

Efficient dynamic allocations in endowment economies with linear utilities and private information*

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Abstract

We revisit the classic problem of designing efficient dynamic allocations in endowment economies with private information, which was studied by Atkeson and Lucas (1992). They showed that for some strictly concave utility functions, the efficient allocation requires immiseration: The wealth of almost all consumers becomes arbitrarily small over time, while a vanishing fraction of consumers becomes infinitely rich. We study linear utility functions, which make the problem of finding the efficient allocations analytically tractable. We show that there always exist efficient allocations with immiseration. However, there may also exist *more fair* efficient allocations, that guarantee a minimal positive continuation utility to all consumers across all time periods. Depending on the parameters, more fair efficient allocations may or may not be supported as competitive market equilibria. Finally, we also show that more fair efficient allocations always exist if an endowment economy has access to fair international credit markets.

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1 Introduction

Following Atkeson and Lucas (1992), we study an economy with a mass of infinitely lived consumers. Each consumer represents a dynasty, in which current members inherit the wealth of their predecessors. In every period, the economy is endowed with a mass of perishable good. In every period, each consumer experiences a privately observed taste shock that determines her current utility of consumption. We study dynamic allocations of the good across the consumers, as depending on their taste shocks. Any feasible allocation should be incentive-compatible, with each consumer willing to select the consumption stream designed for her type. We are interested in efficient feasible allocations, that is, the feasible allocations that maximize the aggregate ex ante life-time payoff of all consumers.

Atkeson and Lucas (1992) show numerically that in some parametric settings in which the current utility of consumption is strictly concave, the efficient dynamic allocation of resources “terminates” almost every consumer (dynasty) over time, that is, the continuation payoff of the consumer converges to zero, and the consumer is effectively removed from future resource allocation. In other words, a vanishing fraction of consumers becomes infinitely rich.¹ This property is often named as an immiseration result. This result implies a trade-off between efficiency and fairness, since fairness considerations would perhaps suggest less inequality, such as, each consumer be guaranteed a minimal positive continuation payoff in each period independent of its history.

We explore to what extent efficiency is in conflict with fairness in the variant of the Atkeson-Lucas model with linear utility of current consumption. Linearity makes the analysis more tractable, and allows for completely characterizing some efficient dynamic allocations analytically.² We focus on the case with two possible taste shock levels in each period: high and low, with high-shock consumers having a higher marginal utility of consumption as compared to low-shock consumers.

For a range of parameters of the model with linear utility, the efficient allocation is essentially unique,³ and requires immiseration. The efficient allocation distributes a positive mass of endowment to low-shock consumers. For some other parameters, it is possible to achieve the first-best optimum, that is, endowment is distributed only among consumers with high-taste shocks. Perhaps surprisingly, in this case, there exist multiple efficient allocations, some of which exhibit a higher level of fairness: Each such an allocation guarantees a minimal positive continuation payoff to each consumer across all periods. We refer to such

¹Related findings were reported in Green (1987), Taub (1990), Thomas and Worrall (1990), and Phelan and Townsend (1991).

²Linear utility is often considered to be a less realistic assumption, as compared to strictly concave utility. We believe, however, that the properties of the optimal resource allocation derived in this paper are simple and intuitive, and might be useful when designing relevant schemes in practice, such as optimal taxation. Likewise, immiseration result with arbitrary large amounts of consumption for certain dynasties is not realistic, yet it is used to predict the increasing level of inequality.

³We only focus on “symmetric” allocations, which prescribe the same consumption for consumers with identical sequences of shocks. For any allocation that prescribes different consumption to such consumers, one can construct a symmetric allocation by averaging their consumption. This operation preserves efficiency.

allocations as *more fair*.

Intuitively, the multiplicity of efficient allocations can be understood as follows: The social planner would ideally distribute the endowment in each period only among the consumers with a current high-taste shock. However, to make the consumers with a low-taste shock self-select and choose to consume nothing in the current period, the social planner must prescribe a higher continuation payoff contingent on reporting a low-taste shock. If the fraction of the high-shock consumers is sufficiently high, each of them consumes a relatively low share of the current endowment. With a relatively low level of current consumption, the low-shock consumers would be willing to wait for a higher consumption in the future, provided that the discount factor is not too low. Moreover, the social planner has a freedom of selecting different levels of continuation payoffs, and can choose to either make the high- or the low-shock consumers indifferent between the two consumption levels, or to make each type of consumers strictly willing to select the consumption level designed for that type.

The multiplicity itself does not yet guarantee the existence of more fair efficient allocations. A more fair allocation can be constructed as follows: Take an allocation under which no incentive constraint is binding, that is, consumers of both types are strictly willing to choose their prescribed consumption levels. Modify this allocation so that if a consumer's continuation payoff drops below a certain cutoff level, the consumer will stop consuming (or consume little). That is, the consumption originally prescribed for this consumer when she faces a high-taste shock is now distributed among the other high-shock consumers with continuation payoffs above the cutoff level. Then, the continuation payoff of the consumer below the cutoff will gradually increase over time until it raises above the cutoff, and the consumer will resume her consumption. Such an allocation guarantees a minimal continuation payoff to each consumer (dynasty), after any possible history of taste shocks. Hence, this allocation exhibits a higher level of fairness in terms of continuation payoffs, as compared to the allocations in Atkeson and Lucas (1992).

In an interpretation, an allocation with immiseration prescribes little or no consumption to a generation of a dynasty if its predecessors faced a large number of high-taste shocks. Our more fair allocations put some bound on the extent to which children are responsible for the consumption of their parents. They prescribe little or no consumption in the later periods of lives to generations who faced a large number of high-taste shocks in the earlier periods of their lives, and guarantee a positive continuation payoff to all their descendants.

The multiplicity of optimal allocations naturally raises the question whether all these allocations can be supported as the outcomes of competitive market equilibria.⁴ To explore this, we consider a sequence of prices, one price for each period, and allow consumers (dynasties) to trade their endowments across periods at these prices. Each consumer is initially endowed with an equal share of all future endowments. Consumers would choose to trade as to maximize their expected continuation payoffs given their taste shocks. A sequence of prices constitutes a market equilibrium if the market clearing condition holds in every period.

⁴From now on, we drop the adjective "competitive", and will use the term market equilibria.

We show that (when the utility function is linear), there always exist market equilibria which induce the efficient dynamic allocations with immiseration. However, depending on the parameters of the model, the more fair efficient allocations may or may not be supported as market equilibria. That is, the necessary and sufficient condition for the existence of “more fair” efficient market equilibria is stronger than that for the existence of more fair efficient dynamic allocations. In other words, a benevolent social planner can (in some cases) make the dynamic allocation more fair than the markets, without losing any efficiency.

The advantage of a social planner comes from the fact that in any market equilibrium, if any high-shock consumer strictly prefers consuming today over saving for tomorrow, then all high-shock consumers spend all their wealth on current consumption, hence obtaining a zero continuation payoff from the next period onwards. In contrast, the social planner can limit the current consumption of some high-shock consumers (for example, the ones with a sufficiently low continuation payoff). In other words, the planner is capable of selecting both the “price” of current consumption (by choosing the appropriate levels of continuation payoff as dependent on the taste shock), and the quantity for each consumer; and discriminate across the consumers with different history. The competitive market can only assign the price.

We extend our analysis by introducing an access to international markets. We consider the case of a small open economy that can lend to and borrow from outside parties. Such economies have been studied in Kocherlakota (2010). In the most interesting case, the interest rate R of lending and borrowing offsets the consumers’ discount rate δ (that is, $\delta R = 1$). In this case, it is always possible to achieve the first-best solution, that is, to distribute consumption only among the consumers with high-taste shocks. Moreover, unlike the case of economy with no trade (that is, a closed economy), it is also always possible to achieve more fair efficient allocations. Intuitively, the possibility of lending and borrowing enables the planner to distribute consumption among high-shock consumers across periods, and, in particular, to make the level of current consumption low enough, which guarantees incentives for low-shock consumers, and hence, efficiency. With this observation, the arguments that establish the existence of more fair efficient allocations are analogous to those for the closed economy.

Several conclusions can be reached from our results. First, efficiency requires immiseration for some strictly concave utility functions, but not necessarily for linear utility functions. This is, perhaps, a surprising conclusion, because one could guess that when utility is linear, the social planner has stronger incentives to allocate the good to high-shock consumers, hence accelerating the process of terminating consumers, and allocating the good to a vanishing fraction of remaining consumers. However, there is another effect: with linear utility, the social planner might be able to redistribute consumption among the high-shock consumers in order to lower future levels of inequality, with no reduction in aggregate payoff. So, there may be no conflict between efficiency and fairness, in the case of linear utility. We believe that this result could be extended: For a small degree of concavity of utility function, the conflict between efficiency and fairness should be modest. Second, the social planner concerned with fairness might be able to correct the market outcome, without any loss in terms of efficiency (for linear utility), or with a modest loss of efficiency (for

modestly concave utility). Finally, fair international lending and borrowing enable the planner to guarantee that the continuation payoffs of all generations of all dynasties will be bounded away from 0.

The rest of the paper is organized as follows. We discuss the relevant literature in the next subsection. In Sections 2, 3 we formally define the economy, and introduce the social planner's problem. We avoid stating any Bellman equation, since our analysis never explicitly refers to this equation. Actually, we believe that departing from the traditional approach of explicitly studying Bellman equations facilitated our analysis. In Section 4, we explore the optimal dynamic allocations (in the case when the consumers have a linear utility of consumption). We study the outcomes of market equilibria in Section 5. In Section 6 we analyze the case when the economy can exchange its resources with the outside world. Finally, we conclude and describe some conjectures, follow-up ideas and plans for future research in Section 7.

1.1 Literature

Atkeson and Lucas consider the problem of optimizing the welfare (that is, continuation payoff) of the first generation of dynasties. The welfare of future generations is taken into account only indirectly, since it affects the first generation. In contrast, Phelan (2006) and Farhi and Werning (2007) consider an alternative setting in which positive weights are placed directly on the welfare of each future generation in the planning problem. Consequently, all dynasties are guaranteed a minimum positive continuation payoff in each period independent of their histories. In Phelan (2006), equal weights are placed on all generations, and in Farhi and Werning (2007), the society places positive but vanishing weights on the welfare of future generations. Those weights make the effective discount factor of the society higher than that of the citizens, and this higher discount factor induces a mean-reversing force on the continuation payoffs at the optimal allocations. Both models avoid the immiseration result. More specifically, if there exist steady state distributions over continuation payoffs, then they place no mass at zero payoff. Farhi and Werning show that under some conditions such steady states exist. In the case of logarithmic utility, if the society assigns an equal ex ante life-time payoff to all dynasties (as we do), the distribution over continuation payoffs converges to a unique steady state, which places no mass at zero payoff.

