# Optimal Deadlines for Agreements

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Abstract. Two players repeatedly try to agree on a joint decision between two alternatives. Costly negotiations in continuous time last until either an agreement or the deadline is reached. Each player either knows that the state is an agreement state corresponding to his ex ante favorite alternative, or is unsure whether the state is an agreement state corresponding to his opponent's favorite or a disagreement state with each player preferring his own favorite choice. We show that the optimal deadline is positive if only if the ex ante probability of the disagreement state is neither too high nor too low. Moreover, the optimal deadline is finite and increasing in the ex ante probability of the disagreement state.

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## 1. Introduction

In this paper we study the deadline effects in a repeated negotiation game. Although the model is quite specific, it captures essential ingredients of a general environment of strategic information aggregation, where: agents have imperfect and private information about the state; in some states of the world the agents agree on what is the best choice to make if they share information, while in others they disagree; strategic information aggregation can lead to poor decisions when the choice must be made without delay; and in repeated negotiation with costly delay, the agents gain endogenous information from their failure to agree so far, but without arrival of any exogenous new information.

We model repeated negotiation as a continuous time repeated proposal game. At any instant two players simultaneously choose one of two choices to propose, paying a flow cost of delay, until either they agree, at which point the agreement is implemented and the game ends, or the deadline expires. The two players favor different choices: each is willing to go along with the other player's favorite choice only if he is sufficiently convinced that the state is an agreement state supporting that choice. At any point of the game, each player is either "informed," meaning that he knows that the state is the agreement state corresponding to his favorite; or "uninformed," meaning that he is unsure whether the state is the agreement state corresponding to his opponent's favorite, or the state is the disagreement state with each player preferring his own favorite choice. Deadline is modeled in this paper as a commitment to short circuiting the information aggregation process of negotiation by flipping a coin.

With the same information structure but with different configurations of the payoff structure, we can interpret the above repeated proposal game as the following standard adoption game. Two firms in some industry repeatedly try to agree whose standard to adopt as the industry standard. Each firm naturally prefers its own standard, unless the state is the agreement state favoring its rival's standard. The disagreement state corresponds to a situation where neither standard is the better one in terms of a greater sum of payoffs of the two firms, or both standards are equally good.

We construct a symmetric equilibrium in which the informed type always "persists" by proposing his favorite alternative, and the uninformed type's behavior depends on the time left before the expiration of the deadline. When the time to deadline is sufficiently long, in equilibrium the uninformed type "concedes" by proposing the opponent's favorite alternative at some flow rate. When the time to deadline is short, in equilibrium, the uninformed "persists" by proposing his own favorite until the deadline is reached. At the deadline, the uninformed may concede with a strictly positive probability. We show that these are the only three types of equilibrium behavior, and characterize a unique symmetric equilibrium for any initial belief of the uninformed type that the state is a disagreement state. This characterization implies a critical deadline for each belief of the uninformed, such that whether the uninformed type concedes at some flow rate or persists depends only on whether the time remaining is longer or shorter than the critical deadline corresponding to his belief.

The deadline effects on the equilibrium strategy of the uninformed have strong implications to the deadline effects on the ex ante payoffs of both the uninformed and the informed. For any initial belief of the uninformed, extending the deadline only hurts both the informed and the uninformed if the starting point is shorter than the critical deadline corresponding to the initial belief, because the uninformed would just persist a little longer with no change in the equilibrium play when the deadline arrives. On the other hand, extending the deadline beyond the critical deadline does not affect the welfare of the uninformed, whose equilibrium play in the concession phase is the same as when there is no deadline, but affects the informed both by prolonging the concession phase and by potentially changing the equilibrium play at the deadline. We show that when the uninformed initially has a low belief that the state is the disagreement state, an extension of the deadline hurts the informed, because once the concession phase is over the uninformed persists until the deadline and then concedes with probability 1. The opposite is true for high initial beliefs: the informed benefits from a longer deadline because the uninformed concedes with probability 0 after finishing the concession phase and persisting until the deadline. For intermediate initial beliefs, an extension of the deadline increases the probability of concession by the uninformed once the deadline is reached, and as a result benefits the informed.

We provide a complete characterization of the "optimal deadline" that maximizes the ex ante probability-weighted sum of expected payoffs of the two players. The optimal deadline is 0 when the initial belief of the uninformed that the state is the disagreement state is either too low or too high. That is, when the two players are either too pessimistic or optimistic about the prospect of an agreement, the optimal deadline is zero, so that the decision is made by flipping a coin without delay when the players are pessimistic, or by trying to reach an agreement immediately. For intermediate initial beliefs of the uninformed, the optimal deadline is such that after the shortest concession phase the uninformed persists until the deadline and then concedes with probability 1. Thus, when positive, the optimal deadline is necessarily finite, because given that the uninformed concedes with probability 1 at the deadline, extending it further would only hurt the informed by prolonging the concession phase. Moreover, when positive, the optimal deadline is bounded away from 0, because it has to be long enough for the uninformed to concede with probability 1 after the concession and persistence phases. Finally, when positive, the optimal deadline is increasing in the initial belief of the uninformed, because the concession phase needs to be longer when the prospect of an agreement is lower. Coincidentally, the intermediate degrees of conflicts for which the optimal deadline is positive are also the same environments in which the quality of strategy information aggregation is the poorest without delay and negotiation.

This paper is organized as follows. A continuous time repeated proposal game is introduced in the next section. In section 3, we establish key properties of a symmetric equilibrium in which the informed types always persist. Section 4 uses the characterization in section 3 to construct a symmetric equilibrium that exhibits deadline effects on the play of the uninformed types. This symmetric equilibrium is shown to be unique. We characterize the optimal deadline for any initial degree of conflict. Section 5 assumes that the negotiation is either terminated by an agreement, or an exogenous event that occurs with a constant flow rate. We interpret this setup as a model of stochastic deadlines. We show that stochastic deadlines are not useful in terms of improving the ex ante welfare of the players relative to the case of no deadlines. That is, the optimal constant exit rate is either 0, which means that they should just keep on negotiation as long as there is no agreement, or infinity, implying that the two players are better off committing to make a decision immediately. In section 6, we allow for an additional penalty incurred by the two players if they fail to reach an agreement. We show that our result of positive and increasing deadline for an intermediate degrees of conflict is robust to modifications of the deadline payoffs of the game. Section 7 concludes with a brief discussion of related literature and future research questions.

## 2. A Repeated Proposal Game

Two players, called LEFT and RIGHT, have to make a joint choice between two alternatives, $l$ and $r$. We refer to $l$ as the favorite alternative of LEFT; and $r$ the favorite of RIGHT. There are three possible states of the world: $L$, $M$, and $R$. Both state $L$ and state $R$ are "agreement states" in which the mutually preferred alternative is $l$ and $r$ correspondingly; and state $M$ is disagreement state in which LEFT prefers $l$ and RIGHT prefers $r$. For player LEFT, the payoffs in state $L$ are $\overline{\pi}_F$ if $l$ is chosen and $\underline{\pi}_F$ if $r$ is chosen, with $\overline{\pi}_F > \underline{\pi}_F$; the payoffs in state $R$ are $\overline{\pi}_D$ if $r$ is chosen and $\underline{\pi}_D$ if $l$ is chosen, with $\overline{\pi}_D > \underline{\pi}_D$; and the payoffs are $\overline{\pi}_M$ if $l$ is chosen and $\underline{\pi}_M$ if $r$ is chosen, with $\overline{\pi}_M > \underline{\pi}_M$. The payoffs to RIGHT are symmetrically defined.

Each player is either "informed" or "uninformed." If informed, LEFT knows that the state is $L$. If uninformed, he knows that the state is either $M$ or $R$. The information structure for RIGHT is symmetric. Let $\gamma$ be the common belief of the uninformed types that the state is $M$; it captures the degree of conflict in our model. Based on his own information, the preference between $l$ and $r$ of an uninformed LEFT depends on the value of $\gamma$. If LEFT could dictate the outcome, he strictly prefers his favorite $l$ to the disfavored alternative $r$ if and only if

$$\gamma > \gamma_* \equiv \frac{\overline{\pi}_D - \underline{\pi}_D}{\overline{\pi}_D - \underline{\pi}_D + \overline{\pi}_M - \underline{\pi}_M}.$$

By assumption, $\gamma_* \in (0,1)$. Symmetrically, an uninformed RIGHT strictly prefers $r$ to $l$ if and only if $\gamma > \gamma_*$.

The repeated proposal game is modeled in continuous time, running from $t = 0$ to the deadline $T$. We allow $T$ to be infinite. The two players simultaneously propose $l$ or $r$ at each instant $t$, until the game ends. The game may end before the deadline if the two proposals by the two players agree. In this case, the agreed alternative is implemented immediately. Alternatively, if the deadline $T$ is is reached, the game ends with the decision made by a coin flip. During the game, each player incurs an additive payoff loss due to delay. Let $\delta$ be the flow rate of delay cost.

To simplify the analysis, we assume that for each player the benefit of implementing his favorite alternative relative to implementing the other alternative is at least as large in the corresponding agreement state as in the disagreement state.

ASSUMPTION 1. $\overline{\pi}_F - \underline{\pi}_F \geq \overline{\pi}_M - \underline{\pi}_M$.

The above assumption ensures that regardless of the delay cost it is optimal for the informed types to insist on their favorite alternative. This allows us to focus on a particularly simple, and natural, equilibrium. It will be clear from our analysis that this assumption is sufficient but not necessary.

