat{F}(u) - R(u-x)'\theta \right)^2 \frac{1}{h} K \left( \frac{u-x}{h} \right) \, dG(u), \quad \hat{F}(u) = \frac{1}{n} \sum_{i=1}^{n} \mathds{1}(x_i \leq u), \tag{1}
\]

where \( K(\cdot) \) is a kernel function, \( \mathcal{X} \) is the support of \( F(\cdot) \), and \( G(\cdot) \) is a known weighting function to be specified later. The local \( L^2 \) estimator (1) is closely related to another estimator, which is constructed by local regression:

\[
\hat{\theta} = \arg\min_{\theta} \sum_{i=1}^{n} \left( \hat{F}(x_i) - R(x_i-x)'\theta \right)^2 \frac{1}{h} K \left( \frac{x_i-x}{h} \right). \tag{2}
\]

The local regression estimator can be equivalently expressed as \( \hat{\theta}_F \), meaning that it can be viewed as a special case of the local \( L^2 \) estimator, with \( G(\cdot) \) in (1) replaced by the empirical distribution function \( \hat{F}(\cdot) \).

For future reference, we first discuss some of the notation we use in the main paper and this Supplemental Appendix (SA). For a function \( g(\cdot) \), we denote its \( j \)-th derivative as \( g^{(j)}(\cdot) \). For simplicity, we also use the “dot” notation to denote the first derivative: \( \dot{g}(\cdot) = g^{(1)}(\cdot) \). Assume \( g(\cdot) \) is suitably smooth on \([x-\delta, x) \cup (x, x+\delta]\) for some \( \delta > 0 \), but not necessarily continuous or differentiable at \( x \). If \( g(\cdot) \) and its one-sided derivatives can be continuously extended to \( x \) from the two sides, we adopt the following special notation:

\[
g_u^{(j)} = \mathds{1}(u < 0)g^{(j)}(x-) + \mathds{1}(u \geq 0)g^{(j)}(x+).
\]

With \( j = 0 \), the above is simply \( g_u = \mathds{1}(u < 0)g(x-) + \mathds{1}(u \geq 0)g(x+) \). Also for \( j = 1 \), we use \( \dot{g}_u = g_u^{(1)} \). Convergence in probability and in distribution are denoted by \( \Rightarrow \) and \( \rightsquigarrow \), respectively, and limits are taken with respect to the sample size \( n \) going to infinity unless otherwise specified. We use \( | \cdot | \) to denote the Euclidean norm.

The following matrices will feature in asymptotic expansions of our estimators:

\[
\Gamma_{h,x} = \int_{-\infty}^{x} \frac{R(u)R(u)'K(u)g(x+hu)}{h^2} \, du = \int_{-\infty}^{x} R(u)R(u)'K(u)g_u \, du + O(h) = \Gamma_{1h,x} + O(h),
\]

3
\[ \Sigma_{h,x} = \int_{X} R(u)R(v) \left[ F(x + h (u \wedge v)) - F(x + h u)F(x + hv) \right] K(u) K(v) g(x + hu)g(x + hv) dudv \]

\[ = F(x)(1 - F(x)) \left( \int_{X} R(u)K(u)g_u du \right) \left( \int_{X} R(u)K(u)g_u du \right) \]

\[ + h \int_{X} R(u)R(v)K(u)K(v) \left[ f(x) (uf_u + vf_v)g_u g_v + F(x)(1 - F(x)) (u\dot{g}_u g_v + v\dot{g}_v g_u) \right] dudv \]

\[ + h \int_{X} R(u)R(v)K(u)K(v)(u \wedge v)f_{u \wedge v} g_u g_v dudv + O(h^2) \]

\[ = \Sigma_{1h,x} + h \Sigma_{2h,x} + O(h^2). \]

Now we list the main assumptions.

**Assumption 1.** \( x_1, \ldots, x_n \) is a random sample from a distribution \( F(\cdot) \) supported on \( X \subseteq \mathbb{R} \), and \( x \in X \).

(i) For some \( \delta > 0 \), \( F(\cdot) \) is absolutely continuous on \( [x - \delta, x + \delta] \) with a density \( f(\cdot) \) admitting constants \( f(x-) \), \( f(x+) \), \( \dot{f}(x-) \), and \( \dot{f}(x+) \), such that

\[ \sup_{u \in [-\delta,0]} \frac{f(x + u) - f(x) - uf(x-)}{u^2} + \sup_{u \in (0,\delta]} \frac{f(x + u) - f(x+) - u\dot{f}(x+)}{u^2} < \infty. \]

(ii) \( K(\cdot) \) is nonnegative, symmetric, and continuous on its support \([ -1, 1 ]\), and integrates to 1.

(iii) \( R(\cdot) \) is locally bounded, and there exists a positive-definite diagonal matrix \( \Upsilon_h \) for each \( h > 0 \), such that \( \Upsilon_h R(u) = R(u/h) \)

(iv) For all \( h \) sufficiently small, the minimum eigenvalues of \( \Gamma_{h,x} \) and \( h^{-1} \Sigma_{h,x} \) are bounded away from zero.

**Assumption 2.** For some \( \delta > 0 \), \( G(\cdot) \) is absolutely continuous on \([ x - \delta, x + \delta ]\) with a derivative \( g(\cdot) \geq 0 \) admitting constants \( g(x-) \), \( g(x+) \), \( \dot{g}(x-) \), and \( \dot{g}(x+) \), such that

\[ \sup_{u \in [-\delta,0]} \frac{g(x + u) - g(x) - u\dot{g}(x-)}{u^2} + \sup_{u \in (0,\delta]} \frac{g(x + u) - g(x+) - u\dot{g}(x+)}{u^2} < \infty. \]

**Example 1 (Local Polynomial Estimator).** Before closing this section, we briefly introduce the local polynomial estimator of Cattaneo, Jansson, and Ma (2020), which is a special case of our local regression distribution estimator. The local polynomial estimator employs the following polynomial basis:

\[ R(u) = \left( 1, u, \frac{1}{2} u^2, \ldots, \frac{1}{p!} u^p \right) \]

for some \( p \in \mathbb{N} \). As a result, it estimates the distribution function, the density function, and
derivatives thereof. To be precise,
\[ \theta(x) = \left( F(x), f(x), f^{(1)}(x), \ldots, f^{(p-1)}(x) \right)'. \]

With \( R(\cdot) \) being a polynomial basis, it is straightforward to characterize the approximation bias \( \varrho(h, x) \). Assuming the distribution function \( F(\cdot) \) is at least \( p+1 \) times continuously differentiable in a neighborhood of \( x \), one can employ a Taylor expansion argument and show that \( \varrho(h, x) = O(h^{p+1}) \).

We will revisit this local polynomial estimator below as a leading example when we discuss pointwise and uniform asymptotic properties of our local distribution estimator.

\[ \square \]

2 Pointwise Distribution Theory

We discuss pointwise (i.e., for a fixed evaluation point \( x \in X \)) large-sample properties of the local \( L^2 \) estimator (1), and that of the local regression estimator (2). For ease of exposition, we suppress the dependence on the evaluation point \( x \) whenever possible.

2.1 Local \( L^2 \) Distribution Estimation

With simple algebra, the local \( L^2 \) estimator in (1) takes the following form
\[
\hat{\theta}_G = \left( \int_X R(u - x) R(u - x)' \frac{1}{h} K \left( \frac{u - x}{h} \right) dG(u) \right)^{-1} \left( \int_X R(u - x) \hat{F}(u) \frac{1}{h} K \left( \frac{u - x}{h} \right) dG(u) \right).
\]

We can further simplify the above. First note that the “denominator” can be rewritten as
\[
\int_X R(u - x) R(u - x)' \frac{1}{h} K \left( \frac{u - x}{h} \right) dG(u) = \Upsilon_h^{-1} \left( \int_X \Upsilon_h R(u - x) R(u - x)' \Upsilon_h \frac{1}{h} K \left( \frac{u - x}{h} \right) g(u) du \right) \Upsilon_h^{-1} = \Upsilon_h^{-1} \Gamma_h \Upsilon_h^{-1}.
\]

The same technique can be applied to the “numerator”, which leads to
\[
\hat{\theta}_G - \theta = \Upsilon_h \Gamma_h^{-1} \left( \int_{\frac{X-x}{h}} R(u) \left[ \hat{F}(x + hu) K(u) g(x + hu) du \right] - \theta \right)
\]
\[
= \Upsilon_h \Gamma_h^{-1} \int_{\frac{X-x}{h}} R(u) \left[ F(x + hu) - \theta R(u) \Upsilon_h^{-1} K(u) g(x + hu) du \right]
\]
\[
+ \Upsilon_h \frac{1}{n} \sum_{i=1}^{n} \Gamma_h^{-1} \int_{\frac{X-x}{h}} R(u) \left[ 1(x_i \leq x + hu) - F(x + hu) \right] K(u) g(x + hu) du. \quad (3)
\]

The above provides a further expansion of the local \( L^2 \) estimator into a term that contributes as bias, and another term that contributes asymptotically to the variance.

The large-sample properties of the local \( L^2 \) estimator (1) are as follows.
Theorem 1 (Local $L^2$ Distribution Estimation: Asymptotic Normality). Assume Assumptions 1 and 2 hold, and that $h \to 0$, $nh \to \infty$ and $n\varrho(h)^2/h \to 0$. Then (i) (3) satisfies

$$\left| \int_{-\infty}^{x-x/h} R(u) \left[ F(x+h) - \theta' R(u) \Upsilon_h^{-1} \right] K(u) du \right| = O(\varrho(h)).$$

(ii) (4) satisfies

$$\mathbb{V} \left[ \int_{-\infty}^{x-x/h} R(u) \left[ I(x_i \leq x+h) - F(x+h) \right] K(u) g(x+h) du \right] = \Sigma_h,$$

and

$$\Sigma_h^{-1/2} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \int_{-\infty}^{x-x/h} R(u) \left[ I(x_i \leq x+h) - F(x+h) \right] K(u) g(x+h) du \right) \sim \mathcal{N}(0, I).$$

(iii) The local $L^2$ estimator is asymptotically normally distributed

$$\sqrt{n} \left( \Gamma_h^{-1} \Sigma_h \Gamma_h^{-1} \right)^{-1/2} \Upsilon_h^{-1} (\widehat{\theta}_G - \theta) \sim \mathcal{N}(0, I).$$

For valid inference, one needs to construct standard errors. To start, note that $\Gamma_h$ is known, and hence we only need to estimate $\Sigma_h$. Consider the following:

$$\hat{\Sigma}_h = \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{x-x/h} R(u) R(v) \left[ I(x_i \leq x+h) - \hat{F}(x+h) \right] \left[ I(x_i \leq x+h) - \hat{F}(x+h) \right] K(u) K(v) g(x+h) g(x+h) dv du,$$

where $\hat{F}(\cdot)$ is the empirical distribution function. The following theorem shows that standard errors constructed using $\hat{\Sigma}_h$ are consistent.

Theorem 2 (Local $L^2$ Distribution Estimation: Standard Errors). Assume Assumptions 1 and 2 hold, and that $h \to 0$ and $nh \to \infty$. Let $c$ be a nonzero vector of suitable dimension, then

$$\left| \frac{c' \hat{\Sigma}_h c}{c' \Sigma_h c} - 1 \right| = O_p \left( \sqrt{\frac{1}{nh}} \right).$$

If, in addition that $n\varrho(h)^2/h \to 0$, then

$$\frac{c' (\hat{\theta}_G - \theta)}{\sqrt{c' \Upsilon_h \Gamma_h^{-1} \hat{\Sigma}_h \Gamma_h^{-1} \Upsilon_h c/n}} \sim \mathcal{N}(0, 1).$$
2.2 Local Regression Distribution Estimation

The local regression distribution estimator (2) can be understood as a special case of the local $L^2$ estimator by setting $G = \hat{F}$ (i.e., using the empirical distribution as the design). However, the empirical measure $\hat{F}$ is not smooth, so that large-sample properties of the local regression estimator cannot be deduced directly from Theorem 1. In this subsection, we will show that estimates obtained by the two approaches, (1) and (2), are asymptotically equivalent under suitable regularity conditions. To be precise, we establish the equivalence of the local regression distribution estimator, $\hat{\theta}$, and the (infeasible) local $L^2$ distribution estimator, $\hat{\theta}_F$ (i.e., using $F$ as the design weighting in (1)). As before, we suppress the dependence on the evaluation point $x$.

First, the local regression estimator can be written as

$$\hat{\theta} - \theta = \left( \frac{1}{n} \sum_{i=1}^{n} R(x_i - x) R(x_i - x)' \frac{1}{h} K \left( \frac{x_i - x}{h} \right) \right)^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} R(x_i - x) \left[ \hat{F}(x_i) - R(x_i - x)' \theta \right] \frac{1}{h} K \left( \frac{x_i - x}{h} \right) \right)$$

$$= \Upsilon_h \hat{\Gamma}_h^{-1} \Gamma_h \Gamma_h^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} \Upsilon_h R(x_i - x) \left[ \hat{F}(x_i) - R(x_i - x)' \theta \right] \frac{1}{h} K \left( \frac{x_i - x}{h} \right) \right),$$

where

$$\hat{\Gamma}_h = \frac{1}{n} \sum_{i=1}^{n} \Upsilon_h R(x_i - x) R(x_i - x)' \Upsilon_h \frac{1}{h} K \left( \frac{x_i - x}{h} \right),$$

and $\Gamma_h$ is defined as before with $G = F$.

To proceed, we further expand as follows

$$\frac{1}{n} \sum_{i=1}^{n} \Upsilon_h R(x_i - x) \left[ \hat{F}(x_i) - R(x_i - x)' \theta \right] \frac{1}{h} K \left( \frac{x_i - x}{h} \right)$$

$$= \frac{1}{n^2} \sum_{i,j=1, i \neq j}^{n} \Upsilon_h R(x_j - x) \left[ 1(x_i \leq x_j) - F(x_j) \right] \frac{1}{h} K \left( \frac{x_j - x}{h} \right)$$

$$+ \frac{1}{n^2} \sum_{j=1}^{n} \Upsilon_h R(x_j - x) \left[ 1 - F(x_j) \right] \frac{1}{h} K \left( \frac{x_j - x}{h} \right)$$

$$+ \frac{1}{n} \sum_{j=1}^{n} \Upsilon_h R(x_j - x) \left[ F(x_j) - R(x_j - x)' \theta \right] \frac{1}{h} K \left( \frac{x_j - x}{h} \right).$$

(6)

The last two terms correspond to the leave-in bias and the approximation bias, respectively. We
further decompose the first term with conditional expectation:

\[
\frac{1}{n^2} \sum_{i,j=1, i \neq j}^{n} \Upsilon_h R(x_j - x) \left[ \mathbb{1}(x_i \leq x_j) - F(x_j) \right] \frac{1}{h} K \left( \frac{x_j - x}{h} \right)
\]

\[
= \frac{1}{n^2} \sum_{i,j=1, i \neq j}^{n} \mathbb{E} \left[ \Upsilon_h R(x_j - x) \left[ \mathbb{1}(x_i \leq x_j) - F(x_j) \right] \frac{1}{h} K \left( \frac{x_j - x}{h} \right) \right] |x_i|
\]

\[+
\frac{1}{n^2} \sum_{i,j=1, i \neq j}^{n} \Upsilon_h R(x_j - x) \left[ \mathbb{1}(x_i \leq x_j) - F(x_j) \right] \frac{1}{h} K \left( \frac{x_j - x}{h} \right)
\]

\[- \mathbb{E} \left[ \Upsilon_h R(x_j - x) \left[ \mathbb{1}(x_i \leq x_j) - F(x_j) \right] \frac{1}{h} K \left( \frac{x_j - x}{h} \right) \right] |x_i|
\]

\[=
\frac{n-1}{n^2} \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} R(u) \left[ \mathbb{1}(x_i \leq x + hu) - F(x + hu) \right] K(u) f(x + hu) du \quad (8)
\]

\[+ \frac{1}{n^2} \sum_{i,j=1, i \neq j}^{n} \Upsilon_h R(x_j - x) \left[ \mathbb{1}(x_i \leq x_j) - F(x_j) \right] \frac{1}{h} K \left( \frac{x_j - x}{h} \right)
\]

\[- \int_{x_i}^{x_{i+1}} R(u) \left[ \mathbb{1}(x_i \leq x + hu) - F(x + hu) \right] K(u) f(x + hu) du. \quad (9)
\]

The following theorem studies the large-sample properties of each term in the above decomposition, and shows that the local regression distribution estimator is asymptotically equivalent to the local $L^2$ estimator by setting $G = F$, and hence it is asymptotically normally distributed.

**Theorem 3 (Local Regression Distribution Estimation: Asymptotic Normality).** Assume Assumption 1 holds, and that $h \to 0$, $nh^2 \to \infty$ and $n(\varrho(h))^2/h \to 0$. Then

(i) \( \hat{\Gamma}_h \) satisfies

\[
|\hat{\Gamma}_h - \Gamma_h| = O_P \left( \sqrt{\frac{1}{nh}} \right).
\]

(ii) (6) and (7) satisfy

\[
(6) = O_P \left( \frac{1}{n} \right), \quad (7) = O_P (\varrho(h)).
\]

(iii) (9) satisfies

\[
(9) = O_P \left( \sqrt{\frac{1}{n^2h}} \right).
\]

(iv) The local regression distribution estimator (2) satisfies

\[
\sqrt{n} \left( \Gamma_h^{-1} \Sigma_h \Gamma_h^{-1} \right)^{-1/2} \Upsilon_h^{-1} \left( \hat{\theta} - \theta \right) = \sqrt{n} \left( \Gamma_h^{-1} \Sigma_h \Gamma_h^{-1} \right)^{-1/2} \Upsilon_h^{-1} \left( \hat{\theta}_F - \theta \right) + o_P(1) \sim \mathcal{N}(0, I).
\]

We now discuss how to construct standard errors in the local regression framework. Note

\[
\end{equation}
\]
that $\Gamma_h$ can be estimated by $\hat{\Gamma}_h$, whose properties have already been studied in Theorem 3(i). To estimate $\Sigma_h$, we propose the following

\[
\hat{\Sigma}_h = \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{1}{n} \sum_{j=1}^{n} Y_h R(x_j - x) \left( \mathbb{1}(x_i \leq x_j) - \hat{F}(x_j) \right) \frac{1}{h} K \left( \frac{x_j - x}{h} \right) \right] \cdot \left[ \frac{1}{n} \sum_{j=1}^{n} Y_h R(x_j - x) \left( \mathbb{1}(x_i \leq x_j) - \hat{F}(x_j) \right) \frac{1}{h} K \left( \frac{x_j - x}{h} \right) \right]'.
\]

where $\hat{F}(\cdot)$ is the empirical distribution function. The following theorem shows that standard errors constructed using $\hat{\Sigma}_h$ are consistent.

