

Skewness Correction for Asset Pricing

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Skewness Correction for Asset Pricing

It is shown that, for CRRA agents, the sensitivity of risk correction for any cumulant depends on the cumulant of the next order. This result is then used to derive some interesting approximations for variance and skewness correction. The first corollary is that negative skewness alone leads to higher variance swap rates since the variance swap contract provides insurance against sudden market drops. Thus, high variance swap rates are not necessarily an indication of high variance risk premia. When the results are extended to the multifactor case, we are able to disentangle the swap rate premia into their skewness and stochastic variance premia components. Finally, we contribute to the understanding of option skews by showing that only $1 - u$ percent of excess kurtosis contributes to negative skewness correction, where u is a newly introduced statistic that normalizes skewness with kurtosis.

Introduction

There is a long line of literature that establishes the importance of systematic return skewness, as well as co-skewness of individual returns with the market, in the formation of asset prices¹. The contribution of this paper is to first recover the exact *sensitivity* of risk correction to risk aversion, and second by using this relation to provide linear approximations for variance and skewness risk correction without specifying exact return dynamics. Even though results are approximate, this approach avoids to pre-specify a stochastic process that imposes a particular functional relation between volatility, skewness, and kurtosis. The results are developed in continuous-time within the framework of the Lévy processes, which are now given considerable attention since they nest the Brownian motion by incorporating jumps that arrive at some, potentially infinite, rate². Lévy models are used to model time

¹This research started with the early co-skewness models of Rubinstein (1973) and Kraus and Litzenberger (1976). Harvey and Siddique (2000) show that systematic skewness can explain an average 3.6 percent risk premium. Recently, Carr and Wu (2006) find that skewness is also highly variable over time.

²Examples of infinite jump activity models are the normal inverse Gaussian by Barndorff-Nielsen (1998), the generalized hyperbolic by Eberlein et al. (1998), the Variance Gamma by Madan and Milne (1991), and the generalization in Carr et al. (2002)

changes (Clark, 1973), capture higher moments, generate flexible volatility surfaces, and discuss market incompleteness³. Ait-Sahalia (2004) suggests that disentangling the pure jump from the diffusive component may be at the core of risk management, since the diffusive risks are hedgeable.

In the simplest case, the risk premium of a security, the difference between its expected return and the risk free rate, is driven by variance. Since the risk free rate is equal to the risk neutrally expected return, we may think of variance as the sensitivity of drift correction to changes in risk aversion. Our main theorem generalizes this idea by showing that when agents exhibit constant relative risk aversion, the magnitude of risk correction for the n^{th} cumulant depends on the $(n+1)^{th}$ cumulant.

Widespread interest for direct exposure to variance risk has led to the introduction of variance swaps that provide payoffs driven by the differential between realized variance and an ex-ante swap rate. Since variance swaps can be initiated at zero cost, the no arbitrage condition implies that the variance swap rate equals a *risk neutral* expectation of the realized variance for the underlying security. Variance swap rates (as well as option implied

³Carr, Jin and Madan (2001), Cvitanic, Polimenis and Zapatero (2005).

volatilities squared⁴) tend to be higher than historical variance rates, and it is almost universally suggested that the *entire* rate differential is due to the pricing of stochastic variance risk. The often cited explanation is that if there is no variance risk (or it is not priced), the variance rate under the historical and risk neutral measures should equal. In the first corollary of the main theorem it is shown that this reasoning is not always valid, since when skewness is negative, variance is upwards adjusted. The novel insight is that when SKEW is negative, a long position in the variance swap contract is more valuable as insurance against extreme negative movements in the underlying, and swap rates will be upwards adjusted according to

$$K_{var} \approx \sigma^2 - \gamma \text{SKEW} \sigma^3 \quad (1)$$

When we model the individual stocks as having a beta exposure to the market plus a Gaussian idiosyncratic part, it is shown that the rates at which individual variance swaps may be entered is

$$K_{var}^i \approx \sigma_i^2 - \frac{\gamma}{b_i} \text{SKEW}_i \sigma_i^3 \quad (2)$$

In a related paper, Demeterfi, Derman, Kamal and Zou (1999) approximate the effect of volatility skew, defined as the slope of the implied volatility

⁴Jackwerth and Rubinstein (1996).

curve, on variance swaps.

Nevertheless, variance is stochastically changing, and there is great interest in the academic and practitioner communities in pricing variance risk.⁵ For example, by analyzing the gains of delta hedged strategies, Bakshi and Kapadia (2003a,b) find evidence of negative market volatility premia. When the main theorem is later extended to a multifactor setting, we are also able to measure the effect of stochastic variance premia on the variance swap rate. Thus, we *disentangle* the variance correction into separate skewness, and stochastic variance premia components. More exactly, it is shown that it is not the leverage effect alone (i.e. a negative return-volatility correlation) that is responsible for higher swap rates. Rather, *a bias in the strength* of the leverage effect, that makes it more pronounced in negatively moving (falling) markets, is responsible for such high variance swap rates. The intuition is that, due to the biased leverage effect, large negative returns tend to increase volatility more than positive returns tend to decrease it. Thus large payoffs to the long position of a variance swap will tend to arrive at states where volatility has been upwards updated since inception, and thereby provide

⁵Carr and Wu (2004) propose a new method for the estimation of variance risk premia from options data.

insurance against such undesirable volatility increases.

It is known that the Black-Scholes-Merton implied volatility for deeply out of the money put options is higher than that for out of the money calls. Pan (2002) finds that jump and volatility premia are significant for explaining option "smirks". There is an almost unanimous agreement that these volatility smirks are signs of a *strongly negative risk neutral skewness*. Since empirical return skewness is not high enough, risk neutral skewness should then be the result of *skew correction*. This is highlighted in a related paper by Bakshi, Kapadia and Madan (2003) who derive theoretical links from risk aversion, and actual returns' higher moments to risk neutral skewness and option prices.

Even though, it is widely recognized that skewness and kurtosis (and the jumps that generate them) are significant in pricing non-linear payoffs such as the ones generated by options, the question of whether it is skewness or kurtosis the most important factor in determining option smiles is still open. The third corollary of the central theorem provides some new insight by defining a new statistic u that normalizes skewness for excess kurtosis. It is shown that leftward risk correction for market skewness, ΔSKEW , is

driven by the $(1 - u)$ percent of the excess kurtosis,

$$\Delta\text{SKEW} \approx -\gamma(1 - u)(\text{KURT} - 3)\sigma \quad (3)$$

Thus, a fat-tailed return distribution generates an increasingly negative skewness, only to the extent that *kurtosis-normalized* skewness u is less than 100%. This result generalizes Theorem 2 in Bakshi, Kapadia and Madan (2003), which argues that the source of the negative risk neutral skewness is total excess kurtosis, and is valid in the special case of symmetric distributions for which the kurtosis-normalized skew u is zero. We show that for skewed processes, the u fraction can be quite high, and skewness correction small, or even zero. A counter-intuitive consequence is that, for a given kurtosis, skewness is more heavily corrected for more symmetric processes.

In section 1, the central result for Lévy cumulants is developed, and we derive the approximate variance swap rate for the index and individual stocks. In section 2, the relation of kurtosis to skewness correction is developed. In section 3, the results are extended for many risk factors, and the relation of stochastic volatility to variance swap rates is developed. In section 4, the simpler case of stochastic volatility as an independent time change is discussed.