If $T = 0$, the game described above becomes a strategic game where each player can propose either $l$ or $r$, and the outcome is that an agreement is implemented immediately and a disagreement results in a decision made by a coin flip. For any belief of the uninformed $\gamma > \gamma_*$, this strategic game has a unique equilibrium where each player proposes his favorite alternative.[1] For any belief of the uninformed $\gamma < \gamma_*$, there is a unique equilibrium where the informed types propose their favorite and the uninformed propose the favorite alternative of his rival. At $\gamma = \gamma_*$, there is a continuum of equilibria, where the informed always propose their favorite while the uninformed types propose their favorite with a probability between 0 and 1. Let $U^0(\gamma)$ and $V^0(\gamma)$ be the equilibrium payoff functions of the uninformed and informed types respectively. We have

$$U^0(\gamma) = \begin{cases} \frac{1}{2}\gamma(\overline{\pi}_M + \underline{\pi}_M) + (1-\gamma)\overline{\pi}_D & \text{if } \gamma \in [0, \gamma_*), \\ \in \left[\gamma_*\underline{\pi}_M + (1-\gamma_*)\overline{\pi}_D, \frac{1}{2}\gamma_*(\underline{\pi}_M + \overline{\pi}_M) + (1-\gamma_*)\overline{\pi}_D\right] & \text{if } \gamma = \gamma_*, \\ \frac{1}{2}\gamma(\overline{\pi}_M + \underline{\pi}_M) + \frac{1}{2}(1-\gamma)(\overline{\pi}_D + \underline{\pi}_D) & \text{if } \gamma \in (\gamma_*, 1]; \end{cases}$$

[1] There is no mechanism that Pareto improves on this outcome. More precisely, for any $\gamma > \gamma_*$, in any incentive compatible outcome of a direct mechanism (without transfers) the probability of implementing a fixed alternative is constant across the three states. See Damiano, Li and Suen (2008) for a formal argument.

and

$$V^0(\gamma) = \begin{cases} \overline{\pi}_F & \text{if } \gamma \in [0, \gamma_*), \\ \in \left[\frac{1}{2}(\underline{\pi}_F + \overline{\pi}_F), \overline{\pi}_F\right] & \text{if } \gamma = \gamma_*, \\ \frac{1}{2}(\overline{\pi}_F + \underline{\pi}_F) & \text{if } \gamma \in (\gamma_*, 1]. \end{cases}$$

Note that at $\gamma = \gamma_*$, both $U^0$ and $V^0$ are upper hemicontinuous but not lower hemicontinuous.[2]

Due to the symmetry of the model, we define the efficient outcome in the disagreement state as a fair coin flip. Given this, both the informed and the uninformed in equilibrium receive their efficient payoffs for $\gamma \in [0, \gamma_*)$ in expectation, while the equilibrium outcome is inefficient for $\gamma \in (\gamma_*, 1]$.

A notable feature of the no-delay game with $T = 0$ is that the equilibrium behavior of the uninformed, and the payoffs of both the uninformed and the informed change discontinuously as $\gamma$ increases from below $\gamma_*$ to above. Corresponding to this discontinuity, there is a continuum of equilibria at $\gamma = \gamma_*$ when $T = 0$. This particular feature is not critical for our result of the optimal deadline being positive and increasing for intermediate values of $\gamma$. In section 6, we extend the model by allowing for a disagreement penalty. In the corresponding no-delay game, the discontinuity disappears. The continuum of equilibria at $\gamma_*$ is replaced by an interval of values of $\gamma$ for which there is a unique equilibrium where the uninformed concedes with a probability ranging from 0 to 1.

## 3. Preliminary Analysis

We focus on symmetric perfect Bayesian equilibria where at any $t$, the informed types "persist" with probability 1, that is, they propose their favorite alternative, while the uninformed types may randomize between persisting and "conceding," or, proposing their rival's favorite alternative. In any such equilibrium, only "regular" disagreements happen on the equilibrium path, where each player proposes his favorite alternative. It turns out

[2] If $\overline{\pi}_M + \underline{\pi}_M \leq \overline{\pi}_D + \underline{\pi}_D$, then the payoff function for the uninformed $U^0(\gamma)$ is strictly decreasing in $\gamma$ for $\gamma \in [0, \gamma_*)$ and weakly decreasing for $\gamma \in (\gamma_*, 1]$. In this case, our model has the interpretation that conflicts between the two players as captured by the belief of the uninformed types reduce their payoffs. However, such interpretation is not needed for any result in this paper.

how the game is played after a "reverse" disagreement in which each player proposes his rival's favorite does not matter to the equilibrium construction.[3] We will only specify the strategy of the uninformed after regular disagreements. Let $x(t)$ be the flow rate of concession by the uninformed types once the game reaches time $t$. Upon reaching time $t$, the probability of an uninformed type proposing his rival's alternative in the time interval $\mathrm{d}t$ is $x(t)\mathrm{d}t$, if this is defined. We allow for the possibility that at some $t$, the uninformed concedes with a strictly positive probability at the instant $t$; this is referred to an "atom" in the flow rate of concession, and denoted as $y(t) > 0$. An equilibrium in any game considered below is then described by the flow concession rate function $x(t)$ and a collection of atoms, with the understanding that $x(t)$ is defined everywhere except at each $t$ such that $y(t) > 0$.

### 3.1. Differential equations

Suppose in equilibrium $x(t)$ is strictly positive and finite for some time interval $(\underline{t}, \overline{t})$. The uninformed must be indifferent between persisting and conceding at any $t \in (\underline{t}, \overline{t})$. Let $\gamma(t)$ be the belief of the uninformed at time $t$ that the state is $M$. The payoff from conceding at time $t$ to the uninformed is

$$\gamma(t)\Big((1 - x(t)\mathrm{d}t)\underline{\pi}_M + x(t)\mathrm{d}t\ U_r\Big) + (1 - \gamma(t))\overline{\pi}_D, \tag{3.1}$$

where $U_r$ is the continuation payoff after a reverse disagreement. Since the uninformed is indifferent between persisting and conceding, the value function $\mathcal{U}(t)$ of the uninformed is the limit of the above expression as $\mathrm{d}t$ goes to 0. We have

$$\mathcal{U}(t) = \gamma(t)\underline{\pi}_M + (1 - \gamma(t))\overline{\pi}_D.$$

Note $\mathcal{U}(t)$ depends on $t$ only through $\gamma(t)$. We define a payoff function

$$U(\gamma) = \gamma\underline{\pi}_M + (1 - \gamma)\overline{\pi}_D, \tag{3.2}$$

[3] For convenience, we can imagine that the game ends and the decision is made by a coin toss when a reverse disagreement occurs.

which is valid for all $\gamma = \gamma(t)$ for some $t \in (\underline{t}, \bar{t})$. The expected payoff at time $t$ to the uninformed from persisting is

$$\gamma(t)x(t)\mathrm{d}t\ \overline{\pi}_M + \Big(\gamma(t)(1 - x(t)\mathrm{d}t) + (1 - \gamma(t))\Big)(-\delta \mathrm{d}t + \mathcal{U}(t + \mathrm{d}t))$$

where the continuation payoff $\mathcal{U}(t+\mathrm{d}t)$ after a regular disagreement is given by $U(\gamma(t+\mathrm{d}t))$ as in (3.2). Subtracting $\mathcal{U}(t+\mathrm{d}t)$ both from the payoff from conceding and the payoff from persisting and then equating the two, we have

$$\begin{aligned}
&\gamma(t)\Big((1 - x(t)\mathrm{d}t)\ \underline{\pi}_M + x(t)\mathrm{d}t\ U_r\Big) + (1 - \gamma(t))\overline{\pi}_D - \mathcal{U}(t + \mathrm{d}t) \\
=&\gamma(t)x(t)\mathrm{d}t\ \overline{\pi}_M - \Big(\gamma(t)(1 - x(t)\mathrm{d}t) + (1 - \gamma(t))\Big)\delta \mathrm{d}t - \gamma(t)x(t)\mathrm{d}t\ \mathcal{U}(t + \mathrm{d}t).
\end{aligned}$$

Diving both sides by d$t$ and taking the limit as d$t$ goes to 0, and noting that the left-hand-side becomes $(\overline{\pi}_D - \underline{\pi}_M)\dot{\gamma}(t)$, we have:

$$\dot{\gamma}(t) = \gamma(t)x(t)\left(\gamma(t) + \frac{\overline{\pi}_M - \overline{\pi}_D}{\overline{\pi}_D - \underline{\pi}_M}\right) - \frac{\delta}{\overline{\pi}_D - \underline{\pi}_M}.$$

By Bayes' rule,

$$\gamma(t + \mathrm{d}t) = \frac{\gamma(t)(1 - x(t)\mathrm{d}t)}{\gamma(t)(1 - x(t)\mathrm{d}t) + (1 - \gamma(t))}.$$

Subtracting $\gamma(t)$ from both sides, dividing by d$t$ and taking the limit as d$t$ goes to 0, we have

$$\dot{\gamma}(t) = -\gamma(t)(1 - \gamma(t))x(t).$$

Thus, using the definition of $\gamma_*$, we have the following first-order, autonomous differential equation for $\gamma(t)$:

$$-\frac{\dot{\gamma}(t)}{1 - \gamma(t)} = \frac{\delta}{\overline{\pi}_M - \underline{\pi}_M}. \tag{3.3}$$

From the above differential equation and Bayes' rule, we immediately have

$$x(t) = \frac{1}{\gamma(t)} \frac{\delta}{\overline{\pi}_M - \underline{\pi}_M}.$$

Note that the above implies that the flow rate of concession $x$ is a function $t$ only through $\gamma(t)$.

For any $t \in (\underline{t}, \bar{t})$, Let $\mathcal{V}(t)$ be the expected payoff of the informed types at time $t$. Since the informed always persist, the payoff function satisfies the following Bellman equation:

$$\mathcal{V}(t) = x(t)\mathrm{d}t\ \overline{\pi}_F + (1 - x(t)\mathrm{d}t)(-\delta \mathrm{d}t + \mathcal{V}(t + \mathrm{d}t)).$$

Subtracting $V(t + \mathrm{d}t)$ from both sides, dividing by $\mathrm{d}t$ and taking $\mathrm{d}t$ to 0, we have the following first order differential equation for $\mathcal{V}(t)$:

$$-\dot{\mathcal{V}}(t) = x(t)(\overline{\pi}_F - \mathcal{V}(t)) - \delta.$$

Since $\gamma(t)$ is determined by an autonomous differential equation (with some boundary condition), and since $x(t)$ depends on $t$ only through $\gamma(t)$, there exists a function $V(\gamma)$ such that

$$V(\gamma) = \mathcal{V}(t)$$

whenever $\gamma = \gamma(t)$ for all $t \in (\underline{t}, \bar{t})$, implying that $\dot{\mathcal{V}}(t) = V'(\gamma(t))\dot{\gamma}(t)$. Further, using the differential equation for $\gamma(t)$ and the expression of $x$ as a function of $\gamma$, we have the following differential equation for $V(\gamma)$:

$$V'(\gamma) = \frac{\overline{\pi}_F - V(\gamma)}{\gamma(1-\gamma)} - \frac{\overline{\pi}_M - \underline{\pi}_M}{1-\gamma}. \tag{3.4}$$

For any belief $\gamma$ of the uninformed, we define the ex ante payoff of each player as the weighted average of the informed and the uninformed, given by

$$W(\gamma) = \frac{1}{2-\gamma}U(\gamma) + \frac{1-\gamma}{2-\gamma}V(\gamma).$$

### 3.2. Equilibrium with no deadline

Now suppose that $T = \infty$. Fix any initial belief $\gamma^0 > 0$. Let $g(t;\gamma^0)$ be the unique solution to the differential equation (3.3) with the initial condition $g(0;\gamma^0) = \gamma^0$, given by

$$g(t;\gamma^0) = 1 - (1-\gamma^0)e^{\delta_* t}, \tag{3.5}$$

where for notational brevity we define

$$\delta_* \equiv \frac{\delta}{\overline{\pi}_M - \underline{\pi}_M}.$$

Of course the solution is a decreasing function of time. The corresponding flow rate of concession of the uninformed is given by $x(t) = \delta_*/g(t;\gamma^0)$, which is an increasing function of time, implying that the uninformed types are increasingly more willing to concede as regular disagreements continue. Let $Q(\gamma^0)$ solve

$$g(Q(\gamma^0);\gamma^0) = 0.$$

Then, we have

$$Q(\gamma^0) = -\frac{\ln(1-\gamma^0)}{\delta_*}.$$

We refer to $Q(\gamma^0)$ as the "terminal time" of the equilibrium play. We have the following proposition.