**Theorem 4 (Local Regression Distribution Estimation: Standard Errors).** Assume Assumption 1 holds. In addition, assume $h \to 0$ and $nh^2 \to \infty$. Let $c$ be a nonzero vector of suitable dimension. Then

\[
\left| \frac{c' \hat{\Gamma}_h^{-1} \hat{\Sigma}_h \hat{\Gamma}_h^{-1} c}{c' \hat{\Gamma}_h^{-1} \Sigma_h \hat{\Gamma}_h^{-1} c} - 1 \right| = O_p \left( \sqrt{\frac{1}{nh^2}} \right).
\]

If, in addition that $n\varrho(h)^2/h \to 0$, one has

\[
\frac{c' (\hat{\theta} - \theta)}{\sqrt{c' Y_h \hat{\Gamma}_h^{-1} \Sigma_h \hat{\Gamma}_h^{-1} Y_h c/n}} \sim \mathcal{N}(0, 1).
\]


3 Efficiency

For ease of presentation, we focus on the (infeasible) local $L^2$ distribution estimator $\hat{\theta}_F$,

\[
\hat{\theta}_F = \arg\min_{\theta} \int_X \left( \hat{F}(u) - R(u - x)^\prime \theta \right)^2 \frac{1}{h} K \left( \frac{u - x}{h} \right) dF(u),
\]

but all the results in this section are applicable to the local regression distribution estimator $\hat{\theta}$, as we showed earlier that it is asymptotically equivalent to $\hat{\theta}_F$. In addition, we consider a specific basis:

\[
R(u) = \left( 1, \ P(u)\prime, \ Q(u)\prime \right)\prime,
\]

where $P(u)$ is a polynomial basis of order $p$:

\[
P(u) = \left( u, \ \frac{1}{2}u^2, \ \cdots, \ \frac{1}{p!}u^p \right)\prime,
\]
and \( Q(u) \) is a scalar function, and hence is a “redundant regressor.” Without \( Q(\cdot) \), the above reduces to the local polynomial estimator of Cattaneo, Jansson, and Ma (2020). See Section 1 and Example 1 for an introduction.

We consider additional regressors because they may help improve efficiency (i.e., reduce the asymptotic variance). Following Assumption 1, we assume there exists a scalar \( \upsilon_h \) (depending on \( h \)) such that \( \upsilon_h Q(u) = Q(u/h) \). Therefore, \( \Upsilon_h \) is a diagonal matrix containing \( h^{-1}, h^{-2}, \cdots, h^{-p}, \upsilon_h \).

As we consider a (local) polynomial basis, it is natural to impose smoothness assumptions on \( F(\cdot) \).

**Assumption 3.** For some \( \delta > 0 \), \( F(\cdot) \) is \((p+1)\)-times continuously differentiable in \( \mathcal{X} \cap [x-\delta, x+\delta] \) for some \( p \geq 1 \), and \( G(\cdot) \) is twice continuously differentiable in \( \mathcal{X} \cap [x-\delta, x+\delta] \).

Under the above assumption, the approximation error satisfies \( \varrho(h) = O(h^{p+1}) \), and the parameter \( \theta \) can be partitioned into the following:

\[
\theta = \left( \theta_1, \theta_P, \theta_Q \right) = \left( F(x), f(x), \cdots, f^{(p-1)}(x), 0 \right).
\]

We first state a corollary, which specializes Theorem 1 to the polynomial basis (11).

**Corollary 5 (Local Polynomial \( L^2 \) Distribution Estimation: Asymptotic Normality).** Assume Assumptions 1 and 3 hold, and that \( h \to 0, nh \to \infty \), and \( n \varrho(h)^2/h \to 0 \). Then the local polynomial \( L^2 \) distribution estimator in (10) satisfies

\[
\sqrt{n} \left( \Gamma_h^{-1} \Sigma_h \Gamma_h^{-1} \right)^{-1/2} \Upsilon_h^{-1} (\hat{\theta}_F - \theta) \sim \mathcal{N}(0, I).
\]

### 3.1 Effect of Orthogonalization

To start, consider the following (sequentially) orthogonalized basis:

\[
R^\perp(u) = \left( 1, P^\perp(u)', Q^\perp(u)' \right)',
\]

where

\[
P^\perp(u) = P^\perp(u) - \int_{x-x}^{x+x} K(u)P(u)du,
\]

\[
Q^\perp(u) = Q(u) - \left( 1, P(v)' \right) \left( \int_{x-x}^{x+x} K(v) \left( 1, P(v)' \right) \left( 1, P(v)' \right)^{-1} \left( \int_{x-x}^{x+x} K(v) \left( 1, P(v)' \right)^{-1} Q(v)dv \right) \right) \left( 1, P(v)' \right)'.
\]

The above transformation can be represented by the following:

\[
R^\perp(u) = \Lambda_h R(u),
\]

where \( \Lambda_h \) is a nonsingular upper triangular matrix. (Note that the matrix \( \Lambda_h \) depends on the bandwidth only because we would like to handle both interior and boundary evaluation points. If,
for example, we fix the evaluation point to be in the interior of the support of the data, then \( \Lambda_h \) is a fixed matrix and no longer depends on \( h \). Alternatively, one could also use the notation \( \Lambda_x \) to denote such dependence.) Now consider the following orthogonalized local polynomial \( L^2 \) estimator

\[
\hat{\theta}_F^\perp = \arg\min_{\theta} \int_X \left( \hat{F}(u) - \Lambda_h^t R(u - x)^t \theta \right)^2 \frac{1}{h} K \left( \frac{u - x}{h} \right) dF(u).
\] (13)

To discuss its properties, we partition the estimator and the target parameter as

\[
\hat{\theta}_F^\perp = \left( \hat{\theta}_{1,F}^\perp, (\hat{\theta}_{P,F}^\perp, \hat{\theta}_{Q,F}^\perp) \right),
\]

where \( \hat{\theta}_{1,F}^\perp \) is the first element of \( \hat{\theta}_F^\perp \) and \( \hat{\theta}_{Q,F}^\perp \) is the last element of \( \hat{\theta}_F^\perp \). Similarly, we can partition the target parameter,

\[
\theta^\perp = \Lambda_h^{-1} \theta = \left( \theta_{1}^\perp, (\theta_{P}^\perp, \theta_{Q}^\perp) \right),
\]

so that \( \theta_{1}^\perp \) is the first element of \( \Lambda_h^{-1} \theta \) and \( \theta_{Q}^\perp \) is the last element of \( \Lambda_h^{-1} \theta \). As \( \theta_Q = 0 \), simple least squares algebra implies

\[
\theta^\perp = \left( \theta_{1}^\perp, \theta_{P}^\perp, 0 \right)' = \left( \theta_{1}^\perp, f(x), f^{(1)}(x), \ldots, f^{(p-1)}(x), 0 \right)',
\]

Note that, in general, \( \theta_{1}^\perp \neq \theta_1 \), meaning that after orthogonalization, the intercept of the local polynomial estimator no longer estimates the distribution function \( F(x) \).

The following corollary gives the large-sample properties of the orthogonalized local polynomial estimator, excluding the intercept.

**Corollary 6 (Orthogonalized Local Polynomial \( L^2 \) Distribution Estimation: Asymptotic Normality).** Assume Assumptions 1 and 3 hold, and that \( h \to 0, nh \to \infty \), and \( n \theta(h)^2/h \to 0 \). Then the orthogonalized local polynomial \( L^2 \) distribution estimator in (13) satisfies

\[
\left[ \left( \Gamma_{P,h} \right)^{-1} \Sigma_{P,P,h} \left( \Gamma_{P,h} \right)^{-1} \right]^{-1/2} \left[ \left( \Gamma_{Q,h} \right)^{-1} \Sigma_{Q,Q,h} \left( \Gamma_{Q,h} \right)^{-1} \right]^{-1/2} \sqrt{\frac{n}{h f(x)}} Y^{-1} \left[ \hat{\theta}_{P,E}^\perp - \theta_P^\perp \right] \sim \mathcal{N}(0, I),
\]

where

\[
\Gamma_{P,h} = \int_{X-h}^{X+h} P(u)^t P(u)' K(u) du, \quad \Gamma_{Q,h} = \int_{X-h}^{X+h} Q(u)^t Q(u) K(u) du,
\]

\[
\Sigma_{P,P,h} = \int_{X-h}^{X+h} K(u) K(v) P(u)^t P(v)' (u \wedge v) dv du,
\]

\[
\Sigma_{Q,Q,h} = \int_{X-h}^{X+h} K(u) K(v) Q(u)^t Q(v)' (u \wedge v) dv du,
\]

\[
\Sigma_{P,Q,h} = (\Sigma_{Q,P,h})' = \int_{X-h}^{X+h} K(u) K(v) P(u)^t Q(v) (u \wedge v) dv du,
\]

11
3.2 Optimal $Q$

Now we discuss the optimal choice of $Q$, which minimizes the asymptotic variance of the minimum distance estimator. Recall from the main paper that, with orthogonalized basis, the minimum distance estimator of $f(\ell)(x)$, for $0 \leq \ell \leq p - 1$, has an asymptotic variance

$$f(x) \left[ e^\ell (\Gamma^\perp_{P,h})^{-1}\Sigma^\perp_{PP,h}(\Gamma^\perp_{P,h})^{-1} - e^\ell (\Gamma^\perp_{P,h})^{-1}\Sigma^\perp_{QQ,h}(\Sigma^\perp_{QQ,h})^{-1}\Sigma^\perp_{QP,h}(\Gamma^\perp_{P,h})^{-1} e^\ell \right],$$

where $e^\ell$ is the $(\ell + 1)$-th standard basis vector. In subsequent analysis, we drop the multiplicative factor $f(x)$.

Let $p_\ell(u)$ be defined as

$$p_\ell(u) = e^\ell (\Gamma^\perp_{P,h})^{-1} P^\perp(u),$$

then the objective is to maximize

$$\left( \int_{\mathcal{X}^\perp x} K(u)K(v)Q^\perp(u)Q^\perp(v)(u \land v)dudv \right)^{-1} \left( \int_{\mathcal{X}^\perp x} K(u)K(v)p_\ell(u)Q^\perp(v)(u \land v)dudv \right)^2.$$

Alternatively, we would like to solve (recall that $Q(u)$ is a scaler function)

$$\text{maximize} \quad \left( \frac{\int_{\mathcal{X}^\perp x} K(u)K(v)p_\ell(u)q(v)(u \land v)dudv}{\int_{\mathcal{X}^\perp x} K(u)K(v)q(u)q(v)(u \land v)dudv} \right)^2, \quad \text{subject to} \quad \int_{\mathcal{X}^\perp x} K(u)q(u)(1, P(u)')du = 0.$$

To proceed, define the following transformation for a function $g(\cdot)$:

$$\mathcal{H}(g)(u) = \int_{\mathcal{X}^\perp x} 1(v \geq u)K(v)g(v)dv.$$

This transformation satisfies two important properties, which are summarized in the following lemma.

**Lemma 7 (\mathcal{H}-transformation).**

(i) If $g_1(\cdot)$ and $g_2(\cdot)$ are bounded, and that either $\int_{\mathcal{X}^\perp x} K(u)g_1(u)du$ or $\int_{\mathcal{X}^\perp x} K(u)g_2(u)du$ is zero, then

$$\int_{\mathcal{X}^\perp x \cap [-1,1]} \mathcal{H}(g_1)(u)\mathcal{H}(g_2)(u)du = \int_{\mathcal{X}^\perp x} K(u)K(v)g_1(u)g_1(v)(u \land v)dudv.$$

(ii) If $g_1(\cdot)$ and $g_2(\cdot)$ are bounded, $g_2(\cdot)$ is continuously differentiable with a bounded derivative,
and that either \( \int_{x - h}^{x - x} K(u)g_1(u)du \) or \( \int_{x - x}^{x - x} K(u)g_2(u)du \) is zero, then

\[
\int_{x - h}^{x - x} \mathcal{H}(g_1)(u)\hat{g}_2(u)du = \int_{x - x}^{x - x} K(u)g_1(u)g_2(u)du.
\]

With the previous lemma, we can rewrite the maximization problem as

\[
\max_{\mathcal{H}(q)} \left( \int_{x - x}^{x - x} \mathcal{H}(g_1)(u)\hat{g}_1(u) \mathcal{H}(q)(u)du \right)^2
\]

subject to

\[
\int_{x - h}^{x - x} \mathcal{H}(q)(u)du = 0, \quad \mathcal{H}(q) \left( \frac{\inf \mathcal{X} - x}{h} \vee (-1) \right) = 0. \tag{14}
\]

**Theorem 8 (Variance Bound of the Minimum Distance Estimator).** An upper bound of the maximization problem (14) is

\[
e'(\ell) (\Gamma_{P,h}^\perp)^{-1}\Gamma_{P,h}^\perp \mathcal{H}(q)(u)du - e'(\ell) \left( \int_{x - h}^{x - x} \hat{P}(u)\hat{P}(u)'du \right)^{-1} e'.
\]

Therefore, the asymptotic variance of the minimum distance estimator is bounded below by

\[
f(x)e'(\ell) \left( \int_{x - h}^{x - x} \hat{P}(u)\hat{P}(u)'du \right)^{-1} e',
\]

where \( \hat{P}(u) = (1, \ u, \ u^2/2, \ u^3/3!, \ \cdots, \ u^{p-1}/(p-1)!)' \).

**Example 2 (Local Linear/Quadratic Minimum Distance Density Estimation).** Consider a simple example where \( \ell = 0 \) and \( P(u) = u \), which means we focus on the asymptotic variance of the estimated density in a local linear regression. Also assume we employ a uniform kernel: \( K(u) = \frac{1}{2} \mathbb{1}(|u| \leq 1) \), and that the integration region is \( \mathcal{X} = \mathbb{R} \) (i.e., \( x \) is an interior evaluation point). Note that this example also applies to local quadratic regressions, as \( u \) and \( u^2 \) are orthogonal for interior evaluation points.

Taking \( P(u) = u \), the variance bound in Theorem 8 is easily found to be

\[
f(x) \left( \int_{-1}^{1} \hat{P}(u)\hat{P}(u)'du \right)^{-1} = f(x) \frac{1}{2}.
\]

We now calculate the asymptotic variance of the minimum distance estimator. To be specific, we choose \( Q(u) = u^{2j+1} \), which is a higher-order polynomial function. With tedious calculation, one can show that the minimum distance estimator has the following asymptotic variance

\[
\text{Asy} \mathbb{V}[\hat{f}_{MD}(x)] = f(x) \frac{11 + 4j}{20 + 8j},
\]

which asymptotes to \( f(x)/2 \) as \( j \to \infty \). As a result, it is possible to achieve the maximum amount of
Figure 1. Equivalent Kernel of the Local Linear Minimum Distance Density Estimator.

Notes: The basis function $R(u)$ consists of an intercept, a linear term $u$ (i.e., local linear regression), and an odd higher-order polynomial term $u^{2j+1}$ for $j = 1, 2, \cdots, 30$. Without the higher-order polynomial regressor, the local linear density estimator using the uniform kernel is equivalent to the kernel density estimator using the Epanechnikov kernel (black line). Including a higher-order redundant regressor leads to an equivalent kernel that approaches the uniform kernel as $j$ tends to infinity (red).

In Figure 1, we plot the equivalent kernel of the local linear minimum distance density estimator using a uniform kernel. Without the redundant regressor, it is equivalent to the kernel density estimator using the Epanechnikov kernel. As $j$ gets larger, however, the equivalent kernel of the minimum distance estimator becomes closer to the uniform kernel, which is why, as $j \to \infty$, the minimum distance estimator has an asymptotic variance the same as the kernel density estimator using the uniform kernel.

Example 3 (Local Cubic Minimum Distance Estimation). We adopt the same setting in Example 2, i.e., local polynomial density estimation with the uniform kernel at an interior evaluation point. The difference is that we now consider a local cubic regression: $P(u) = (u, \frac{1}{2}u^2, \frac{1}{3!}u^3)'$.

As before, the variance bound in Theorem 8 is easily found to be

$$f(x) \left( \int_{-1}^{1} \hat{P}(u)\hat{P}(u)'du \right)^{-1} = f(x) \begin{bmatrix} \frac{9}{8} & 0 & -\frac{15}{4} \\ 0 & \frac{3}{2} & 0 \\ -\frac{15}{4} & 0 & \frac{45}{2} \end{bmatrix}.$$  

Again, we compute the asymptotic variance of our minimum distance estimator. Note, however, that now we have both odd and even order polynomials in our basis $P(u)$, therefore we include two higher-order polynomials, that is, we set $Q(u) = (u^{2j}, u^{2j+1})'$. The asymptotic variance of our
Table 1. Variance Comparison.

(a) Density $f(x)$

<table>
<thead>
<tr>
<th>Kernel Function</th>
<th>$p = 1$</th>
<th>$p = 2$</th>
<th>$p = 3$</th>
<th>$p = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform</td>
<td>0.600</td>
<td>0.600</td>
<td>1.250</td>
<td>1.250</td>
</tr>
<tr>
<td>Triangular</td>
<td>0.743</td>
<td>0.743</td>
<td>1.452</td>
<td>1.452</td>
</tr>
<tr>
<td>Epanechnikov</td>
<td>0.714</td>
<td>0.714</td>
<td>1.407</td>
<td>1.407</td>
</tr>
</tbody>
</table>

| MD Variance Bound | 0.500 | 0.500 | 1.125 | 1.125 |

(b) Density Derivative $f^{(1)}(x)$

<table>
<thead>
<tr>
<th>Kernel Function</th>
<th>$p = 2$</th>
<th>$p = 3$</th>
<th>$p = 4$</th>
<th>$p = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform</td>
<td>2.143</td>
<td>2.143</td>
<td>11.932</td>
<td>11.932</td>
</tr>
<tr>
<td>Triangular</td>
<td>3.498</td>
<td>3.498</td>
<td>17.353</td>
<td>17.353</td>
</tr>
<tr>
<td>Epanechnikov</td>
<td>3.182</td>
<td>3.182</td>
<td>15.970</td>
<td>15.970</td>
</tr>
</tbody>
</table>

| MD Variance Bound | 1.500 | 1.500 | 9.375 | 9.375 |

Notes: Panel (a) compares asymptotic variance of the local polynomial density estimator of Cattaneo, Jansson, and Ma (2020) for different polynomial orders ($p = 1, 2, 3, \text{ and } 4$) and different kernel functions (uniform, triangular and Epanechnikov). Also shown are the variance bound of the minimum distance estimator (MD Variance Bound), calculated according to Theorem 8. Panel(b) provides the same information for the estimated density derivative. All comparisons assume an interior evaluation point $x$.

minimum distance estimator is

$$\text{AsyV} \begin{bmatrix} \hat{f}_{MD}'(x) \\ \hat{f}_{MD}''(x) \\ \hat{f}_{MD}'''(x) \end{bmatrix} = f(x) \begin{bmatrix} 0 & -\frac{15(4j+17)}{8(2j+1)} \\ -\frac{15(4j+17)}{8(2j+1)} & 0 \\ \frac{9(4j+15)}{16(2j+1)} & \frac{12j+39}{8j+20} \end{bmatrix},$$

which, again, asymptotes to the variance bound as $j \to \infty$. See also Table 1 for the efficiency gain of employing the minimum distance technique.

Example 4 (Local $p = 5$ Minimum Distance Estimation). We consider the same setting in Example 2 and 3, but with $p = 5$: $P(u) = (u, \frac{1}{2}u^2, \cdots, \frac{1}{5!}u^5)'$. 