1 Correcting market cumulants

I assume the market index returns X_t are generated by a Lévy process,

$$X_t = \eta w_t + \int_0^t \int_{-\infty}^{\infty} x N(ds, dx) \quad (4)$$

where w_t is a diffusion, and $N(dt, dx)$ is the jump counter with Lévy measure $\pi(dx)$. I further assume that $1 \wedge |x|$ is π -integrable.⁶ In the moment generating function, $\mathcal{M}(s)$, of a Lévy process, time is factored out,

$$\mathcal{M}(s) = E e^{sX_t} = e^{t\mathcal{K}(s)} \quad (5)$$

where $\mathcal{K}(s)$ is the cumulant generator of the Lévy process.

For agents who exhibit a constant relative risk aversion γ , the risk neutral index process is an exponentially tilted version⁷ of the original process X_t

$$\left(\frac{dQ}{dP} \right)_t = e^{-\gamma X_t - t\mathcal{K}(-\gamma)} \quad (6)$$

Given (6), the risk neutral⁸ cumulant function of X_t is a first difference of the actual $\mathcal{K}(s)$,

$$\mathcal{K}^*(s) = \mathcal{K}(s - \gamma) - \mathcal{K}(-\gamma) \quad (7)$$

⁶This is stronger than the general condition $\int (1 \wedge x^2) \pi(dx) < \infty$.

⁷See, for example, Carr and Wu (2004).

⁸A star superscript denotes a risk neutral quantity.

The cumulants of a Lévy process are horizon-scaled derivatives of its cumulant function at zero. From (7), risk-neutral cumulants are recovered by differentiating at $s = -\gamma$. The n^{th} order risk neutral cumulant is thus a *function of risk aversion*,

$$c_n(\gamma) = \frac{\partial^n \mathcal{K}^*(0)}{\partial s^n} = \frac{\partial^n \mathcal{K}(-\gamma)}{\partial s^n} \quad (8)$$

When risk neutral cumulants are explicitly written as functions of γ , we may think of actual cumulants as risk corrected cumulants *for risk neutral agents*,

$$c_n = c_n(0) \quad (9)$$

The central goal of the paper is to provide linear approximations to variance and skewness *risk correction* for Lévy processes. By risk correction for a quantity f we mean the difference between the risk neutral and actual quantities, $\Delta f = f^* - f$. Since economic theory anticipates risk correction due to risk aversion,

$$f^* = f(\gamma) \quad (10)$$

the natural approximation to the risk-adjusted quantity is the linear approximation with respect to the risk aversion parameter (or price of risk) γ ,

$$\Delta f \approx \left(\frac{\partial f}{\partial \gamma} \right)_{\gamma=0} \times \gamma$$

The linear approximation will be exact for linear (CAPM-style) risk corrections, but in the general case it will be of the type

$$f(\gamma) = f + \frac{\partial f(0)}{\partial \gamma} \gamma + o(\gamma)$$

where the little o notation shows that only sub-linear terms are discarded⁹.

In Merton's benchmark case, the market risk premium $\mu - r$ equals $\gamma\sigma^2$, or equivalently, since the risk neutral drift μ^* equals r

$$\mu^* - \mu = -\gamma\sigma^2 \tag{11}$$

which implies that the sensitivity of drift equals

$$\frac{\partial \mu^*}{\partial \gamma} = -\sigma^2 \tag{12}$$

That is, the drift correction, $\mu^* - \mu$, is driven by the variance (i.e. the next order cumulant). The above discussion is generalized for cumulants of higher order in the central result for this paper:

Theorem 1. *The risk aversion sensitivity of the n^{th} risk corrected market cumulant equals the negative $(n+1)^{\text{th}}$ cumulant,*

$$\left(\frac{\partial c_n(\gamma)}{\partial \gamma} \right)_{\gamma=0} = -c_{n+1} \tag{13}$$

⁹ $\lim_{\gamma \rightarrow 0} \frac{o(\gamma)}{\gamma} = 0.$

Proof: From (8),

$$\frac{\partial c_n(\gamma)}{\partial \gamma} = -\frac{\partial^{n+1} \mathcal{K}(-\gamma)}{\partial s^{n+1}} = -c_{n+1}(\gamma) \quad (14)$$

Since $c_{n+1}(0) = c_{n+1}$, we have

$$\frac{\partial c_n(0)}{\partial \gamma} = -c_{n+1} \quad (15)$$

1.1 The approximate variance swap rate

Variance correction is less straightforward than drift correction because the next cumulant can be positive or negative depending on the sign of skewness. That is, while negative market skewness will increase risk neutral variance, a positive skewness *will lower* variance. This observation has implications for the formation of *variance swap rates*.

One way to take a position in volatility is to have a delta-neutral position on the market. A more direct facility for volatility trading, available to large investors, is a variance swap that pays the difference between a realized estimate of return variability and a fixed variance rate determined at time zero. Since variance swaps have zero initial cost, the rate at which variance swaps can be entered equals the risk neutrally expected value of the future realized quadratic variation.

Lemma 1. *The linear approximation of the variance swap rate for a constant volatility market is*

$$K_{var} \approx \sigma^2 - \gamma \text{SKEW} \sigma^3 \quad (16)$$

Proof: From Theorem 1, $\Delta c_2 \approx -\gamma c_3$, and also $\text{SKEW} = c_3/\sigma^3$ \square

Informally, it is almost universally argued that variance swap rates are higher than stock variance rates to reflect the stochastic nature of volatility (or variance), that is to capture negative volatility premia¹⁰. The novel insight here is that the main force behind variance correction that leads to higher swap rates, is not negative volatility premia but negative skewness, since negative SKEW alone generates higher swap rates even for a constant volatility¹¹. For example, for $\gamma = 3$, $\text{SKEW} = -1.5$ and $\sigma = 20\%$ the SKEW correction 3.6% which is added to the actual variance of 4%. For $\gamma = 3$, $\text{SKEW} = -1.5$ and $\sigma = 30\%$ the skewness-related correction is 12.15%, actually bigger than σ^2 which is 9%. The intuition is that when $\text{SKEW} < 0$, a long position on a variance swap is more valuable because the positive payoffs tend to arrive due to extremely negative returns, and thus provide insurance against extreme negative movements in the underlying.

¹⁰The volatility premia connection to high swap rates is developed at the last section.

¹¹I am grateful to a referee for pointing this out.

1.2 Individual stock swap rates.

Individual stocks are assumed to have a beta exposure to systematic risk,

$$X_t^i = a_i t + b_i X_t + \varepsilon_i w_t^i \quad (17)$$

with X_t the market risk and w_t^i an idiosyncratic diffusion orthogonal to the market. In this case, $R_i^2 = \frac{b_i^2 \sigma^2}{\sigma_i^2}$ of the total quadratic variation rate

$$\sigma_i^2 = b_i^2 \eta^2 + b_i^2 \int x^2 \pi(dx) + \varepsilon_i^2 = b_i^2 \sigma^2 + \varepsilon_i^2 \quad (18)$$

of the i^{th} stock is systematic.

The first key observation is that the diffusive idiosyncratic risk *does not enter in higher cumulants*

$$c_n^i = b_i^n c_n \quad \text{for } n \geq 3 \quad (19)$$

As the next corollary shows, the individual stock in (17) conforms with recent empirical findings (e.g. Bakshi, Kapadia and Madan (2003)) that most individual stocks seem to be less left-skewed than the market:

Corollary 1. *(proof in appendix) For individual stocks in (17)*

$$\text{SKEW}_i = \text{SKEW} \times R_i^3 \quad (20)$$

The fact that the entire market is more left skewed than its individual components seems counter-intuitive at first, but should not surprise since portfolio

skewness is *not* a weighted sum of individual skews.