Proposition 1. *Suppose that $T = \infty$ and Assumption 1 holds. For any initial belief $\gamma^0 \in (0,1)$, there exists an equilibrium in which the informed types always persist, and at any time $t < Q(\gamma^0)$, the uninformed types concede with a flow rate $x(t) = \delta_*/g(t;\gamma^0)$ and concede with probability 1 at $t = Q(\gamma^0)$.*

Given the construction through the differential equation, the uninformed types are indifferent between conceding and persisting at any time $t < Q(\gamma^0)$. Since the belief of the uninformed that the state is $M$ is 0 at $t = Q(\gamma^0)$, the expected payoff from conceding is $\overline{\pi}_D$, which is at least as great as the expected payoff from persisting regardless of the continuation. For the informed types, for any $t < Q(\gamma^0)$, the expected payoff from the deviation to conceding is

$$x(t)\mathrm{d}t\ V_r + (1 - x\mathrm{d}t)\underline{\pi}_F,$$

where $V_r$ is the continuation payoff after a reverse disagreement. Taking d$t$ to 0 implies that the deviation payoff is $\underline{\pi}_F$. On the other hand, the equilibrium expected payoff is $V(g(t;\gamma^0))$, given by the solution to the differential equation for $V(\gamma)$, which can be solved explicitly. The general solution is

$$V(\gamma) = \overline{\pi}_F - \frac{1-\gamma}{\gamma}\ln(1-\gamma)(\overline{\pi}_M - \underline{\pi}_M) + \frac{1}{\gamma}\Big((1-\gamma)(C + \overline{\pi}_F) - (\overline{\pi}_M - \underline{\pi}_M)\Big), \quad (3.6)$$

where $C$ is a constant to be determined by a boundary condition. The boundary condition we use is $V(0) = \overline{\pi}_F$; this follows because when $\gamma = 0$ the uninformed types concede with probability 1 and hence it is optimal for the informed to persist with probability 1. Thus

$$C = -\overline{\pi}_F + (\overline{\pi}_M - \underline{\pi}_M),$$

and the explicit solution for $V(\gamma)$ is then

$$V(\gamma) = \overline{\pi}_F - \left(1 + \frac{1-\gamma}{\gamma}\ln(1-\gamma)\right)(\overline{\pi}_M - \underline{\pi}_M). \tag{3.7}$$

It is immediate from the solution that the equilibrium payoff for the informed is greater than or equal to $\overline{\pi}_F - (\overline{\pi}_M - \underline{\pi}_M)$ for all $\gamma$, which by Assumption 1 is greater than or equal to $\underline{\pi}_F$.[4] Thus it is optimal for the informed to persist.

### 3.3. Hazard rates and duration

Fix an initial belief $\gamma_0 \in (0,1)$ of the uninformed. Suppose that for some $\bar{t} \in (0,T]$ in equilibrium $x(t)$ is given by Proposition 1 for all $t \in [0,\bar{t})$.

Let $t_I < \bar{t}$ be the random variable representing the ending time of the game in equilibrium, conditional on that one player is informed, i.e., the state is $L$ or $R$. Let $F_I(t;\gamma^0)$ be its cumulative distribution function and $f_I(t;\gamma^0)$ be its probability density function. Since $x(t)\mathrm{d}t$ is the probability that the game ends at time interval $[t, t+\mathrm{d}t]$ conditional on it having survived up to time $t$, or the conditional hazard rate, we have

$$x(t) = \frac{f_I(t;\gamma^0)}{1 - F_I(t;\gamma^0)} = -\frac{\mathrm{d}}{\mathrm{d}t}\ln(1 - F_I(t;\gamma^0)).$$

Integrating both sides, we get

$$F_I(t;\gamma^0) = 1 - e^{-\int_0^t x(\tau)\mathrm{d}\tau}.$$

The above holds for all $t < \bar{t}$. Using the solution to (3.3) in (3.5) and $x(t) = \delta_*/g(t;\gamma^0)$, we have

$$\int_0^t x(\tau)\mathrm{d}\tau = \delta_* t - \ln(1 - (1-\gamma^0)e^{\delta_* t}) + \ln\gamma^0,$$

[4] Assumption 1 ensures that there is no profitable deviation for the informed even if the belief of the uninformed is arbitrarily close to 1. Since the solution $V(\gamma)$ is decreasing in $\gamma$, Assumption 1 is sufficient for Proposition 1 but not necessary given any fixed initial belief $\gamma^0$.

implying that

$$1 - F_I(t;\gamma) = \frac{1-\gamma^0}{\gamma^0}\frac{g(t;\gamma^0)}{1-g(t;\gamma^0)}. \tag{3.8}$$

Differentiate with respect to $\gamma^0$ to get:

$$\frac{\partial}{\partial\gamma^0}(1 - F_I(t;\gamma^0)) = \frac{\gamma^0 - g(t;\gamma^0)}{(\gamma^0)^2(1-g(t;\gamma^0))} > 0.$$

Thus, greater $\gamma^0$ stochastically increases $t_I$.

Similarly, let $t_U < \bar{t}$ be the ending time of the game conditional on that both players are uninformed, i.e., the state is $M$. Let $F_U(t;\gamma^0)$ be its cumulative distribution function. We have

$$1 - F_U(t;\gamma^0) = (1 - F_I(t;\gamma^0))^2,$$

implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln(1 - F_U(t;\gamma^0)) = 2\frac{\mathrm{d}}{\mathrm{d}t}\ln(1 - F_I(t;\gamma^0)).$$

Thus, the hazard rate conditional on the state being $M$ is simply twice the concession rate the uninformed player. Further, the derivative of $1 - F_U$ with respect to $\gamma^0$ is also positive, and so greater $\gamma^0$ stochastically increases the distribution of duration conditional on the uninformed state as well.

Finally, for any $t < \bar{t}$ the unconditional distribution of duration is given by

$$1 - F(t;\gamma^0) = \frac{2(1-\gamma^0)}{2-\gamma^0}(1 - F_I(t;\gamma^0)) + \frac{\gamma^0}{2-\gamma^0}(1 - F_U(t;\gamma^0)).$$

Differentiate with respect to $\gamma^0$ to get:

$$\begin{aligned}&\frac{2}{(2-\gamma^0)^2}((1 - F_U(t;\gamma^0)) - (1 - F_I(t;\gamma^0))) + \frac{2(1-\gamma^0)}{2-\gamma^0}\frac{\gamma^0 - g(t;\gamma^0)}{(\gamma^0)^2(1-g(t;\gamma^0))}\\ &\quad + \frac{2\gamma^0}{2-\gamma^0}\frac{\gamma^0 - g(t;\gamma^0)}{(\gamma^0)^2(1-g(t;\gamma^0))}(1 - F_I(t;\gamma^0)).\end{aligned}$$

Using the formula (3.8), we can simplify the above as

$$\frac{2(1-\gamma^0)(\gamma^0 - g(t;\gamma^0))(2-\gamma^0 - g(t;\gamma^0))}{(\gamma^0)^2(2-\gamma^0)^2(1-g(t;\gamma^0))^2},$$

which is positive. Thus, a greater initial belief stochastically increases the unconditional duration.

Now, suppose that $T = \infty$. To compute the upperbound of the support of $F_I$, note that as $t$ approaches $Q(\gamma^0)$, $F_I$ becomes arbitrarily close to 1. This implies that the upperbound of the support of $F_I$ is $Q(\gamma^0)$. Thus, in the game without deadline, regardless the initial belief $\gamma^0$ the game ends with probability 1 before the belief of the uninformed types as given by (3.5) reaches 0. The expected duration $\mathrm{E}[t_I]$, conditional on that the state is $L$ or $R$, is given by

$$\int_0^{Q(\gamma^0)} (1 - F_I(t;\gamma^0))\mathrm{d}t = \frac{1}{\delta_*}\Big(1 + \frac{1-\gamma^0}{\gamma^0}\ln(1-\gamma^0)\Big).$$

From the expression of $V(\gamma^0)$ in (3.7), the payoff loss for the informed types due to delay is precisely the expected duration times the flow delay cost $\delta$. The support of $t_U$ is the same as that of $t_I$. The expected duration in this case is

$$\mathrm{E}[t_U] = \int_0^{Q(\gamma^0)} e^{-\int_0^{t_U} 2x(\tau)\mathrm{d}\tau}\mathrm{d}t_U.$$

Of course, $\mathrm{E}[t_U] < \mathrm{E}[t_I]$.

## 4. Finite Deadlines

This section contains the main results of the paper. We first use the analysis in the previous section to construct a symmetric equilibrium in which the informed type always persists, and the uninformed type behaves as if there is no deadline when the time to the deadline is sufficiently long, and may stop and persist until just before the deadline is reached, followed by an equilibrium play of the no-delay game ($T = 0$) depending on the stopping belief. We then show that this equilibrium is unique. The welfare effects of marginally extending the deadline are shown to depend on the initial belief of the uninformed that the state is the disagreement state $M$. We establish that the optimal deadline is 0 when this initial belief is either below the critical belief $\gamma_*$ or sufficiently close to 1, and is otherwise just long enough so that the uninformed persists as soon as the equilibrium belief reaches $\gamma_*$ and then concedes with probability 1 upon the deadline. The optimal deadline is thus an increasing function of the initial belief when it is positive.