Green (1987), Thomas and Worrall (1990), and Phelan (1998) consider a related problem of optimal allocation with the possibility of lending and borrowing across periods. They analyze a dual problem of minimizing the costs of achieving a certain value of ex ante life-time payoff for a risk-averse agent with private information. Both Green, and Thomas and Worrall show immiseration result: The continuation payoff of the agent almost surely becomes arbitrarily negative. Phelan analyzes the dual problem further, and shows that the presence of immiseration depends on the characteristics of a (strictly concave) utility function. The critical feature is whether the marginal utility is bounded away from zero as consumption increases. With the positive lower bound on marginal utility, the immiseration result does not hold: The continuation payoff becomes either arbitrarily large or arbitrarily negative, with a positive probability of each event.

An important technique used in the analysis of strictly concave utilities is the martingale convergence theorem. The first-order conditions for the optimum yield that the marginal cost of achieving continuation payoff is a martingale and hence has to converge almost surely. At the same time, the value of continuation payoff must almost never converge to a finite value, otherwise one would not differentiate between different types of consumers. Hence, for the marginal cost to converge, the continuation payoff has to either become arbitrarily negative, or arbitrarily large. With a positive lower bound on marginal utility, as in Phelan (1998), the chances of both types of convergence are positive, which yields no immiseration.

Our paper performs a similar exercise to Phelan (1998) by considering yet another type of utility functions. However, due to linearity, we cannot use the first-order conditions, and, respectively, the martingale convergence theorem. Instead we use linearity to obtain analytically tractable recursive solutions. Unlike in Phelan (1998), there may exist a positive lower bound on continuation payoff across all time periods.

2 Closed endowed economy

We consider an economy with a mass of infinitely-lived consumers (dynasties); this mass is normalized to 1. In every period $t = 0, 1, \dots$, the economy is endowed with a mass of η units of some perishable (nonstorable) good. This is the only good in the economy. The endowment has the form of a “helicopter drop” to the economy.

Each consumer gets a flow utility from the consumption of c units of the good equal to $\theta u(c)$, where θ represents the consumer’s privately observed taste shock. The function $u(c)$ is strictly increasing over c , is the same across all consumers and all time periods. In all our results, u is assumed to be linear. The shock value θ is i.i.d. over consumers and over time, and takes only one of two values: θ^- and θ^+ , where $0 < \theta^- < \theta^+$, with probabilities ρ^- and ρ^+ , respectively. Consumers discount future payoffs by a common discount factor δ . So, the life-time (normalized) payoff of a consumer who faced a sequence of shocks $\theta_0, \theta_1, \dots$ is

$$(1 - \delta) \sum_{t=0}^{\infty} \delta^t \theta_t u(c_t),$$

where c_t is the consumption at time t .

Our setting is basically the model of Atkeson and Lucas (1992). Our analysis differs from theirs as follows: (a) They study all Pareto efficient dynamic allocations, not - as we do - only the ex-ante symmetric allocations that maximize the life-time payoff of consumers; (b) They characterize the allocations that attain an exogenously given (Pareto efficient) distribution of expected life-time utilities with the minimal per-period endowment η . This dual approach is more abstract, but it is sometimes technically more convenient; (c) Atkeson and Lucas allow (again because it is technically convenient) for negative consumption, so that the property of dynamic allocations that they describe is formulated slightly differently. They claim that “in an efficient allocation the degree of inequality continually increases, with a diminishing fraction of the population receiving an increasing fraction of the resources.”

3 Social planner's problem

We explore the optimal (also called efficient, or second-best) dynamic allocations of endowment, that is, the allocations which would be implemented by a benevolent social planner who maximizes the consumers' expected life-time payoff, subject to the consumers' incentives constraints to report their taste shocks truthfully.

We assume that the social planner aims to provide the same expected life-time payoff to all consumers. However, in periods other than $t = 0$, different consumers will typically be prescribed different continuation payoffs. For example, if the social planner prescribes a higher consumption contingent on reporting a higher taste shock, then, to satisfy the incentives constraints for truthful reporting, the planner must prescribe a higher continuation payoff to a consumer who reports a lower taste shock.

We denote by w the expected life-time payoff, that is, the continuation payoff of each consumer beginning in period $t = 0$, before consumers learn their taste shocks. In each period, a consumer reports a shock value $\hat{\theta} \in \Theta = \{\theta^+, \theta^-\}$. A history of taste shocks reported in periods $0, \dots, t - 1$ is denoted as $h_{t-1} \in \Theta^t$. For a consumer with history h_{t-1} , we denote by $w_{h_{t-1}}$ her continuation payoff beginning in period t .

An allocation divides in each period the endowment among the consumers. It is convenient to represent any allocation as a pair of history-dependent functions $[f_t(h_{t-1}, \hat{\theta}), g_t(h_{t-1}, \hat{\theta})]$: a function f_t , which prescribes the consumption in period t , and a function g_t which represents the continuation payoff beginning from period $t + 1$.⁵ Both f_t and g_t are functions of history h_{t-1} , and taste shock $\hat{\theta}$ reported in period t . For $t = 0$, we use notations $f_0(\hat{\theta})$, and $g_0(\hat{\theta})$, respectively. That is, a consumer with history h_{t-1} , who reports $\hat{\theta}$ in period t , gets to consume $f_t(h_{t-1}, \hat{\theta})$ in period t , and her continuation payoff beginning from period $t + 1$ equals $g_t(h_{t-1}, \hat{\theta}) = w_{h_{t-1}, \hat{\theta}}$.

4 Efficient dynamic allocations

Assume that $u(c) = ac$, with $a > 0$ being a constant marginal utility. This case is tractable analytically, and allows to obtain some potentially interesting results. Without loss of generality, we assume $a = 1$, that is, $u(c) = c$.

We restrict attention to deterministic functions f_t and g_t , because any random assignment which satisfies the feasibility and incentives constraints can be replaced with its expected values. The expected values also satisfy the feasibility and incentives constraints, and yield the same expected life-time payoff.⁶

Note that an optimal allocation exists, because, after restricting attention to deterministic allocations, the social planner's problem has the form of maximizing a continuous objective function over a compact domain.

⁵We will sometimes suppress the dependence of f_t and g_t on $h_{t-1}, \hat{\theta}$.

⁶One can restrict attention to deterministic functions f_t and g_t even when the marginal utility of consumption is diminishing.

4.1 Optimal allocations with recursive structure

We will first show that there is an optimal allocation which has what we call a *recursive structure*. Suppose that in a (perhaps, not unique) optimal dynamic allocation, represented by two functions $[f_t, g_t]$, the expected life-time payoff of each consumer is w . We claim that there is a (perhaps different) optimal allocation, $[f_t^*, g_t^*]$, that gives the same w , such that in all subsequent periods $t = 1, 2, \dots$, the consumption f_t^* and the continuation payoff g_t^* are proportional to f_0^* and g_0^* , respectively.

Lemma 1 *Given any optimal dynamic allocation, $[f_t, g_t]$, there exists a feasible and incentive-compatible allocation, $[f_t^*, g_t^*]$, that yields the same expected life-time payoff w , such that for any period $t > 0$, a history h_{t-1} of taste shocks reported up to period $t - 1$, and a report $\hat{\theta} \in \{\theta^+, \theta^-\}$ in period t ,*

$$f_t^*(h_{t-1}, \hat{\theta}) = \frac{g_{t-1}^*(h_{t-1})}{w} f_0^*(\hat{\theta}),$$

and

$$g_t^*(h_{t-1}, \hat{\theta}) = \frac{g_{t-1}^*(h_{t-1})}{w} g_0^*(\hat{\theta}),$$

where $g_{t-1}^*(h_{t-1})$ is the continuation payoff at the beginning of period t .

Intuitively, this lemma says that some optimal allocation has a “regular pattern”. We can construct such an allocation due to the constant marginal utility of consumption. Indeed, we initially only know that an optimal allocation exists, but we have no information regarding its structure. We obtain from this allocation a (possibly) different optimal allocation with a regular pattern by moving consumption from some consumers to other consumers in some periods, and moving consumption in the opposite direction in other periods. This reshuffling is possible, because the utility of an additional unit of consumption is the same across all consumers (of a given taste). If the marginal utility of consumption would vary with the consumption level, the reshuffling would be difficult, because we would have no control over the value of an additional unit of consumption without any information regarding the structure of the optimal allocation from which we begin.

Proof. Under any given incentive-compatible allocation, $[f_t, g_t]$, in $t = 0$ a mass ρ^+ of consumers report high shocks, and the remaining mass ρ^- of consumers report low shocks. Starting in period $t = 1$, the consumers are prescribed continuation payoffs of $g_0(\theta^+)$ and $g_0(\theta^-)$, respectively. Thus, in period $t = 1$, the social planner faces the problem of allocating the endowment so that ρ^+ consumers will each have a continuation payoff of $g_0(\theta^+)$, while ρ^- consumers will each have a continuation payoff of $g_0(\theta^-)$. We will show that

$$\rho^+ g_0(\theta^+) + \rho^- g_0(\theta^-) = w. \tag{1}$$

That is, the aggregate continuation payoff, beginning from period $t = 1$ is the same as that beginning from period $t = 0$. This property will be the key argument in this proof. It will no longer hold true for strictly concave utility.

Starting in period $t = 1$, each consumer i is prescribed one of the two consumption streams, as dependent on i 's report, $\hat{\theta}_i \in \{\theta^+, \theta^-\}$, in $t = 0$. We consider a weighted average of these two consumption streams, and

refer to this as a *new* allocation. That is, for any $t > 0$, any history $h_{1,t-1}$ of reports *in periods* $1, \dots, t-1$, and any report $\widehat{\theta}$ in period t , the new allocation prescribes the consumption of

$$f_t^{\text{new}}(h_{1,t-1}, \widehat{\theta}) \equiv \rho^+ f_t(\theta^+, h_{1,t-1}, \widehat{\theta}) + \rho^- f_t(\theta^-, h_{1,t-1}, \widehat{\theta}),$$

and a continuation payoff equal to

$$g_t^{\text{new}}(h_{1,t-1}, \widehat{\theta}) = \rho^+ g_t(\theta^+, h_{1,t-1}, \widehat{\theta}) + \rho^- g_t(\theta^-, h_{1,t-1}, \widehat{\theta}),$$

Note that the new allocation f_t^{new} prescribes consumption from $t = 1$, while the original allocation f_t prescribes consumption from $t = 0$, hence the latter depends on a longer history of reports.