15
The variance bound in Theorem 8 is

\[
f(x) \left( \int_{-1}^{1} \dot{P}(u) \dot{P}(u)\,du \right)^{-1} = f(x) \begin{bmatrix}
225 \\
128 \\
0 \\
-99225
\end{bmatrix} \begin{bmatrix}
16 \\
8 \\
4 \\
2
\end{bmatrix}.
\]

Again, we include two higher order polynomials: \( Q(u) = (u^{2j}, u^{2j+1})'. \) The asymptotic variance of our minimum distance estimator is

\[
\text{Asy}V \begin{bmatrix}
\hat{f}_{\text{MD}}^{(0)}(x) \\
\hat{f}_{\text{MD}}^{(1)}(x) \\
\hat{f}_{\text{MD}}^{(2)}(x) \\
\hat{f}_{\text{MD}}^{(3)}(x) \\
\hat{f}_{\text{MD}}^{(4)}(x)
\end{bmatrix} = f(x) \begin{bmatrix}
225(4j+19) \\
256(2j+9) \\
0 \\
525(4j+21) \\
64(2j+9)
\end{bmatrix} \begin{bmatrix}
16(2j+7) \\
(2j+7) \\
0 \\
0 \\
(2j+7)
\end{bmatrix} = f(x) \begin{bmatrix}
0 \\
75(4j+17) \\
0 \\
0 \\
8(2j+7)
\end{bmatrix} \begin{bmatrix}
525(4j+21) \\
64(2j+9) \\
0 \\
0 \\
1575(4j+21)
\end{bmatrix} \begin{bmatrix}
0 \\
0 \\
16(2j+9) \\
0 \\
8j+28
\end{bmatrix} = f(x) \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} \begin{bmatrix}
2835(4j+23) \\
32(2j+9) \\
0 \\
0 \\
99225(4j+27)
\end{bmatrix} \begin{bmatrix}
32(2j+9) \\
0 \\
0 \\
0 \\
8j+36
\end{bmatrix},
\]

which converges to the variance bound as \( j \to \infty. \) See also Table 1 for the efficiency gain of employing the minimum distance technique.

Before closing this section, we make several remarks on the variance bound derived in Theorem 8, as well as to what extent it is achievable.

**Remark 1 (Achievability of the Variance Bound).** The previous two examples suggest that the variance bound derived in Theorem 8 can be achieved by employing a minimum distance estimator with two additional regressors, one higher-order even polynomial and one higher-order odd polynomial. With analytic calculation, we verify that this is indeed the case for \( p \leq 10 \) when a uniform kernel function is used.

**Remark 2 (Optimality of the Variance Bound).** Granovsky and Müller (1991) discuss the problem of finding the optimal kernel function for kernel-type estimators. To be precise, consider the following

\[
\frac{1}{nh^{\ell+1}} \sum_{i=1}^{n} \phi_{\ell,k} \left( \frac{x_i - x}{h} \right),
\]

where \( \phi_{\ell,k}(u) \) is a function satisfying

\[
\int_{-1}^{1} u^j \phi_{\ell,k}(u)\,du = \begin{cases} 
0 & 0 \leq j < k, \ j \neq \ell \\
\ell! & j = \ell
\end{cases}, \quad \int_{-1}^{1} u^k \phi_{\ell,k}(u)\,du \neq 0.
\]
Then it is easy to see that, with a Taylor expansion argument,

$$
E \left[ \frac{1}{nh^{\ell+1}} \sum_{i=1}^{n} \phi_{\ell,k} \left( \frac{x_i - x}{h} \right) \right] = \frac{1}{h^{\ell+1}} \int_{-1}^{1} \phi_{\ell,k} \left( \frac{u - x}{h} \right) f(u) du
$$

$$
= \frac{1}{h^\ell} \int_{-1}^{1} \phi_{\ell,k}(u) f(x + hu) du
$$

$$
= \frac{1}{h^\ell} \int_{-1}^{1} \phi_{\ell,k}(u) \left[ \sum_{j=0}^{k-1} \frac{(hu)^j}{j!} f^{(j)}(x) + u^k O(h^k) \right] du
$$

$$
= f^{(\ell)}(x) + O(h^{k-\ell}).
$$

That is, the kernel $\phi_{\ell,k}(u)$ facilitates estimating the $\ell$-th derivative of the density function with a leading bias of order $h^{k-\ell}$. Asymptotic variance of this kernel-type estimator is easily found to be

$$
\text{Asy} \text{Var} \left[ \frac{1}{nh^{\ell+1}} \sum_{i=1}^{n} \phi_{\ell,k} \left( \frac{x_i - x}{h} \right) \right] = f(x) \int_{-1}^{1} \phi_{\ell,k}(u)^2 du.
$$

Granovsky and Müller (1991) provide the exact form of the kernel function $\phi_{\ell,k}(u)$ that minimizes the asymptotic variance subject to the order of the leading bias.

Take $\ell = 0$ and $k = 2$, $\phi_{\ell,k}(u)$ takes the following form:

$$
\phi_{\ell,k}(u) = \frac{1}{2} \mathbb{I}(|u| \leq 1),
$$

which is the uniform kernel and minimizes variance among all second order kernels for density estimation. As illustrated in Example 2, our variance bound matches $f(x) \int_{-1}^{1} \phi_{\ell,k}(u)^2 du$.

Now take $\ell = 1$ and $k = 3$. This will give an estimator for the density derivative $f^{(1)}(x)$ with a leading bias of order $O(h^2)$. The optimal choice of $\phi_{\ell,k}(u)$ is

$$
\phi_{\ell,k}(u) = \frac{3}{2} u \mathbb{I}(|u| \leq 1).
$$

to match the order of bias, we consider the minimum distance estimator with $p = 3$. Again, the variance bound in Theorem 8 matches $f(x) \int_{-1}^{1} \phi_{\ell,k}(u)^2 du$.

As a final illustration, take $\ell = 1$ and $k = 5$, which gives an estimator for the density derivative $f^{(1)}(x)$ with a leading bias of order $O(h^4)$. The optimal choice of $\phi_{\ell,k}(u)$ is

$$
\phi_{\ell,k}(u) = \left( \frac{75}{8} u - \frac{105}{8} u^3 \right) \mathbb{I}(|u| \leq 1).
$$

It is easy to see that $f(x) \int_{-1}^{1} \phi_{\ell,k}(u)^2 du = 75f(x)/8$. To match the bias order, we take $p = 5$ for our minimum distance estimator. The variance bound is $75f(x)/8$, which is the same as $f(x) \int_{-1}^{1} \phi_{\ell,k}(u)^2 du$.

With analytic calculations, we verify that the variance bound stated in Theorem 8 is the same as
the minimum variance found in Granovsky and Müller (1991). Together with the previous remark, we reach a much stronger conclusion: including two higher-order polynomials in our minimum distance estimator can help achieve the variance bound in Theorem 8, which, in turn, is the smallest variance any kernel-type estimator can achieve (given a specific leading bias order).

**Remark 3 (Another Density Estimator Which Achieves the Variance Bound).** The following estimator achieves the bound of Theorem 8, although it does not belong to the class of estimators we consider in this paper.

\[
\hat{\theta}_{\text{ND}} = \left( \int_X \hat{P}(u-x) \hat{P}(u-x) \frac{1}{h} K \left( \frac{u-x}{h} \right) \, du \right)^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} \hat{P}(x_i-x) \frac{1}{h} K \left( \frac{x_i-x}{h} \right) \right),
\]

where \( \hat{P}(u) = (1, u, u^2/2, \ldots, w^{p-1}/(p-1)!)' \) is the \((p-1)\)-th order polynomial basis. The subscript represents “numerical derivative,” because the above estimator can be understood as

\[
\hat{\theta}_{\text{ND}} = \left( \int_X \hat{P}(u-x) \hat{P}(u-x) \frac{1}{h} K \left( \frac{u-x}{h} \right) \, du \right)^{-1} \left( \int_X \hat{P}(u-x) \frac{1}{h} K \left( \frac{u-x}{h} \right) \frac{d \hat{F}(u)}{du} \, du \right)
= \arg\min_{\theta} \int_X \left( \frac{d \hat{F}(u)}{du} - \hat{P}(u-x)' \theta \right)^2 \frac{1}{h} K \left( \frac{u-x}{h} \right) \, du,
\]

where the derivative \( d \hat{F}(u)/du \) is interpreted in the sense of generalized functions. From the above, it is clear that this estimator requires the knowledge of the boundary position (that is, the knowledge of \( X \)).

With straightforward calculations, this estimator has a leading bias

\[
\mathbb{E}[\hat{\theta}_{\text{ND}}] = \left( \int_X \hat{P}(u-x) \hat{P}(u-x) \frac{1}{h} K \left( \frac{u-x}{h} \right) \, du \right)^{-1} \mathbb{E} \left[ \hat{P}(x_i-x) \frac{1}{h} K \left( \frac{x_i-x}{h} \right) \right]
= \theta + h^p \Upsilon_h f^{(p)}(x) \left( \int_{X \times \mathbb{R}} \hat{P}(u) \hat{P}(u)' K(u) \, du \right)^{-1} \int_{X \times \mathbb{R}} \hat{P}(u) u^p K(u) \, du + o(h^p \Upsilon_h),
\]

where \( \Upsilon_h \) is a diagonal matrix containing 1, \( h^{-1} \), \( \ldots, h^{-(p-1)} \). Its leading variance is also easy to establish:

\[
\nabla[\hat{\theta}_{\text{ND}}] = \frac{1}{nh} \Upsilon_h f(x) \left( \int_{X \times \mathbb{R}} \hat{P}(u) \hat{P}(u)' K(u) \, du \right)^{-1} \left( \int_{X \times \mathbb{R}} \hat{P}(u) \hat{P}(u)' K(u)^2 \, du \right)
\cdot \left( \int_{X \times \mathbb{R}} \hat{P}(u) \hat{P}(u)' K(u) \, du \right)^{-1} \Upsilon_h
+ o \left( \frac{1}{nh} \Upsilon_h^2 \right).
\]

To reach the efficiency bound in Theorem 8, it suffices to set \( K(\cdot) \) to be the uniform kernel. Section 5.1.1 in Loader (2006) also discussed this estimator, although it seems its efficiency property has
4 Uniform Distribution Theory

We establish distribution approximation for \( \{\hat{\theta}_G(x), x \in I\} \) and \( \{\hat{\theta}(x), x \in I\} \), which can be viewed as processes indexed by the evaluation point \( x \) in some set \( I \subseteq X \). Recall the definition of \( \Gamma_{h,x} \) and \( \Sigma_{h,x} \) from Section 1, and we define \( \Omega_{h,x} = \Gamma_{h,x}^{-1} \Sigma_{h,x} \Gamma_{h,x}^{-1} \).

We first study the following (infeasible) centered and Studentized process:

\[
T_G(x) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} c_h' x \frac{R(u) \left[ 1(x_i \leq x + hu) - F(x + hu) \right] K(u) g(x + hu) du}{\sqrt{c_h' x \Omega_{h,x} x c_h}}, \quad x \in I,
\]

where we consider linear combinations through a (known) vector \( c_{h,x} \), which can depend on the sample size through the bandwidth \( h \), and can depend on the evaluation point. Again, we use the subscript \( G \) to denote the local \( L^2 \) approach with \( G \) being the design distribution. To economize notation, let

\[
K_{h,x}(x) = \frac{c_h' x \Gamma_{h,x}^{-1} \int_{x-x}^{x+hu} R(u) \left[ 1(x_i \leq x + hu) - F(x + hu) \right] K(u) g(x + hu) du}{\sqrt{c_h' x \Omega_{h,x} x c_h}},
\]

then we can conveniently rewrite (15) as

\[
\Xi_G(x) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} K_{h,x}(x_i),
\]

and hence the centered and Studentized process \( \Xi_G(\cdot) \) takes a kernel form. The difference compared to standard kernel density estimators, however, is that the (equivalent) kernel in our case changes with the evaluation point, which is why our estimator is able to adapt to boundary points automatically. From the pointwise distribution theory developed in Section 2, the process \( \Xi_G(x) \) has variance

\[
\mathbb{V}[\Xi_G(x)] = \mathbb{E}[K_{h,x}(x_i)^2] = 1.
\]

We can also compute the covariance as

\[
\text{Cov} [\Xi_G(x), \Xi_G(y)] = \mathbb{E}[K_{h,x}(x_i)K_{h,y}(x_i)] = \frac{c_h' x \Omega_{h,x,y} x c_{h,y}}{\sqrt{c_h' x \Omega_{h,x} x c_h} \sqrt{c_h' y \Omega_{h,y} y c_{h,y}}} + O(h),
\]

not been realized in the literature.
where $\Omega_{h,x,y} = \Gamma_{h,x}^{-1} \Sigma_{h,x,y} \Gamma_{h,y}^{-1}$, and
\[
\Sigma_{h,x,y} = \int_{X^{-y}} \int_{X^{-x}} R(u) R(v)' \left[ F((x + hu) \land (y + hv)) - F(x + hu)F(y + hv) \right]
K(u)K(v)g(x + hu)g(y + hv)du dv.
\]

Of course one can further expand the above, but this is unnecessary for our purpose.

For future reference, let
\[
r_1(\varepsilon, h) = \sup_{x,y \in I, |x-y| \leq \varepsilon} \left| c'_{h,x}\Upsilon_h - c'_{h,y}\Upsilon_h \right|, \quad r_2(h) = \sup_{x \in \mathbb{Z}} \frac{1}{|c'_{h,x}\Upsilon_h|}.
\]

Remark 4 (On the Order of $r_1(\varepsilon, h)$, $r_2(h)$ and $\sup_{x \in I} \varrho(h, x)$). In general, it is not possible to give precise orders of the quantities introduced above. In this remark, we consider the local polynomial estimator of Cattaneo, Jansson, and Ma (2020) (see Section 1 for an introduction). The local polynomial estimator employs a polynomial basis, and hence estimates the density function and higher-order derivatives by (it also estimates the distribution function)
\[
\hat{F}^{(\ell)}(x) = c'_{\ell} \hat{\theta}(x),
\]
where $\epsilon_\ell$ is the $(\ell + 1)$-th standard basis vector. As a result, $c_{h,x} = \epsilon_\ell$, which does not depend on the evaluation point. For the scaling matrix $\Upsilon_h$, we note that it is diagonal with elements $1, h^{-1}, \ldots, h^{-p}$, and hence it does not depend on the evaluation point either. Therefore, we conclude that, for density (and higher-order) derivative estimation using the local polynomial estimator, $r_1(\varepsilon, h)$ is identically zero. Similarly, we have that $r_2(h) = h^\ell$. Finally, given the discussion in Section 1, the bias term generally has order $\sup_{x \in I} \varrho(h, x) = h^{\mu+1}$ for the local polynomial density estimator.

The above discussion restricts to the local polynomial density estimator, but more can be said about $r_2(h)$. We will argue that, in general, one should expect $r_2(h) = O(1)$. Recall that the leading variance of $c'_{h,x} \hat{\theta}(x)$ and $c'_{h,x} \hat{\theta}_G(x)$ is $\frac{1}{n} c'_{h,x} \Upsilon_h \Omega_{h,x} \Upsilon_h c_{h,x}$, and that the maximum eigenvalue of $\Omega_{h,x}$ is bounded. Therefore, the variance has order $O(1/(nr_2(h)^2))$. In general, we do not expect the variance to shrink faster than $1/n$, which is why $r_2(h)$ is usually bounded. In fact, for most interesting cases, $c'_{h,x} \hat{\theta}(x)$ and $c'_{h,x} \hat{\theta}_G(x)$ will be “nonparametric” estimators in the sense that they estimate local features of the distribution function. If this is the case, we may even argue that $r_2(h)$ will be vanishing as the bandwidth shrinks.

We also make some additional assumptions.

Assumption 4. Let $I$ be a compact interval.

(i) The density function is twice continuously differentiable and bounded away from zero in $I$.

(ii) There exists some $\delta > 0$ and compactly supported kernel functions $K^{\dagger}(\cdot)$ and $\{K^{\dagger,d}(\cdot)\}_{d \leq \delta}$, such that (ii.1) $\sup_{u \in \mathbb{R}} |K^{\dagger}(u)|, \sup_{d \leq \delta, u \in \mathbb{R}} |K^{\dagger,d}(u)| < \infty$; (ii.2) the support of $K^{\dagger,d}(\cdot)$ has Lebesgue
measure bounded by $Cd$, where $C$ is independent of $d$; and (ii.3) for all $u$ and $v$ such that $|u - v| \leq \delta$,

$$|K(u) - K(v)| \leq |u - v| \cdot K^\dagger(u) + K^{\dagger u-v}(u).$$

(iii) The basis function $R(\cdot)$ is Lipschitz continuous in $[-1, 1]$.

(iv) For all $h$ sufficiently small, the minimum eigenvalues of $\Gamma_{h,x}$ and $h^{-1}\Sigma_{h,x}$ are bounded away from zero uniformly for $x \in \mathcal{I}$.

(v) $h \to 0$ and $nh/\log n \to \infty$ as $n \to \infty$.

(vi) For some $C_1 > 0$ and $C_2, C_3 \geq 0$,

$$r_1(\varepsilon, h) = O\left(\varepsilon^{C_1} h^{-C_2}\right), \quad r_2(h) = O\left(h^{C_3}\right).$$

In addition,

$$\sup_{x \in \mathcal{I}} |\mathcal{Y}_{h}\mathcal{T}_h| \leq O(1).$$

**Assumption 5.** The design density function $g(\cdot)$ is twice continuously differentiable and is bounded away from zero in $\mathcal{I}$.

For any $h > 0$ (and fixed $n$), we can define a centered Gaussian process, $\{\mathcal{B}_G(x) : x \in \mathcal{I}\}$, which has the same variance-covariance structure as the process $\mathcal{T}_G(\cdot)$. The following lemma shows that it is possible to construct such a process, and that $\mathcal{T}_G(\cdot)$ and $\mathcal{B}_G(\cdot)$ are “close in distribution.”

**Theorem 9 (Strong Approximation).** Assume Assumptions 1, 2, 4 and 5 hold. Then on a possibly enlarged probability space there exist two processes, $\{\mathcal{T}_G(x) : x \in \mathcal{I}\}$ and $\{\mathcal{B}_G(x) : x \in \mathcal{I}\}$, such that (i) $\mathcal{X}_G(\cdot)$ has the same distribution as $\mathcal{T}_G(\cdot)$; (ii) $\mathcal{B}_G(\cdot)$ is a Gaussian process with the same covariance structure as $\mathcal{T}_G(\cdot)$; and (iii)

$$\mathbb{P}\left[\sup_{x \in \mathcal{I}} |\mathcal{X}_G(x) - \mathcal{B}_G(x)| > \frac{C_4(u + C_5 \log n)}{\sqrt{nh}}\right] \leq C_5 e^{-C_5 u},$$

where $C_5$ is some constant that does not depend on $h$ or $n$.