The second key observation is that since the idiosyncratic risk *is not priced*, its cumulants are independent of γ and thus any cumulant correction is the result of correction on the market risk:

Corollary 2. *The n^{th} risk corrected cumulant for an individual stock equals,*

$$\frac{\partial c_n^i(0)}{\partial \gamma} = -b_i^n c_{n+1} \quad (21)$$

Thus, the risk neutral quadratic variation of the i^{th} stock grows at a rate

$$\sigma_i^2(\gamma) = \sigma_i^2 - \gamma b_i^2 c_3 + o(\gamma) \quad (22)$$

which implies that

Lemma 2. *The swap rate for the i^{th} individual stock follows*

$$K_{var}^i \approx \sigma_i^2 - \frac{\gamma}{b_i} \text{SKEW}_i \sigma_i^3 \quad (23)$$

Demeterfi, Derman, Kamal and Zou (1999) approximate the effect of volatility skew (the slope of the implied volatility curve) on variance swaps. Instead, Lemmas 1 and 2 measure the direct effect of actual market skewness on the swap rates for both the index and individual stocks.

The above results provide an indirect method of estimating market prices (i.e. γ 's) without assuming an exact return generating process. After having estimated individual stock betas we may recover γ as the slope of a cross-sectional regression of variance swap rate differentials $K_{var}^i - \sigma_i^2$ on $\frac{\text{SKEW}_i \sigma_i^3}{b_i}$.

2 Skewness correction

The contemporary empirical option pricing literature agrees that the so called volatility smiles are signs of a *strongly negative risk neutral skewness*. Since empirical return skewness is not high enough, risk neutral skewness should then be the result of *risk correction*. It is informally believed that excess kurtosis is related to the risk neutral skewness implicit in option smiles.

Furthermore, it is already recognized that fat tails are indeed responsible for option smiles.¹² Here we take this analysis one step further, by showing that only a fraction $1 - u$ of excess kurtosis generates skewness correction, where we define the *horizon-independent kurtosis-normalized skewness* u statistic as follows

$$u = \frac{3}{2} \frac{c_3^2}{c_2 c_4} = \frac{3}{2} \frac{\text{SKEW}^2}{\text{KURT} - 3} \quad (24)$$

¹²Theorem 2 (page 109) in Bakshi, Kapadia and Madan (2003).

Lemma 3. (proof in the appendix) *The index skewness is corrected to the left only by the $1-u$ percent of excess kurtosis*

$$\Delta\text{SKEW} = -\gamma(1-u)(\text{KURT} - 3)\sigma + o(\gamma) \quad (25)$$

2.1 Why only a fraction of excess kurtosis?

Lemma 3 shows that fat tails are indeed responsible for skew smiles. Since the full variance is responsible for drift correction, and the full skewness (Lemma 1) for variance correction, it is tempting to ask why when it comes to skewness correction only $1-u$ percent of excess kurtosis participates. The source of the confusion is that while for $n = 1, 2$ the central moment, m_n , follows

$$m_n(\gamma) \approx m_n - \gamma m_{n+1} \quad (26)$$

for $n > 2$ the recursive equation is *only* valid for cumulants, c_n .

Actually, from Theorem 1, the third moment is corrected as follows,

$$\frac{\partial m_3(0)}{\partial \gamma} = \frac{\partial c_3(0)}{\partial \gamma} = -c_4 = -m_4 + 3\sigma^4 \quad (27)$$

So the correct relation becomes

$$\Delta m_3 = -\gamma m_4 \underbrace{+ 3\sigma^4 \gamma}_{\text{not in Eq. (26)}} + o(\gamma) \quad (28)$$

In the special case of symmetric distributions, the u statistic vanishes, and the correction implied by the entire excess kurtosis still applies. Thus, keeping kurtosis constant, the more symmetric the market returns are, the smaller u implies a more aggressive skewness correction.

Corollary 3. *When market returns are symmetric, the entire excess kurtosis generates risk-neutral skewness*

$$\text{SKEW}(\gamma) \approx -\gamma(\text{KURT} - 3)\sigma \quad (29)$$

2.2 Individual stock skews

We have shown in (20) that individual stocks in (17) will have less pronounced skews. From

$$u_i = \frac{3 (c_3^i)^2}{2 c_4^i \sigma_i^2}$$

and $c_n^i = b_i^n c_n$ for $n \geq 3$, we have that for the i^{th} stock

$$u_i = u \times R_i^2 \quad (30)$$

which implies that a larger percentage of the kurtosis is responsible for skew correction than in the index case, $1 - u_i > 1 - u$:

Lemma 4. *Individual skew corrections are given by*

$$\Delta\text{SKEW}_i = -\gamma(1 - u_i)(\text{KURT}_i - 3) \frac{\sigma_i}{b_i} + o(\gamma) \quad (31)$$

Add figure 1 around here

The fact that a larger *fraction of kurtosis* will generate skew correction does not imply that individual stocks have more overall skew, because they start with a smaller actual skew (20), and, since idiosyncratic risk is assumed not to have fat-tails

$$\text{KURT}_i - 3 = \frac{c_4^i}{\sigma_i^4} = \frac{b_i^4 c_4}{b_i^4 \sigma^4 / R_i^4} = (\text{KURT} - 3) \times R_i^4 \quad (32)$$

they also have smaller kurtosis to start with (see Fig.1). Re-write (31) as

$$\Delta\text{SKEW}_i \approx -\gamma(\text{KURT} - 3) \sigma (1 - uR_i^2) R_i^3 \quad (33)$$

and compare to (25), to see that individual stock skews will be corrected more aggressively when $(1 - uR_i^2) R_i^3 > 1 - u$. For small values of the index u , a higher systematic risk R_i^2 implies a steeper skew correction. But for

$u > 60\%$, this is not true anymore; as R_i^2 grows beyond $\frac{60\%}{u}$ the individual stock skewness will receive less correction; i.e. the sensitivity of ΔSKEW_i with respect to R_i^2 becomes positive.

Add figures 2,3,4,5 around here

2.3 The u statistic can be large

Since Lemma 3 is counter-intuitive in asserting that symmetry in the actual returns imparts more asymmetry in the risk corrected ones, a natural question is whether the statistic u , which regulates the intensity of this phenomenon, will attain large enough values for the phenomenon to become significant. As is shown here for a simple pure jump process, the broadly used gamma, $u = 100\%$. For $l, v > 0$, the Lévy measure of the pure jump gamma process, $\gamma_t(l, v)$

$$\pi(dx) = \frac{v}{x} e^{-x/l} dx \quad \text{for } x > 0 \quad (34)$$

generates an infinite arrival rate of small jumps, in the sense that the arrival rate of jumps away from zero for any $\epsilon > 0$ is finite, $\pi(\epsilon, \infty) < \infty$. It is well

known that the n^{th} cumulant of the gamma equals

$$c_n = (n - 1)!l^n v \quad (35)$$

and it is thus clear that $u = 100\%$. In other words, the significant heavy tails of gamma do generate any skewness correction, $\Delta\text{SKEW} = 0$.