Assume the new allocation is prescribed to every consumer from $t = 1$. By construction, the resource constraint is satisfied. Moreover, due to linearity of utility, the new allocation is incentive compatible. Indeed, starting from $t = 1$, in the original allocation, the consumption streams for the two groups of consumers with weights ρ^+ and ρ^- provide incentives to report truthfully, and their weighted average yield incentives for the new allocation.

Since the new allocation is incentive-compatible, and uses the amount η of resource in each period, the continuation payoff, $\rho^+ g_0(\theta^+) + \rho^- g_0(\theta^-)$, provided by this allocation at the beginning of period $t = 1$, cannot exceed w —the continuation payoff provided by the original efficient allocation at the beginning of $t = 0$. Otherwise, one could use the new allocation (from $t = 0$) instead of the original one.

Similarly, one cannot have that $\rho^+ g_0(\theta^+) + \rho^- g_0(\theta^-) < w$. Otherwise, starting from period $t = 1$, one could prescribe a “scaled” variant of the original allocation (from $t = 0$) to both groups of ρ^+, ρ^- of consumers, in a way that preserves incentives and saves resources. Namely, to each consumer in the group of mass ρ^+ (respectively, ρ^-), for any $t > 0$, any history $h_{1,t-1}$ of reports in periods $1, \dots, t-1$, and any report $\widehat{\theta}$ in period t , one could prescribe consumption $\frac{g_0(\theta^+)}{w} f_{t-1}(h_{1,t-1}, \widehat{\theta})$ (respectively, $\frac{g_0(\theta^-)}{w} f_{t-1}(h_{1,t-1}, \widehat{\theta})$). Each of the scaled allocations is incentive-compatible, and yields the original values of continuation payoffs of $g_0(\theta^+), g_0(\theta^-)$ at the start of $t = 1$. Moreover, the aggregate amount of resource used in the scaled allocations in each period is less than endowment: $[\rho^+ \frac{g_0(\theta^+)}{w} + \rho^- \frac{g_0(\theta^-)}{w}] \eta < \eta$. One could distribute the remaining resource uniformly among consumers, hence not violating their incentives, and increasing their payoffs.

Thus, $\rho^+ g_0(\theta^+) + \rho^- g_0(\theta^-) = w$. So, we can replace the prescriptions of any given optimal allocation, $[f_t, g_t]$, from period $t = 1$ with the corresponding prescriptions of the given allocation from period $t = 0$ scaled by a factor of $\frac{g_0(\theta^+)}{w}$ for the mass ρ^+ of consumers who reported θ^+ in period 0, and scaled by a factor of $\frac{g_0(\theta^-)}{w}$ for the mass ρ^- of consumers who reported θ^- in period 0. The lemma follows from analogous arguments applied to periods $t = 2, 3, \dots$ ■

Remark 1 *1. Lemma 1 easily generalizes to more than two values of shocks, which allows for extending the entire analysis to this more general case.*

2. Lemma 1 also generalizes to the shock processes that are first-order irreducible Markov chains, provided that the distribution of shocks in period $t = 0$ is the ergodic (stable) distribution for the Markov chain. However, we have not attempted to generalize our analysis to Markov shock processes.

3. Similarly to the proof of expression (1), one can show that the aggregate continuation payoff of consumers remains the same in each time period, $t = 0, 1, \dots$. Hence, the efficient allocations are somewhat robust to renegotiation: One cannot switch to a new allocation with a higher aggregate payoff. In contrast, with strictly concave utility, for a given efficient allocation, the aggregate continuation payoff in $t > 0$ is strictly smaller than in $t = 0$. One could have incentives to renegotiate and start a new efficient allocation, undermining the credibility of the original one.

The social planner's problem becomes much simpler if we are looking for a recursive solution. Consider the following reduced problem in which the social planner aims to maximize the expected payoff:

$$w = \rho^+[(1 - \delta)\theta^+ f_0(\theta^+) + \delta g_0(\theta^+)] + \rho^-[(1 - \delta)\theta^- f_0(\theta^-) + \delta g_0(\theta^-)]$$

subject to the incentives constraints:

$$\begin{aligned} (1 - \delta)\theta^+ f_0(\theta^+) + \delta g_0(\theta^+) &\geq (1 - \delta)\theta^+ f_0(\theta^-) + \delta g_0(\theta^-) \\ (1 - \delta)\theta^- f_0(\theta^-) + \delta g_0(\theta^-) &\geq (1 - \delta)\theta^- f_0(\theta^+) + \delta g_0(\theta^+) \end{aligned}$$

and the following two resource constraints:

$$\rho^+ f_0(\theta^+) + \rho^- f_0(\theta^-) = \eta$$

$$w = \rho^+ g_0(\theta^+) + \rho^- g_0(\theta^-)$$

The objective of the social planner is to maximize the expected life-time payoff w , which is expressed in the first display in terms of the current consumption f_0 and the continuation payoff g_0 . The first inequality constraint represents the incentive constraint of the consumers with the high-taste shock, and the second inequality represents the incentive constraint of the consumers with the low-taste shock. The first equality constraint says that the economy consumes the available endowment, and the second equality constraint ensures that the aggregated continuation payoff is equal to the expected life-time payoff. All constraints except the last one must be satisfied by any feasible and incentives-compatible allocation, and Lemma 1 guarantees that some optimal allocation satisfies also the last constraint.

The variables $f_0(\theta^+)$, $f_0(\theta^-)$, $g_0(\theta^+)$ and $g_0(\theta^-)$ from any solution of the social planner's problem with the recursive structure satisfy the four constraints of the reduced problem, and also any $f_0(\theta^+)$, $f_0(\theta^-)$, $g_0(\theta^+)$ and $g_0(\theta^-)$ which solve the reduced problem determine a solution to the social planner's problem with the recursive structure. Therefore, the solutions of the reduced problem can be identified with the solutions of the original problem that have the recursive structure.

To simplify notation, we use: f^+ (f^-) instead of $f_0(\theta^+)$ ($f_0(\theta^-)$), and use g^+ (g^-) instead of $g_0(\theta^+)$ ($g_0(\theta^-)$). Then, the reduced problem becomes:

$$w = \rho^+[(1 - \delta)\theta^+ f^+ + \delta g^+] + \rho^-[(1 - \delta)\theta^- f^- + \delta g^-]$$

$$(1 - \delta)\theta^+ f^+ + \delta g^+ \geq (1 - \delta)\theta^+ f^- + \delta g^-$$

$$(1 - \delta)\theta^- f^- + \delta g^- \geq (1 - \delta)\theta^- f^+ + \delta g^+$$

$$\rho^+ f^+ + \rho^- f^- = \eta$$

$$w = \rho^+ g^+ + \rho^- g^-$$

Using the second resource constraint to eliminate δ from the objective function, and writing the incentives constraints as a double inequality, we obtain

$$w = \rho^+ \theta^+ f^+ + \rho^- \theta^- f^- \tag{2}$$

$$(1 - \delta)\theta^+(f^+ - f^-) \geq \delta(g^- - g^+) \geq (1 - \delta)\theta^-(f^+ - f^-) \tag{3}$$

$$\rho^+ f^+ + \rho^- f^- = \eta \tag{4}$$

$$w = \rho^+ g^+ + \rho^- g^- \tag{5}$$

The following result characterizes the solutions of the reduced problem, and so all optimal dynamic allocations with recursive structure:

Theorem 1 1. *If $\delta\rho^+\theta^+ > (1 - \delta)\rho^-\theta^-$, then there exist multiple solutions f^+ , f^- , g^+ and g^- of the reduced problem; in each of them $f^- = 0$ and $f^+ = \eta/\rho^+$; moreover,*

$$g^+ \in \left[\max \left\{ 0, \theta^+ \eta - \frac{\rho^-}{\rho^+} \frac{1 - \delta}{\delta} \theta^+ \eta \right\}, \theta^+ \eta - \frac{\rho^-}{\rho^+} \frac{1 - \delta}{\delta} \theta^- \eta \right],$$

and

$$g^- = \frac{\theta^+ \eta - \rho^+ g^+}{\rho^-}.$$

2. *If $\delta\rho^+\theta^+ = (1 - \delta)\rho^-\theta^-$, then there is a unique solution, in which $f^- = 0$, $f^+ = \eta/\rho^+$, $g^+ = 0$, and $g^- = \theta^+ \eta/\rho^-$.*

3. *If $\delta\rho^+\theta^+ < (1 - \delta)\rho^-\theta^-$, then there is a unique solution, in which*

$$f^- = \frac{\eta}{\rho^-} \left[\frac{(1 - \delta)\theta^- \rho^- - \delta\rho^+\theta^+}{(1 - \delta)\theta^- \rho^- + \theta^- \rho^+ - \delta\rho^+\theta^+} \right] > 0,$$

$$f^+ = \frac{\theta^- \eta}{(1 - \delta)\theta^- \rho^- + \theta^- \rho^+ - \delta\rho^+\theta^+},$$

$g^+ = 0$ and $g^- = \theta^+ \eta/\rho^-$.

Intuitively, one can understand Theorem 1 as follows: The social planner would ideally distribute the current endowment only among the consumers with the high-taste shock. However, to make the consumers with the low-taste shock report truthfully, the social planner must prescribe a larger share of future endowment contingent on reporting the low-taste shock. If the fraction of the high-shock consumers is large (as in case 1), each of them consumes a relatively low share of the current endowment, even if all endowment is distributed among them. Hence, assuming that the discount factor is sufficiently large, the social planner has enough future endowment to provide incentives to the low-shock consumers.

The multiplicity of optimal allocations comes from the fact that if there is more than enough future endowment to give incentives to the low-shock consumers, the remaining future endowment can be distributed in various ways: the social planner can make the low-shock consumers just indifferent between reporting truthfully and reporting the high-taste shock; she can make the high-shock consumers just indifferent between reporting truthfully and reporting the low-taste shock; or she can distribute the endowment in any way in between the two.

However, if the fraction of the high-shock consumers is small (as in case 3), and if the social planner distributed the current endowment only among the consumers with the high-taste shock, their consumption would have to be too big, and the social planner would not have enough future endowment to provide incentives to the low-shock consumers for reporting truthfully. In this case, the social planner prescribes as much of the current endowment as possible to the high-shock consumers, and prescribes all future endowment to the low-shock consumers in order to make them indifferent between reporting truthfully, and reporting the high-taste shock.