### 4.1. Equilibrium

Let $n(T)$ be the initial belief $\gamma^0$ of the uninformed such that

$$g(T; n(T)) = 0;$$

this will correspond to the largest initial belief such that the uninformed types play as if there is no deadline. Using the solution $g(t; \gamma^0)$, we have

$$n(T) = 1 - e^{-\delta_* T}.$$

Define $s(T)$ be the lowest initial belief of the uninformed below $\gamma_*$ such that it is an equilibrium for the uninformed to persist until the deadline, at which point the uninformed types switch to concede, as follows. At the belief $\gamma^0$, the payoff from conceding right away is $U(\gamma^0)$ as in (3.2); the equilibrium payoff is

$$\gamma^0 \frac{\overline{\pi}_M + \underline{\pi}_M}{2} + (1 - \gamma^0)\overline{\pi}_D - \delta T.$$

This is an equilibrium if

$$\gamma^0 \geq s(T) \equiv 2\delta_* T.$$

In addition, we need $s(T) \leq \gamma_*$, which is equivalent to

$$T \leq \frac{\gamma_*}{2\delta_*} \equiv T_*.$$

We claim that $s(T) > n(T)$ for all $T > 0$. To see this, note that $s(T)$ increases linearly at the rate of $2\delta_*$, while $n(T)$ is concave, with $n(0) = s(0)$, and grows at the rate of $\delta_*$ at $T = 0$. Further,

$$-\frac{\mathrm{d}s(T-t)}{\mathrm{d}t} = 2\delta_* > (1 - g(t; \gamma^0))\delta_* = -\frac{\mathrm{d}g(t; \gamma^0)}{\mathrm{d}t}.$$

This implies that $s(T-t) - g(t; \gamma^0)$ can cross 0 at most once and only from above for any initial belief $\gamma^0$. It follows that for any $\gamma^0 < n(T)$, we have $g(t; \gamma^0) < s(T-t)$ for all $t$; for any $\gamma^0 > s(T)$, we have $g(t; \gamma^0) > s(T-t)$ for all $t$ ; for any $\gamma^0 \in (n(T), s(T))$, there exists $t < T$, denoted as $J^s(\gamma^0)$, at which point $s(T-t)$ joins $g(t; \gamma^0)$, i.e.,

$$g(J^s(\gamma^0); \gamma^0) = s(T - J^s(\gamma^0)).$$

Next, let $k(T)$ as the largest initial belief $\gamma^0$ of the uninformed above the critical belief $\gamma_*$ such that, the evolution of the belief according to $g(t;\gamma^0)$ is truncated by the deadline $T$ before it reaches $\gamma_*$. By definition,

$$g(T;k(T)) = \gamma_*,$$

which gives

$$k(T) = 1 - (1-\gamma_*)e^{-\delta_* T}.$$

For each $\gamma^0 \in (\gamma_*, k(T))$, define $J^k(\gamma^0)$ as the time that $\gamma(t)$ reaches $\gamma_*$ when it evolves according to $g(t;\gamma^0)$. That is,

$$g(J^k(\gamma^0);\gamma^0) = \gamma_*.$$

Finally, define $p(T)$ be lowest initial belief of the uninformed for which it is an equilibrium to persist for the entire duration of the game. The equilibrium payoff for the uninformed is

$$\gamma^0 \frac{\overline{\pi}_M + \underline{\pi}_M}{2} + (1-\gamma^0)\frac{\overline{\pi}_D + \underline{\pi}_D}{2} - \delta T;$$

the payoff from deviating to conceding right away is $U(\gamma^0)$ as in (3.2). Thus,

$$p(T) = \gamma_* + 2(1-\gamma_*)\delta_* T.$$

Note that $p(0) = \gamma_*$ and $p(T) < 1$ for all $T \leq T_*$. Also, $p(T) > k(T)$ for all $T$. To see this, note that $p(T)$ increases linearly at the rate of $2(1-\gamma_*)\delta_*$, while $k(T)$ is concave, with $k(0) = p(0)$, and grows at the rate of $(1-\gamma_*)\delta_*$ at $T = 0$. Further,

$$-\frac{\mathrm{d}p(T-t)}{\mathrm{d}t} = 2(1-\gamma_*)\delta_* > (1-g(t;\gamma^0))\delta_* = -\frac{\mathrm{d}g(t;\gamma^0)}{\mathrm{d}t}$$

for any $\gamma^0$ and $t$ such that $g(t;\gamma^0) > \gamma_*$. This implies that $p(T-t) - g(t;\gamma^0)$ can cross 0 at most once and only from above for any initial belief $\gamma^0$. It follows that for any $\gamma^0 < k(T)$, we have $g(t;\gamma^0) < p(T-t)$ for all $t$; for any $\gamma^0 > p(T)$, we have $g(t;\gamma^0) > p(T-t)$ for all $t$; for any $\gamma^0 \in (k(T), p(T))$, there exists $t < T$, denoted as $J^p(\gamma^0)$, at which point $p(T-t)$ joins $g(t;\gamma^0)$, i.e.,

$$g(J^p(\gamma^0);\gamma^0) = p(T - J^p(\gamma^0)).$$

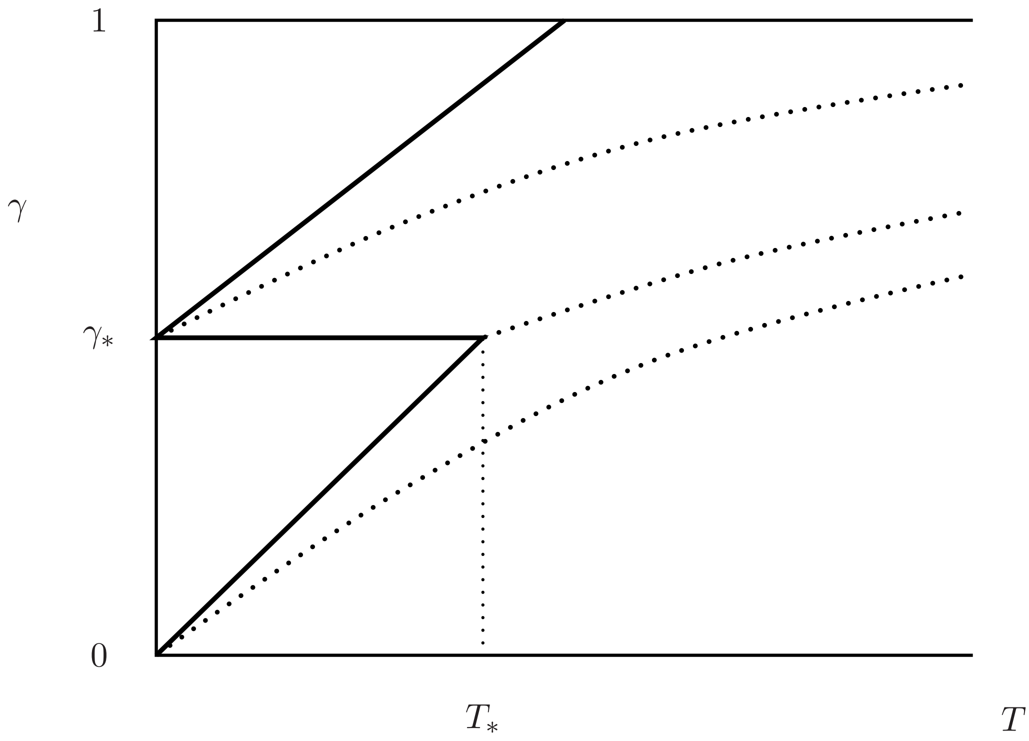


Figure 1

Proposition 2. *Suppose that $T \le T_*$ and Assumption 1 holds. For any initial belief $\gamma^0 \in [0, 1)$, there is an equilibrium in which the informed always persists, and the strategy of the uninformed is: (i) if $\gamma^0 \in [0, n(T)]$, concede with a flow rate $\delta_*/g(t;\gamma^0)$ and concede with probability 1 at $t = Q(\gamma^0)$; (ii) if $\gamma^0 \in (n(T), s(T)]$, concede with a flow rate $\delta_*/g(t;\gamma^0)$ for $t < J^s(\gamma^0)$, persist for all $t \in [J^s(\gamma^0), T)$, and concede with probability 1 at $t = T$; (iii) if $\gamma^0 \in (s(T), \gamma_*)$, persist for all $t < T$ and concede with probability 1 at $t = T$; (iv) if $\gamma^0 = \gamma_*$, persist for all $t < T$, and concede with any probability between $2\delta_* T/\gamma_*$ and 1; (v) if $\gamma^0 \in (\gamma_*, k(T)]$, concede with a flow rate $\delta_*/g(t;\gamma^0)$ for $t \le J^k(\gamma^0)$, persist for all $t \in (J^k(\gamma^0), T)$, and concede with probability $2\delta_*(T - J^k(\gamma^0))/\gamma_*$ at $t = T$; (vi) if $\gamma^0 \in (k(T), p(T))$, concede with a flow rate $\delta_*/g(t;\gamma^0)$ for $t < J^p(\gamma^0)$, and persist for $t$ larger; (vii) if $\gamma^0 \in [p(T), 1)$, persist for all $t$.*

The seven cases in Proposition 1 are illustrated in Figure 1. They are numbered in increasing order by $\gamma$ for each $T \le T_*$: the dotted curve starting at $T = 0$, $\gamma = 0$ represents $n(T)$; the line segment starting at at the same point and ending at $T = T_*$, $\gamma = \gamma_*$ represents $s(T)$; the dotted curve starting at $T = 0$, $\gamma = \gamma_*$ represents $k(T)$; and the line segment starting at the same point represents $p(T)$.

The proof of case (i) is identical to the no deadlines case, because by construction $Q(\gamma^0) < T$ for all $\gamma^0 \in [0, n(T)]$.

For case (iii), it suffices to verify that the informed types do not want to deviate to conceding. The equilibrium payoff is $\overline{\pi}_F - \delta T$; the payoff from conceding right away is $\underline{\pi}_F$, which is smaller because

$$T \leq T_* = \frac{\gamma_*(\overline{\pi}_M - \underline{\pi}_M)}{2\delta} < \frac{\overline{\pi}_F - \underline{\pi}_F}{\delta},$$

by Assumption 1.