2.4 Two-sided jumps generate more skew correction

The simple gamma of the previous section is not a good candidate since it won't generate jumps of both signs. We may easily correct this by combining two gammas that generate jumps of opposite signs

$$X_t = \gamma_t^+(l_+, v_+) - \gamma_t^-(l_-, v_-) \quad , \quad \text{where } l_{\pm}, v_{\pm} > 0 \quad (36)$$

When the $v_+ = v_-$, this process is called a Variance Gamma (e.g. Madan, Carr and Chang, 1998). The next lemma is proved in the appendix,

Lemma 5. *For a Variance Gamma process there will always be some skew correction ($u < 100\%$).*

Lemma 5 is counter-intuitive, but motivates an important general observation that provides intuition about the skewness correction mechanism: when two-sided jump processes are involved we *always* get leftward skew correction. To understand this general observation we have to consider what

happens to the one-sided jump measures $\pi_{\pm}(dx)$ when we correct for risk a *two-sided* process that combines positive and negative jumps,

$$X_t = X_t^+ - X_t^- = \int_0^t \int_0^{+\infty} x N^+(ds, dx) - \int_0^t \int_0^{+\infty} x N^-(ds, dx) \quad (37)$$

The two-sided cumulant function equals

$$\mathcal{K}(s) = \int_{-\infty}^{+\infty} (e^{sx} - 1) \pi(dx) \quad (38)$$

with

$$\pi(dx) = \begin{cases} \pi_+(dx) & \text{for } x > 0 \\ \pi_-(dx) & \text{for } x < 0 \end{cases} \quad (39)$$

From (7) we have that

$$\mathcal{K}^*(s) = \int_{-\infty}^{+\infty} (e^{(s-\gamma)x} - 1) \pi(dx) - \int_{-\infty}^{+\infty} (e^{-\gamma x} - 1) \pi(dx) \quad (40)$$

$$= \int_{-\infty}^{+\infty} (e^{sx} - 1) e^{-\gamma x} \pi(dx) \quad (41)$$

which implies that

Lemma 6. *The corrected Lévy measure equals*

$$\pi^*(dx) = e^{-\gamma x} \pi(dx) \quad (42)$$

Lemma 6 implies that while the positive jumps are arriving at slower rates under the risk neutral measure,

$$\text{for } x > 0 : \pi_+^*(dx) = \pi^*(dx) = e^{-\gamma x} \pi(dx) < \pi(dx) = \pi_+(dx) \quad (43)$$

negative jumps are accelerated,

$$\text{for } x < 0 : \pi_-^*(dx) = \pi^*(dx) = e^{-\gamma x} \pi(dx) > \pi(dx) = \pi_-(dx) \quad (44)$$

This asymmetry on the treatment of opposite signed jumps, i.e. the acceleration of negative jump arrivals combined with the deceleration of the positive jumps, generates left skew correction.

3 The multifactor case

Since in practical applications, index returns may be exposed to multiple risks, it is useful to extend Theorem 1 to the two-factor case. The extension to any number of factors is straightforward. The index returns are now assumed to be exposed to two risks

$$X_t = \eta_x w_t^x + \int_0^t \iint x N(ds, (dx, dy)) \quad (45)$$

where the double integral extends over the entire jump support region, and

$$Y_t = \eta_y w_t^y + \int_0^t \iint y N(ds, (dx, dy)) \quad (46)$$

where w_t^x and w_t^y have correlation ρ_{xy} . Further correlation between the risks may arise from the jump parts and no restriction is placed here. Furthermore, it is assumed that risks are potentially priced differently, γ_x being the aversion

to exposure to risk of type X , while γ_y being the aversion to risk of type Y .

In this case, the change of measure is

$$\left(\frac{dQ}{dP}\right)_t = \exp(-\gamma_x X_t - \gamma_y Y_t - t\mathcal{K}(-\gamma_x, -\gamma_y)) \quad (47)$$

where for simplicity we assume that X and Y are of finite variation

$$\mathcal{K}(s, q) = \frac{1}{2}\eta_x^2 s^2 + \frac{1}{2}\eta_y^2 q^2 + \rho_{xy}\eta_x\eta_y sq + \iint (e^{sx+qy} - 1) \pi(dx, dy) \quad (48)$$

Following the same reasoning as in Theorem 1, we anticipate that the risk corrected cumulant of i^{th} and j^{th} order with respect to X and Y respectively, is a function of the risk prices

$$c^{i,j}(\gamma_x, \gamma_y) = \frac{\partial^{i+j}\mathcal{K}(-\gamma_x, -\gamma_y)}{\partial s^i \partial q^j} \quad (49)$$

and, that the sensitivity of the cumulant with respect to risk aversion depends on the cumulants of higher order as follows

$$\frac{\partial c^{i,j}(0,0)}{\partial \gamma_x} = -c^{i+1,j} \text{ and } \frac{\partial c^{i,j}(0,0)}{\partial \gamma_y} = -c^{i,j+1} \quad (50)$$

Expanding and keeping only terms linear in the gammas, we get

$$c^{i,j}(\gamma_x, \gamma_y) \approx c^{i,j} - \gamma_x c^{i+1,j} - \gamma_y c^{i,j+1} \quad (51)$$

3.1 Volatility premia and swap rates

Besides risk premia, there is now a growing interest in the practitioner and academic communities for the incorporation of *variance risk premia*, due to the stochastic nature of variance itself. When the first factor X_t captures index returns, and Y_t is a stochastic volatility factor, (51) has a direct implication for the formation of the variance swap rates, that is the risk neutral rate of return variation, $K_{var} = c^{2,0}(\gamma_x, \gamma_y)$

$$K_{var} \approx \sigma_x^2 - \gamma_x \text{SKEW}_x \sigma_x^3 - \gamma_y \text{co-SKEW}_{xy} \sigma_x^2 \sigma_y \quad (52)$$

where, all the terms are as in Lemma 1, except for the new term

$$\text{co-SKEW}_{xy} = \frac{c^{2,1}}{\sigma_x^2 \sigma_y} \quad (53)$$

It is a well known empirical fact that returns and their volatilities are negatively correlated, the so-called leverage effect. The co-SKEW term captures a new effect and should not be confused with that negative correlation since it captures volatility updates related to stock price jumps only. More specifically, since

$$c^{2,1} = \frac{\partial^3 \mathcal{K}(0, 0)}{\partial s^2 \partial q} \quad (54)$$

the continuous path dynamics of X_t and Y_t cannot survive in $c^{2,1}$. Thus, the $c^{2,1}$ cumulant is only driven by the pure jump components in X_t and Y_t ,

$$c^{2,1} = \iint x^2 y \pi(dx, dy) \quad (55)$$

and captures the tendency of relatively large returns (of both signs) to occur together with *positive volatility updates*.

If *large negative* returns tend to *upwards* update volatility with the same strength that *large positive* returns tend to *lower* it, the $c^{2,1}$ cumulant is zero, and the co-SKEW effect in swap rates will disappear. On the other hand, if there is a bias, in the sense that the strength of the volatility update effect is more pronounced when the market goes down, which implies a tendency for large returns to coincide with positive volatility updates, co-SKEW_{xxy} will be positive. In the biased case and if, as is empirically suggested (e.g. Bakshi and Kapadia, 2003), volatility premia are negative (here this is captured by $\gamma_y < 0$), the co-SKEW related factor will tend to further increase variance swap rates on top of the negative return SKEW effect captured by Lemma 1. The underlying intuition¹³ is that the higher payoff to the variance swap contract will tend to arrive with positive volatility updates and thus provide insurance against undesirable high volatility states (since $\gamma_y < 0$).