Proof. Observe first that in any solution of the reduced problem, we must have either that the low-shock consumers are prescribed no consumption: $f^- = 0$, or that the incentive constraint for the low-shock consumers binds: $\delta(g^- - g^+) = (1 - \delta)\theta^-(f^+ - f^-)$. Otherwise, the planner could slightly decrease f^- and increase f^+ (so that (4) is still satisfied), which would strictly increase the value of w .

Thus, there are two cases to be considered, depending on whether $f^- = 0$ or $\delta(g^- - g^+) = (1 - \delta)\theta^-(f^+ - f^-)$. (These two equalities hold simultaneously only if condition $\delta\rho^+\theta^+ = (1 - \delta)\rho^-\theta^-$ is satisfied.) First, assume that $f^- = 0$. Then, (4) yields $f^+ = \eta/\rho^+$, and (2) yields $w = \theta^+\eta$. The remaining two conditions (3), (5) become:

$$(1 - \delta)\theta^+\frac{\eta}{\rho^+} \geq \delta(g^- - g^+) \geq (1 - \delta)\theta^-\frac{\eta}{\rho^+}, \quad (6)$$

and

$$\theta^+\eta = \rho^+g^+ + \rho^-g^-. \quad (7)$$

Thus, the reduced problem has a solution with $f^- = 0$, if there exist nonnegative g^+ and g^- that satisfy (6) and (7). Computing g^- as a function of g^+ from (7), one can express the four conditions required for existence in terms of variable g^+ . The constraints that g^+ and g^- are nonnegative become

$$0 \leq g^+ \leq \frac{\theta^+\eta}{\rho^+}, \quad (8)$$

and the two inequalities in (6) are equivalent to:

$$\theta^+\eta - \frac{\rho^-}{\rho^+} \frac{1-\delta}{\delta} \theta^+\eta \leq g^+ \leq \theta^+\eta - \frac{\rho^-}{\rho^+} \frac{1-\delta}{\delta} \theta^-\eta. \quad (9)$$

The second inequality in (8) is implied by the second inequality in (9). Moreover, if $\delta\rho^+\theta^+ > (1-\delta)\rho^-\theta^-$, then

$$\theta^+\eta - \frac{\rho^-}{\rho^+} \frac{1-\delta}{\delta} \theta^-\eta > 0,$$

and obviously

$$\theta^+\eta - \frac{\rho^-}{\rho^+} \frac{1-\delta}{\delta} \theta^-\eta > \theta^+\eta - \frac{\rho^-}{\rho^+} \frac{1-\delta}{\delta} \theta^-\eta.$$

So, we obtain an interval of values of g^+ , as described in part 1 of Theorem 1, that satisfy both the first inequalities in both (8) and (9), and the second inequality in the latter condition. If $\delta\rho^+\theta^+ = (1-\delta)\rho^-\theta^-$, $g^+ = 0$ is the unique value that satisfies (8) and (9). If $\delta\rho^+\theta^+ < (1-\delta)\rho^-\theta^-$, no g^+ satisfies (8) and (9).

Assume now that $\delta(g^- - g^+) = (1-\delta)\theta^-(f^+ - f^-)$, but $f^- > 0$. In this case, the incentive constraint for the high-shock consumers is automatically satisfied. Note that in any solution $g^+ = 0$, because otherwise one could increase f^+ and decrease f^- (so that (4) is still satisfied), and thus increase w , and simultaneously decrease g^+ and increase g^- to ensure that the incentive constraint for the low-shock consumers still binds.

Setting $g^+ = 0$ and solving the system of equations (2), (4), (5), and $\delta(g^- - g^+) = (1-\delta)\theta^-(f^+ - f^-)$, we obtain: $g^- = \theta^+\eta/\rho^-$ and

$$f^+ = \frac{\theta^-\eta}{(1-\delta)\theta^-\rho^- + \theta^-\rho^+ - \delta\rho^+\theta^+}$$

and

$$f^- = \frac{\eta}{\rho^-} \left[\frac{(1-\delta)\theta^-\rho^- - \delta\rho^+\theta^+}{(1-\delta)\theta^-\rho^- + \theta^-\rho^+ - \delta\rho^+\theta^+} \right]$$

If $\delta\rho^+\theta^+ < (1-\delta)\rho^-\theta^-$, then both the numerator and the denominator of the expression for f^- are positive, so that f^- (as well as f^+) are positive. That is, there is a solution in which $g^+ = 0$, which is described in part 3 of Theorem 1. If condition $\delta\rho^+\theta^+ \geq (1-\delta)\rho^-\theta^-$ is satisfied, then the numerator of the expression for f^- is nonpositive, and so we cannot have both: $f^- \geq 0$ and $f^+ \geq 0$. Thus, there is no solution in this case. ■

4.2 Immiseration

Atkeson and Lucas (1992) showed, quoting their abstract, that “in an efficient allocation the degree of inequality continually increases, with a diminishing fraction of the population receiving an increasing fraction of the resources.”⁷ In this section, we address the question whether the optimal dynamic allocations have this feature, that is, immiseration, in the setting with linear utility.

Our first result says that the optimal allocations with the recursive structure lead to immiseration. More specifically, we establish the following proposition.

⁷More specifically, they proved this result for the CARA, CRRA, and logarithmic utilities.

Proposition 1 *The optimal dynamic allocations with the recursive structure have the following properties:*

(i) *If $\delta\rho^+\theta^+ \leq (1-\delta)\rho^-\theta^-$, then the consumption of consumers who experience the high-taste shock is terminated in the following period, and the entire endowment is distributed to a vanishing fraction of consumers who never experienced the high-taste shock.*

(ii) *If $\delta\rho^+\theta^+ > (1-\delta)\rho^-\theta^-$, then for every $\varepsilon > 0$, there is a period T such that for $t > T$ the continuation payoff of a fraction of at least $1-\varepsilon$ consumers is smaller than ε , and the remaining endowment is distributed to a fraction ε of consumers.*

Proof. Part (i) follows immediately from parts 2 and 3 of Theorem 1. Indeed, $g^+ = 0$ in these cases.

To show part (ii), note first that by part 1 of Theorem 1 and the Law of Large Numbers, for any sufficiently large t , the continuation payoff of a fraction of at least $1-\varepsilon$ consumers is equal to

$$w \left(\frac{g^+}{w} \right)^{\rho^+ t} \left(\frac{g^-}{w} \right)^{\rho^- t} \left(\frac{g^+}{g^-} \right)^{o(t)}$$

where $o(t)$ can be positive or negative, and $o(t)/t \rightarrow 0$ as $t \rightarrow \infty$. So, it suffices to show that

$$\left(\frac{g^+}{w} \right)^{\rho^+} \left(\frac{g^-}{w} \right)^{\rho^-} < 1 \quad (10)$$

for $w = \theta^+\eta$, any

$$g^+ \in \left[\max \left\{ 0, \theta^+\eta - \frac{\rho^-}{\rho^+} \frac{1-\delta}{\delta} \theta^+\eta \right\}, \theta^+\eta - \frac{\rho^-}{\rho^+} \frac{1-\delta}{\delta} \theta^-\eta \right],$$

and

$$g^- = \frac{\theta^+\eta - \rho^+ g^+}{\rho^-}.$$

We can represent (10) as

$$(1-x)^{\rho^+} \left(1 + \frac{\rho^+}{\rho^-} x \right)^{\rho^-} < 1, \quad (11)$$

where

$$x \in \left[\frac{\rho^-}{\rho^+} \frac{1-\delta}{\delta} \frac{\theta^-}{\theta^+}, \min \left\{ \frac{\rho^-}{\rho^+} \frac{1-\delta}{\delta}, 1 \right\} \right].$$

The derivative of (11) with respect to x is

$$-\rho^+ (1-x)^{\rho^+-1} \left(1 + \frac{\rho^+}{\rho^-} x \right)^{\rho^-} + \rho^+ (1-x)^{\rho^+} \left(1 + \frac{\rho^+}{\rho^-} x \right)^{\rho^- - 1} < 0,$$

therefore the LHS of (11) is lower than its value at $x = 0$, which is 1. ■

Perhaps surprisingly, if $\delta\rho^+\theta^+ > (1-\delta)\rho^-\theta^-$, it is possible to construct another optimal allocation which has a positive lower bound on a continuation payoff for any consumer. According to the alternative allocation, if a consumer's continuation payoff drops below a certain cutoff level, the consumer will stop consuming, the consumer's continuation payoff will gradually increase until it raises back above the cutoff level. This implies that the continuation payoff can never drop below a certain value, and thus, the optimal allocation is more fair: No matter what the history of each consumer (dynasty), all dynasties are guaranteed a minimal positive continuation payoff.

In an interpretation, this more fair allocation puts some bound on the extent to which children are responsible for the consumption of their parents. If parents experience lots of high-taste shocks, and consume much in early periods of their lives, they consume little in later periods of their lives, independent of the shocks that they experience. In contrast, the optimal dynamic allocations with the recursive structure let parents consume much as long as they experience high-taste shocks, at the expense of the consumption of their children.

Theorem 2 *Suppose that $\delta\rho^+\theta^+ > (1 - \delta)\rho^-\theta^-$. Then, there exists an optimal allocation such that the continuation payoffs of all consumer in all periods exceed some $Z > 0$.*

Before giving the proof, it is worth pointing out that the condition on the parameters required by Theorem 2 seems quite mild. The condition is obviously more likely to be satisfied when the discount factor is higher, the fraction of consumers facing the high-taste shock is greater, and the difference between the two types consumers is greater.

Proof. If $\delta\rho^+\theta^+ > (1 - \delta)\rho^-\theta^-$, then by part 1 of Theorem 1 there exists an optimal allocation with the recursive structure, such that $g^+, g^- > 0$, and both incentives constraints in (3) are not binding. We will call this optimal allocation *original* in this proof. According to this allocation, any consumer with the high-taste shock in period $t = 0$ consumes $f^+ = \eta/\rho^+$, and any consumer with the low-taste shock consumes $f^- = 0$. If a small positive value $\Delta > 0$ is added to the consumption of such a high-shock consumer, so that the consumption is increased to $\eta/\rho^+ + \Delta$, and the promised continuation payoff is decreased to $g^+ - \Delta\theta^+(1 - \delta)/\delta$, then the life-time payoff remains unchanged. In addition, if Δ is small enough, $g^+ - \Delta\theta^+(1 - \delta)/\delta$ is still strictly positive, and the incentives constraints (3) still hold. It will be convenient to represent the higher consumption by saying that the original consumption was multiplied by

$$\frac{\frac{\eta}{\rho^+} + \Delta}{\frac{\eta}{\rho^+}} = \frac{\eta + \Delta\rho^+}{\eta}. \quad (12)$$

Let

$$X = w \frac{\Delta\rho^+}{\Delta\rho^+ + \eta},$$

where w is the expected life-time payoff under the original allocation. Call *poor* the consumers whose continuation payoff is below X , and call *rich* all other consumers. Consider the following perturbation of the original allocation: all poor consumers are prescribed no consumption in the current period, and their promised continuation payoffs (at the beginning of the next period) are increased accordingly to keep the continuation payoffs (at the beginning of the current period) unchanged. The consumption “taken away” from the poor consumers is then redistributed among the rich consumers with the high-taste shock, proportionally to their continuation payoff (at the beginning of the current period), and on top of their consumption prescribed by the original allocation. This new allocation is incentive compatible for the poor consumers (since their shock report does not affect their allocation). So, it is left to show that the new allocation remains incentive compatible for the rich consumers.