Next we consider the continuity property of the implied (equivalent) kernel of the process $\mathcal{T}_G(\cdot)$, which will help control the complexity of the Gaussian process $\mathcal{B}_G(\cdot)$. To be precise, define the pseudo-metric $\sigma_G(x, y)$ as

$$\sigma_G(x, y) = \sqrt{\mathbb{V}[\mathcal{X}_G(x) - \mathcal{X}_G(y)]} = \sqrt{\mathbb{E}[(K_{h,x}(x_i) - K_{h,y}(x_i))^2]},$$

we would like to provide an upper bound of $\sigma_G(x, y)$ in terms of $|x - y|$ (at least for all $x$ and $y$ such that $|x - y|$ is small enough).
**Lemma 10 (VC-type Property).** Assume Assumptions 1, 2, 4 and 5 hold. Then for all \( x, y \in \mathcal{I} \) such that \( |x - y| = \varepsilon \leq h \),

\[
\sigma_G(x, y) = O \left( \frac{\varepsilon}{\sqrt{h}} \cdot \frac{1}{h} \right) + \frac{1}{h} r_1(\varepsilon, h)r_2(h) + \frac{1}{h} r_1(\varepsilon, h)^2 r_2(h)^2 \right).
\]

Therefore,

\[
\mathbb{E} \left[ \sup_{x \in \mathcal{I}} |B_G(x)| \right] = O \left( \sqrt{\log n} \right), \quad \text{and} \quad \mathbb{E} \left[ \sup_{x \in \mathcal{I}} |T_G(x)| \right] = O \left( \sqrt{\log n} \right).
\]

**4.1 Local \( L^2 \) Distribution Estimation**

We first discuss the covariance estimator. For the local \( L^2 \) distribution estimator, let \( \hat{\Omega}_{h,x,y} = \Gamma_{h,x}^{-1} \hat{\Sigma}_{h,x,y} \Gamma_{h,y}^{-1} \) with \( \hat{\Sigma}_{h,x,y} \) given by

\[
\hat{\Sigma}_{h,x,y} = \frac{1}{n} \sum_{i=1}^{n} \int_{x}^{x+h} \int_{y}^{y+h} R(u) R(v) \left[ \mathbf{1}(x_i \leq x + hu) - \tilde{F}(x + hu) \right] \left[ \mathbf{1}(x_i \leq y + hv) - \tilde{F}(y + hv) \right] K(u) K(v) g(x + hu) g(y + hv) du dv.
\]

The next lemma characterizes the convergence rate of \( \hat{\Omega}_{h,x,y} \).

**Lemma 11 (Local \( L^2 \) Distribution Estimation: Covariance Estimation).** Assume Assumptions 1, 2, 4 and 5 hold, and that \( nh^2 / \log n \to \infty \). Then

\[
\sup_{x,y \in \mathcal{I}} \left| \frac{c'_{h,x} \Gamma^{-1}_h \hat{\Omega}_{h,x,y} \Gamma^{-1}_h \hat{\Sigma}_{h,x,y} \Gamma^{-1}_h \hat{\Omega}_{h,y} \Gamma^{-1}_h \hat{\Sigma}_{h,y}}{c'_{h,x} \Gamma^{-1}_h \hat{\Omega}_{h,x} \Gamma^{-1}_h \hat{\Sigma}_{h,x} \Gamma^{-1}_h \hat{\Omega}_{h,y} \Gamma^{-1}_h \hat{\Sigma}_{h,y}} \right| = O_P \left( \sqrt{\frac{\log n}{nh^2}} \right).
\]

We now consider the estimator \( c'_{h,x} \hat{\theta}_G(x) \). From (3) and (4), one has

\[
T_G(x) = \sqrt{n} \frac{c'_{h,x} \left( \hat{\theta}_G(x) - \theta(x) \right)}{\sqrt{c'_{h,x} \Gamma^{-1}_h \hat{\Omega}_{h,x} \Gamma^{-1}_h \hat{\Sigma}_{h,x}}}
\]

\[
= \sqrt{n} \frac{c'_{h,x} \Gamma^{-1}_h \int_{x}^{x+h} R(u) \left[ F(x + hu) - \theta' R(u) \Gamma^{-1}_h \right] K(u) g(x + hu) du}{\sqrt{c'_{h,x} \Gamma^{-1}_h \hat{\Omega}_{h,x} \Gamma^{-1}_h \hat{\Sigma}_{h,x}}}
\]

\[
+ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{c'_{h,x} \Gamma^{-1}_h \int_{x}^{x+h} R(u) \left[ \mathbf{1}(x_i \leq x + hu) - F(x + hu) \right] K(u) g(x + hu) du}{\sqrt{c'_{h,x} \Gamma^{-1}_h \hat{\Omega}_{h,x} \Gamma^{-1}_h \hat{\Sigma}_{h,x}}}.
\]

In the following lemma, we analyze the two terms in the above decomposition.
Lemma 12. Assume Assumptions 1, 2, 4 and 5 hold, and that \( nh^2 / \log n \to \infty \). Then
\[
\sup_{x \in I} \left| (16) \right| = O_P \left( \frac{\log n}{\sqrt{nh^2}} \right), \quad \sup_{x \in I} \left| (17) - \mathcal{I}_G(x) \right| = O_P \left( \frac{\log n}{\sqrt{nh^2}} \right). \]

Now we state the main result on uniform distributional approximation.

Theorem 13 (Local \( L^2 \) Distribution Estimation: Uniform Distributional Approximation). Assume Assumptions 1, 2, 4 and 5 hold, and that \( nh^2 / \log n \to \infty \). Then on a possibly enlarged probability space there exist two processes, \( \{ \tilde{T}_G(x) : x \in I \} \) and \( \{ B_G(x) : x \in I \} \), such that (i) \( \tilde{T}_G(\cdot) \) has the same distribution as \( T_G(\cdot) \); (ii) \( B_G(\cdot) \) is a Gaussian process with the same covariance structure as \( T_G(\cdot) \); and (iii)
\[
\sup_{x \in I} \left| T_G(x) - \mathcal{I}_G(x) \right| + \sup_{x \in I} \left| \tilde{T}_G(x) - B_G(x) \right| = O_P \left( \frac{\log n}{\sqrt{nh^2}} + \sqrt{\frac{n}{h} \sup_{x \in I} \varrho(h, x)} \right). \]

Remark 5 (On the Remainders in Theorems 13 and 14). Recall that the local polynomial density estimator employs a polynomial basis, which implies that \( \sup_{x \in I} \varrho(h, x) = h^{p+1} \), where \( p \) is the highest polynomial order. Then the error in Theorem 13 reduces to
\[
\sqrt{nh^{2p+1}} + \frac{\log n}{\sqrt{nh^2}}.
\]
Therefore, a sufficient set of conditions for both errors to be negligible is \( nh^{2p+1} \to 0 \) and \( nh^2 / \log^5 n \to \infty \).
4.2 Local Regression Distribution Estimation

Now we consider the local regression estimator \{\hat{\theta}(x), x \in \mathcal{I}\}. As before, we first discuss the construction of the covariance \Omega_{h,x,y}. Let \hat{\Omega}_{h,x,y} = \hat{\Gamma}_{h,x}^{-1}\hat{\Sigma}_{h,x,y}\hat{\Gamma}_{h,y}^{-1}. Construction of \hat{\Gamma}_{h,x} is given in Section 2.2. The following lemma shows that \hat{\Gamma}_{h,x} is uniformly consistent.

**Lemma 15 (Uniform Consistency of \hat{\Gamma}_{h,x}).** Assume Assumptions 1 and 4 hold. Then

$$\sup_{x \in \mathcal{I}} \left| \hat{\Gamma}_{h,x} - \Gamma_{h,x} \right| = O_p\left( \sqrt{\frac{\log n}{nh}} \right).$$

Construction of \hat{\Sigma}_{h,x,y} also mimics that in Section 2.2. To be precise, we let

$$\hat{\Sigma}_{h,x,y} = \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{1}{n} \sum_{j=1}^{n} \Upsilon_h R(x_j - x) \left[ 1(x_i \leq x_j) - \hat{F}(x_j) \right] \frac{1}{h} K \left( \frac{x_j - x}{h} \right) \right]$$

$$\left[ \frac{1}{n} \sum_{j=1}^{n} \Upsilon_h R(x_j - y) \left[ 1(x_i \leq x_j) - \hat{F}(x_j) \right] \frac{1}{h} K \left( \frac{x_j - y}{h} \right) \right]^\prime,$$

where \hat{F}(\cdot) remains to be the empirical distribution function. The following result justifies consistency of \hat{\Omega}_{h,x,y}.

**Lemma 16 (Local Regression Distribution Estimation: Covariance Estimation).** Assume Assumptions 1 and 4 hold, and that \(nh^2/\log n \to \infty\). Then

$$\sup_{x,y \in \mathcal{I}} \left| \frac{c'_{h,x} \Upsilon_h (\hat{\Omega}_{h,x,y} - \Omega_{h,x,y}) \Upsilon_h c_{h,y}}{\sqrt{c'_{h,x} \Upsilon_h \Omega_{h,x} \Upsilon_h c_{h,x} \sqrt{c'_{h,y} \Upsilon_h \Omega_{h,y} \Upsilon_h c_{h,y}}}} \right| = O_p\left( \sqrt{\frac{\log n}{nh^2}} \right).$$

24
The following is an expansion of $T(\cdot)$.

\[
T(x) = \sqrt{n} c_{h,x} \left( \frac{\hat{\theta}(x) - \theta(x)}{\sqrt{\hat{\gamma}_{h,x} \hat{\Omega}_{h,x} \hat{\gamma}_{h,x}}} \right)
\]

\[= \frac{1}{n \sqrt{n}} \sum_{i=1}^{n} c_{h,x} \hat{\gamma}_{h,x} \hat{\gamma}^{-1}_{h,x} \hat{\gamma}_{h,x} R(x_i - x) \left[ 1 - F(x_i) \right] \frac{1}{h} K \left( \frac{x - x}{h} \right)
\]

\[+ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} c_{h,x} \hat{\gamma}_{h,x} \hat{\gamma}^{-1}_{h,x} \hat{\gamma}_{h,x} R(x_i - x) \left[ F(x_i) - \theta(x) R(x_i - x) \right] \frac{1}{h} K \left( \frac{x - x}{h} \right)
\]

\[+ \frac{1}{n \sqrt{n}} \sum_{i=1}^{n} c_{h,x} \hat{\gamma}_{h,x} R(x_i - x) \left[ 1 \left( x_i \leq x \right) - F(x_j) \right] \frac{1}{h} K \left( \frac{x - x}{h} \right)
\]

\[+ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} c_{h,x} \hat{\gamma}_{h,x} R(x_i - x) \left[ 1 \left( x_i \leq x \right) - F(x + hu) \right] K(u) f(x + hu) du
\]

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(18)</td>
<td></td>
</tr>
<tr>
<td>(19)</td>
<td></td>
</tr>
<tr>
<td>(20)</td>
<td></td>
</tr>
<tr>
<td>(21)</td>
<td></td>
</tr>
</tbody>
</table>

**Lemma 17.** Assume Assumptions 1 and 4 hold, and that $nh^2 / \log n \to \infty$. Then

\[\sup_{x \in I} \left| (18) \right| = O_p \left( \frac{1}{\sqrt{nh}} \right), \quad \sup_{x \in I} \left| (19) \right| = O_p \left( \sqrt{\frac{\log n}{nh}} \sup_{x \in I} \varphi(h, x) \right), \quad \sup_{x \in I} \left| (20) \right| = O_p \left( \frac{\log n}{\sqrt{nh^2}} \right).\]

**Lemma 18.** Assume Assumptions 1 and 4 hold, and that $nh^2 / \log n \to \infty$. Then

\[\sup_{x \in I} \left| (21) - \xi_F(x) \right| = O_p \left( \frac{\log n}{\sqrt{nh^2}} \right).
\]

Finally we have the following result on uniform distributional approximation for the local regression distribution estimator, as well as a feasible approximation by simulating from a Gaussian process with estimated covariance.

**Theorem 19 (Local Regression Distribution Estimation: Uniform Distributional Approximation).** Assume Assumptions 1 and 4 hold, and that $nh^2 / \log n \to \infty$. Then on a possibly enlarged probability space there exist two processes, \{\tilde{\xi}_F(x) : x \in I\} and \{\mathcal{B}_F(x) : x \in I\}, such that (i) \tilde{\xi}_F(\cdot) has the same distribution as \xi_F(\cdot); (ii) \mathcal{B}_F(\cdot) is a Gaussian process with the same covariance structure as \xi_F(\cdot); and (iii)

\[\sup_{x \in I} \left| T(x) - \xi_F(x) \right| + \sup_{x \in I} \left| \tilde{\xi}_F(x) - \mathcal{B}_F(x) \right| = O_p \left( \frac{\log n}{\sqrt{nh^2}} + \sqrt{\frac{\log n}{h}} \sup_{x \in I} \varphi(h, x) \right).
\]
Theorem 20 (Local Regression Distribution Estimation: Feasible Distributional Approximation). Assume Assumptions 1 and 4 hold, and that \( nh^2 / \log n \to \infty \). Then conditional on the data there exists a centered Gaussian process \( \hat{B}_F(\cdot) \) with covariance

\[
\text{Cov}^* \left[ \hat{B}_F(x), \hat{B}_F(y) \right] = \frac{c'_{h,x} \hat{\Omega}_{h,x,y} \hat{\Omega}_{h,y} c_{h,y}}{\sqrt{c'_{h,x} \hat{\Omega}_{h,x,x} \hat{\Omega}_{h,x,x} \sqrt{c'_{h,y} \hat{\Omega}_{h,y,y} \hat{\Omega}_{h,y,y}}}}
\]

such that

\[
\sup_{u \in \mathbb{R}} \left| \mathbb{P} \left[ \sup_{x \in I} |\hat{B}_F(x)| \leq u \right] - \mathbb{P}^* \left[ \sup_{x \in I} |\hat{B}_F(x)| \leq u \right] \right| = O_p \left( \left( \frac{\log^5 n}{nh^2} \right)^{\frac{1}{4}} \right).
\]
5 Proofs

5.1 Proof of Theorem 1

Part (i)

The bias term can be bounded by
\[
\left| \int_{\frac{x-x}{h}} R(u) \left[ F(x + hu) - \theta R(u) \right] K(u) \, du \right| \leq \sup_{u \in [-1, 1]} \left| F(x + hu) - \theta' R(u) \right| \int_{\frac{x-x}{h}} |R(u)| K(u) \, du
\]
\[= g(h) \int_{\frac{x-x}{h}} |R(u)| K(u) \, du.\]

Part (ii)

The variance can be found as
\[
\mathbb{V} \left[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \int_{\frac{x-x}{h}} R(u) \left[ 1(x_i \leq x + hu) - F(x + hu) \right] K(u) g(x + hu) \, du \right]
\]
\[= \int \int_{\frac{x-x}{h}} R(u) R(v) K(u) K(v) \left[ F(x + h(u \wedge v)) - F(x + hu) F(x + hv) \right] g(x + hu) g(x + hv) \, du \, dv.\]

To establish asymptotic normality, we verify the Lyapunov condition with a fourth moment calculation. Take \(c\) to be a nonzero vector of conformable dimension, and we employ the Cramer-Wold device:
\[
\frac{1}{n} (c' \Sigma_h c)^{-2} \mathbb{E} \left[ \left( \int_{\frac{x-x}{h}} c' R(u) \left[ 1(x_i \leq x + hu) - F(x + hu) \right] K(u) g(x + hu) \, du \right)^4 \right].
\]
If \(c' \Sigma_h c\) is bounded away from zero as the bandwidth decreases, the above will have order \(n^{-1}\), as \(K(\cdot)\) is bounded and compactly supported and \(R(\cdot)\) is locally bounded. Therefore, the Lyapunov condition holds in this case. The more challenging case is when \(c' \Sigma_h c\) is of order \(h\). In this case, it implies
\[
F(x)(1 - F(x)) \left| \int \int_{\frac{x-x}{h}} c' R(u) K(u) g(u) \, du \right|^2 = O(h).
\]

Now consider the fourth moment. The leading term is
\[
F(x)(1 - F(x))(3F(x)^2 - 3F(x) + 1) \left| \int \int_{\frac{x-x}{h}} c' R(u) K(u) g(x + hu) \, du \right|^4 = O(h),
\]
meaning that for the Lyapunov condition to hold, we need the requirement that \(nh \to \infty\).

Part (iii)

This follows immediately from Part (i) and (ii).
5.2 Proof of Theorem 2

To study the property of $\hat{\Sigma}_h$, we make the following decomposition:

$$\hat{\Sigma}_h = \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{x} R(u)R(v) \left[ I(x_i \leq x + hu) - F(x + hu) \right] K(u)K(v)g(x + hu)g(x + hv) \, du \, dv$$

(\text{I})

$$- \int_{x-h}^{x} R(u)R(v) \left[ \hat{F}(x + hu) - F(x + hu) \right] \left[ \hat{F}(x + hv) - F(x + hv) \right] K(u)K(v)g(x + hu)g(x + hv) \, du \, dv.$$  \hspace{1cm} (\text{II})

First, it is obvious that term (II) is of order $O_P(1/n)$. Term (I) requires more delicate analysis. Let $c$ be a vector of unit length and suitable dimension, and define

$$c_i = \int_{x-h}^{x} c' R(u)R(v) \left[ I(x_i \leq x + hu) - F(x + hu) \right] \left[ I(x_i \leq x + hv) - F(x + hv) \right] K(u)K(v)g(x + hu)g(x + hv) \, du \, dv.$$

Then

$$c'(I)c = E[c'(I)c] + O_P \left( \sqrt{\text{E}[c'(I)c]} \right) = E[c_i] + O_P \left( \sqrt{\frac{1}{n}} \left( \text{E}[c_i^2] - (E[c_i])^2 \right) \right),$$

which implies that

$$\frac{c'(I)c}{E[c'(I)c]} = O_P \left( \sqrt{\frac{1}{n}} \left( E[c_i^2] - (E[c_i])^2 \right) \right).$$

With the same argument used in the proof of Theorem 1, one can show that

$$\frac{E[c_i^2]}{(E[c_i])^2} = O \left( \frac{1}{h} \right),$$

which implies

$$\frac{c'(I)c}{c'(\Sigma_h)c} = O_P \left( \sqrt{\frac{1}{nh}} \right).$$

5.3 Proof of Theorem 3

Part (i)

For the “denominator,” its variance is bounded by

$$\left| \text{V} \left[ \frac{1}{n} \sum_{i=1}^{n} \Upsilon_h R(x_i - x) R(x_i - x)' \Upsilon_h \frac{1}{h} K \left( \frac{x_i - x}{h} \right) \right] \right| \leq \frac{1}{n} \text{E} \left[ \left| \Upsilon_h R(x_i - x) R(x_i - x)' \Upsilon_h \right|^2 \frac{1}{h^2} K \left( \frac{x_i - x}{h} \right)^2 \right]$$

$$= \frac{1}{n} \int_{x} \left| \Upsilon_h R(u - x) R(u - x)' \Upsilon_h \right|^2 \frac{1}{h^2} K \left( \frac{u - x}{h} \right)^2 f(u) \, du = \frac{1}{nh} \int_{x-h}^{x+h} |R(u)R(u')|^2 K(u)^2 f(x + hu) \, du$$

$$= O \left( \frac{1}{nh} \right).$$

Therefore, under the assumption that $h \to 0$ and $nh \to \infty$, we have

$$\left| \frac{1}{n} \sum_{i=1}^{n} \Upsilon_h R(x_i - x) R(x_i - x)' \Upsilon_h \frac{1}{h} K \left( \frac{x_i - x}{h} \right) - \Gamma_h \right| = O_P \left( \sqrt{\frac{1}{nh}} \right).$$
which further implies that
\[
\hat{\theta} - \theta = \Upsilon h \Gamma^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} \Upsilon h R(x_i - x) \left[ \hat{F}(x_i) - R(x_i - x)' \theta_0 \right] \frac{1}{h} K \left( \frac{x_i - x}{h} \right) \right) (1 + o_P(1)).
\]

**Part (ii)**

The order of the leave-in bias is clearly 1/n. For the approximation bias (7), we obtained its mean in the proof of Theorem 1 by setting \( G = F \), which has an order of \( g(h) \). The approximation bias has a variance of order

\[
\mathbb{E} \left[ \left( \frac{1}{n} \sum_{j=1}^{n} \Upsilon h R(x_j - x) \left[ F(x_j) - R(x_j - x)' \theta_0 \right] \frac{1}{h} K \left( \frac{x_j - x}{h} \right) \right)^2 \right] 
\leq \frac{1}{n} \mathbb{E} \left[ \Upsilon h R(x_j - x) \left[ F(x_j) - R(x_j - x)' \theta_0 \right] \frac{1}{h} K \left( \frac{x_j - x}{h} \right) \right]^2 
\leq \frac{1}{nh} \int \left| R(u) \right|^2 K(u)^2 f(x + hu) du 
\leq \frac{1}{nh} \rho(h)^2 \int \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} |R(u)|^2 K(u)^2 f(x + hu) du = O \left( \frac{\rho(h)^2}{nh} \right).
\]

Therefore,

\[
(7) = O_P \left( \rho(h) + \rho(h) \sqrt{\frac{1}{nh}} \right) = O_P(\rho(h)),
\]

provided that \( nh \to \infty \).