¹³A similar explanation is discussed in Carr and Wu (2005).

4 Stochastic volatility as a simple time change

It is known that volatilities that vary stochastically over time can be treated as a stochastic time change (e.g. Carr and Wu, 2003). As usual, assume that there is an instantaneous nonnegative *business activity rate* v_t , and index returns are generated by a process Y obtained by evaluating X in (4) at random stopping times $Y_t = X_{\tau_t}$ where

$$\tau_t = \int_0^t v_{s-} ds \quad (56)$$

Generally, changes in the business activity rate are correlated to returns and one needs to use the methodology developed previously. The simplest case, where the random time τ_t is *independent* of X_t , is considerably simpler and we treat this case here. Under independence, the instantaneous quadratic variation rate follows

$$\sigma_t^2 = (\eta^2 + \int x^2 \pi(x) dx) v_t = \sigma^2 v_t \quad (57)$$

Analogously to the *time invariant* factor c_n in the conditional cumulant

$$c_{n,t} = c_n \times v_t \quad (58)$$

we may identify the time invariant components for skewness and excess kurtosis, SKEW and KURT-3, through the following relations

$$\text{SKEW}_t = \frac{c_3 v_t}{(c_2 v_t)^{3/2}} = \text{SKEW} \times v_t^{-1/2} \quad (59)$$

$$\text{KURT}_t - 3 = \frac{c_4 v_t}{(c_2 v_t)^2} = (\text{KURT} - 3) \times v_t^{-1} \quad (60)$$

For CRRA agents, Theorem 1 still applies for conditional cumulants

$$\frac{\partial c_{n,t}(0)}{\partial \gamma} = -c_{n+1,t} \quad (61)$$

and their *time-invariant* factors, $\frac{\partial c_n(0)}{\partial \gamma} = -c_{n+1}$. Appropriately modifying Lemma 1, for the case of an independent business rate, and since all cumulants are v -scaled versions of their time invariant components, we get

$$\sigma_t^2(\gamma) \approx (\sigma^2 - \gamma \text{SKEW} \sigma^3) v_t \quad (62)$$

while the approximate stochastic swap rate for a period between time t and T equals the expected risk neutral quadratic variation over this period

$$(\sigma^2 - \gamma \text{SKEW} \sigma^3) \int_t^T v_s ds \quad (63)$$

Furthermore, since the u statistic is *non-stochastic*

$$u_t = u = \frac{3}{2} \frac{c_3^2}{c_4 c_2} \quad (64)$$

and Lemma 3 still applies for conditional skew, using (59) and (60) we get

Corollary 4. *Index skew is less prevalent at high volatility states*

$$\text{SKEW}_t(\gamma) \approx (\text{SKEW} - \gamma(\text{KURT} - 3)(1 - u)\sigma) / \sqrt{v_t} \quad (65)$$

5 Conclusion

The first result of the paper is that, when agents exhibit a constant degree of relative risk aversion, risk correction for any cumulant depends on the next order cumulant. This result is then used to develop some general results for skewness correction. Firstly, it is shown that negative return skewness implies an increase in the variance swap rates. It is then shown that only $1 - u$ percent of the excess kurtosis generates skewness correction, where u is a new kurtosis-normalized skew measure. Finally, the results are extended for stochastic volatility and multi-factor risks. In the last application, we are able to disentangle the variance swap rate differential into a skewness component, and a second component due to negative volatility premia.

A Appendix

Proof of (20). The diffusive idiosyncratic risk does not enter in higher cumulants

$$\text{SKEW}_i = \frac{c_3^i}{\sigma_i^3} = \frac{b_i^3 c_3}{b_i^3 \sigma^3 / R_i^3} = \text{SKEW} R_i^3$$

Proof of Lemma 3. Using (13), the derivative of the corrected skewness with respect to risk aversion at zero equals

$$\begin{aligned} \left(\frac{\partial \text{SKEW}(\gamma)}{\partial \gamma} \right)_{\gamma=0} &= \frac{\partial}{\partial \gamma} \left(\frac{c_3(\gamma)}{c_2^{3/2}(\gamma)} \right)_{\gamma=0} \\ &= \left(\frac{\partial c_3(\gamma)}{\partial \gamma} c_2^{-3/2}(\gamma) - \frac{3}{2} c_3(\gamma) c_2^{-5/2}(\gamma) \frac{\partial c_2(\gamma)}{\partial \gamma} \right)_{\gamma=0} \\ &= - \left(c_4 \sigma^{-3} - \frac{3}{2} c_3^2 \sigma^{-5} \right) = - \frac{c_4}{\sigma^4} \left(1 - \frac{3}{2} \frac{c_3^2}{c_4 c_2} \right) \sigma \end{aligned}$$

Finally expanding $\text{SKEW}(\gamma)$ around zero and using the above value for $\frac{\partial \text{SKEW}}{\partial \gamma} \Big|_{\gamma=0}$ we recover (25).

Proof of Lemma 4. The proof follows the same steps as above, with the added observation that for individual stocks $\frac{\partial c_2^i(0)}{\partial \gamma} = -b_i^n c_{n+1}$

$$\begin{aligned} \left(\frac{\partial \text{SKEW}_i(\gamma)}{\partial \gamma} \right)_{\gamma=0} &= \frac{\partial}{\partial \gamma} \left(\frac{c_3^i(\gamma)}{(c_2^i(\gamma))^{3/2}} \right)_{\gamma=0} \\ &= \left(\frac{\partial c_3^i(\gamma)}{\partial \gamma} (c_2^i(\gamma))^{-3/2} - \frac{3}{2} c_3^i(\gamma) (c_2^i(\gamma))^{-5/2} \frac{\partial c_2^i(\gamma)}{\partial \gamma} \right)_{\gamma=0} \\ &= - \left(b_i^3 c_4 \sigma_i^{-3} - \frac{3}{2} c_3^i \sigma_i^{-5} b_i^2 c_3 \right) = - \frac{c_4^i}{\sigma_i^4} \left(1 - \frac{3}{2} \frac{(c_3^i)^2}{c_4^i c_2^i} \right) \frac{\sigma_i}{b_i} \end{aligned}$$

Proof of Lemma 5. From (35) we can recover the cumulants of the VG

process

$$\begin{aligned}c_2 &= (l_+^2 + l_-^2)v \\c_3 &= 2(l_+^3 - l_-^3)v \\c_4 &= 6(l_+^4 + l_-^4)v\end{aligned}\tag{66}$$

Even though for the one-sided u statistics

$$u_{\pm} = \frac{3}{2} \frac{c_3^{\pm 2}}{c_4^{\pm} c_2^{\pm}} = 100\%\tag{67}$$

the two-sided u , which depends on total cumulants, is smaller

$$u = \frac{3}{2} \frac{(c_3^+ - c_3^-)^2}{(c_4^+ + c_4^-)(c_2^+ + c_2^-)} = \frac{(l_+^3 - l_-^3)^2}{(l_+^4 + l_-^4)(l_+^2 + l_-^2)} < 100\%\tag{68}$$

as simple algebra shows. \square

References

- [1] Yacine Ait-Sahalia. Disentangling Volatility From Jumps. *Journal of Financial Economics*, 74:487–528, 2004.
- [2] Gurdip Bakshi and Nikunj Kapadia. Delta-Hedged Gains and the Negative Market Volatility Risk Premium. *Review of Financial Studies*, 16:527–566, 2003.
- [3] Gurdip Bakshi and Nikunj Kapadia. Volatility Risk Premiums Embedded in Individual Equity Options: Some New Insights. *The Journal of Derivatives*, 11(1):45–54, 2003.
- [4] Gurdip Bakshi, Nikunj Kapadia, and Dilip Madan. Stock Return Characteristics, Skew Laws, and the Differential Pricing of Individual Equity Options. *Review of Financial Studies*, 16:101–143, 2003.
- [5] Ole E. Barndorff-Nielsen. Processes of Normal Inverse Gaussian Type. *Finance and Stochastics*, 2:41–68, 1998.
- [6] Peter Carr, Hélyette Geman, Dilip Madan, and Marc Yor. The Fine Structure of Asset Returns: An Empirical Investigation. *Journal of Business*, 75:305–332, 2002.