Denote the aggregate continuation payoff of the poor consumers (under the original allocation) at the beginning of period t by V_t . Notice that the aggregate continuation payoff of all consumers in all periods is equal to w . Indeed, this follows from (1), which was the main step in the proof of Lemma 1, and then by induction from the recursive structure of the original allocation. Thus, the aggregate continuation payoff of rich consumers is equal to $w - V_t$. Note that $V_t \leq X$, since the mass of the poor consumers cannot exceed 1. Under the original allocation, the aggregate consumption of the rich consumers is $\eta \frac{w - V_t}{w}$. (This is so, since the original allocation is recursive.) Under the new allocation, the aggregate consumption of the rich consumers is η . So, the new allocation is incentive compatible, if the relative increase in consumption, $\frac{w}{w - V_t}$, is smaller than (12), that is, if

$$\frac{w}{w - V_t} \leq \frac{\frac{\eta}{\rho^+} + \Delta}{\frac{\eta}{\rho^+}},$$

which holds by the definition of X , and since $V_t \leq X$.

Thus, the new allocation is incentive compatible. Notice now that the relative drop in the continuation payoff of a high-shock consumer is bounded by

$$\frac{g^+ - \Delta \theta^+ \frac{(1-\delta)}{\delta}}{w}.$$

Therefore, since the continuation payoff increases once it drops below X , its lowest possible value is at least

$$Z \equiv X \frac{g^+ - \Delta \theta^+ \frac{(1-\delta)}{\delta}}{w} > 0.$$

■

5 Market Equilibria

In this section, we consider competitive trading of consumption across different periods, assuming that initially each consumer is entitled to an equal share of the endowments. We show that, when utility is linear, at least some market equilibria achieve the same expected life-time payoff as the social planner would. This is in contrast with the claim of Atkeson and Lucas, made for some strictly concave utility functions (see, the discussion on pp. 444-445 of their paper), that market equilibria cannot support the efficient allocation. This is also one of the special cases in which money is a sufficient carrier of memory, unlike in the general case as shown by Kocherlakota (1998).

For some parameters of the model with linear utility, there also exist dynamically optimal market equilibria without immiseration. However, the condition on the parameters of the model required for the existence of these “more fair” equilibria is stronger than that required for the existence of a dynamically optimal more fair allocation. So, the market not always attains what the social planner can, namely, an allocation that is second-best and such that the total wealth is not accumulated in the hands of a vanishing fraction of consumers.

5.1 Definition of market equilibrium

We define market equilibrium as i) a sequence of positive prices p_0, p_1, \dots such that

$$\sum_{t=0}^{\infty} p_t < \infty,$$

where p_t stands for the price in period 0 of one unit of consumption to be obtained in period t ; together with ii) quantities consumed by consumers, which satisfy some optimality and market-clearing conditions that will be introduced next. This is equivalent to the Atkeson and Lucas concept of unmonitored trade by means of a one-period real bond entitling the holder to one unit of consumption tomorrow.

We assume that each consumer holds the same share of the rights to all future endowments, so that she can afford to consume η units of the good in each period. That is, the monetary wealth of each consumer in period $t = 0$ is

$$W_0 \equiv \sum_{t=0}^{\infty} p_t \eta.$$

Consumers are allowed to exchange consumption across different periods at prices p_0, p_1, \dots . Recall that these prices are expressed in the terms of period 0, that is, by paying p_t in period 0 a consumer acquires the right to consume one unit of the good in period t .

Denote by $f_t(\theta_0, \theta_1, \dots, \theta_t, \omega_0, \omega_1, \dots, \omega_t) \geq 0$ the demand in period t coming from a consumer with shock history $\theta_0, \theta_1, \dots, \theta_t$ who maximizes the expected life-time payoff taking prices as given. A consumer who is indifferent between several consumption plans is allowed to randomize across those plans. This randomness is represented by variable ω_t . It will be sufficient for our purposes to assume that i) the range of every ω_t is finite, that ii) the realizations of ω_t across different consumers are independent, and that iii) all consumers with the same history of shocks, and the same realizations $\omega_1, \dots, \omega_{t-1}$ of all previous random choices of consumption plans, use the same random device ω_t . Since there is a continuum of consumers, the distribution of choices made contingent on a shock history $\theta_0, \theta_1, \dots, \theta_t, \omega_0, \omega_1, \dots, \omega_{t-1}$ coincides with the distribution used by an individual consumer.

The demands $\{f_t(\theta_0, \theta_1, \dots, \theta_t, \omega_0, \omega_1, \dots, \omega_t)\}_{(\theta_0, \theta_1, \dots, \theta_t, \omega_0, \omega_1, \dots, \omega_t)}_{t=0}^{\infty}$ maximize the consumer's expected life-time payoff, subject to the budget constraint:

$$\forall_{\theta_0, \theta_1, \dots, \theta_t, \omega_0, \omega_1, \dots, \omega_t, \dots} \sum_{t=0}^{\infty} p_t f_t(\theta_0, \theta_1, \dots, \theta_t, \omega_0, \omega_1, \dots, \omega_t) \leq W_0. \quad (13)$$

Finally, the following market-clearing condition must be satisfied in each period t :

$$E_{\omega_0, \omega_1, \dots, \omega_t} \sum_{\theta_0, \theta_1, \dots, \theta_t} f_t(\theta_0, \theta_1, \dots, \theta_t, \omega_0, \omega_1, \dots, \omega_t) \rho(\theta_0, \theta_1, \dots, \theta_t) = \eta \quad (14)$$

where $\rho(\theta_0, \theta_1, \dots, \theta_t)$ denotes the mass of consumers with shock history $\theta_0, \theta_1, \dots, \theta_t$.

5.2 Characterization of equilibria

When the utility of current consumption is linear, we are able to explicitly characterize some market equilibria.

Theorem 3 1. If $\rho^+ \geq 1 - \delta$, then $p_t = \delta^t$ are market equilibrium prices. In this equilibrium, all endowment is consumed by the high-shock consumers. In addition, the high-shock consumers are indifferent between spending their wealth on the current consumption and saving for the future consumption.

2. If $\rho^+ < 1 - \delta$, and $\delta\rho^+\theta^+ > (1 - \delta)\rho^-\theta^-$, then $p_t = (\rho^-)^t$ are market equilibrium prices. In this equilibrium, all endowment is consumed by the high-shock consumers, who spend all their wealth on the current consumption.

3. If $\delta\rho^+\theta^+ \leq (1 - \delta)\rho^-\theta^-$, then $p_t = \Pi^t$, where $\Pi = \frac{\delta[\theta^+\rho^+ + \theta^-\rho^-]}{\theta^-}$, are market equilibrium prices. In this equilibrium, the endowments are shared between the high- and low-shock consumers. The high-shock consumers spend all their wealth on the current consumption, and the low-shock consumers are indifferent between spending their wealth on the current consumption and saving for the future consumption.

Notice that the inequalities in cases 1-3 are mutually exclusive, because $\delta\rho^+\theta^+ \leq (1 - \delta)\rho^-\theta^-$ implies that $\rho^+ < 1 - \delta$.

Proof. To prove Theorem 3, we must find the demands which maximize the consumer's life-time expected payoff (subject to the budget constraint), and satisfy the market-clearing conditions.

Consider a consumer who has wealth W_t in period t . To determine the consumer's optimal behavior, we will compare her continuation payoffs from the following four strategies, given the current taste shock: (i) The consumer has the high-taste shock, and spends all her wealth in period t . Then, her continuation payoff is equal to

$$V = (1 - \delta)\theta^+ \frac{W_t}{p_t}. \quad (15)$$

(ii) The consumer has the low-taste shock, and spends all her wealth in period t . Then, her continuation payoff is equal to

$$V = (1 - \delta)\theta^- \frac{W_t}{p_t}. \quad (16)$$

In the other two strategies, the consumer does not spend anything in period t , so her current taste shock does not have any effect on the continuation payoff.

(iii) If the consumer spends all her wealth in period $t + 1$, her continuation payoff is

$$V = (1 - \delta)\delta[\rho^+\theta^+ + \rho^-\theta^-] \frac{W_t}{p_{t+1}} = (1 - \delta)\delta[\rho^+\theta^+ + \rho^-\theta^-] \frac{W_t}{\pi p_t}, \quad (17)$$

where $\pi = \frac{p_{t+1}}{p_t}$ denotes the price ratio, which does not change over time in any of the cases 1-3.

(iv) If in the future the consumer spends all her wealth at the first time she experiences the high-taste shock, her continuation payoff is

$$V = (1 - \delta)\delta\rho^+\theta^+ \frac{W_t}{p_{t+1}} + (1 - \delta)\delta^2\rho^+\rho^-\theta^+ \frac{W_t}{p_{t+2}} + (1 - \delta)\delta^3\rho^+(\rho^-)^2\theta^+ \frac{W_t}{p_{t+3}} + \dots$$

$$\begin{aligned}
&= (1 - \delta)\delta\rho^+\theta^+\frac{W_t}{p_t\pi} + (1 - \delta)\delta^2\rho^+\rho^-\theta^+\frac{W_t}{p_t\pi^2} + (1 - \delta)\delta^3\rho^+(\rho^-)^2\theta^+\frac{W_t}{p_t\pi^3} + \dots \\
&= \frac{(1 - \delta)\delta\rho^+\theta^+\frac{W_t}{p_t\pi}}{1 - \frac{\delta\rho^-}{\pi}}.
\end{aligned} \tag{18}$$

Comparing values (15)-(18) will determine the consumers' optimal behavior. Indeed, since in each of the three parts of Theorem 3 the price ratio $\frac{p_{t+1}}{p_t}$ is the same in all periods t , consumers have the same incentives in each period for spending their wealth on the current consumption versus saving it for the future consumption. Thus, to determine the consumer's optimal strategy, one can without loss of generality consider only two continuation strategies of future consumption: spending all her wealth immediately in the following period, and spending all her wealth in the first period in which she experiences the high-taste shock.