For case (ii), first note that the continuation play is in equilibrium for any $\gamma^0$ at $t = J^s(\gamma^0)$ by case (iii). The continuation payoff for the uninformed is thus

$$\mathcal{U}(J^s(\gamma^0)) = g(J^s(\gamma^0); \gamma^0)\frac{\overline{\pi}_M + \underline{\pi}_M}{2} + (1 - g(J^s(\gamma^0); \gamma^0)\overline{\pi}_D - \delta(T - J^s(\gamma^0)),$$

which is equal to $U(g(J^s(\gamma^0); \gamma^0))$ by the definition of $J^s(\gamma^0)$. It follows that the strategy of the uninformed specified in Proposition 2 is an equilibrium for $t < J^s(\gamma^0)$. For the informed, the equilibrium continuation payoff at $t = J^s(\gamma^0)$ is $\overline{\pi}_F - \delta(T - J^s(\gamma^0))$. The equilibrium payoff function $V(\gamma)$ at any $\gamma = g(t; \gamma^0)$ for $t < J^s(\gamma^0)$ solves the differential equation for $V$ with the boundary condition

$$V(g(J^s(\gamma^0); \gamma^0)) = \overline{\pi}_F - \delta(T - J^s(\gamma^0)).$$

The general solution (3.6) can be written as

$$\frac{\gamma}{1-\gamma}V(\gamma) = \frac{\overline{\pi}_F - (\overline{\pi}_M - \underline{\pi}_M)}{1-\gamma} - \ln(1-\gamma)(\overline{\pi}_M - \underline{\pi}_M) + C,$$

where $C$ is determined by the boundary condition. We claim that the right-hand-side of the above expression is at least as great as $\underline{\pi}_F\gamma/(1-\gamma)$ for all $\gamma \in [g(J^s(\gamma^0); \gamma^0), \gamma^0]$, so that it is optimal for the informed to persist. To see this, note that this is the case at $\gamma = g(J^s(\gamma^0); \gamma^0)$ from case (iii). For all larger values of $\gamma$, the claim follows from rewriting the inequality as

$$\frac{(\overline{\pi}_F - \underline{\pi}_F) - (\overline{\pi}_M - \underline{\pi}_M)}{1-\gamma} - \ln(1-\gamma)(\overline{\pi}_M - \underline{\pi}_M) + C \geq \underline{\pi}_F,$$

and noting the left-hand-side is increasing in $\gamma$ by Assumption 1.

For case (iv), where $\gamma$ is exactly $\gamma_*$, there is a continuum of equilibria, depending on the probability of concession at the deadline by the uninformed. The equilibrium payoff of uninformed, by persisting until the deadline and then conceding, is

$$\gamma_* \left( y \frac{\overline{\pi}_M + \underline{\pi}_M}{2} + (1-y)\underline{\pi}_M \right) + (1-\gamma_*)\overline{\pi}_D - \delta T,$$

where $y$ varies from $2\delta_* T/\gamma_*$ to 1. The payoff from conceding right away is $U(\gamma_*)$. By construction, the equilibrium payoff is greater than or equal to $U(\gamma_*)$, with equality at $y = 2\delta_* T/\gamma_*$. The equilibrium payoff of the informed is

$$y\overline{\pi}_F + (1-y)\underline{\pi}_F - \delta T \geq \underline{\pi}_F,$$

the payoff from conceding, because $y \geq 2\delta_* T/\gamma_*$ together with Assumption 1.

For case (v), first note that for any $\gamma^0$ at $t = J^k(\gamma^0)$, the continuation play is in equilibrium for any $\gamma^0$ at $t = J^k(\gamma^0)$ by the equilibrium with the lowest deadline concession probability $y$ in case (iv). The continuation payoff for the uninformed is thus

$$\begin{aligned}\mathcal{U}(J^k(\gamma^0)) =& \gamma_* \left( \frac{2\delta_*(T - J^k(\gamma^0))}{\gamma_*} \frac{\overline{\pi}_M + \underline{\pi}_M}{2} + \left(1 - \frac{2\delta_*(T - J^k(\gamma^0))}{\gamma_*}\right) \underline{\pi}_M \right) + (1-\gamma_*)\overline{\pi}_D \\ & - \delta(T - J^k(\gamma^0)).\end{aligned}$$

By the construction in case (iv), the above is equal to $U(\gamma_*)$. It follows immediately that the uninformed strategy is an equilibrium for $t < J^k(\gamma^0)$. For the informed, since there is no incentive to deviate to concession at $t = J^k(\gamma^0)$ by case (iv), the same argument as in case (ii) implies that there is no incentive to deviate for any smaller $t$.

For case (vii), the equilibrium play is such that the uninformed types persist all along until the game ends. By construction, this strategy of the uninformed is in equilibrium. For the informed, the equilibrium payoff is

$$\frac{\overline{\pi}_F + \underline{\pi}_F}{2} - \delta T \geq \underline{\pi}_F,$$

which is the expected payoff from conceding right away, where the inequality follows because $T \leq T_*$ together with Assumption 1.

$$T \leq T_* = \frac{\gamma_*(\overline{\pi}_M - \underline{\pi}_M)}{2\delta} \leq \frac{\overline{\pi}_F - \underline{\pi}_F}{2\delta},$$

by Assumption 1.

Finally, for case (vi), the equilibrium play is the same as without deadline until $t = J^p(\gamma^0)$ and the updated belief is equal to $\gamma_*$, at which point it continues as in case in (vii), which we have already argued is in equilibrium. Since by construction the continuation payoff for the uninformed is $\mathcal{U}(J^p(\gamma^0)) = U(\gamma_*)$, the proposed play by the uninformed is in equilibrium for all $t < J^p(\gamma^0)$. For the informed, since there is no incentive to deviate to concession at $t = J^p(\gamma^0)$ by case (vii), the same argument as in case (ii) implies that there is no incentive to deviate for any smaller $t$.

Now suppose that $T > T_*$. The above analysis immediately implies that for any $\gamma^0 < \min\{1, p(T)\}$, the equilibrium is given by the uninformed playing as in the case of no deadlines, until time to deadline is given by $T_*$, after which point the equilibrium continues as described in Proposition 2, with the initial belief is given by $g(T - T_*; \gamma^0)$ and the deadline $T_*$. For any $\gamma^0 \in [\min\{1, p(T)\}, 1)$, the equilibrium is given by the uninformed persisting for all $t$.

Corollary 1. *Suppose that $T > T_*$ and Assumption 1 holds. There is an equilibrium in which the informed always persists, and for all $\gamma^0 < \min\{1, p(T)\}$ the uninformed types concede with a flow rate $x(t) = \delta_*/g(t; \gamma^0)$ for all $t < T_*$, continued by the equilibrium play given by Proposition 2 with the initial belief $g(T - T_*; \gamma^0)$ and the deadline $T_*$; and for all $\gamma^0 \in [\min\{1, p(T)\}, 1)$ the uninformed types persist for all $t$.*

## 4.2. Uniqueness

First, we show that in any symmetric equilibrium where the informed type always persists there is no atom with a strictly positive probability of concession $y(t) > 0$ at any time $t < T$. Suppose there is one. Upon reaching $t$, with his belief that the state is $M$ given by $\gamma(t)$, the equilibrium payoff to the uninformed from conceding is

$$\gamma(t)y(t)U_r + \gamma(t)(1 - y(t))\underline{\pi}_M + (1 - \gamma(t))\overline{\pi}_D.$$

For any small and positive $\eta$, either (i) $y(\tau) > 0$ for all $\tau \in (t, t + \eta)$, or (ii) there exists some $\tau(\eta) \in (t, t + \eta)$ such that $y(\tau(\eta)) = 0$. In the first case, consider the following deviation strategy for the uninformed: persist in the time interval $[t, t+\eta)$ and concede with probability 1 at time $t+\eta$. Since $y(\tau) > 0$ for all $\tau \in [t, t+\eta)$, the equilibrium probability that the opposing uninformed type concedes in this time interval is 1. The expected payoff from this deviation strategy to the uninformed is therefore bounded from below by $U(\gamma(t)) - \eta\delta$. In the second case, consider instead the following deviation strategy for the uninformed: persist in the time interval $[t, \tau(\eta))$ and concede with probability 1 at time $\tau(\eta)$. This strategy ensures that the probability of opposing uninformed type concedes in the time interval $[t, \tau(\eta))$ is at least $y(t)$. Further, regardless of whether at time $\tau(\eta)$ the opposing uninformed type persists or continuously randomizes, the probability of concession is 0. Thus, The expected payoff from this deviation strategy to the uninformed is therefore bounded from below by

$$\gamma(t)y(t)\overline{\pi}_M + \gamma(t)(1-y(t))\underline{\pi}_M + (1-\gamma(t))\overline{\pi}_D - (\tau(\eta) - t)\delta.$$

Since $U_r$ is strictly less than $\overline{\pi}_M$ by symmetry, in either case there is a profitable deviation for the uninformed type for sufficiently small $\eta$, a contradiction.

Second, we show that if in equilibrium at some time $t$ the flow rate of concession for the uninformed is $x(t) = 0$, then $x(\tau) = 0$ for all $\tau \in (t, T)$. Suppose not. Then, define $t' = \inf_{\tau > t}\{\tau : x(\tau) > 0\}$, and we have $t' \in (t, T)$. Since by the previous claim there is atom in the rate of concession at $t'$ , the equilibrium payoff for the uninformed type upon reaching $t'$ is $U(\gamma(t')$, with $\gamma(t') = \gamma(t)$. Thus, the equilibrium expected payoff to uninformed at time $t$ is $U(\gamma(t)) - (t' - t)\delta$. This is strictly smaller than the payoff from conceding at $t$ instead.

Third, we claim that for any initial belief $\gamma$ of the uninformed, there is a unique value $B(\gamma)$ of the deadline such that in equilibrium the uninformed persists until the deadline is reached if $T < B(\gamma)$ and must initially concede with a positive flow rate if $T > B(\gamma)$.[5] This value is given by the boldfaced, piece-wise linear function in Figure 1, or

$$B(\gamma) = \begin{cases} s^{-1}(\gamma) & \text{if } \gamma \leq \gamma_*, \\ [0, T^*] & \text{if } \gamma = \gamma_*, \\ p^{-1}(\gamma) & \text{if } \gamma > \gamma_*. \end{cases}$$

[5] When $T = B(\gamma)$ for $\gamma \neq \gamma_*$, or $T \in [0, T_*]$ for $\gamma = \gamma_*$, the equilibrium is not unique because $x(0)$ could be either positive or 0.