- [7] Peter Carr, Hélyette Geman, Dilip Madan, and Marc Yor. Stochastic Volatility for Lévy Processes. *Mathematical Finance*, 13:345–382, 2003.
- [8] Peter Carr, Hélyette Geman, Dilip Madan, and Marc Yor. Pricing Options on Realized Variance. *Finance and Stochastics*, 9, 2005.
- [9] Peter Carr, Xing Jin, and Dilip B. Madan. Optimal Investment in Derivative Securities. *Finance and Stochastics*, 5:33–59, 2001.
- [10] Peter Carr and Liuren Wu. The Finite Moment Log Stable Process and Option Pricing. *Journal of Finance*, 58:753–777, 2003.
- [11] Peter Carr and Liuren Wu. What Type of Process Underlies Options? A Simple Robust Test. *Journal of Finance*, 58:2581–2610, 2003.
- [12] Peter Carr and Liuren Wu. Time-changed Levy Processes and Option Pricing. *Journal of Financial Economics*, 71:113–141, 2004.
- [13] Peter Carr and Liuren Wu. Variance Risk Premia. *working paper*, pages –, 2004.
- [14] Peter Carr and Liuren Wu. Stochastic Skew for Currency Options. *Journal of Financial Economics*, forthcoming:–, 2006.

- [15] Peter K. Clark. A Subordinated Stochastic Process Model with Finite Variance for Speculative Prices. *Econometrica*, 41:135–155, 1973.
- [16] Ernst Eberlein, Ulrich Keller, and Karsten Prause. New Insights into Smile, Mispricing, and Value at Risk: The Hyperbolic Model. *Journal of Business*, 71:371–405, 1998.
- [17] Hélyette Geman, Dilip Madan, and Marc Yor. Time Changes for Lévy Processes. *Mathematical Finance*, 11:79–96, 2001.
- [18] Hélyette Geman, Dilip Madan, and Marc Yor. Stochastic Volatility, Jumps and Hidden Time Changes. *Finance and Stochastics*, 6:63–90, 2002.
- [19] C. Harvey and A. Siddique. Conditional skewness in asset pricing tests. *Journal of Finance*, 55:1263–1295, 2000.
- [20] Steve Heston. Invisible parameters in option prices. *Journal of Finance*, 48:933–947, 1993.
- [21] Jing-Zhi Huang and Liuren Wu. Specification Analysis of Option Pricing Models Based on Time Changed Levy Processes. *Journal of Finance*, 59:forthcoming, 2004.

- [22] J. Jackwerth and M. Rubinstein. Recovering Probability Distributions from Option Prices. *Journal of Finance*, 51:1611–1631, 1996.
- [23] K. Demeterfi, E. Derman, M. Kamal, and J. Zou. A Guide to Volatility and Variance Swaps. *The Journal of Derivatives*, 6:9–32, 1999.
- [24] Alan Kraus and Robert Litzenberger. Skewness preference and the valuation of risk assets. *Journal of Finance*, 31:1085–1100, 1976.
- [25] Dilip Madan, Peter Carr, and E. Chang. The variance gamma process and option pricing. *European Finance Review*, 2:79–105, 1998.
- [26] Dilip Madan and Frank Milne. Option Pricing with VG Martingale Components. *Mathematical Finance*, 1:39–55, 1991.
- [27] Jun Pan. The jump-risk premia implicit in options: Evidence from an integrated time-series study. *Journal of Financial Economics*, 63:3–50, 2002.
- [28] Mark Rubinstein. The fundamental theorem of parameter-preference security valuation. *Journal of Financial and Quantitative Analysis*, 8:61–69, 1973.

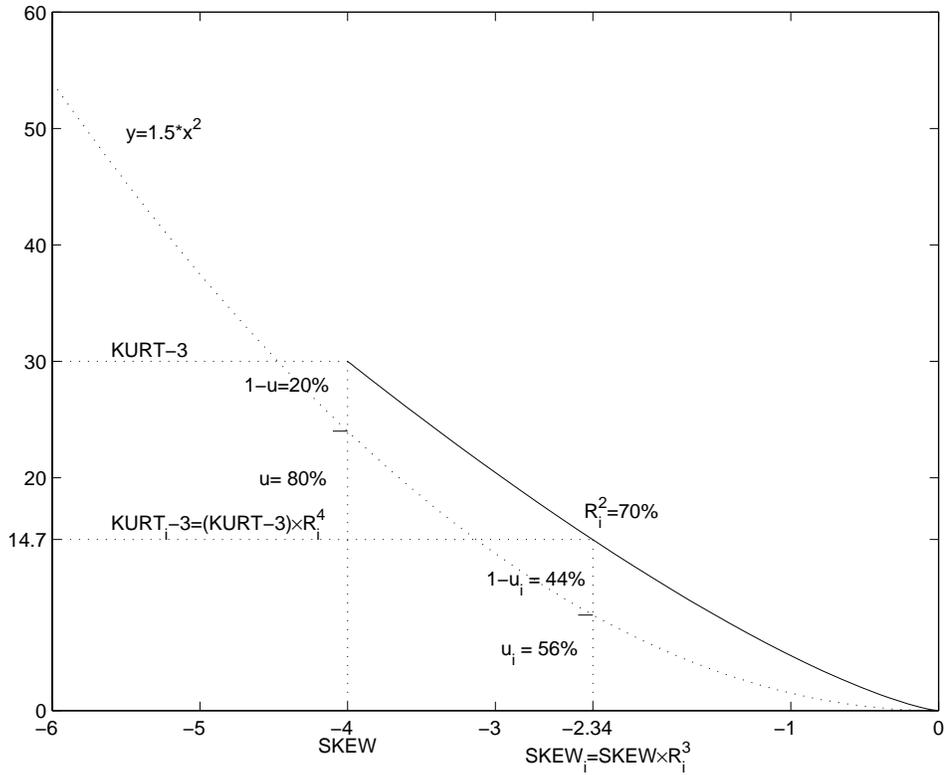


Figure 1. For points inside the parabola a larger fraction of kurtosis $(1 - u)$ is available for skew correction. All individual stocks plot further inside the parabola than the index ($u_i = uR_i^2$). In the example here market $SKEW=-4$ and $KURT-3=30$.