1. In part 1, $\pi = \delta$. Expressions (15) and (18) yield the same value of $(1 - \delta)\theta^+\frac{W_t}{p_t}$, which is strictly greater than (16) and (17). That is, the consumer's optimal strategy is to consume only when she experiences the high-taste shock. Moreover, when she experiences a high-taste shock, she is indifferent between spending her wealth in the current period, and waiting to spend her wealth when she experiences the next high-taste shock.

For the market-clearing condition to be satisfied, the total wealth of the consumers with the high-taste shock has to be higher than the cost of the whole current endowment, so that they can afford to buy the whole current endowment in each period. The total cost of endowment is $p_0\eta = \eta$, and the total wealth of high-shock consumers is

$$\rho^+W_0 = \rho^+ \sum_{t=0}^{\infty} p_t\eta = \rho^+ \sum_{t=0}^{\infty} \delta^t\eta = \frac{\rho^+}{1 - \delta}\eta,$$

which is indeed higher than η because $\rho^+ \geq 1 - \delta$.

2. In part 2, $\pi = \rho^-$. Expression (18) takes value $\frac{\delta\rho^+\theta^+W_t}{\rho^-p_t}$, which is strictly smaller than (15) due to inequality $\rho^+ < 1 - \delta$ (which is equivalent to $\rho^- > \delta$). At the same time, (18) is strictly greater than both (16) and (17) since $\delta\rho^+\theta^+ > (1 - \delta)\rho^-\theta^-$. That is, the optimal choice of a consumer is to spend all money when she experiences the high-taste shock, and does not spend anything when she experiences the low-taste shock.

The total wealth of the consumers with the high-taste shock is equal to

$$\rho^+W_0 = \rho^+ \sum_{t=0}^{\infty} p_t\eta = \rho^+ \sum_{t=0}^{\infty} (\rho^-)^t\eta = \eta,$$

which equals the total cost of consumption.

3. In part 3, $\pi = \frac{\delta[\theta^+\rho^+ + \theta^-\rho^-]}{\theta^-}$, and the value in expression (18) is $\frac{(1 - \delta)\theta^-W_t}{p_t}$, which is equal to that in expressions (16) and (17), and strictly smaller than that in expression (15). That is, the optimal choice for all consumers is to spend all their money on the current consumption if they experience the high-taste shock. The consumers experiencing the low-taste shock are indifferent.

To check the market-clearing condition, it is enough to show that the total wealth of the high-shock consumers is smaller than the cost of the endowment, because the low-shock consumers can buy the excess endowment. The total wealth of the high-shock consumers is

$$\rho^+ W_0 = \rho^+ \sum_{t=0}^{\infty} \pi^t \eta.$$

By substituting $\pi = \frac{\delta[\theta^+ \rho^+ + \theta^- \rho^-]}{\theta^-}$, and summing up the geometric progression, we have that $\rho^+ W_0 \leq \eta$ since $\delta \rho^+ \theta^+ \leq (1 - \delta) \rho^- \theta^-$. ■

The equilibria described in Theorem 3 attain the second-best outcome.

Proposition 2 *For any set of parameters, a market equilibrium attains the optimal expected life-time payoff.*

Proof. In the equilibria described in parts 1 and 2 of Theorem 3, all current endowment is consumed in every period by high-shock consumers, as it was in parts 1 and 2 of Theorem 1. Moreover, parts 1 and 2 of Theorem 3 and parts 1 and 2 in Theorem 1 require the same condition $\delta \rho^+ \theta^+ > (1 - \delta) \rho^- \theta^-$. In part 3 of Theorem 3, the share of the endowment consumed by the high-shock consumers is the same in each period, and is equal to

$$\frac{\rho^+ W_0}{\eta} = \rho^+ \sum_{t=0}^{\infty} \Pi^t = \frac{\rho^+}{1 - \Pi} = \frac{\rho^+}{1 - \frac{\delta[\theta^+ \rho^+ + \theta^- \rho^-]}{\theta^-}} = \frac{\rho^+ \theta^-}{(1 - \delta) \theta^- \rho^- + \theta^- \rho^+ - \delta \rho^+ \theta^+}.$$

Exactly the same share is consumed by the high-shock consumers in part 3 of Theorem 1. Thus, the social planner cannot attain a higher expected life-time payoff than that in the market equilibrium. ■

So, the market mechanism can achieve the second-best outcomes in terms of payoffs. We will next address the question whether the market can also reduce inequality, without losing any efficiency in the cases in which the social planner can do so. The social planner requires that $\delta \rho^+ \theta^+ > (1 - \delta) \rho^- \theta^-$ to achieve the second-best outcome through a more fair allocation. That is, if this condition is satisfied, the social planner can prevent all endowment to be consumed by a vanishing fraction of consumers. Our next proposition shows that the market requires a stronger condition, namely, that $\rho^+ > 1 - \delta$. Intuitively, this stronger condition is required, because in any market equilibrium, the high-shock consumers must be indifferent between spending their wealth immediately, and saving for the future consumption. The social planner has more freedom, she may assign no consumption to some high-shock consumers (e.g., the ones with sufficiently low continuation payoff) even though they would strictly prefer to consume.

Thus, the key feature of market equilibria responsible for the failure of reducing inequality is that the price cannot depend on the current wealth, or the history of consumption. This suggests that policy measures that increase the cost of financing the current consumption for the consumers with higher past consumption should have a chance of delivering more fair allocations.

Proposition 3 *1. If $\rho^+ > 1 - \delta$, then there exists a market equilibrium in which the total endowment is not distributed among a vanishing fraction of consumers.*

2. If $\rho^+ \leq 1 - \delta$, then in any market equilibrium the total endowment is distributed among a vanishing fraction of consumers.

Before giving the proof, it is worth pointing out that greater is the difference between the two types of consumers, higher is the chance that $\delta\rho^+\theta^+ > (1 - \delta)\rho^-\theta^-$ and $\rho^+ \leq 1 - \delta$, that is, higher is the chance that the social planner has an advantage over competitive markets in achieving the second-best outcome through an allocation without immiseration.

Proof. 1. The prices in Theorem 3, part 1, induce a market equilibrium, in which the high-shock consumers are indifferent between spending their wealth for the current consumption and saving it for the future consumption. In a way similar to that from the proof of Theorem 2, we specify the market-clearing consumption such that the wealth of any consumer is bounded away from zero. Intuitively, since the total wealth of the high-shock consumers is strictly higher than the total cost of the current endowment, one can prescribe zero consumption to the consumers with relatively low wealth, and the wealth of the remaining high-shock consumers will still exceed the total cost of endowment. Then, the wealth of any consumer will stay bounded below from zero, and there will be no vanishing fraction of consumers acquiring all wealth.

2. Consider any market equilibrium, with prices p_0, p_1, \dots . Note that in any period in which the high-shock consumers are strictly willing to spend their wealth, the mass of consumers with positive wealth decreases (at least) by the rate of ρ^- . So, if in the limit the total wealth is not distributed among a vanishing fraction of consumers, then there must exist a period T such that for all periods $t > T$, the high-shock consumers are indifferent between consuming in the current period and saving their wealth for the future. Moreover, it means that no low-shock consumer spends anything in all periods $t > T$. Thus, in any period $t > T$ any high-shock consumer is indifferent between spending all wealth in period t , and spending all her wealth next time she experiences the high-taste shock.

Denote by $\pi_s \equiv \frac{p_s}{p_{s-1}}$ (for $s \geq 1$) the rate at which prices change. Fix some $t > T$, and consider a consumer with wealth W_t in period t . Since any high-shock consumer in period t is indifferent between spending all her wealth W_t in period t and waiting to spend it when she experiences the next high-taste shock, we have that

$$(1 - \delta)\theta^+ \frac{W_t}{p_t} = (1 - \delta)\delta\rho^+\theta^+ \frac{W_t}{p_{t+1}} + (1 - \delta)\delta^2\rho^+\rho^-\theta^+ \frac{W_t}{p_{t+2}} + \dots, \quad (19)$$

and if we divide the equality above by $(1 - \delta)\theta^+ W_t$, express prices in periods $t + 1, t + 2, \dots$ as $p_{t+1} = p_t\pi_{t+1}, p_{t+2} = p_t\pi_{t+1}\pi_{t+2}, \dots$, and multiply equality (19) by p_t , we obtain that

$$1 = \delta\rho^+ \frac{1}{\pi_{t+1}} + \delta^2\rho^+\rho^- \frac{1}{\pi_{t+1}\pi_{t+2}} + \dots \quad (20)$$

Similarly, since any high-shock consumer in period $t + 1$ is indifferent between spending all her wealth W_{t+1} in period t and waiting to spend it when she experiences the next high-taste shock, we obtain a counterpart of (20) for period $t + 1$

$$1 = \delta\rho^+ \frac{1}{\pi_{t+2}} + \delta^2\rho^+\rho^- \frac{1}{\pi_{t+2}\pi_{t+3}} + \dots \quad (21)$$

Multiplying (21) by $\frac{\delta\rho^-}{\pi_{t+1}}$, and subtracting it from (20) yields

$$1 - \frac{\delta\rho^-}{\pi_{t+1}} = \frac{\delta\rho^+}{\pi_{t+1}}$$

and, since $\rho^+ + \rho^- = 1$, we obtain that $\pi_{t+1} = \delta$. By the same argument, one can show that $\pi_{t+2} = \pi_{t+3} = \dots = \delta$, that is, the rate at which prices change is fixed over time at δ for all $t > T$. In the proof of Theorem 3, part 1, it is shown that with such a rate, the market-clearing condition requires the total wealth of high-shock consumers to be weakly higher than the total cost of the current endowment, which is equivalent to $\rho^+ \geq 1 - \delta$. Thus, if $\rho^+ < 1 - \delta$, the market-clearing condition fails, while if $\rho^+ = 1 - \delta$, the high-shock consumers spend all their wealth on the current consumption (while being indifferent). That is, the total wealth must be owned by a vanishing fraction of consumers. ■

6 Open economy

Thus far we have assumed that the economy is closed. It cannot lend to and borrow from outside parties. In this section, we allow for the possibility of trading, as in Chapter 3 of Kocherlakota (2010). The social planner faces a fixed gross interest rate $R > 1$ of lending and borrowing of the nonperishable good. We assume the case of a small open economy, that is, the outside parties have an unlimited amount of the good to trade, and the planner's actions have no effect on R .