Note that by construction, for each $\gamma \neq \gamma_*$, by construction at $T = B(\gamma)$, given the opposing uninformed type persisting until the deadline and then playing according to the unique equilibrium in the no-delay game with $T = 0$, the uninformed type is indifferent between following the same strategy and conceding immediately, getting a payoff of $U(\gamma)$ as in (3.2). For $\gamma = \gamma_*$, there is a continuum of equilibria in the no-delay game with $T = 0$, and at any $T \leq T_*$, given the selection of equilibrium strategy in the no-delay game characterized by case (iv) in Proposition 2, the uninformed type is again indifferent between persisting until the deadline and then conceding, and conceding immediately. Fix any initial belief $\gamma$. If in some equilibrium $x(0) = 0$ for some deadline $T > B(\gamma)$ and $\gamma \neq \gamma_*$, or $T > T_*$ and $\gamma = \gamma_*$, then by the claim established above, $x(t) = 0$ for all $t \in [0, T)$. The payoff to the uninformed in this posited equilibrium is then $-T\delta + U^0(\gamma)$ where $U^0$ is the payoff function of the no-delay game (with the best equilibrium payoff corresponding to the uninformed conceding with probability 1 in the case of $\gamma = \gamma_*$). By the construction of $B(\gamma)$, this payoff is strictly less than the payoff from conceding immediately at $t = 0$. This contradiction establishes that in any equilibrium $x(0) > 0$ for any deadline $T > B(\gamma)$. Next, suppose that in some equilibrium $x(0) > 0$ for some deadline $T < B(\gamma)$ and $\gamma \neq \gamma_*$. The expected payoff to the uninformed from this posited equilibrium is equal to the payoff from conceding immediately, which is $U(\gamma)$. Suppose instead the uninformed persists until the deadline and then play the unique equilibrium strategy in the no-delay game corresponding to $\gamma$. Since the payoff to the uninformed increases whenever the uninformed opponent concedes, and in the no-delay game the equilibrium probability of concession is decreasing in the belief of the uninformed, the payoff from this deviation is at least as large as when the opposing uninformed type follows the same deviation strategy. It follows then from the construction of $B$ that this is a profitable deviation.

The proceeding claims establish that $B(\gamma)$ is the unique boundary in the $T$-$\gamma$ space that separates the region where on equilibrium path the uninformed type persists and the region where the uninformed concedes with a positive flow rate. It then follows from the uniqueness of the solution to the differential equation for equilibrium evolution of belief $\gamma(t)$ that the equilibrium constructed in Proposition 2 is unique in the class where the players use symmetric strategies and the informed types always persist.

As an immediate corollary, the equilibrium constructed in Proposition 1 is also the unique symmetric equilibrium in which the informed types always persist.

### 4.3. Deadline effects

From the preliminary analysis in Section 3, for any fixed initial belief $\gamma^0$ of the uninformed, the equilibrium payoff of the informed $V(\gamma^0)$ varies one-to-one with a constant $C$ in the solution (3.6) to the differential equation for $V$. If the boundary is given by the value of $V(g(t;\gamma^0))$, then the constant $C$ satisfies

$$C = \frac{g(t;\gamma^0)}{1-g(t;\gamma^0)} V(g(t;\gamma^0)) - \frac{\overline{\pi}_F - (\overline{\pi}_M - \underline{\pi}_M)}{1-g(t;\gamma^0)} + \ln(1-g(t;\gamma^0))(\overline{\pi}_M - \underline{\pi}_M). \qquad (4.1)$$

Further, the deadline effect of an increase in $T$ on the equilibrium payoff $V(\gamma^0)$ of the informed for the initial belief $\gamma^0$ is the same as its effect on the constant $C$.

We distinguish two cases. First, assume that $T \leq T_*$. For the uninformed, the deadline effect is strictly negative when either $\gamma^0 \in (s(T), \gamma_*)$, or $\gamma^0 \in (p(T), 1)$, and otherwise there is no effect. For the informed, there are 6 intervals of initial beliefs.

For $\gamma^0 \in [0, n(T))$, there is no deadline effect.

For $\gamma^0 \in [n(T), s(T))$, the boundary condition is

$$V(g(J^s(\gamma^0);\gamma^0)) = \overline{\pi}_F - \delta(T - J^s(\gamma^0)),$$

with

$$g(J^s(\gamma^0);\gamma^0) = 2\delta_*(T - J^s(\gamma^0)) = 1 - (1-\gamma^0)e^{\delta_* J^s(\gamma^0)}.$$

Differentiate the second equation with respect to $T$ to get:

$$\frac{\partial J^s(\gamma^0)}{\partial T} = \frac{2}{2-(1-\gamma^0)e^{\delta_* J^s(\gamma^0)}} > 0.$$

Now, substitute in the boundary condition

$$V(g(J^s(\gamma^0);\gamma^0) = \overline{\pi}_F - \frac{g(J^s(\gamma^0);\gamma^0)\delta}{2\delta_*}$$

in the expression (4.1) for the constant $C$ to get

$$C = -\overline{\pi}_F + \left( \frac{2 - g^2(J^s(\gamma^0);\gamma^0)}{2(1-g(J^s(\gamma^0);\gamma^0))} + \ln(1-g(J^s(\gamma^0);\gamma^0)) \right) (\overline{\pi}_M - \underline{\pi}_M).$$

Thus, $C$ depends on $T$ only through $g(J^s(\gamma^0);\gamma^0)$. The derivative of $C$ with respect to $g(J^s(\gamma^0);\gamma^0)$ is

$$\left(\frac{g(J^s(\gamma^0);\gamma^0)}{1-g(J^s(\gamma^0);\gamma^0)}\right)^2 \frac{\overline{\pi}_M - \underline{\pi}_M}{2} > 0.$$

Thus, $C$ is increasing in $g(J^s(\gamma^0);\gamma^0)$, implying that $C$ is decreasing in $J^s(\gamma^0)$. Since we already know that $J^s(\gamma^0)$ is increasing in $T$, it follows that $C$ decreases with $T$ and thus $V(\gamma^0)$ decreases with $T$. The deadline effect is negative.

For $\gamma^0 \in [s(T), \gamma_*)$, the deadline effect is negative.

For $\gamma^0 \in (\gamma_*, k(T)]$, the boundary condition is

$$\begin{aligned} V(g(J^k(\gamma^0);\gamma^0)) =& \frac{2\delta_*(T - J^k(\gamma^0))}{\gamma_*}\overline{\pi}_F + \left(1 - \frac{2\delta_*(T - J^k(\gamma^0))}{\gamma_*}\right)\frac{\overline{\pi}_F + \underline{\pi}_F}{2} \\ & - \delta(T - J^k(\gamma^0)), \end{aligned}$$

with

$$g(J^k(\gamma^0);\gamma^0) = \gamma_*.$$

Since $J^k(\gamma^0)$ is independent of $T$, the constant $C$ varies with $T$ only because the boundary value $V(\gamma_*)$ depends on $T$. We have

$$\frac{\mathrm{d}V(\gamma_*)}{\mathrm{d}T} = \frac{\delta}{\gamma_*}\frac{\overline{\pi}_F - \underline{\pi}_F}{\overline{\pi}_M - \underline{\pi}_M} - \delta,$$

which is positive by Assumption 1. Thus, in this range $V(\gamma^0)$ increases with $T$, and the deadline effect is positive.

In the case $\gamma^0 \in [k(T), p(T))$, the boundary condition is

$$V(g(J^p(\gamma^0);\gamma^0)) = \frac{\overline{\pi}_F + \underline{\pi}_F}{2} - \delta(T - J^p(\gamma^0)),$$

with

$$g(J^p(\gamma^0);\gamma^0) = \gamma_* + 2(1-\gamma_*)\delta_*(T - J^p(\gamma^0)) = 1 - (1-\gamma^0)e^{\delta_* J^p(\gamma^0)}.$$

Differentiating the second equation with respect to $T$, we have

$$\frac{\partial J^p(\gamma^0)}{\partial T} = \frac{2(1-\gamma_*)}{2(1-\gamma_*) - (1-\gamma^0)e^{\delta_* J^p(\gamma^0)}} > 0.$$

Substituting in the boundary condition

$$V(g(J^p(\gamma^0);\gamma^0) = \frac{\overline{\pi}_F + \underline{\pi}_F}{2} - \frac{(g(J^p(\gamma^0);\gamma^0) - \gamma_*)\delta}{2(1-\gamma_*)\delta_*} \tag{4.2}$$

in the expression (4.1) for the constant $C$, we get

$$\begin{aligned} C = -\overline{\pi}_F - \frac{g(J^p(\gamma^0);\gamma^0)}{1-g(J^p(\gamma^0);\gamma^0)} \left( \frac{\overline{\pi}_F - \underline{\pi}_F}{2} + \frac{g(J^p(\gamma^0);\gamma^0) - \gamma_*}{1-\gamma_*} \frac{\overline{\pi}_M - \underline{\pi}_M}{2} \right) \\ + \left( \frac{1}{1-g(J^p(\gamma^0);\gamma^0)} + \ln(1 - g(J^p(\gamma^0);\gamma^0)) \right) (\overline{\pi}_M - \underline{\pi}_M). \end{aligned}$$

Thus, $C$ depends on $T$ only through $g(J^p(\gamma^0);\gamma^0)$. The derivative of $C$ with respect to $g(J^p(\gamma^0);\gamma^0)$ has the same sign as

$$-(\overline{\pi}_F - \underline{\pi}_F) + \left( \gamma_* - \frac{1-g(J^p(\gamma^0);\gamma^0)}{1-\gamma_*} (g(J^p(\gamma^0);\gamma^0) - \gamma_*) \right) (\overline{\pi}_M - \underline{\pi}_M).$$

Since $g(J^p(\gamma^0);\gamma^0) > \gamma_*$, the above is negative by Assumption 1. Thus, $C$ is increasing in $J^p(\gamma^0)$. Since we already know that $J^p(\gamma^0)$ is increasing in $T$, we have $C$ increases with $T$ and thus $V(\gamma^0)$ increases with $T$. The deadline effect is positive.

Finally, when $\gamma^0 \in [p(T), 1]$, the deadline effect is negative.