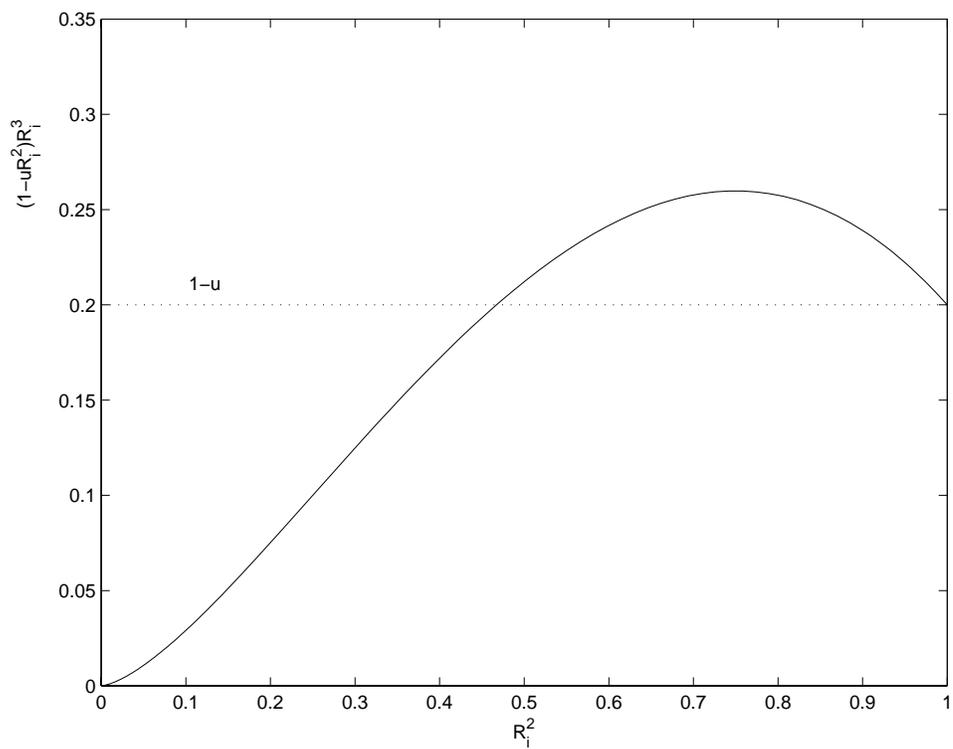


Figure 2. When index u is high, individual skew corrections can exceed index skew correction (here $u = 80\%$ and $60\%/u = 75\%$).

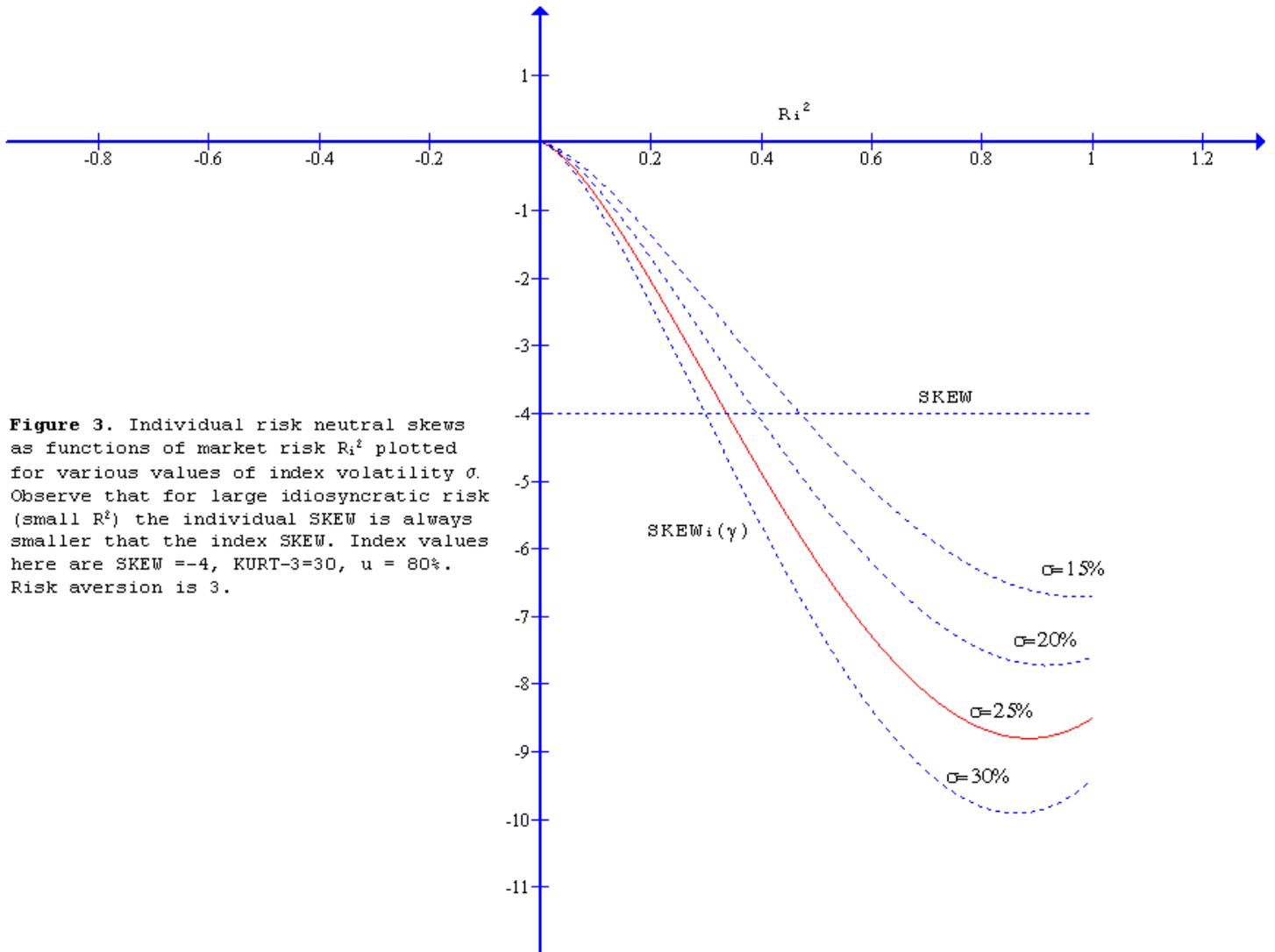


Figure 3. Individual risk neutral skews as functions of market risk R_i^2 plotted for various values of index volatility σ . Observe that for large idiosyncratic risk (small R_i^2) the individual SKEW is always smaller than the index SKEW. Index values here are SKEW = -4, KURT = 30, $u = 80\%$. Risk aversion is 3.