**Part (iii)**

We compute the variance of the U-statistic (9). For simplicity, define

\[
u_{ij} = \Upsilon h R(x_j - x) \left[ \mathbb{I}(x_i \leq x_j) - F(x_j) \right] \frac{1}{h} K \left( \frac{x_j - x}{h} \right) - \int_{\frac{x-h}{h}}^{\frac{x}{h}} R(u) \left[ \mathbb{I}(x_i \leq x + hu) - F(x + hu) \right] K(u) f(x + hu) du,
\]

which satisfies \( \mathbb{E}[u_{ij}] = \mathbb{E}[u_{ij} | x_i] = \mathbb{E}[u_{ij} | x_j] = 0 \). Therefore

\[
\mathbb{E} \left[ \left( \frac{1}{n^2 h} \sum_{i,j=1}^{n} u_{ij} u_{ij}' \right) \right] = \frac{1}{n^2 h} \sum_{i,j=1}^{n} \mathbb{E} \left[ u_{ij} u_{ij}' \right] = \frac{1}{n^2 h} \sum_{i,j=1}^{n} \mathbb{E} \left[ u_{ij} u_{ij}' \right] + \mathbb{E} \left[ u_{ij} u_{ij}' \right],
\]

meaning that

\[
(9) = O_P \left( \sqrt{\frac{1}{n^2 h}} \right).
\]

**Part (iv)**

This follows immediately from Part (i)–(iii) and Theorem 1.
5.4 Proof of Theorem 4

We first decompose ˆΣₜ into two terms,

\[(I) = \frac{1}{n^3} \sum_{i,j,k=1}^{n} \Upsilon_h R_i R'_i \Upsilon_h W_j W_k \left( \mathbb{1}(x_i \leq x_j) - F(x_j) \right) \left( \mathbb{1}(x_i \leq x_k) - F(x_k) \right) \]

and

\[(II) = -\frac{1}{n^2} \sum_{i,j,k=1}^{n} \Upsilon_h R_i R'_i \Upsilon_h W_j W_k \left( \hat{F}(x_j) - F(x_j) \right) \left( \hat{F}(x_k) - F(x_k) \right),\]

where we use \(R_i = R(x_i - x)\) and \(W_i = K((x_i - x)/h)/h\) to conserve space.

(II) satisfies

\[|\text{(II)}| \leq \sup_x |\hat{F}(x) - F(x)|^2 \frac{1}{n^2} \sum_{j,k=1}^{n} |\Upsilon_h R_i R'_i \Upsilon_h W_j W_k|.
\]

It is obvious that

\[\sup_x |\hat{F}(x) - F(x)|^2 = o_p \left( \frac{1}{n} \right).
\]

As for the second part, we have

\[
\frac{1}{n^3} \sum_{j,k=1}^{n} \mathbb{E} \left[ |\Upsilon_h R_j R'_j \Upsilon_h W_j W_k| \right] = \frac{n-1}{n} \mathbb{E} \left[ |\Upsilon_h R_j R'_j \Upsilon_h W_j W_k| \big| j \neq k \right] \frac{1}{n} \mathbb{E} \left[ |\Upsilon_h R_k R'_k \Upsilon_h W_k W_k| \right] \\
= o_p \left( 1 + \frac{1}{nh} \right) = o_p \left( 1 \right),
\]

which holds as long as \(nh \to \infty\). Then it further implies that

\[(II) = o_p \left( \frac{1}{n} \right).
\]

To analyze (I), we further expand this term into “diagonal” and “non-diagonal” sums:

\[(I) = \frac{1}{n^3} \sum_{i,j,k=1}^{n} \Upsilon_h R_i R'_i \Upsilon_h W_j W_k \left( \mathbb{1}(x_i \leq x_j) - F(x_j) \right) \left( \mathbb{1}(x_i \leq x_k) - F(x_k) \right) \quad (I.1)
\]

\[
+ \frac{1}{n^2} \sum_{i,k=1}^{n} \Upsilon_h R_i R'_i \Upsilon_h W_i W_k \left( \mathbb{1}(x_i \leq x_i) - F(x_i) \right) \left( \mathbb{1}(x_i \leq x_k) - F(x_k) \right) \quad (I.2)
\]

\[
+ \frac{1}{n^2} \sum_{i,j=1}^{n} \Upsilon_h R_j R'_j \Upsilon_h W_j W_i \left( \mathbb{1}(x_i \leq x_j) - F(x_j) \right) \left( \mathbb{1}(x_i \leq x_i) - F(x_i) \right) \quad (I.3)
\]

\[
+ \frac{1}{n^2} \sum_{i,j=1}^{n} \Upsilon_h R_j R'_j \Upsilon_h W_j W_j \left( \mathbb{1}(x_i \leq x_j) - F(x_j) \right) \left( \mathbb{1}(x_i \leq x_j) - F(x_j) \right) \quad (I.4)
\]

\[
+ \frac{1}{n^3} \sum_{i} \Upsilon_h R_i R'_i \Upsilon_h W_i W_i \left( \mathbb{1}(x_i \leq x_i) - F(x_i) \right) \left( \mathbb{1}(x_i \leq x_i) - F(x_i) \right) \quad (I.5)
\]

By calculating the expectation of the absolute value of the summands above, it is straightforward to show

\[(I.2) = o_p \left( \frac{1}{n} \right), \quad (I.3) = o_p \left( \frac{1}{n} \right), \quad (I.4) = o_p \left( \frac{1}{nh} \right), \quad (I.5) = o_p \left( \frac{1}{n^2h} \right).
\]

30
Therefore, we have
\[ \hat{\Sigma}_h = (I.1) + O_P \left( \frac{1}{n h} \right) \]
\[ = \frac{1}{n^3} \sum_{i,j,k=1}^{n} \mathcal{T}_h R_j R_k' \mathcal{T}_h W_j W_k \left( \mathbb{1}(x_i \leq x_j) - F(x_j) \right) \left( \mathbb{1}(x_i \leq x_k) - F(x_k) \right) + O_P \left( \frac{1}{n h} \right). \]

To proceed, define
\[ u_{ij} = \mathcal{T}_h R_j W_j \left( \mathbb{1}(x_i \leq x_j) - F(x_j) \right) \quad \text{and} \quad \bar{u}_i = \mathbb{E}[u_{ij} | x_i; i \neq j]. \]

Then we can further decompose (I.1) into

\[ (I.1) = \frac{1}{n^3} \sum_{i,j,k=1}^{n} u_{ij} u'_{ik} = \frac{1}{n^3} \sum_{i,j,k=1}^{n} \mathbb{E}[u_{ij} u'_{ik} | x_i] + \frac{1}{n^3} \sum_{i,j,k=1}^{n} \left( u_{ij} u'_{ik} - \mathbb{E}[u_{ij} u'_{ik} | x_i] \right). \]

We have already analyzed (I.1.1) in Theorem 2, which suggests
\[ (I.1.1) = \Sigma_h + O_P \left( \frac{1}{\sqrt{n}} \right). \]

Now we study (I.1.2), which satisfies

\[ (I.1.2) = \frac{n-2}{n^3} \sum_{i,j=1}^{n} \left( u_{ij} - \bar{u}_i \right) \tilde{u}_j' + \frac{n-2}{n^3} \sum_{i,j=1}^{n} \bar{u}_i \left( u_{ij} - \bar{u}_i \right)' + \frac{1}{n^3} \sum_{i,j,k=1}^{n} \left( u_{ij} - \bar{u}_i \right) \left( u_{ik} - \bar{u}_i \right)'. \]

With variance calculation, it is easy to see that
\[ (I.1.2.3) = O_P \left( \frac{1}{n h} \right). \]

Therefore we have
\[ \frac{c' \left( \hat{\Sigma}_h - \Sigma_h \right) c}{c' \Sigma_h c} = O_P \left( \frac{1}{\sqrt{n h^2}} \right) + 2 \frac{c' (I.1.2.1) c}{c' \Sigma_h c}, \]

since (I.1.2.1) and (I.1.2.2) are transpose of each other. To close the proof, we calculate the variance of the last term in the above.

\[ \mathbb{V} \left[ \frac{c' (I.1.2.1) c}{c' \Sigma_h c} \right] = \frac{1}{(c' \Sigma_h c)^2} \frac{(n-2)^2}{n^6} \mathbb{E} \left[ \sum_{i,j=1}^{n} \sum_{i',j'=1}^{n} c' \left( u_{ij} - \bar{u}_i \right) \tilde{u}_j' c' \left( u_{i'j'} - \bar{u}_{i'} \right) \tilde{u}_{j'} c \right] \]
\[ = \frac{1}{(c' \Sigma_h c)^2} \frac{(n-2)^2}{n^6} \mathbb{E} \left[ \sum_{i,j,i',j'=1}^{n} c' u_{ij} \tilde{u}_{i'} c' u_{i'j'} \tilde{u}_{j'} c \right] + \text{higher order terms.} \]

31
The expectation is further given by (note that \(i, j\) and \(i'\) are assumed to be distinct indices)

\[
E \left[ c'u_{ij}u'_{i,j}u'_{i,\nu}c \right] = E \int \int \int \frac{W^2 \left[ c' \mathcal{T}_h R_i R(u) c c' \mathcal{T}_h R_j R(v) c \right] K(u) K(v)}{\sqrt{2\pi \sigma^2}} dx dy dz
\]

\[
= \int \int \int \frac{\left[ F(x_j \wedge (x + hu)) - F(x_j)F(x + hu) \right] \left[ F(x_j \wedge (x + hv)) - F(x_j)F(x + hv) \right] f(x + hu)f(x + hv) du dv}{\sqrt{2\pi \sigma^2}}
\]

\[
= \frac{1}{h} \int \int \int \frac{\left[ F(x + h(w \wedge u)) - F(x + hu)F(x + hv) \right] \left[ F(x + h(w \wedge v)) - F(x + hu)F(x + hv) \right] f(x + hu)f(x + hv) du dv}{\sqrt{2\pi \sigma^2}}
\]

\[
= \frac{1}{h} F(x)^2 (1 - F(x))^2 \int \int \int \left[ c' R(w) R(u) c c' R(w) R(v) c \right] K(u) K(v) K(w)^2 f_u f_v f_w dudv + \text{higher-order terms}.
\]

If \(c'\Sigma_h c = O(1)\), then the above will have order \(h\), which means

\[
\forall \left[ \frac{c' \left( I.1.2.1 \right) c}{c' \Sigma_h c} \right] = O \left( \frac{1}{nh} \right).
\]

If \(c'\Sigma_h c = O(h)\), however, \(E \left[ c'u_{ij}u'_{i,j}u'_{i,\nu}c \right]\) will be \(O(1)\), which will imply that

\[
\forall \left[ \frac{c' \left( I.1.2.1 \right) c}{c' \Sigma_h c} \right] = O \left( \frac{1}{nh^2} \right).
\]

As a result, we have

\[
\frac{c' \left( \tilde{\Sigma}_h - \Sigma_h \right) c}{c' \Sigma_h c} = O_p \left( \frac{1}{\sqrt{nh^2}} \right).
\]

Now consider

\[
\frac{c' \Gamma^{-1}_h \tilde{\Sigma}_h \Gamma^{-1}_h c}{c' \Gamma^{-1}_h \Sigma_h \Gamma^{-1}_h c} - 1 = \frac{c' \Gamma^{-1}_h \left( \tilde{\Sigma}_h - \Sigma_h \right) \Gamma^{-1}_h c}{c' \Gamma^{-1}_h \Sigma_h \Gamma^{-1}_h c} + \frac{c' \left( \Gamma^{-1}_h - \Gamma^{-1}_h \right) \Sigma_h \Gamma^{-1}_h c}{c' \Gamma^{-1}_h \Sigma_h \Gamma^{-1}_h c} + \frac{c' \left( \Gamma^{-1}_h \tilde{\Sigma}_h - \Gamma^{-1}_h \right) \Sigma_h \Gamma^{-1}_h c}{c' \Gamma^{-1}_h \Sigma_h \Gamma^{-1}_h c} + \frac{c' \left( \Gamma^{-1}_h - \Gamma^{-1}_h \right) \Sigma_h \Gamma^{-1}_h c}{c' \Gamma^{-1}_h \Sigma_h \Gamma^{-1}_h c}.
\]

From the analysis of \(\tilde{\Sigma}_h\), we have

\[
\frac{c' \Gamma^{-1}_h \left( \tilde{\Sigma}_h - \Sigma_h \right) \Gamma^{-1}_h c}{c' \Gamma^{-1}_h \Sigma_h \Gamma^{-1}_h c} = O_p \left( \frac{1}{\sqrt{nh^2}} \right).
\]

For the second term, we have

\[
\left| \frac{c' \left( \Gamma^{-1}_h - \Gamma^{-1}_h \right) \Sigma_h \Gamma^{-1}_h c}{c' \Gamma^{-1}_h \Sigma_h \Gamma^{-1}_h c} \right| \leq \frac{|c' \left( \Gamma^{-1}_h - \Gamma^{-1}_h \right) \Sigma_h \Gamma^{-1}_h c|}{|c' \Gamma^{-1}_h \Sigma_h \Gamma^{-1}_h c|} \leq O_p \left( \sqrt{\frac{1}{nh^2}} \right).
\]

The third term has order

\[
\left| \frac{c' \left( \Gamma^{-1}_h - \Gamma^{-1}_h \right) \Sigma_h \left( \tilde{\Sigma}_h - \Gamma^{-1}_h \right) c}{c' \Gamma^{-1}_h \Sigma_h \Gamma^{-1}_h c} \right| = O_p \left( \frac{1}{nh^2} \right).
\]

5.5 Proof of Corollary 5

This follows directly from Theorem 1.
5.6 Proof of Corollary 6

To understand (13), note that
\[
\hat{\theta}_\perp F = \left( \int_X \Lambda_h R(u-x)R(u-x)^T \Lambda_h \frac{1}{h} K \left( \frac{u-x}{h} \right) dF(u) \right)^{-1} \left( \int_X \Lambda_h R(u-x)\hat{F}(u) \frac{1}{h} K \left( \frac{u-x}{h} \right) dF(u) \right)
\]

\[
= \Lambda_h^{-1} \left( \int_X R(u-x)R(u-x)^2 \frac{1}{h} K \left( \frac{u-x}{h} \right) dF(u) \right)^{-1} \left( \int_X R(u-x)\hat{F}(u) \frac{1}{h} K \left( \frac{u-x}{h} \right) dF(u) \right),
\]

which means \( \hat{\theta}_\perp F = \Lambda_h^{-1} \hat{\theta}_F \). Then we have (up to an approximation bias term)

\[
\hat{\theta}_F - \Lambda_h^{-1} \theta_0 = \Lambda_h^{-1} (\hat{\theta}_F - \theta_0)
\]

\[
= \Lambda_h^{-1} \left( \int_X R(u-x)R(u-x)^2 \frac{1}{h} K \left( \frac{u-x}{h} \right) dF(u) \right)^{-1} \left( \int_X R(u-x)(\hat{F}(u) - F(u)) \frac{1}{h} K \left( \frac{u-x}{h} \right) dF(u) \right)
\]

\[
= \Lambda_h^{-1} \left( \int_X R(u-x)R(u-x)^2 \frac{1}{h} K \left( \frac{u-x}{h} \right) J(u) K(u) f(x+hu) du \right)^{-1} \left( \int_X R(u-x)(\hat{F}(x+hu) - F(x+hu)) K(u) f(x+hu) du \right)
\]

We first discuss the transformed parameter vector \( \Lambda_h^{-1} \theta_0 \). By construction, the matrix \( \Lambda_h \) takes the following form:

\[
\Lambda_h = \begin{bmatrix}
1 & c_{1,2} & c_{1,3} & \cdots & c_{1,p+2} \\
0 & 1 & 0 & \cdots & c_{2,p+2} \\
0 & 0 & 1 & \cdots & c_{2,p+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\]

where \( c_{i,j} \) are some constants (possibly depending on \( h \)). Therefore, the above matrix differs from the identity matrix only in its first row and in the last column. This observation also holds for \( \Lambda_h^{-1} \). Since the last component of \( \theta_0 \) is zero (because the extra regressor \( Q_h(\cdot) \) is redundant), we conclude that, except for the first element, \( \Lambda_h \theta \) and \( \theta \) are identical. More specifically, let \( I_{-1} \) be the identity matrix excluding the first row:

\[
I_{-1} = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\]

which is used to extract all elements of a vector except for the first one, then by Theorem 1,

\[
\sqrt{n} \left( I_{-1} (\Lambda_h^{-1} \Upsilon_h \Lambda_h) (\Gamma_h^\perp)^{-1} \Sigma_h (\Gamma_h^\perp)^{-1} (\Lambda_h^{-1} \Upsilon_h \Lambda_h)^T I_{-1} \right)^{-1/2} \begin{bmatrix}
\hat{\theta}_F - \theta_F \\
\hat{\theta}_Q - \theta_Q
\end{bmatrix} \approx \mathcal{N}(0, I),
\]

where \( \hat{\theta}_F \) contains the second to the \( p + 1 \)-th element of \( \theta_F \), and \( \hat{\theta}_Q \) is the last element.