Given the above analysis, and given Corollary 1, the deadline effects of increasing $T$ on the equilibriums of the informed and uninformed for the case of when $T > T_*$ are now straightforward to obtain. Note that $s(T_*) = \gamma_*$. For both the uninformed and the informed, the deadline effects are negative for any $\gamma^0 \in [\min\{1, p(T)\}, 1)$. For the uninformed, there are no deadline effects for any $\gamma^0 < \min\{1, p(T)\}$. For the informed, the deadline effects for any $\gamma^0 < \min\{1, p(T)\}$ are determined by the value of $g(T - T_*;\gamma^0)$ relative to the thresholds $n(T_*)$, $\gamma_*$, $k(T_*)$ and $p(T_*)$: there are no deadline effects if $g(T - T_*;\gamma^0) \in [0, n(T_*))$; otherwise, the deadline effect is negative if $g(T-T_*;\gamma^0) \in [n(T_*)), \gamma_*)$, positive if $g(T-T_*;\gamma^0) \in (\gamma_*, p(T_*))$, and negative if $g(T-T_*;\gamma^0) \in [p(T_*), \min\{1, p(T)\})$.

### 4.4. Optimal deadline

The optimal deadline is 0 for any initial belief of the uninformed $\gamma^0 \leq \gamma_*$. This follows because the efficient outcome is obtained with no delay, that is, $T = 0$.

For any $\gamma^0 \geq \gamma_*$, let $S(\gamma^0)$ be the deadline greater than $T_*$ such that when the belief of the uninformed $g(t;\gamma^0)$ reaches $\gamma_*$, the time remaining is $T_*$, illustrated by the dotted curve in Figure 1 starting at $T = T_*$ and $\gamma = \gamma_*$. That is,

$$g(S(\gamma^0) - T_*; \gamma^0) = \gamma_*.$$

By definition $S(\gamma_*) = T_*$, and $S(\gamma^0)$ is an increasing function. We claim that the optimal deadline for any $\gamma^0 > \gamma_*$ is either 0, or $S(\gamma^0)$.

First, for any $\gamma^0 \in (\gamma_*, p(T_*))$, the analysis for the case of $T \leq T_*$ in the last subsection implies that both the payoff to the informed $V(\gamma^0)$ and the payoff to the uninformed are decreasing in deadline $T$ for all $T$ such that $p(T) < \gamma^0$; and then $U(\gamma^0)$ is independent of $T$ and $V(\gamma^0)$ increases with $T$ for all $T < T_*$ such that $p(T) > \gamma^0$. The analysis for the case of $T > T_*$ implies that $V(\gamma^0)$ continues to increase in $T$ and $U(\gamma^0)$ stays unchanged for $T \in (T_*, S(\gamma^0))$; and for both $V(\gamma^0)$ and $U(\gamma^0)$ weakly decrease with $T$ for all longer deadlines. It follows immediately that the optimal deadline for any $\gamma^0 \in (\gamma_*, p(T_*))$ is either 0 or $S(\gamma^0)$.

Second, for any $\gamma^0 > p(T_*)$, the deadline effects on both $U(\gamma^0)$ and $V(\gamma^0)$ are negative for all $T < T_*$. The deadline effects continue to be negative for all longer deadline $T$, until $T$ satisfies $g(T - T_*; \gamma^0) = p(T_*)$, at which point $U(\gamma^0)$ becomes a constant and $V(\gamma^0)$ starts to increase. This continues until $T$ reaches $S(\gamma^0)$, after which point both $V(\gamma^0)$ and $U(\gamma^0)$ weakly decrease with $T$. Thus, the optimal deadline is also either 0 or $S(\gamma^0)$.

Thus for any initial belief $\gamma^0 > \gamma_*$ we have 0 and $S(\gamma^0)$ are the only local maxima for deadline. Now we determine whether the optimal deadline is strictly positive. Let $W^0(\gamma^0)$ denote ex ante welfare with a deadline of $T = 0$. We have

$$W^0(\gamma^0) = \frac{1}{2-\gamma^0} U^0(\gamma^0) + \frac{1-\gamma^0}{2-\gamma^0} V^0(\gamma^0).$$

For $\gamma^0 \in (\gamma_*, 1]$, this is

$$W^0(\gamma^0) = \frac{1}{2(2-\gamma^0)} \left( \gamma^0(\overline{\pi}_M + \underline{\pi}_M) + (1-\gamma^0)(\overline{\pi}_D + \underline{\pi}_D) + (1-\gamma^0)(\overline{\pi}_F + \overline{\pi}_F) \right).$$

Let $U^S(\gamma^0)$ and $V^S(\gamma^0)$ be the welfare of the uninformed and the informed when $T = S(\gamma^0)$. We have $U^S(\gamma^0) = U(\gamma^0)$ as given by (3.2), and using (3.6), (4.1) and the boundary condition

$$V(\gamma_*) = \overline{\pi}_F - \delta T_*,$$

we have

$$V^S(\gamma^0) = \overline{\pi}_F - \frac{1-\gamma^0}{\gamma^0}\left(\ln\left(\frac{1-\gamma^0}{1-\gamma_*}\right) + \frac{1}{1-\gamma^0} - \frac{2-\gamma_*^2}{2(1-\gamma_*)}\right)(\overline{\pi}_M - \underline{\pi}_M).$$

The difference in ex ante welfare under $T = S(\gamma^0)$ and under $T = 0$ is

$$W^S(\gamma^0) - W^0(\gamma^0) = \frac{1}{2(2-\gamma^0)}\Delta(\gamma^0),$$

where

$$\begin{aligned}\Delta(\gamma^0) =& (1-\gamma^0)(\overline{\pi}_F - \underline{\pi}_F + \overline{\pi}_D - \underline{\pi}_D) - \gamma^0(\overline{\pi}_M - \underline{\pi}_M) \\ & - \frac{2(1-\gamma^0)^2}{\gamma^0}\left(\ln\left(\frac{1-\gamma^0}{1-\gamma_*}\right) + \frac{1}{1-\gamma^0} - \frac{2-\gamma_*^2}{2(1-\gamma_*)}\right)(\overline{\pi}_M - \underline{\pi}_M).\end{aligned}$$

Take derivative of $\Delta$ with respect to $\gamma^0$ to obtain:

$$\begin{aligned}\Delta'(\gamma^0) = & -(\overline{\pi}_F - \underline{\pi}_F + \overline{\pi}_D - \underline{\pi}_D) - 3(\overline{\pi}_M - \underline{\pi}_M) \\ & + \frac{2(1-(\gamma^0)^2)}{(\gamma^0)^2}\left(\ln\left(\frac{1-\gamma^0}{1-\gamma_*}\right) + \frac{1}{1-\gamma^0} - \frac{2-\gamma_*^2}{2(1-\gamma_*)}\right)(\overline{\pi}_M - \underline{\pi}_M).\end{aligned}$$

The limit of the last term as $\gamma^0$ goes to 1 is equal to $4(\overline{\pi}_M - \underline{\pi}_M)$. Further, it is increasing for all $\gamma^0 > \gamma_*$: the derivative has the same sign as

$$-1 - (1+\gamma^0)^2 - 2\ln\left(\frac{1-\gamma^0}{1-\gamma_*}\right) + \frac{2-\gamma_*^2}{1-\gamma_*},$$

which is an increasing function of $\gamma^0$; at $\gamma^0 = \gamma_*$, this derivative is equal to $\gamma_*^3/(1-\gamma_*)$, which is positive. Thus,

$$\Delta'(\gamma^0) \leq -(\overline{\pi}_F - \underline{\pi}_F + \overline{\pi}_D - \underline{\pi}_D) + \overline{\pi}_M - \underline{\pi}_M,$$

which is negative by Assumption 1. We have proved that $\Delta(\gamma^0) = 0$ implies $\Delta'(\gamma^0) < 0$ for all $\gamma^0 > \gamma_*$. Finally,[6]

$$\lim_{\gamma^0 \downarrow \gamma_*} \Delta(\gamma_*) = (1-\gamma_*)(\overline{\pi}_F - \underline{\pi}_F - \gamma_*(\overline{\pi}_M - \underline{\pi}_M)),$$

[6] At $\gamma^0 = \gamma_*$ and $T = 0$, there is a continuum of equilibria. For any $\gamma^0 > \gamma_*$, the equilibrium is unique, and thus the limit of $\Delta(\gamma^0)$ as $\gamma^0$ converging to $\gamma_*$ from above is well-defined.

which is positive by Assumption 1, and

$$\lim_{\gamma^0 \to 1} \Delta(1) = -(\overline{\pi}_M - \underline{\pi}_M) < 0.$$

By the intermediate value theorem, there exists a $\overline{\gamma} \in (\gamma_*, 1)$ such that $W^S(\overline{\gamma}) - W^0(\overline{\gamma}) = 0$. Moreover, the single-crossing property of $W^S - W^0$ implies that such $\overline{\gamma}$ is unique, with $W^S(\gamma^0) > W^0(\gamma^0)$ if and only if $\gamma^0 \in (\gamma_*, \overline{\gamma})$.

## 5. Stochastic Deadlines

We modify the repeated proposal game in section 2 by adding exogenous breakdowns, interpreted as stochastic deadlines. Let $\epsilon > 0$ be the constant rate of exogenous exit, so that upon reaching time $t$, the probability that the game ends exogenously in the next time interval $\mathrm{d}t$ is $\epsilon \mathrm{d}t$. In this event, we assume that the decision is made by a fair coin flip. For simplicity we assume that $T = \infty$.