With the possibility of lending and borrowing, the planner faces a budget constraint, rather than the more restrictive per-period resource constraint. Denoting by A_t the aggregate consumption in the economy, the budget constraint is written as

$$\sum_{t=0}^{\infty} \frac{A_t}{R^t} \leq \sum_{t=0}^{\infty} \frac{\eta}{R^t} = \eta \frac{R}{R-1}.$$

6.1 The benchmark case of no taste shock

It is useful to describe the planner's optimal solution in the case when all consumers have the same fixed value of θ across all periods. The structure of the solution depends on the relation between the discount factor δ and the interest rate R . If $\delta R < 1$, then decreasing the consumption in period t by a unit in exchange for an increase in the consumption in period $t+1$ by R units reduces the period- t continuation payoff. Hence, the optimal solution prescribes consuming all the wealth in the initial period: $A_0 = \eta \frac{R}{R-1}$, $A_t = 0$ for all $t > 0$.

If $\delta R = 1$, then saving one unit of the current consumption for the consumption in the following period does not change the continuation payoff, therefore *any* aggregate consumption stream $\{A_t\}_{t=0}^{\infty}$ yields the optimal outcome, as long as the budget constraint is satisfied with equality.

Finally, if $\delta R > 1$, then saving one unit of the current consumption for the consumption in the following period increases the continuation payoff. Thus, there is no upper bound on the feasible continuation payoff. In order to show this, consider a consumption stream with a fixed growth rate: In period 0 the aggregate

consumption is equal to A_0 , while in period t it is equal to $\alpha^t A_0$, with a fixed parameter $\alpha \in (\frac{1}{\delta}, R)$. The budget constraint would yield

$$\sum_{t=0}^{\infty} A_0 \frac{\alpha^t}{R^t} = A_0 \frac{R}{R-\alpha} = \eta \frac{R}{R-1},$$

or, equivalently, $A_0 = \eta \frac{R-\alpha}{R-1}$. Such values of A_0 and α would yield the aggregate life-time payoff

$$\sum_{t=0}^{\infty} A_0 \alpha^t (1-\delta) \delta^t = A_0 (1-\delta) \frac{1}{1-\alpha\delta} = \eta \frac{(R-\alpha)(1-\delta)}{(R-1)(1-\alpha\delta)}.$$

Note that if one takes α arbitrary close to $\frac{1}{\delta}$, the life-time payoff becomes arbitrarily large.

6.2 Main result

Having established the optimal solution in the benchmark case of no taste shock, we proceed with our main environment with two values of taste shocks. The structure of the solution depends on the values of δ and R . If $\delta R > 1$, then, similarly to the benchmark case, there is no upper bound on the continuation payoff. Through lending and borrowing, the planner can arrange the aggregate consumption to be $A_t = (\eta \frac{R-\alpha}{R-1}) \alpha^t$ for all t , for $\alpha \in (\frac{1}{\delta}, R)$, and distribute the good uniformly across all consumers, regardless of their shock values. As α limits to $\frac{1}{\delta}$, the life-time payoff of each consumer will become arbitrarily large.

If $\delta R \leq 1$, then, in any optimal dynamic allocation, the value of the life-time payoff is finite. To characterize optimal allocations, we analyze the dual problem of the planner: Given an ex-ante life-time payoff w of a single consumer, we seek to minimize the net expected cost of an incentive-compatible consumption stream that yields w .

Denote by $C(w)$ this minimal value of net expected cost, expressed in the terms of the good in period 0. Due to linearity of utilities, the function $C(w)$ is linear as well:

Proposition 4 *Function $C(w)$ is linear: $C(w) = \beta w$, where coefficient β is independent of w .*

Proof. Indeed, assume the minimal net expected cost required for achieving a life-time payoff of 1 is β . Then, in order to achieve a life-time payoff of w , the planner can take an optimal allocation achieving a life-time payoff of 1, and “scale” it by w , period by period. The new dynamic allocation is incentive-compatible, and costs βw .

Similarly, the planner cannot achieve w with cost lower than βw , since then she could also achieve a life-time payoff of 1 with cost lower than β by period-by-period scaling by $1/w$ an allocation achieving w , which would contradict the optimality of β . ■

Due to linearity of $C(w)$, it is without loss of generality to consider a recursive solution for the planner, that is, the solution in which the consumption in any period t is proportional to the continuation payoff w_t . Respectively, we can write a similar reduced problem, as in the case of the closed economy. For a given life-time payoff w (that is, the payoff at time $t = 0$), the planner needs to determine four values of f^+ , f^-

of current consumption for high and low shock values, and of g^+, g^- of continuation payoffs, such that the promise-keeping condition and incentive constraints are satisfied:

$$w = \rho^+[(1 - \delta)\theta^+ f^+ + \delta g^+] + \rho^-[(1 - \delta)\theta^- f^- + \delta g^-]$$

$$\begin{aligned} (1 - \delta)\theta^+ f^+ + \delta g^+ &\geq (1 - \delta)\theta^+ f^- + \delta g^- \\ (1 - \delta)\theta^- f^- + \delta g^- &\geq (1 - \delta)\theta^- f^+ + \delta g^+ \end{aligned}$$

Denote by C_0 the (not necessarily optimal) net expected costs in period 0 of achieving the life-time payoff w , estimated in terms of the good in period 0, as determined by values of f^+, f^-, g^+, g^- . Since the allocation is recursive, the net expected costs in period 1, estimated in terms of the good in period 1, are equal to either $C_0 \frac{g^+}{w}$ if the consumer had a high shock in $t = 0$, or $C_0 \frac{g^-}{w}$ if the consumer had a low shock in $t = 0$. One can express C_0 as the cost of the consumption in $t = 0$ and the net expected cost in $t = 1$:

$$C_0 = [\rho^+ f^+ + \rho^- f^-] + \frac{1}{R} [\rho^+ C_0 \frac{g^+}{w} + \rho^- C_0 \frac{g^-}{w}] \quad (22)$$

The goal of the planner is to choose values of f^+, f^-, g^+, g^- as to minimize C_0 .

Proposition 5 1. *If $\delta R = 1$, then there exist multiple solutions f^+, f^-, g^+ and g^- of the reduced problem; each of them achieves the first-best allocation: $f^- = 0$. There is an upper bound on consumption f^+ : $\bar{f}^+ = \frac{w}{(1-\delta)(\rho^+\theta^+ + \rho^-\theta^-)}$. Any value $0 < f^+ \leq \bar{f}^+$ yields multiple solutions, with*

$$g^+ \in \left[\max \left\{ 0, \frac{w}{\delta} - \frac{1-\delta}{\delta} f^+ \theta^+ \right\}, \frac{w}{\delta} - \frac{1-\delta}{\delta} f^+ (\rho^+ \theta^+ + \rho^- \theta^-) \right],$$

and

$$g^- = \frac{w - \rho^+ [(1 - \delta)\theta^+ f^+ + \delta g^+]}{\delta \rho^-}. \quad (23)$$

For all solutions, $C(w) = \frac{w}{(1-\delta)\theta^+}$.

2. *If $\delta R < 1$, then there is a unique solution, in which $f^+ = \frac{w}{(1-\delta)(\rho^+\theta^+ + \rho^-\theta^-)}$, and $g^+ = 0$. Furthermore:*

2.1. *If $\rho^+ \theta^+ + \rho^- \theta^- - \frac{\theta^-}{\delta R} > 0$, then $f^- = 0$, $g^- = \frac{1-\delta}{\delta} \theta^- f^+$, and*

$$C(w) = \frac{w}{1-\delta} \cdot \frac{\rho^+}{\frac{\rho^+ \theta^+}{\delta R} - (\rho^+ \theta^+ + \rho^- \theta^-) \left(\frac{1}{\delta R} - 1 \right)}.$$

2.2. *If $\rho^+ \theta^+ + \rho^- \theta^- - \frac{\theta^-}{\delta R} < 0$, then $f^- = f^+$, $g^- = 0$. That is, any consumption takes place only in period 0. In this case,*

$$C(w) = \frac{w}{1-\delta} \cdot \frac{1}{\rho^+ \theta^+ + \rho^- \theta^-}$$

2.3. *If $\rho^+ \theta^+ + \rho^- \theta^- - \frac{\theta^-}{\delta R} = 0$, then any weighted average of the allocations described in 2.1 and 2.2 is optimal.*

Intuitively, one can understand Proposition 5 as follows. If $\delta R = 1$, then, similar to the case when utility was not subject to shocks, the planner would be intrinsically indifferent regarding when to spend the “wealth” of the economy. In addition, Theorem 1 implies that in order to achieve the first-best in the closed economy, it must be that $\delta/(1 - \delta) \geq \rho^- \theta^- / \rho^+ \theta^+$. That is, the relative weight of future to current consumption should be high enough. In the open economy, by lending enough wealth in the early periods (that is, by setting a low enough value of f^+), the planner can attain a high growth rate of consumption due to interests, which ensures that the relative weight of future to current consumption is high enough.

If $\delta R < 1$, then, on the one hand, the planner has an incentive to spend wealth as quickly as possible, similarly to the benchmark case, because the interest rate is low compared to the rate at which consumers discount the future. On the other hand, in order to prescribe more consumption to the high-shock consumers compared to that prescribed to the low-shock consumers, the planner has an incentive to save wealth for the future. The condition $\rho^+ \theta^+ + \rho^- \theta^- - \frac{\theta^-}{\delta R} < 0$, which is equivalent to $\delta R(\rho^+ \theta^+ + \rho^- \theta^-) < \theta^-$ implies that the former incentive is stronger, while the condition $\rho^+ \theta^+ + \rho^- \theta^- - \frac{\theta^-}{\delta R} > 0$ implies that the latter incentive is stronger.

Proof. Using the promise-keeping condition for w , one can substitute $\rho^+ g^+ + \rho^- g^- = \frac{1}{\delta}[w - (1 - \delta)(\rho^+ \theta^+ f^+ + \rho^- \theta^- f^-)]$ into (22) and obtain

$$C_0 = \frac{\rho^+ f^+ + \rho^- f^-}{1 - \frac{1}{R\delta} + \frac{1-\delta}{R\delta} \frac{\rho^+ \theta^+ f^+ + \rho^- \theta^- f^-}{w}} \quad (24)$$

It follows from (24) that if one increases f^+ and decreases f^- , while keeping fixed the single-period aggregate consumption $\rho^+ f^+ + \rho^- f^-$, then the value of C_0 decreases. Thus, the planner will ideally choose $f^- = 0$, and achieve the first-best allocation, if she can choose the three other variables: f^+ , g^+ , g^- so that the incentives constraints are satisfied.