Now we discuss the covariance matrix in the above display. Due to orthogonalization, \( \Gamma_h^\perp \) is block diagonal. To be precise,

\[
\Gamma_h^\perp = f(x) \begin{bmatrix}
\Gamma_{1,h}^\perp & 0 & 0 \\
\Gamma_{2,h}^\perp & 0 & 0 \\
0 & 0 & \Gamma_{Q,h}^\perp
\end{bmatrix}, \quad \Gamma_{1,h}^\perp = \int_{\mathbb{X}} K(u) du, \quad \Gamma_{2,h}^\perp = \int_{\mathbb{X}} P(u) P(u)^T K(u) du, \quad \Gamma_{Q,h}^\perp = \int_{\mathbb{X}} Q(u)^2 K(u) du.
\]
Finally, using the structure of $\Lambda_h$ and $\Upsilon_h$, we have

$$I_{-1}(\Lambda_{-1}h \Upsilon_h \Lambda_h)(\Gamma_{\perp}^{-1}) = I_{-1} \Upsilon_h(\Gamma_{\perp}^{-1}).$$

The form of $\Sigma_{\perp}^{-1}h$ is quite involved, but with some algebra, and using the fact that the basis $R(\cdot)$ (or $R^+(\cdot)$) includes a constant and polynomials, one can show the following:

$$(\Lambda_{-1}h \Upsilon_h \Lambda_h)(\Gamma_{\perp}^{-1}) - \Upsilon_h(\Gamma_{\perp}^{-1}) = h f(x) \Upsilon_{-1,h}(\Gamma_{-1,h})^{-1} \Sigma_{-1,h}(\Gamma_{-1,h})^{-1} \Upsilon_{-1,h},$$

where $\Upsilon_{-1,h}$, $\Gamma_{-1,h}$ and $\Sigma_{-1,h}$ are obtained by excluding the first row and the first column of $\Upsilon_h$, $\Gamma_{-1,h}$ and $\Sigma_{-1}h$, respectively:

$$\Upsilon_{-1,h} = \begin{bmatrix} h^{-1} & 0 & 0 & \cdots & 0 \\ 0 & h^{-2} & 0 & \cdots & 0 \\ 0 & 0 & h^{-3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & v_h \end{bmatrix}, \quad \Gamma_{-1,h} = f(\chi) \begin{bmatrix} \Gamma_{\perp}^{P,h} & 0 \\ 0 & \Gamma_{\perp}^{Q,h} \end{bmatrix}, \quad \Sigma_{-1,h} = f(\chi)^3 \begin{bmatrix} \Sigma_{\perp}^{P,P,h} & \Sigma_{\perp}^{P,Q,h} \\ \Sigma_{\perp}^{Q,P,h} & \Sigma_{\perp}^{Q,Q,h} \end{bmatrix},$$

and

$$\Sigma_{\perp}^{P,P,h} = \int_{X_{-\infty}^{\infty}} K(u)K(v)P^+(u)P^+(v)(u \wedge v) du dv, \quad \Sigma_{\perp}^{P,Q,h} = \int_{X_{-\infty}^{\infty}} K(u)K(v)Q^+(u)Q^+(v)(u \wedge v) du dv \\Sigma_{\perp}^{Q,P,h} = (\Sigma_{\perp}^{P,P,h})', \quad \Sigma_{\perp}^{Q,Q,h} = \int_{X_{-\infty}^{\infty}} K(u)K(v)P^+(u)Q^+(v)(u \wedge v) du dv.$$

### 5.7 Proof of Lemma 7

**Part (i)**

To start,

$$\int_{X_{-\infty}^{\infty}} H(g_1)(u)H(g_2)(u) du = \int_{X_{-\infty}^{\infty}} \chi_{[-1,1]}(u) \int_{X_{-\infty}^{\infty}} \chi_{[-1,1]}(v_1 \geq u)K(v_1)g(v_1)dv_1 \left( \int_{X_{-\infty}^{\infty}} \chi_{[-1,1]}(v_2 \geq u)K(v_2)g(v_2)dv_2 \right) du = \int_{X_{-\infty}^{\infty}} K(v_1)K(v_2)g(v_1)g(v_2) \left( \int_{X_{-\infty}^{\infty}} \chi_{[-1,1]}(v_1 \geq u) \chi_{[-1,1]}(v_2 \geq u) du \right) dv_1 dv_2 = \int_{X_{-\infty}^{\infty}} K(v_1)K(v_2)g(v_1)g(v_2) \left( (v_1 \wedge v_2) \wedge \left( \frac{X - x}{h} \wedge 1 \right) - \left( \frac{X - x}{h} \vee (-1) \right) \right) dv_1 dv_2 = \int_{X_{-\infty}^{\infty}} K(v_1)K(v_2)g(v_1)g(v_2)(v_1 \wedge v_2)dv_1 dv_2,$$

where to show the last equality, we used the fact that $v_1 \leq \frac{X - x}{h} \wedge 1$ and $v_2 \leq \frac{X - x}{h} \wedge 1$ for the outer double integral.
Part (ii)

For this part,
\[
\int_{\frac{x-x}{h} \cap [-1,1]} \mathcal{H}(g_1)(u) g_2(u) du = \int_{\frac{x-x}{h} \cap [-1,1]} \left( \int_{\frac{x-x}{h}} 1(v \geq u) K(v) g_1(v) dv \right) g_2(u) du
\]
\[
= \int_{\frac{x-x}{h} \cap [-1,1]} K(v) g_1(v) \left( \int_{\frac{x-x}{h}} 1(v \geq u) g_2(u) du \right) dv
\]
\[
= \int_{\frac{x-x}{h} \cap [-1,1]} K(v) g_1(v) \left[ g_2 \left( v \wedge \frac{x-x}{h} \wedge 1 \right) - g_2 \left( \frac{x-x}{h} \vee (-1) \right) \right] dv
\]
\[
= \int_{\frac{x-x}{h} \cap [-1,1]} K(v) g_1(v) g_2(v) dv.
\]

Again, to show the last equality, we used the fact that \( v \leq \frac{x-x}{h} \wedge 1 \) for the outer integral.

5.8 Proof of Theorem 8

To find a bound of the maximization problem, we note that for any \( c \in \mathbb{R}^{b-1} \), one has
\[
\int_{\frac{x-x}{h} \cap [-1,1]} \mathcal{H}(p_c)(u) \mathcal{H}(q)(u) du = \int_{\frac{x-x}{h} \cap [-1,1]} \left[ \mathcal{H}(p_c)(u) + c' \hat{P}(u) \right] \mathcal{H}(q)(u) du,
\]
due to the constraint. Therefore, an upper bound of the objective function is (due to the Cauchy-Schwartz inequality)
\[
\inf_{c} \int_{\frac{x-x}{h} \cap [-1,1]} \left[ \mathcal{H}(p_c)(u) + c' \hat{P}(u) \right]^2 du
\]
\[
= \inf_{c} \int_{\frac{x-x}{h} \cap [-1,1]} \left[ \mathcal{H}(p_c)(u)^2 + 2c' \hat{P}(u) \mathcal{H}(p_c)(u) + c' \hat{P}(u) p_c(u) \right] du
\]
\[
= \int_{\frac{x-x}{h} \cap [-1,1]} \mathcal{H}(p_c)(u)^2 du + \inf_{c} \int_{\frac{x-x}{h} \cap [-1,1]} \left[ 2c' \hat{P}(u) \mathcal{H}(p_c)(u) + c' \hat{P}(u) p_c(u) \right] du
\]
\[
= \int_{\frac{x-x}{h} \cap [-1,1]} \mathcal{H}(p_c)(u)^2 du + \inf_{c} \left[ 2c' \left( \int_{\frac{x-x}{h}} K(u) p_c(u) du \right) + c' \left( \int_{\frac{x-x}{h} \cap [-1,1]} \hat{P}(u) p_c(u) du \right) \right],
\]
which is minimized by setting
\[
c = - \left( \int_{\frac{x-x}{h} \cap [-1,1]} \hat{P}(u) p_c(u) du \right)^{-1} \left( \int_{\frac{x-x}{h}} K(u) p_c(u) du \right).
\]

As a result, an upper bound of (14) is
\[
\int_{\frac{x-x}{h} \cap [-1,1]} \mathcal{H}(p_{c})(u)^2 du - \left( \int_{\frac{x-x}{h}} K(u) p_c(u) du \right)^T \left( \int_{\frac{x-x}{h} \cap [-1,1]} \hat{P}(u) p_c(u) du \right)^{-1} \left( \int_{\frac{x-x}{h}} K(u) p_c(u) du \right).
\]

We may further simplify the above. First,
\[
\int_{\frac{x-x}{h} \cap [-1,1]} \mathcal{H}(p_{c})(u)^2 du = e'_t (\Gamma^T_{\hat{P},h})^{-1} \Sigma \hat{P}_{P,h} (\Gamma^T_{\hat{P},h})^{-1} e_t.
\]

Second, note that
\[
\int_{\frac{x-x}{h}} K(u) p_c(u) du = \left( \int_{\frac{x-x}{h}} K(u) P^T(u) du \right) (\Gamma^T_{\hat{P},h})^{-1} e_t = \left( \int_{\frac{x-x}{h}} K(u) P^T(u) P^T(u) du \right) (\Gamma^T_{\hat{P},h})^{-1} e_t = e_t.
\]
As a result, an upper bound of (14) is
\[
e' \ell(\Gamma_{\bot} P,h)^{-1} e\ell - e' \ell \left( \int_{-\infty}^{x} \hat{P}(u) \hat{P}'(u) \, du \right)^{-1} e\ell.
\]

5.9 Additional Preliminary Lemmas

**Lemma 21.** Assume \( \{u_{i,h}(a) : a \in A \subset \mathbb{R}^d \} \) are independent across \( i \), and \( \mathbb{E}[u_{i,h}(a)] = 0 \) for all \( a \in A \) and all \( h > 0 \). In addition, assume for each \( \varepsilon > 0 \) there exists \( \{u_{i,h,c}(a) : a \in A\} \), such that
\[
|a - b| \leq \varepsilon \quad \Rightarrow \quad |u_{i,h}(a) - u_{i,h}(b)| \leq u_{i,h,c}(a).
\]
Define
\[
C_1 = \sup_{a \in A} \max_{1 \leq i \leq n} \mathbb{V}[u_{i,h}(a)], \quad C_2 = \sup_{a \in A} \max_{1 \leq i \leq n} |u_{i,h}(a)|
\]
\[
C_{1,\varepsilon} = \sup_{a \in A} \max_{1 \leq i \leq n} \mathbb{V}[u_{i,h,c}(a)], \quad C_{2,\varepsilon} = \sup_{a \in A} \max_{1 \leq i \leq n} |u_{i,h,c}(a)| - \mathbb{E}[u_{i,h,c}(a)], \quad C_{3,\varepsilon} = \sup_{a \in A} \max_{1 \leq i \leq n} \mathbb{E}[u_{i,h,c}(a)].
\]
Then
\[
\sup_{a \in A} \left| \frac{1}{n} \sum_{i=1}^{n} u_{i,h}(a) \right| = O_P(\gamma + \gamma_{\varepsilon} + C_{3,\varepsilon}),
\]
where \( \gamma \) and \( \gamma_{\varepsilon} \) are any sequences satisfying
\[
\frac{\gamma^2 n}{(C_1 + \frac{1}{2} \gamma C_2) \log N(\varepsilon, A, \cdot | \cdot)} \quad \text{and} \quad \frac{\gamma_{\varepsilon}^2 n}{(C_{1,\varepsilon} + \frac{1}{2} \gamma_{\varepsilon} C_{2,\varepsilon}) \log N(\varepsilon, A, \cdot | \cdot)}
\]
are bounded from below,

and \( N(\varepsilon, A, \cdot | \cdot) \) is the covering number of \( A \).

**Remark 6.** Provided that \( u_{i,h}(\cdot) \) is reasonably smooth, one can always choose \( \varepsilon \) (as a function of \( n \) and \( h \)) small enough, and the leading order will be given by \( \gamma \) (and hence is determined by \( C_1 \) and \( C_2 \)).

**Proof.** Let \( A_{\varepsilon} \) be an \( \varepsilon \)-covering of \( A \), then
\[
\sup_{a \in A} \left| \frac{1}{n} \sum_{i=1}^{n} u_{i,h}(a) \right| \leq \sup_{a \in A_{\varepsilon}} \left| \frac{1}{n} \sum_{i=1}^{n} u_{i,h}(a) \right| + \sup_{a \in A_{\varepsilon}, b \in A, |a - b| \leq \varepsilon} \left| \frac{1}{n} \sum_{i=1}^{n} u_{i,h}(a) - u_{i,h}(b) \right|.
\]
Next we apply the union bound and Bernstein’s inequality:
\[
P \left[ \sup_{a \in A_{\varepsilon}} \left| \frac{1}{n} \sum_{i=1}^{n} u_{i,h}(a) \right| \geq \gamma u \right] \leq N(\varepsilon, A, \cdot | \cdot) \sup_{a \in A} \left[ \left| \frac{1}{n} \sum_{i=1}^{n} u_{i,h}(a) \right| \geq \gamma u \right]
\]
\[
\leq 2 N(\varepsilon, A, \cdot | \cdot) \exp \left\{ - \frac{1}{2} \frac{\gamma^2 n u^2}{C_1 + \frac{1}{2} \gamma C_2 u} \right\}
\]
\[
= 2 \exp \left\{ - \frac{1}{2} \frac{\gamma^2 n u^2}{C_1 + \frac{1}{2} \gamma C_2 u} + \log N(\varepsilon, A, \cdot | \cdot) \right\}.
\]
Now take \( u \) sufficiently large, then the above is further bounded by:
\[
P \left[ \sup_{a \in A_{\varepsilon}} \left| \frac{1}{n} \sum_{i=1}^{n} u_{i,h}(a) \right| \geq \gamma u \right] \leq 2 \exp \left\{ - \log N(\varepsilon, A, \cdot | \cdot) \left[ \frac{1}{2} \frac{1}{\log N(\varepsilon, A, \cdot | \cdot)} \frac{\gamma^2 n}{C_1 + \frac{1}{2} \gamma C_2 u - 1} - 1 \right] \right\},
\]

36
which tends to zero if \( \log N(\varepsilon, A, |·|) \to \infty \) and

\[
\gamma^2_n \left( C_1 + \frac{1}{3} \gamma C_2 \right) \log N(\varepsilon, A, |·|) \text{ is bounded from below,}
\]

in which case we have

\[
\sup_{a \in A} \left| \frac{1}{n} \sum_{i=1}^{n} u_{i,h}(a) \right| = O_P(\gamma) .
\]

We can apply the same technique to the other term, and obtain

\[
\sup_{a \in A, b \in A, |a - b| \leq \varepsilon} \left| \frac{1}{n} \sum_{i=1}^{n} u_{i,h}(a) - u_{i,h}(b) \right| = O_P(\gamma_\varepsilon) ,
\]

where \( \gamma_\varepsilon \) is any sequence satisfying

\[
\gamma^2_n \left( C_1, \varepsilon + \frac{1}{3} \gamma C_2, \varepsilon \right) \log N(\varepsilon, A, |·|) \text{ is bounded from below.}
\]

\[
\text{Lemma 22 (Corollary 5.1 in Chernozhukov, Chetverikov, Kato, and Koike 2019).} \quad \text{Let } z_1, z_2 \in \mathbb{R}^{l_n} \text{ be two mean-zero Gaussian random vectors with covariance matrices } \Omega_1 \text{ and } \Omega_2 \text{, respectively. Further assume that the diagonal elements in } \Omega_1 \text{ are all one. Then}
\]

\[
\sup_{A \subseteq \mathbb{R}^{l_n}, A \text{ rectangular}} |\mathbb{P}[z_1 \in A] - \mathbb{P}[z_2 \in A]| \leq C \sqrt{\|\Omega_1 - \Omega_2\|_\infty} \log \ell_n ,
\]

where \( \|·\|_\infty \) denotes the supremum norm, and \( C \) is an absolute constant.

\[
\text{Lemma 23 (Equation (3.5) in Giné, Latała, and Zinn 2000).} \quad \text{For a degenerate and decoupled second order U-statistic, } \sum_{i,j=1, i \neq j}^{n} h_{ij}(x_i, \hat{x}_j), \text{ the following holds:}
\]

\[
\mathbb{P} \left[ \left| \sum_{i,j \neq j}^{n} u_{ij}(x_i, \hat{x}_j) \right| > t \right] \leq C \exp \left\{ - \frac{1}{C} \min \left[ \frac{t}{D}, \left( \frac{t}{B} \right)^{\frac{2}{3}}, \left( \frac{t}{A} \right)^{\frac{2}{3}} \right] \right\} ,
\]

where \( C \) is some universal constant, and \( A, B, \) and \( D \) are any constants satisfying

\[
A \geq \max_{1 \leq i, j \leq n} \sup_{u, v} |u_{ij}(u, v)|
\]

\[
B^2 \geq \max_{1 \leq i, j \leq n} \left[ \sup_{u} \left| \sum_{i=1}^{n} E u_{ij}(x_i, v) \right|^2 , \sup_{u} \left| \sum_{j=1}^{n} E u_{ij}(u, \hat{x}_j) \right|^2 \right]
\]

\[
D^2 \geq \sum_{i,j=1, i \neq j}^{n} E u_{ij}(x_i, \hat{x}_j)^2 .
\]

where \( \{x_i, 1 \leq i \leq n\} \) are independent random variables, and \( \{\hat{x}_i, 1 \leq i \leq n\} \) is an independent copy of \( \{x_i, 1 \leq i \leq n\} \).

\[
\text{Remark 7. To apply the above lemma, an additional decoupling step is usually needed. Fortunately, the decoupling step only introduces an extra constant, but will not affect the order of the tail probability bound. Formally,}
\]

\[
\text{Lemma 24 (de la Peña and Montgomery-Smith 1995).} \quad \text{Consider the setting of Lemma 23. Then}
\]

\[
\mathbb{P} \left[ \left| \sum_{i,j \neq j}^{n} u_{ij}(x_i, x_j) \right| > t \right] \leq C \cdot \mathbb{P} \left[ \left| \sum_{i,j \neq j}^{n} u_{ij}(x_i, \hat{x}_j) \right| > t \right] ,
\]

37
where $C$ is a universal constant.

As a result, we will apply Lemma 23 without explicitly mentioning the decoupling step or the extra constant it introduces.