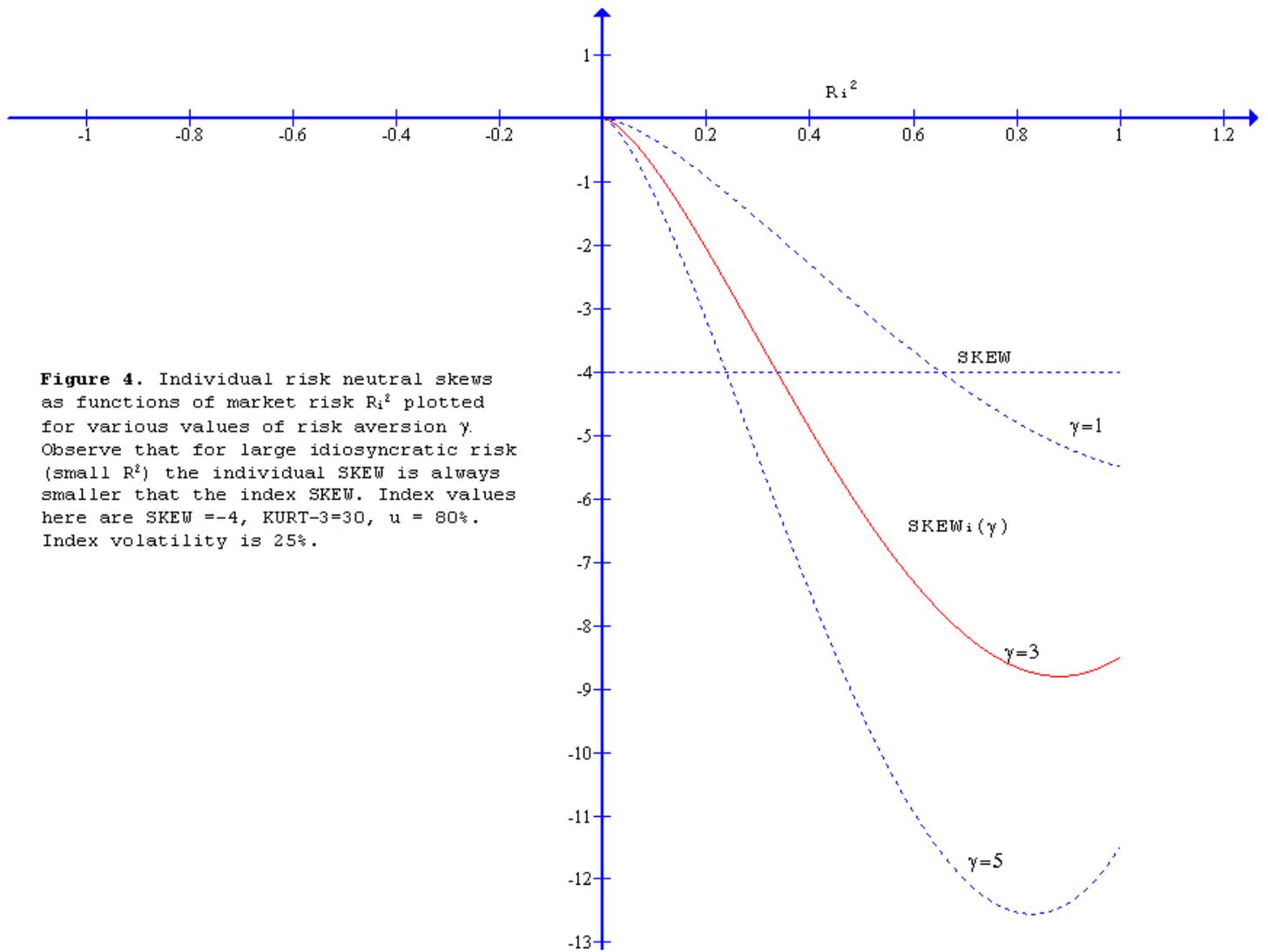


Figure 4. Individual risk neutral skews as functions of market risk R_i^2 plotted for various values of risk aversion γ . Observe that for large idiosyncratic risk (small R^2) the individual SKEW is always smaller than the index SKEW. Index values here are SKEW = -4, KURT-3=30, $u = 80\%$. Index volatility is 25%.

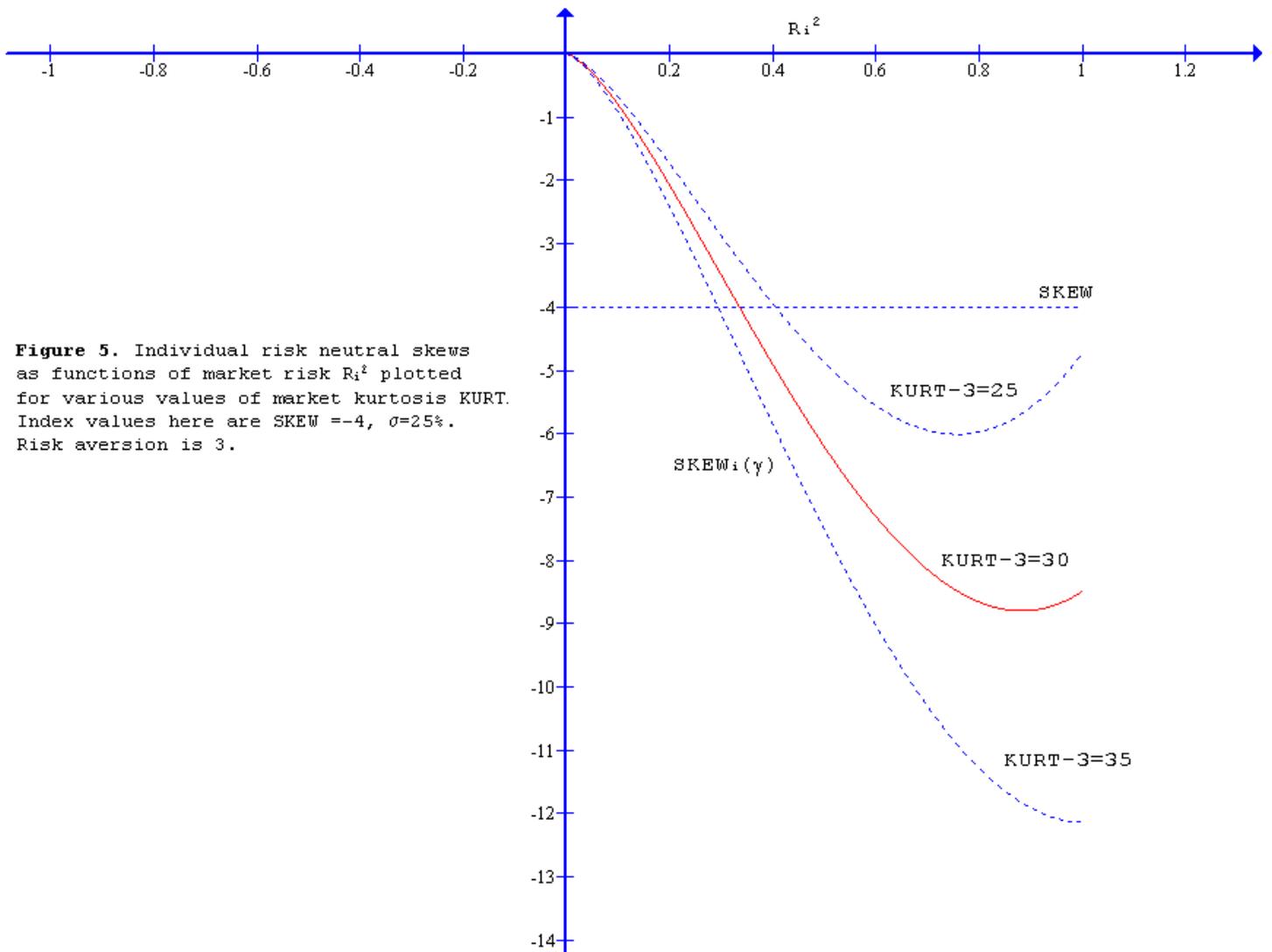


Figure 5. Individual risk neutral skews as functions of market risk R_i^2 plotted for various values of market kurtosis KURT. Index values here are SKEW = -4, $\sigma=25\%$. Risk aversion is 3.