1. Assume that $\delta R = 1$. By plugging in $\delta R = 1$ and $f^- = 0$ into (24), one gets that C_0 , and hence, $C(w)$, is independent of f^+ :

$$C(w) = \frac{w}{(1 - \delta)\theta^+}. \quad (25)$$

Intuitively, it does not matter how exactly the planner distributes consumption among high-shock consumers across periods, similarly to the benchmark case, in which consumers were facing no shocks. Given $\delta R = 1$ and $f^- = 0$, the incentives constraints can be rewritten as

$$\frac{1 - \delta}{\delta} \theta^+ f^+ \geq g^- - g^+ \geq \frac{1 - \delta}{\delta} \theta^- f^+.$$

For a given value of f^+ , one can use the promise-keeping condition to express g^- as a function of f^+ and g^+ . This yields formula (23). This formula can be next used to rewrite the incentives constraints as

$$\frac{1 - \delta}{\delta} \theta^+ f^+ \geq \frac{w - \rho^+(1 - \delta)\theta^+ f^+ - \rho^+ \delta g^+}{\rho^- \delta} - g^+ \geq \frac{1 - \delta}{\delta} \theta^- f^+ \quad (26)$$

The second inequality can be satisfied only if it is satisfied by $g^+ = 0$. This yields an upper bound on f^+ : $f^+ \leq \bar{f}^+ = \frac{w}{(1 - \delta)(\rho^+ \theta^+ + \rho^- \theta^-)}$. Intuitively, as f^+ increases, the aggregate continuation payoff $\rho^+ g^+ + \rho^- g^-$

decreases. However, the planner must leave enough of the continuation payoff to ensure that the low-shock consumers have incentives to report truthfully. This creates an upper bound on f^+ .

Combining (26) with $g^+ \geq 0$ yields the interval for g^+ from the statement 1 of Proposition 5. Thus, the planner can guarantee that the incentives constraints are satisfied by $f^- = 0$, and any values of the three other variables: f^+ , g^+ , g^- that satisfy the conditions from statement 1 of the proposition. The value for $C(w)$ is given by (25)

2. Assume now that $\delta R < 1$. By (24), $\frac{\partial C_0}{\partial f^+} < 0$. Hence, the planner would like to increase f^+ by as much as possible. So, the planner increases f^+ until the incentive constraint for the low-shock consumers binds. If $g^+ > 0$, the planner can decrease g^+ and increase g^- , while keeping fixed the aggregate continuation payoff $\rho^+g^+ + \rho^-g^-$; this would improve the incentives of low-shock consumers, allowing for an increase of f^+ (and a decrease of C_0). Thus, in the optimal allocation, one has that $g^+ = 0$, and that the incentives constraint for the low-shock consumers binds. Substituting these two equalities into the promise-keeping condition yields $f^+ = \frac{w}{(1-\delta)(\rho^+\theta^+ + \rho^-\theta^-)}$.

Substituting this value of f^+ into (24) yields

$$C_0 = \frac{\rho^+w + \rho^-f^-(1-\delta)(\rho^+\theta^+ + \rho^-\theta^-)}{\left(1 - \frac{1}{\delta R}\right)(1-\delta)(\rho^+\theta^+ + \rho^-\theta^-) + \frac{1-\delta}{\delta R}\rho^+\theta^+ + \frac{(1-\delta)^2}{\delta R}(\rho^+\theta^+ + \rho^-\theta^-)\frac{\rho^-\theta^-f^-}{w}}$$

One can now check that C_0 increases with f^- if $\rho^+\theta^+ + \rho^-\theta^- - \frac{\theta^-}{\delta R} > 0$. Then, the planner sets $f^- = 0$, and the g^- such that the incentives constraint of the low-shock consumers is satisfied with equality.

Respectively, C_0 decreases with f^- if $\rho^+\theta^+ + \rho^-\theta^- - \frac{\theta^-}{\delta R} < 0$. Then, the planner sets $f^- = f^+$ (increasing f^- further would violate the incentives constraint of the high-shock consumers). With $f^+ = f^-$, the continuation payoff g^- must be equal to zero. That is, all consumption takes place in period $t = 0$.

Given the values of f^+ , f^- , g^+ and g^- at the optimum, $C(w)$ can be computed from (24). ■

We can finally state and prove the main result of this section.

Theorem 4 *Suppose that $\delta R = 1$. Then, there exists an optimal allocation such that the continuation payoffs of all consumers in all periods exceed some $Z > 0$.*

Proof. Proposition 5 guarantees the existence of recursive optimal allocations such that $g^+, g^- > 0$, and both incentives constraints are not binding. So, one can replicate the proof of Theorem 2. ■

In comparison to Theorem 4, Thomas and Worrall (1990) and Phelan (1998) show that, given a strictly concave utility function $u(c)$, such that $\lim_{c \rightarrow \infty} u'(c) = 0$, the optimal allocation requires immiseration, and the continuation payoff w becomes arbitrarily negative almost surely. Phelan (1998) also shows that with a finite value of $\lim_{c \rightarrow \infty} u'(c) = \alpha$, $0 < \alpha < \infty$, immiseration fails: The continuation payoff would become either arbitrarily negative or arbitrarily large, with a positive probability each. In either case, there is no lower bound on w . Hence, linear utility allows to somewhat restrict the level of inequality in the economy, by guaranteeing a positive lower bound on w .

7 Conclusion

Atkeson and Lucas (1992) discovered that in some endowment economies with private information and strictly concave utility of consumption, the optimal dynamic allocations of resources would “terminate” every citizen (dynasty) over time with certainty, and a vanishing fraction of citizens would become infinitely rich.

In contrast to Atkeson and Lucas, we consider the case of linear utility. We show that their immiseration result might not hold in closed economies, and that it definitely does not hold in small open economies which can exchange their resources with the outside world. This might seem surprising, because with strictly concave utility, and therefore, diminishing marginal utility, the planner should have weaker incentives to allocate the endowment to the high-shock consumers. So, a constant marginal utility should accelerate the process of terminating consumers, and of allocating the good to a vanishing fraction of consumers. This intuition turns out to be incorrect, however, due to another effect. In optimal allocations concavity precludes reallocating consumption from the high-shock consumers with low continuation payoffs to the high-shock consumers with high continuation payoffs. Such reallocations enable the social planner to attain some degree of fairness without losing efficiency in the case of linear utility. In addition, when the planner can lend to and borrow from the outside markets, she can distribute consumption among high-shock consumers across periods without affecting efficiency, which makes it easier to attain some degree of fairness.

Our result for linear utility functions suggests that there may exist some strictly concave utility functions, for which efficiency does not require immiseration, given that the degree of concavity is not too high. The question of whether such functions exist, remains open. We conjecture, though we have not yet managed to show formally, that efficiency does require immiseration for all strictly concave utility functions that satisfy the Inada condition $\lim_{c \rightarrow 0} u'(c) = \infty$.

For linear utility functions, we obtained analytical solution to the social planner’s problem, as well as characterized explicitly some of the market equilibria. The characterized market equilibria yield the same level of efficiency, as in the social planner’s solution, although the social planner may be capable to attain a more fair outcome than the market, without any loss in terms of efficiency.

We explored economies in which consumers’ taste shocks take only two values. The analysis of economies with a higher number of taste-shock values may deliver new insights. For example, in another paper, Olszewski and Safronov (2018), we studied a closely related setting. The analysis suggests that in the case with *more than two* taste-shock values, the social planner might attain a higher expected life-time payoff than the market, even when the discount factor approaches to one.⁸ The advantage of the social planner seems to come from the fact that market prices must be the same for everyone, while the social planner (or a government) can allow the consumers with a lower current wealth to consume more, and reduce the

⁸The fact that the social planner may attain a higher expected life-time payoff than the market for a fixed discount factor was already suggested by Atkeson and Lucas, for the case of strictly concave utilities.

consumption of the consumers with a higher current wealth, compared to what they would consume facing the same price ratio between the current and future consumption.

This last feature, if our intuition is indeed correct, is perhaps surprising in light of our current analysis, where the social planner attains more fair allocations by doing exactly the opposite, namely, making the consumers with a lower current wealth consume less than the consumers with a higher current wealth, even though their shadow price ratios were equal. Intuitively, there is indeed a conflict between the two objectives. Consumers with a lower current wealth are less willing to report higher-taste shocks, to wait for even higher shocks.⁹ So, a lower price for them can enhance efficiency, but can at the same time contribute to immiseration.

8 References

- Atkeson A. and R.E. Lucas, Jr (1992): “On Efficient Distribution with Private Information,” *Review of Economic Studies* 59, 427-453.
- Farhi, E. and I. Werning (2007): “Inequality and Social Discounting,” *Journal of Political Economy* 115, 365-402.
- Green, E. (1987): “Lending and the Smoothing of Uninsurable Income,” in E. Prescott and N. Wallace, Eds., *Contractual Arrangements for Intertemporal Trade*, Minneapolis: University of Minnesota Press.
- Kocherlakota, N. (1998): “Money is Memory,” *Journal of Economic Theory* 81, 232-251.
- Kocherlakota, N. (2010): “The New Dynamic Public Finance,” Princeton University Press.
- Lucas R.E. Jr. (1980): “Equilibrium in Pure Exchange Economy,” *Economic Inquiry* 18, 203-220.
- Olszewski W. and M. Safronov (2018): “Efficient cooperation by exchanging favors,” *Theoretical Economics* 15, 951–978.
- Phelan, C. (1998): “On the Long Run Implications of Repeated Moral Hazard,” *Journal of Economic Theory* 79, 174-191
- Phelan, C. (2006): “Opportunity and Social Mobility,” *Review of Economic Studies* 73, 487-504.
- Phelan, C. and R. M. Townsend (1991): “Computing Multi-Period, Information Constrained Optima,” *Review of Economic Studies* 58, 853-881.
- Taub, B. (1990): “The Equivalence of Lending Equilibria and Signalling-Based Insurance under Asymmetric Information,” *RAND Journal of Economics* 21, 388-408.
- Thomas, J. and T. Worrall, (1990): “Income Fluctuations and Asymmetric Information: An Example of a Repeated Principal-Agent Problem,” *Journal of Economic Theory* 51, 367-390.

⁹Of course, this intuition relies on more than two values of the taste shock, and high values of the discount factor.