5.10 Proof of Theorem 9

To bound the distance between the two processes, $\tilde{\xi}_G(\cdot)$ and $\mathcal{B}_G(\cdot)$, we employ the proof strategy of Giné, Koltchinskii, and Sakhanenko (2004). Recall that $F$ denotes the distribution of $x_i$, and we define

$$K_{h,x} \circ F^{-1}(x) = K_{h,x}(F^{-1}(x)).$$

Take $v < v'$ in $[0, 1]$, we have

$$|K_{h,x} \circ F^{-1}(v) - K_{h,x} \circ F^{-1}(v')|$$

$$= \int_{\mathbb{R}} \frac{c_h' \mathcal{Y}_h \Gamma_{h,x}^{-1} R(u) \left[ \mathbb{1}(F^{-1}(v) \leq x + hu) - \mathbb{1}(F^{-1}(v') \leq x + hu) \right]}{\sqrt{c_h' \mathcal{Y}_h \Omega_{h,x} \mathcal{Y}_h c_h,x}} K(u) g(x + hu) du$$

$$\leq \sqrt{c_h' \mathcal{Y}_h \Omega_{h,x} \mathcal{Y}_h c_h,x} \frac{1}{\sqrt{h}}.$$

Therefore, the function $K_{h,x} \circ F^{-1}(\cdot)$ has a total variation bounded by

$$\frac{\int_{\mathbb{R}} |c_h' \mathcal{Y}_h \Gamma_{h,x}^{-1} R(u) \left[ \mathbb{1}(F^{-1}(v) \leq x + hu) - \mathbb{1}(F^{-1}(v') \leq x + hu) \right]| K(u) g(x + hu) du}{\sqrt{c_h' \mathcal{Y}_h \Omega_{h,x} \mathcal{Y}_h c_h,x}} \leq C_4 \frac{1}{\sqrt{h}}.$$

It is well-known that functions of bounded variation can be approximated (pointwise) by convex combination of indicator functions of half intervals. To be more precise,

$$\left\{ K_{h,x} \circ F^{-1}(\cdot) : x \in \mathcal{I} \right\} \subset C_4 \frac{1}{\sqrt{h}} \text{conv} \left\{ \pm \mathbb{1}(\cdot \leq t), \pm \mathbb{1}(\cdot \geq t) \right\}.$$

Following (2.3) and (2.4) of Giné, Koltchinskii, and Sakhanenko (2004), we have

$$\mathbb{P} \left[ \sup_{x \in \mathcal{I}} |\tilde{\xi}_G(x) - \mathcal{B}_G(x)| > \frac{C_4(u + C_5 \log n)}{\sqrt{nh}} \right] \leq C_5 e^{-C_5 n},$$

where $C_5$ is some universal constant.
5.11 Proof of Lemma 10

Take $|x - y| \leq \varepsilon$ to be some small number, then

$$K_{h,x}(x) - K_{h,y}(x) = \frac{c_h \Gamma_{h,x}^{-1} \int_{x-h}^{x} R(u) \left[ 1 (x \leq x + hu) - F(x + hu) \right] K(u) g(x + hu) du}{\sqrt{c_h \Gamma_{h,x}^{-1} \int_{x-h}^{x} R(u) \left[ 1 (x \leq x + hu) - F(x + hu) \right] K(u) g(x + hu) du}}$$

$$- \frac{c_h \Gamma_{h,y}^{-1} \int_{x-h}^{x} R(u) \left[ 1 (x \leq y + hu) - F(y + hu) \right] K(u) g(y + hu) du}{\sqrt{c_h \Gamma_{h,y}^{-1} \int_{x-h}^{x} R(u) \left[ 1 (x \leq y + hu) - F(y + hu) \right] K(u) g(y + hu) du}}$$

$$= \left( \frac{c_h \Gamma_{h,x}^{-1}}{\sqrt{c_h \Gamma_{h,x}^{-1} \int_{x-h}^{x} R(u) \left[ 1 (x \leq x + hu) - F(x + hu) \right] K(u) g(x + hu) du}} \right) \left( \int_{x-h}^{x} R(u) \left[ 1 (x \leq x + hu) - F(x + hu) \right] K(u) g(x + hu) du \right)$$

$$+ \left( \frac{c_h \Gamma_{h,y}^{-1}}{\sqrt{c_h \Gamma_{h,y}^{-1} \int_{x-h}^{x} R(u) \left[ 1 (x \leq y + hu) - F(y + hu) \right] K(u) g(y + hu) du}} \right) \left( \int_{x-h}^{x} R(u) \left[ 1 (x \leq y + hu) - F(y + hu) \right] K(u) g(y + hu) du \right)$$

$$+ \left( \frac{c_h \Gamma_{h,x}^{-1}}{\sqrt{c_h \Gamma_{h,x}^{-1} \int_{x-h}^{x} R(u) \left[ 1 (x \leq x + hu) - F(x + hu) \right] K(u) g(x + hu) du}} \right) \left( \int_{x-h}^{x} R(u) \left[ 1 (x \leq y + hu) - F(y + hu) \right] K(u) g(y + hu) du \right)$$

$$+ \left( \frac{c_h \Gamma_{h,y}^{-1}}{\sqrt{c_h \Gamma_{h,y}^{-1} \int_{x-h}^{x} R(u) \left[ 1 (x \leq x + hu) - F(x + hu) \right] K(u) g(x + hu) du}} \right) \left( \int_{x-h}^{x} R(u) \left[ 1 (x \leq y + hu) - F(y + hu) \right] K(u) g(y + hu) du \right)$$

For term (I), its variance (replace the placeholder $x$ by $x_j$) is

$$V[\text{I}] = \frac{1}{c_h \Gamma_{h,x}^{-1}} \int_{x-h}^{x} R(u) \left[ 1 (x \leq x + hu) - F(x + hu) \right] K(u) g(x + hu) du = \frac{1}{h} r_1(\varepsilon, h)^2 r_2(h)^2.$$
Next for term (III), we have

\[ V[(III)] = \left( \frac{1}{\sqrt{c_{h,x}^\ast Y_h \Omega_{h,x} T_{h,c_h,x}}} - \frac{1}{\sqrt{c_{h,y}^\ast Y_h \Omega_{h,y} T_{h,c_h,y}}} \right)^2 c_{h,x}^\ast Y_h \Omega_{h,x} T_{h,c_h,x} \]

\[ = \left( 1 - \frac{1 + c_{h,x}^\ast Y_h \Omega_{h,x} T_{h,c_h,x} - c_{h,y}^\ast Y_h \Omega_{h,y} T_{h,c_h,y}}{c_{h,y}^\ast Y_h \Omega_{h,y} T_{h,c_h,y}} \right)^2 \]

\[ \geq \left( \frac{c_{h,x}^\ast Y_h \Omega_{h,x} T_{h,c_h,x} - c_{h,y}^\ast Y_h \Omega_{h,y} T_{h,c_h,y}}{c_{h,y}^\ast Y_h \Omega_{h,y} T_{h,c_h,y}} \right)^2 \]

\[ = \left( \frac{c_{h,x}^\ast Y_h(\Omega_{h,x} - \Omega_{h,y}) T_{h,c_h,x} + (c_{h,x}^\ast Y_h - c_{h,y}^\ast Y_h) \Omega_{h,y} T_{h,c_h,y} + (c_{h,x}^\ast Y_h - c_{h,y}^\ast Y_h) \Omega_{h,y} T_{h,c_h,y}}{c_{h,y}^\ast Y_h \Omega_{h,y} T_{h,c_h,y}} \right)^2. \]

The first term has bound

\[ \frac{c_{h,x}^\ast Y_h(\Omega_{h,x} - \Omega_{h,y}) T_{h,c_h,x}}{c_{h,y}^\ast Y_h \Omega_{h,y} T_{h,c_h,y}} = O \left( \frac{\varepsilon}{h^2} \right). \]

The third term has bound

\[ \frac{(c_{h,x}^\ast Y_h - c_{h,y}^\ast Y_h) \Omega_{h,y} T_{h,c_h,y}}{c_{h,y}^\ast Y_h \Omega_{h,y} T_{h,c_h,y}} \approx \frac{|(c_{h,x}^\ast Y_h - c_{h,y}^\ast Y_h) \Omega_{h,y}^{1/2}|}{\sqrt{c_{h,y}^\ast Y_h \Omega_{h,y} T_{h,c_h,y}}} = O \left( \frac{1}{\sqrt{h}} r_1(\varepsilon, h) r_2(h) \right). \]

Finally, the second term can be bounded as

\[ \frac{(c_{h,x}^\ast Y_h - c_{h,y}^\ast Y_h) \Omega_{h,y} T_{h,c_h,x}}{c_{h,y}^\ast Y_h \Omega_{h,y} T_{h,c_h,x}} = \frac{(c_{h,x}^\ast Y_h - c_{h,y}^\ast Y_h) \Omega_{h,y} T_{h,c_h,x} + (c_{h,x}^\ast Y_h - c_{h,y}^\ast Y_h) \Omega_{h,y} (c_{h,x}^\ast Y_h - c_{h,y}^\ast Y_h)'}{c_{h,y}^\ast Y_h \Omega_{h,y} T_{h,c_h,x}} \]

\[ = O \left( \frac{1}{\sqrt{h}} r_1(\varepsilon, h) r_2(h) + \frac{1}{h} r_1(\varepsilon, h)^2 r_2(h)^2 \right). \]

Overall, we have that

\[ V[(III)] = O \left( \frac{\varepsilon^2}{h^2} + \frac{1}{h} r_1(\varepsilon, h)^2 r_2(h)^2 + \frac{1}{h^2} r_1(\varepsilon, h)^4 r_2(h)^4 \right). \]

Given our assumptions on the basis function and on the kernel function, it is obvious that term (IV) has variance

\[ V[(IV)] = O \left( \frac{1}{h} \left( \frac{\varepsilon}{h} \wedge 1 \right)^2 \right). \]

The bound on \( E[\sup_{x \in \mathcal{I}} |\mathcal{B}_G(x)|] \) can be found by standard entropy calculation, and the bound on \( E[\sup_{x \in \mathcal{I}} |\overline{T}_G(x)|] \) is obtained by the following fact

\[ E \left[ \sup_{x \in \mathcal{I}} |\overline{T}_G(x)| \right] \leq E \left[ \sup_{x \in \mathcal{I}} |\mathcal{B}_G(x)| \right] + E \left[ \sup_{x \in \mathcal{I}} |\overline{T}_G(x) - \mathcal{B}_G(x)| \right], \]

and that

\[ E \left[ \sup_{x \in \mathcal{I}} |\overline{T}_G(x) - \mathcal{B}_G(x)| \right] = \int_0^\infty P \left[ \sup_{x \in \mathcal{I}} |\overline{T}_G(x) - \mathcal{B}_G(x)| > u \right] du = O \left( \frac{\log n}{\sqrt{nh}} \right) = o(\sqrt{\log n}), \]

which follows from Theorem 9 and our assumption that \( \log n/(nh) \to 0 \).
5.12 Proof of Lemma 11

We adopt the following decomposition (the integration is always on $\frac{X-x}{n} \times \frac{X-y}{n}$, unless otherwise specified):

$$\frac{1}{n} \sum_{i=1}^{n} \int \int R(u)R(v) \left[ \mathbb{1}(x_i \leq x + hu) - F(x + hu) \right] \left[ \mathbb{1}(x_i \leq y + hv) - F(y + hv) \right] K(u)K(v)g(x + hu)g(y +hv)dudv$$

(I)

$$- \int \int R(u)R(v) \left[ \hat{F}(x + hu) - F(x + hu) \right] \left[ \hat{F}(y + hv) - F(y + hv) \right] K(u)K(v)g(x + hu)g(y +hv)dudv.$$  

(II)

By the uniform convergence of the empirical distribution function, we have that

$$\sup_{x,y \in \mathcal{I}} |(II)| = O_p \left( \frac{1}{n} \right).$$

From the definition of $\Sigma_{h,x,y}$, we know that

$$\mathbb{E}(I) = \Sigma_{h,x,y}.$$  

As (I) is a sum of bounded terms, we can apply Lemma 21 and easily show that

$$\sup_{x,y \in \mathcal{I}} |(I)| + O_p \left( \sqrt{\frac{\log n}{nh^2}} \right).$$

5.13 Proof of Lemma 12

We rewrite (16) as

$$|(16)| = \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{c_{h,x} \Gamma_{h,x}^{-1} \int_{X-x} R(u) \left[ F(x + hu) - \theta'R(u) \Gamma_{h}^{-1} \right] K(u) g(x + hu) du}{\sqrt{c_{h,x} \Gamma_{h,x} \Gamma_{h,h} x_{h,c_h,x}}} \right|$$

$$\leq \frac{1}{\sqrt{n}} \left[ \sup_{x \in \mathcal{I}} \left| \frac{c_{h,x} \Gamma_{h,x} \Gamma_{h,h} x_{h,c_h,x}}{\sqrt{c_{h,x} \Gamma_{h,x} \Gamma_{h,h} x_{h,c_h,x}}} \right| \sup_{x \in \mathcal{I}} \left| \int_{X-x} R(u) \left[ F(x + hu) - \theta'R(u) \Gamma_{h}^{-1} \right] K(u) g(x + hu) du \right| \right]$$

$$= O_p \left( \sqrt{\frac{\log n}{nh^2}} \sup_{x \in \mathcal{I}} g(h,x) \right),$$

where the final bound holds uniformly for $x \in \mathcal{I}$.

Next, we expand term (17) as

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{c_{h,x} \Gamma_{h,x}^{-1} \int_{X-x} R(u) \left[ \mathbb{1}(x_i \leq x + hu) - F(x + hu) \right] K(u) g(x + hu) du}{\sqrt{c_{h,x} \Gamma_{h,x} \Gamma_{h,h} x_{h,c_h,x}}}$$

$$+ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ 1 - \frac{c_{h,x} \Gamma_{h,x}^{-1} \int_{X-x} R(u) \left[ \mathbb{1}(x_i \leq x + hu) - F(x + hu) \right] K(u) g(x + hu) du}{\sqrt{c_{h,x} \Gamma_{h,x} \Gamma_{h,h} x_{h,c_h,x}}} \right]$$

$$= T_G(x) + \left[ 1 - \frac{c_{h,x} \Gamma_{h,x}^{-1} \int_{X-x} R(u) \left[ \mathbb{1}(x_i \leq x + hu) - F(x + hu) \right] K(u) g(x + hu) du}{\sqrt{c_{h,x} \Gamma_{h,x} \Gamma_{h,h} x_{h,c_h,x}}} \right] T_G(x).$$

(I)

Term (I) can be easily bounded by

$$\sup_{x \in \mathcal{I}} |(I)| = O_p \left( \sqrt{\frac{\log n}{nh^2}} \mathbb{E} \left[ \sup_{x \in \mathcal{I}} |T_G(x)| \right] \right) = O_p \left( \frac{\log n}{\sqrt{nh^2}} \right).$$
5.14 Proof of Theorem 13
The claim follows from Theorem 9 and previous lemmas.

5.15 Proof of Theorem 14
Let \( I_\varepsilon \) be an \( \varepsilon \)-covering (with respect to the Euclidean metric) of \( I \), and assume \( \varepsilon \leq h \). Then the process \( \mathcal{B}_G(\cdot) \) can be decomposed into:
\[
\mathcal{B}_G(x) = \mathcal{B}_G(\Pi_{I_\varepsilon}(x)) + \mathcal{B}_G(x) - \mathcal{B}_G(\Pi_{I_\varepsilon}(x)),
\]
where \( \Pi_{I_\varepsilon} : I \to I_\varepsilon \) is a mapping satisfying:
\[
\Pi_{I_\varepsilon}(x) = \arg\min_{y \in I_\varepsilon} |y - x|.
\]
We first study the properties of \( \mathcal{B}_G(\cdot) - \mathcal{B}_G(\Pi_{I_\varepsilon}(\cdot)) \). With standard entropy calculation, one has:
\[
\mathbb{E} \left[ \sup_{x \in I} \left| \mathcal{B}_G(x) - \mathcal{B}_G(\Pi_{I_\varepsilon}(x)) \right| \right] \leq \mathbb{E} \left[ \sup_{x,y \in I, |x-y| \leq \varepsilon} \left| \mathcal{B}_G(x) - \mathcal{B}_G(y) \right| \right] \leq \mathbb{E} \left[ \sup_{x,y \in I, \sigma(x,y) \leq \delta(\varepsilon)} \left| \mathcal{B}_G(x) - \mathcal{B}_G(y) \right| \right] \leq C \int_0^{\delta(\varepsilon)} \sqrt{\log N(\lambda, I, \sigma_G)} d\lambda,
\]
where
\[
\delta(\varepsilon) = C \left( \frac{1}{\sqrt{h}} + \frac{1}{\sqrt{h}} r_1(\varepsilon, h)r_2(h) + \frac{1}{h} r_1(\varepsilon, h)^2 r_2(h)^2 \right),
\]
for some \( C > 0 \) that does not depend on \( \varepsilon \) and \( h \), and \( N(\lambda, I, \sigma_G) \) is the covering number of \( I \) measured by the pseudo metric \( \sigma_G(\cdot, \cdot) \), which satisfies
\[
N(\lambda, I, \sigma_G) \lesssim \frac{1}{\delta^{-1}(\lambda)}.
\]
Therefore, we have
\[
\mathbb{E} \left[ \sup_{x \in I} \left| \mathcal{B}_G(x) - \mathcal{B}_G(\Pi_{I_\varepsilon}(x)) \right| \right] \lesssim \left( \frac{1}{\sqrt{h}} + \frac{1}{\sqrt{h}} r_1(\varepsilon, h)r_2(h) + \frac{1}{h} r_1(\varepsilon, h)^2 r_2(h)^2 \right) \sqrt{\log n}. \tag{I}
\]
A similar bound holds for the process \( \hat{\mathcal{B}}_G(\cdot) \) due to the uniform consistency of the covariance estimator.

Now consider the discretized version of \( \mathcal{B}_G(\cdot) \) and \( \hat{\mathcal{B}}_G(\cdot) \). By applying Lemmas 11 and 22, we directly obtain the following bound:
\[
\sup_{A \text{ rectangular}} \mathbb{P} \left[ \left\{ \mathcal{B}_G(\Pi_{I_\varepsilon}(x)), x \in I \right\} \in A \right] - \mathbb{P}^* \left[ \left\{ \hat{\mathcal{B}}_G(\Pi_{I_\varepsilon}(x)), x \in I \right\} \in A \right] = O_p \left( \frac{\log n}{nh^2} \right)^{1/4} \log \frac{1}{\varepsilon}. \tag{II}
\]
As \( \varepsilon \) appears in (I) polynomially but only logarithmically in (II), it is possible to choose \( \varepsilon \) sufficiently small so that the discretization error becomes negligible. Therefore,
\[
\sup_{u \in \mathbb{R}} \mathbb{P} \left[ \sup_{x \in I} \left| \mathcal{B}_G(x) \right| \leq u \right] - \mathbb{P}^* \left[ \sup_{x \in I} \left| \hat{\mathcal{B}}_G(x) \right| \leq u \right] = O_p \left( \frac{\log^{3/2} n}{(nh^2)^{1/4}} \right).
\]
5.16 Proof of Lemma 15

We apply Lemma 21. For simplicity, assume $R(\cdot)$ is scalar, and let

$$u_{i,h}(x) = R\left(\frac{x_i - x}{h}\right)^2 \frac{1}{h} K\left(\frac{x_i - x}{h}\right) - \Gamma_{h,x}. \nonumber$$

Then it is easy to see that

$$\sup_{x \in I} \max_{1 \leq i \leq n} \mathcal{V}[u_{i,h}(x)] = O(h^{-1}), \quad \sup_{x \in I} \max_{1 \leq i \leq n} |u_{i,h}(x)| = O(h^{-1}).$$

Let $|x - y| \leq \varepsilon \leq h$, we also have

$$|u_{i,h}(x) - u_{i,h}(y)| \leq \left| R\left(\frac{x_i - x}{h}\right)^2 \frac{1}{h} K\left(\frac{x_i - x}{h}\right) - R\left(\frac{x_i - y}{h}\right)^2 \frac{1}{h} K\left(\frac{x_i - y}{h}\right) \right| + |\Gamma_{h,x} - \Gamma_{h,y}| \nonumber$$

$$\leq \left| R\left(\frac{x_i - x}{h}\right)^2 - R\left(\frac{x_i - y}{h}\right)^2 \right| \frac{1}{h} K\left(\frac{x_i - x}{h}\right) + R\left(\frac{x_i - y}{h}\right)^2 \frac{1}{h} K\left(\frac{x_i - x}{h}\right) - K\left(\frac{x_i - y}{h}\right) \right| + |\Gamma_{h,x} - \Gamma_{h,y}| \nonumber$$

$$\leq M \left[ \frac{\varepsilon}{h} K\left(\frac{x_i - x}{h}\right) + \frac{\varepsilon}{h} K\left(\frac{x_i - y}{h}\right) + \frac{\varepsilon}{h} K\left(\frac{x_i - y}{h}\right) + \frac{\varepsilon}{h} \right]. \nonumber$$

where $M$ is some constant that does not depend on $n$, $h$ or $\varepsilon$. Then it is easy to see that

$$\sup_{x \in I} \max_{1 \leq i \leq n} \mathcal{V}[u_{i,h,x}(x)] = O\left(\frac{\varepsilon}{K^2}\right), \quad \sup_{x \in I} \max_{1 \leq i \leq n} |u_{i,h,x}(x) - E[u_{i,h,x}(x)]| = O(h^{-1}), \quad \sup_{x \in I} \max_{1 \leq i \leq n} |E[u_{i,h,x}(x)]| = O\left(\frac{\varepsilon}{K}\right). \nonumber$$

Now take $\varepsilon = \sqrt{h \log n}/n$, then $\log N(\varepsilon, I, | \cdot |) = O(n \log n)$. Lemma 21 implies that

$$\sup_{x \in I} \left| \frac{1}{n} \sum_{i=1}^{n} R\left(\frac{x_i - x}{h}\right)^2 \frac{1}{h} K\left(\frac{x_i - x}{h}\right) - \Gamma_{h,x} \right| = O\left(\frac{\sqrt{\log n}}{nh}\right). \nonumber$$

5.17 Proof of Lemma 16

Let $R_i(x) = R(x_i - x)$ and $W_i(x) = K((x_i - x)/h)/h$, then we split $\hat{\Sigma}_{h,x}$ into two terms,

$$(I) = \frac{1}{n^2} \sum_{i,j,k} \mathcal{Y}_h R_j(x) R_k(y) \mathcal{Y}_h W_j(x) W_k(y) \left( \mathbb{1}(x_i \leq x_j) - F(x_j) \right) \left( \mathbb{1}(x_i \leq x_k) - F(x_k) \right) \nonumber$$

$$(II) = -\frac{1}{n^2} \sum_{j,k} \mathcal{Y}_h R_j(x) R_k(y) \mathcal{Y}_h W_j(x) W_k(y) \left( \hat{F}(x_j) - F(x_j) \right) \left( \hat{F}(x_k) - F(x_k) \right). \nonumber$$

(II) satisfies

$$\sup_{x,y \in I} |(II)| \leq \sup_{x} |\hat{F}(x) - F(x)|^2 \left( \sup_{x \in I} \frac{1}{n} \sum_{j} |\mathcal{Y}_h R_j(x) W_j(x)| \right)^2. \nonumber$$

It is obvious that

$$\sup_{x} |\hat{F}(x) - F(x)|^2 = O_p\left(\frac{1}{n}\right). \nonumber$$

As for the second part, one can employ the same technique used to prove Lemma 15 and show that

$$\sup_{x \in I} \frac{1}{n} \sum_{j} |\mathcal{Y}_h R_j(x) W_j(x)| = O_p(1). \nonumber$$
implying that
\[ \sup_{x,y \in \mathcal{I}} |(\Pi)| = O_P \left( \frac{1}{n} \right). \]

For (I), we first define
\[ u_{ij}(x) = \Upsilon h R_j(x) W_j(x) \left( \mathbf{1}(x_i \leq x_j) - F(x_j) \right), \]
and
\[ \bar{u}_i(x) = \mathbb{E}[u_{ij}(x) | x_i \neq j], \quad \bar{u}_i(x) = \frac{1}{n} \sum_j u_{ij}(x). \]

Then
\[ (I) = \frac{1}{n} \sum_i \left( \frac{1}{n} \sum_j u_{ij}(x) \right) \left( \frac{1}{n} \sum_j u_{ij}(y) \right)' = \frac{1}{n} \sum_i \bar{u}_i(x) \bar{u}_i(y)' \]
\[ = \frac{1}{n} \sum_i \bar{u}_i(x) \bar{u}_i(y)' + \frac{1}{n} \sum_i (\bar{u}_i(x) - \bar{u}_i(x)) \bar{u}_i(y)' + \frac{1}{n} \sum_i \bar{u}_i(x) (\bar{u}_i(y) - \bar{u}_i(y))' \]
\[ = \frac{1}{n} \sum_i \bar{u}_i(x) \bar{u}_i(y)' + \frac{1}{n} \sum_i (\bar{u}_i(x) - \bar{u}_i(x)) \bar{u}_i(y)' + \frac{1}{n} \sum_i \bar{u}_i(x) (\bar{u}_i(y) - \bar{u}_i(y))' \]
\[ + \frac{1}{n} \sum_i (\bar{u}_i(x) - \bar{u}_i(x)) (\bar{u}_i(y) - \bar{u}_i(y))'. \]

Term (I.1) has been analyzed in Lemma 11, which satisfies
\[ \sup_{x,y \in \mathcal{I}} |(I.1) - \Sigma_{h,x,y}| = O_P \left( \sqrt{\frac{\log n}{n}} \right). \]

Term (I.2) has expansion:
\[ (I.2) = \frac{1}{n^2} \sum_{i,j} (u_{ij}(x) - \bar{u}_i(x)) \bar{u}_i(y)' = \frac{1}{n^2} \sum_{i,j \text{ distinct}} (u_{ij}(x) - \bar{u}_i(x)) \bar{u}_i(y)' + \frac{1}{n} \sum_i (u_{ii}(x) - \bar{u}_i(x)) \bar{u}_i(y)' \]
\[ + \frac{1}{n} \sum_{i,j \text{ distinct}} \left\{ (u_{ij}(x) - \bar{u}_i(x)) \bar{u}_i(y)' - \mathbb{E} \left[ (u_{ij}(x) - \bar{u}_i(x)) \bar{u}_i(y)' \right] x_j \right\}. \]

By the same technique of Lemma 15, one can show that
\[ \sup_{x,y \in \mathcal{I}} |(I.2.2)| = O_P \left( \frac{1}{n} \right). \]

We need a further decomposition to make (I.2.1) a degenerate U-statistic:
\[ (I.2.1) = \frac{n - 1}{n^2} \sum_j \mathbb{E} \left[ (u_{ij}(x) - \bar{u}_i(x)) \bar{u}_i(y)' \right] x_j \]
\[ + \frac{1}{n} \sum_{i,j \text{ distinct}} \left\{ (u_{ij}(x) - \bar{u}_i(x)) \bar{u}_i(y)' - \mathbb{E} \left[ (u_{ij}(x) - \bar{u}_i(x)) \bar{u}_i(y)' \right] x_j \right\}. \]

(I.2.1) has zero mean. By discretizing \( \mathcal{I} \) and apply Bernstein’s inequality, one can show that the (I.2.1.1) has
order $O_p\left(\sqrt{\log n/n}\right)$.

For (I.2.1.2), we first discretize $I$ and then apply a Bernstein-type inequality (Lemma 23) for degenerate U-statistics, which gives an order

$$\sup_{x,y \in I} |(I.2.1.2)| = O_p\left(\frac{\log n}{\sqrt{n^2 h}}\right).$$

Overall, we have

$$\sup_{x,y \in I} |(I.2)| = O_p\left(\frac{1}{n} + \sqrt{\frac{\log n}{n}} + \frac{\log n}{\sqrt{n^2 h}}\right) = O_p\left(\frac{\log n}{n}\right),$$

and the same bound applies to (I.3).

For (I.4), one can show that

$$\sup_{x,y \in I} \left| \frac{1}{n} \sum_j Y_j R_j(x) W_j(x) \left( 1(x \leq x_j) - F(x_j) \right) \left( 1(x \leq x_j) - F(x_j) \right) \right| = O_p\left(\sqrt{\frac{\log n}{n h}}\right),$$

which means

$$\sup_{x,y \in I} |(I.4)| = O_p\left(\frac{\log n}{n h}\right) = O_p\left(\frac{\log n}{n}\right),$$

under our assumption that $\log n/(nh^2) \to 0$.

As a result, we have

$$\sup_{x,y \in I} \left| \hat{\Sigma}_{h,x,y} - \Sigma_{h,x,y} \right| = O_p\left(\frac{\log n}{n}\right).$$

Now take $c$ to be a generic vector. Then we have

$$\frac{c'_{x,h} \hat{\gamma}_{h,x} (\hat{\Omega}_{h,x,y} - \Omega_{h,x,y}) \hat{\gamma}_{h,y} c_{h,y}}{\sqrt{c'_{x,h} \hat{\gamma}_{h,x} \hat{\Omega}_{h,x,y} \hat{\gamma}_{h,y} c_{h,y}}} = \frac{c'_{x,h} \hat{\gamma}_{h,x} \hat{\Omega}_{h,x,y} c_{h,y} c_{h,y}}{\sqrt{c'_{x,h} \hat{\gamma}_{h,x} \hat{\Omega}_{h,x,y} \hat{\gamma}_{h,y} c_{h,y}}} \sqrt{c'_{x,h} \hat{\gamma}_{h,x} \hat{\Omega}_{h,x,y} \hat{\gamma}_{h,y} c_{h,y}} + \frac{c'_{x,h} \hat{\gamma}_{h,x} \hat{\Omega}_{h,x,y} c_{h,y} c_{h,y}}{\sqrt{c'_{x,h} \hat{\gamma}_{h,x} \hat{\Omega}_{h,x,y} \hat{\gamma}_{h,y} c_{h,y}}} \sqrt{c'_{x,h} \hat{\gamma}_{h,x} \hat{\Omega}_{h,x,y} \hat{\gamma}_{h,y} c_{h,y}} + \frac{c'_{x,h} \hat{\gamma}_{h,x} \hat{\Omega}_{h,x,y} c_{h,y} c_{h,y}}{\sqrt{c'_{x,h} \hat{\gamma}_{h,x} \hat{\Omega}_{h,x,y} \hat{\gamma}_{h,y} c_{h,y}}} \sqrt{c'_{x,h} \hat{\gamma}_{h,x} \hat{\Omega}_{h,x,y} \hat{\gamma}_{h,y} c_{h,y}}.$$

From the analysis of $\hat{\Sigma}_{h,x,y}$, we have

$$\sup_{x,y \in I} \left| \frac{c'_{x,h} \hat{\gamma}_{h,x} \hat{\Omega}_{h,x,y} c_{h,y} c_{h,y}}{\sqrt{c'_{x,h} \hat{\gamma}_{h,x} \hat{\Omega}_{h,x,y} \hat{\gamma}_{h,y} c_{h,y}}} \sqrt{c'_{x,h} \hat{\gamma}_{h,x} \hat{\Omega}_{h,x,y} \hat{\gamma}_{h,y} c_{h,y}} \right| = O_p\left(\frac{\log n}{n h^2}\right).$$

For the second term, we have

$$\frac{c'_{x,h} \hat{\gamma}_{h,x} \hat{\Omega}_{h,x,y} c_{h,y} c_{h,y}}{\sqrt{c'_{x,h} \hat{\gamma}_{h,x} \hat{\Omega}_{h,x,y} \hat{\gamma}_{h,y} c_{h,y}}} \sqrt{c'_{x,h} \hat{\gamma}_{h,x} \hat{\Omega}_{h,x,y} \hat{\gamma}_{h,y} c_{h,y}} \leq \frac{\left| c'_{x,h} \hat{\gamma}_{h,x} (\hat{\gamma}_{h,x}^{-1} - \Gamma_{h,x}^{-1}) \Sigma_{h,x,y}^{-1/2} \right| \left| c_{x,h} \hat{\gamma}_{h,x} \hat{\gamma}_{h,x}^{-1/2} \right|}{\sqrt{c'_{x,h} \hat{\gamma}_{h,x} \hat{\Omega}_{h,x,y} \hat{\gamma}_{h,y} c_{h,y}}} \sqrt{c'_{x,h} \hat{\gamma}_{h,x} \hat{\Omega}_{h,x,y} \hat{\gamma}_{h,y} c_{h,y}} \leq O_p\left(\frac{\log n}{n h^2}\right).$$

The same bound holds for the third term.
5.18 Proof of Lemma 17

We decompose (18) as

\[
\sup_{x \in I} |(18)| \leq \frac{1}{\sqrt{n}} \left[ \sup_{x \in I} \left| \frac{c'_{h,x} \hat{Y}_h \hat{\Gamma}_{h,x}^{-1}}{c_{h,x} \hat{Y}_h \hat{\Omega}_{h,x} \hat{Y}_h \hat{\Theta}_{h,x}} \right| \left[ \sup_{x \in I} \left| \frac{1}{n} \sum_{i=1}^{n} R((x_i - x)/h)[1 - F(x_i)] \frac{1}{h} K(\frac{x_i-x}{h}) \right| \right] \right].
\]

As both \(\hat{\Gamma}_{h,x}\) and \(c'_{h,x} \hat{Y}_h \hat{\Omega}_{h,x} \hat{Y}_h \hat{\Theta}_{h,x}\) are uniformly consistent, term (I) has order

\[
(I) = O_p \left( \sqrt{\frac{1}{n}} \right).
\]

For (II), we can employ the same technique used to prove Lemma 15 and show that

\[
(II) = O_p \left( 1 + \frac{\log n}{nh} \right) = O_p(1),
\]

where the leading order in the above represents the mean of \(R((x_i - x)/h)[1 - F(x_i)]\frac{1}{h} K(\frac{x_i-x}{h})\).

Next, term (19) is bounded by

\[
\sup_{x \in I} |(19)| \leq \sqrt{n} \left[ \sup_{x \in I} \left| \frac{c'_{h,x} \hat{Y}_h \hat{\Gamma}_{h,x}^{-1}}{c_{h,x} \hat{Y}_h \hat{\Omega}_{h,x} \hat{Y}_h \hat{\Theta}_{h,x}} \right| \left[ \sup_{x \in I} \left| \frac{1}{n} \sum_{i=1}^{n} R((x_i - x)/h)[F(x_i) - \theta(x) R(x_i - x)] \frac{1}{h} K(\frac{x_i-x}{h}) \right| \right].
\]

Employing the same argument used to prove Lemma 17, we have

\[
(I) = O_p \left( \sqrt{\frac{1}{n}} \right).
\]

To bound term (II), recall that \(K(\cdot)\) is supported on \([-1, 1]\), meaning that

\[
\sup_{x \in I} \left| \frac{1}{n} \sum_{i=1}^{n} R((x_i - x)/h)[F(x_i) - \theta(x) R(x_i - x)] \frac{1}{h} K(\frac{x_i-x}{h}) \right| = \sup_{x \in I} \left| \frac{1}{n} \sum_{i=1}^{n} R((x_i - x)/h)[F(x_i) - \theta(x) R(x_i - x)] \mathbb{I}(|x_i - x| \leq h) \frac{1}{h} K(\frac{x_i-x}{h}) \right|
\]

\[
\leq \left[ \sup_{x \in I} \left| \frac{1}{n} \sum_{i=1}^{n} R((x_i - x)/h) \frac{1}{h} K(\frac{x_i-x}{h}) \right| \right] \left[ \sup_{x \in I} \sup_{u \in [x-h, x+h]} \left| F(u) - \theta(u) R(u - x) \right| \right].
\]

Term (II.2) has the bound \(\sup_{x \in I} \vartheta(h, x)\). Term (II.1) can be bounded by mean and variance calculations and adopting the proof of Lemma 15, which leads to

\[
(II.1) = O_p \left( 1 + \frac{\log n}{nh} \right) = O_p(1).
\]

To show the last conclusion, define the following:

\[
u_{ij}(u) = \mathbf{I}(x_i \leq x) \left[ F(x_i) - \frac{1}{h} K(\frac{x_i-x}{h}) \right] R(u) \mathbb{I}(x_i \leq x + hu) - \int_{\frac{x_i-x}{h}}^{\frac{x_i-x}{h}} R(u) \mathbb{I}(x_i \leq x + hu) - F(x + hu) \frac{1}{h} K(\frac{x_i-x}{h}) (x + hu) du,
\]

46
then $n^{-2} \sum_{i,j=1, i \neq j}^n u_{ij}(x)$ is a degenerate U-statistic. We rewrite (20) as

$$\sup_{x \in \mathcal{I}} \| (20) \| \leq \left( \sup_{x \in \mathcal{I}} \left| \frac{c_{h,x} \tilde{Y}_h \tilde{I}_{h,x}^{-1}}{\sqrt{c_{h,x} \tilde{Y}_h \Omega_{h,x} \tilde{Y}_h c_{h,x}}} \right| \right) \left( \sup_{x \in \mathcal{I}} \left| \frac{1}{n^2} \sum_{i,j=1, i \neq j}^n u_{ij} \right| \right).$$

As before, we have

$$(I) = O_{\mathbb{P}} \left( \sqrt{\frac{1}{h}} \right).$$

Now we consider (II). Let $\mathcal{I}_\varepsilon$ be an $\varepsilon$-covering of $\mathcal{I}$, we have

$$\sup_{x \in \mathcal{I}_\varepsilon} \left| \frac{1}{n^2} \sum_{i,j=1, i \neq j}^n u_{ij}(x) \right| \leq \max_{x \in \mathcal{I}_\varepsilon} \left| \frac{1}{n^2} \sum_{i,j=1, i \neq j}^n u_{ij}(x) \right| + \max_{x \in \mathcal{I}_\varepsilon, y \in \mathcal{I}_\varepsilon, |x-y| \leq \varepsilon} \left| \frac{1}{n^2} \sum_{i,j=1, i \neq j}^n \left( u_{ij}(x) - u_{ij}(y) \right) \right|.$$  

We rely on the concentration inequality in Lemma 23 for degenerate second order U-statistics. By our assumptions, $A$ can be chosen to be $C_1 h^{-1}$ where $C_1$ is some constant that is independent of $x$. Similarly, $B$ can be chosen to be $C_2 \sqrt{nh^{-1}}$ for some constant $C_2$ which is independent of $x$, and $D$ can be chosen as $C_3 \sqrt{nh^{-1}}$ for some $C_3$ independent of $x$. Therefore, by setting $\eta = K \log n / \sqrt{n^2 h}$ for some large constant $K$, we have

$$P[(II.1) \geq \eta] \leq C \varepsilon \max_{x \in \mathcal{I}_\varepsilon} \left| \sum_{i,j=1, i \neq j}^n u_{ij}(x) \right| \geq n^2 \eta$$

$$\leq C \varepsilon \exp \left\{ - \frac{1}{2} \min \left\{ \frac{n^2 h^{1/2} \eta}{nc_3}, \left( \frac{n^2 h \eta}{nc_3} \right)^{1/2}, \left( \frac{n^2 h \eta}{c_3} \right)^{1/2} \right\} \right\}$$

$$= C \varepsilon \exp \left\{ - \frac{1}{2} \min \left\{ \frac{K \log n}{c_3}, \left( \frac{K \sqrt{nh \log n}}{c_2} \right)^{1/2}, \left( \frac{K \sqrt{nh \log n}}{c_1} \right)^{1/2} \right\} \right\}.$$  

As $\varepsilon$ is at most polynomial in $n$, the above tends to zero for all $K$ large enough, which implies

$$(II.1) = O_{\mathbb{P}} \left( \log \frac{n}{\sqrt{n^2 h}} \right).$$

With tedious but still straightforward calculations, it can be shown that

$$(II.2) = O_{\mathbb{P}} \left( \frac{\varepsilon}{h} + \frac{\log n}{\sqrt{n^2 h}} + \frac{\varepsilon \log n}{h \sqrt{n^2 h}} \right),$$

and to match the rates, let $\varepsilon = h \log n / \sqrt{n^2 h}$.

5.19 Proof of Lemma 18

The proof resembles that of Lemma 12.

5.20 Proof of Theorem 19

The proof resembles that of Theorem 13.
5.21 Proof of Theorem 20

The proof resembles that of Theorem 14.

References