### 5.1. Equilibrium

To construct the differential equation for the evolution of the equilibrium belief $\gamma(t)$, note that the expected payoff of the uninformed from conceding is still given by (3.1), and therefore the value function $U(\gamma)$ is unchanged from the case of $\epsilon = 0$. The payoff from persisting becomes[7]

$$\gamma(t)x_\epsilon(t)\mathrm{d}t\,\overline{\pi}_M + \Big(\gamma(t)(1 - x_\epsilon(t)\mathrm{d}t) + (1-\gamma(t))\Big)(1 - \epsilon\mathrm{d}t)(-\delta\mathrm{d}t + \mathcal{U}(t+\mathrm{d}t))$$
$$+ \epsilon\mathrm{d}t\left((1-\gamma(t))\frac{\overline{\pi}_D + \underline{\pi}_D}{2} + \gamma(t)(1 - x_\epsilon(t)\mathrm{d}t)\frac{\overline{\pi}_M + \underline{\pi}_M}{2}\right),$$

where $x_\epsilon(t)$ denotes the flow rate of concession by the uninformed. Following the same steps in deriving the differential equation for $\gamma(t)$ in the case of $\epsilon = 0$, we have

$$-\frac{\dot{\gamma}(t)}{1-\gamma(t)} = \delta_* - \frac{\epsilon}{2(1-\gamma_*)}(\gamma(t) - \gamma_*), \tag{5.1}$$

[7] The expected payoffs from conceding and persisting are computed under the assumption that for each infinitesimally small time interval $(t, t + \mathrm{d}t)$ exogenous exit happens only if an agreement has not just been reached. This assumption is inconsequential as we take $\mathrm{d}t$ to zero.

and

$$x_\epsilon(t) = \frac{1}{\gamma(t)}\left(\delta_* - \frac{\epsilon}{2(1-\gamma_*)}(\gamma(t)-\gamma_*)\right).$$

The differential equation (5.1) for $\gamma(t)$ can be explicitly solved. The general solution is given by

$$\gamma(t) = 1 - \frac{(1-\gamma_*)(2\delta_* - \epsilon)}{Ce^{-t(2\delta_*-\epsilon)/2} - \epsilon},$$

where $C$ is a constant to be determined by an initial condition.

To describe the equilibrium, we distinguish two cases depending on the initial belief $\gamma^0$. It is convenient to define

$$\alpha \equiv \gamma_* + (1-\gamma_*)\frac{2\delta_*}{\epsilon}.$$

Note that $\alpha > \gamma_*$. Further, $\alpha < 1$ is equivalent to $\epsilon > 2\delta_*$. In the first case, $\gamma^0 < \min\{1, \alpha\}$. For any time $t$, the uninformed types are indifferent between persisting and conceding, and the equilibrium play is completely by the above differential equation. Using the initial condition, we can rewrite the solution $g_\epsilon(t;\gamma^0)$ as

$$\frac{1-\gamma^0}{1-g_\epsilon(t;\gamma^0)} = \frac{1-\gamma^0}{1-\alpha} + \frac{\alpha-\gamma^0}{\alpha-1}e^{-t(2\delta_*-\epsilon)/2}.$$

The corresponding flow rate of concession for the uninformed types is given by

$$x_\epsilon(t) = \frac{2\delta_*-\epsilon}{2g_\epsilon(t;\gamma^0)}\frac{\alpha - g_\epsilon(t;\gamma^0)}{\alpha-1}.$$

Note that when $\alpha < 1$, as $\gamma^0$ approaches $\alpha$ from below, we have that $x_\epsilon(0)$ converges to 0. Finally let $Q_\epsilon(\gamma^0)$ be the terminal time, which solves $g_\epsilon(Q_\epsilon(\gamma^0);\gamma^0) = 0$. Then, we have

$$Q_\epsilon(\gamma^0) = -\frac{2}{2\delta_*-\epsilon}\ln\left(\frac{\alpha(1-\gamma^0)}{\alpha-\gamma^0}\right).$$

Note that when $\alpha < 1$, as $\gamma^0$ approaches $\alpha$ from below $Q_\epsilon$ approaches infinity.

In the second case, in equilibrium the uninformed types persist with probability 1 at any time $t$ just as the informed types, with the game ending with an exogenous exit. Note that $\alpha \geq \gamma_*$ and is decreasing in $\epsilon$. Thus the second case occurs only when the exit rate $\epsilon$ is large and the initial belief $\gamma^0$ is high. We have the following proposition.

Proposition 3. *Suppose that $T = \infty$ and $\epsilon > 0$, and Assumption 1 holds. For any initial belief $\gamma^0 \in (0, \min\{1,\alpha\})$, there exists an equilibrium in which at any time $t \in [0, Q_\epsilon(\gamma^0))$,* *the uninformed types concede with a flow rate $x_\epsilon(t)$ and concede with probability 1 at $t = Q_\epsilon(\gamma^0)$, while the informed types always persist. For any initial belief $\gamma^0 \in [\min\{1,\alpha\},1)$, there exists an equilibrium in which both uninformed and informed types always persist.*

For the case of $\gamma^0 \in (0, \min\{1,\alpha\})$, it suffices to verify that the equilibrium payoff of the informed types is at least as large as the payoff from deviating to conceding, which is equal to $\underline{\pi}_F$ regardless of $\epsilon$. The differential equation for the value function of the informed is now

$$V_\epsilon'(\gamma) = -\frac{(\alpha-\gamma_*)(\overline{\pi}_M - \underline{\pi}_M) + (1-\gamma_*)(\overline{\pi}_F - \underline{\pi}_F)}{(1-\gamma)(\alpha-\gamma)} + \frac{\alpha - \gamma + 2(1-\gamma_*)\gamma}{\gamma(1-\gamma)(\alpha-\gamma)}(\overline{\pi}_F - V_\epsilon(\gamma)).$$

The general solution is

$$V_\epsilon(\gamma) = \overline{\pi}_F - \left(1 - \frac{1-\gamma}{\gamma}\left(\frac{\alpha}{2(1-\gamma_*)} - C_\epsilon\left(\frac{1-\gamma}{\alpha-\gamma}\right)^{2\epsilon/(2\delta_*-\epsilon)}\right)\right) \cdot \frac{(\alpha-\gamma_*)(\overline{\pi}_M - \underline{\pi}_M) + (1-\gamma_*)(\overline{\pi}_F - \underline{\pi}_F)}{(\alpha-\gamma_*)+(1-\gamma_*)},$$

where $C_\epsilon$ is a constant to be determined. The boundary condition is $V_\epsilon(0) = \overline{\pi}_F$, as in the case of no deadlines. This implies that $C_\epsilon$ must satisfy

$$\frac{\alpha}{2(1-\gamma_*)} - C_\epsilon\left(\frac{1-\gamma}{\alpha-\gamma}\right)^{2\epsilon/(2\delta_*-\epsilon)} = 0.$$

Substitute the resulting $C_\epsilon$ back into the solution to obtain:

$$V_\epsilon(\gamma) = \overline{\pi}_F - \left(1 - \frac{1-\gamma}{\gamma}\frac{H(\gamma)}{2(1-\gamma_*)}\right)\frac{(\alpha-\gamma_*)(\overline{\pi}_M - \underline{\pi}_M) + (1-\gamma_*)(\overline{\pi}_F - \underline{\pi}_F)}{(\alpha-\gamma_*)+(1-\gamma_*)}, \quad (5.2)$$

where

$$H(\gamma) \equiv \alpha - \alpha\left(\frac{\alpha(1-\gamma)}{\alpha-\gamma}\right)^{2\epsilon/(2\delta_*-\epsilon)}.$$

It can be easily verified that $\lim_{\gamma\to 0} V(\gamma) = \overline{\pi}_F$. Further, we have $H(\gamma) > 0$ for all $\gamma \in (0,\alpha)$, whether $\alpha < 1$ or $\alpha > 1$. Since

$$\frac{(\alpha-\gamma_*)(\overline{\pi}_M - \underline{\pi}_M) + (1-\gamma_*)(\overline{\pi}_F - \underline{\pi}_F)}{(\alpha-\gamma_*)+(1-\gamma_*)} \le \overline{\pi}_F - \underline{\pi}_F,$$

it follows immediately from Assumption 1 that $V_\epsilon(\gamma) \ge \underline{\pi}_F$ for all $\gamma$.

For the case of $\gamma^0 \in [\min\{1,\alpha\},1)$, in equilibrium the game ends with exogenous exit, with a terminal payoff of $\frac{1}{2}(\overline{\pi}_F + \underline{\pi}_F)$ to the informed and

$$\gamma \frac{\overline{\pi}_M + \underline{\pi}_M}{2} + (1-\gamma)\frac{\overline{\pi}_D + \underline{\pi}_D}{2}$$

to the uninformed. Further, the exogenous exit time follows an exponential distribution with parameter $\epsilon$, and hence the expected duration of the game is $1/\epsilon$. Thus, the equilibrium expected payoff loss from delay is $\delta/\epsilon$ for both the informed and the uninformed. If the uninformed types deviate to conceding, the expected payoff is

$$\gamma \underline{\pi}_M + (1-\gamma)\overline{\pi}_D < \gamma \frac{\overline{\pi}_M + \underline{\pi}_M}{2} + (1-\gamma)\frac{\overline{\pi}_D + \underline{\pi}_D}{2} - \frac{\delta}{\epsilon},$$

because $\gamma < \alpha$. For the informed, the expected payoff from concession is instead $\underline{\pi}_F$, which is lower than the equilibrium payoff if

$$\overline{\pi}_F - \underline{\pi}_F \geq \frac{2\delta}{\epsilon}.$$

The above follows from Assumption 1, and from the assumption that $\alpha < 1$.[8]

### 5.2. Deadline effects

Now we examine the deadline effects of an increase in $\epsilon$ on the equilibrium play. In the case of $\gamma^0 \in [\min\{1,\alpha\},1)$, clearly there are no deadline effects. As $\epsilon$ increases, the game ends sooner through faster exogenous exit. For the case of $\gamma^0 \in (0,\min\{1,\alpha\})$, the derivative of the terminal time $Q_\epsilon(\gamma^0)$ with respect to $\epsilon$ has the same sign as

$$\frac{2\delta_* - \epsilon}{2}\left(\frac{Q_\epsilon(\gamma^0)}{2} - \frac{(\alpha-\gamma_*)^2\gamma^0}{\alpha(\alpha-\gamma^0)\delta_*}\right).$$

Since $Q_\epsilon(0) = 0$, the above derivative is 0 when $\gamma^0 = 0$. Further, its derivative with respect to $\gamma^0$ can be shown to have the same sign as $\gamma^0 - \gamma_*$. Thus, the derivative of $Q_\epsilon(\gamma^0)$ with respect to $\epsilon$ is negative for all $\gamma^0 \in (0,\gamma_*)$, and it can cross 0 at most once.[9]

[8] This derivation also shows that when $\alpha < 1$, the equilibrium payoff function for the informed $V(\gamma)$ is continuous at $\gamma = \alpha$.

[9] When $\delta_\epsilon > 0$, which is equivalent to $\alpha > 1$, the derivative of $Q_\epsilon(\gamma^0)$ with respect to $\epsilon$ approaches infinity as $\gamma^0$ approaches 1.

Now we turn to the deadline effects on the evolution of the equilibrium beliefs of the uninformed types $g_\epsilon(t;\gamma^0)$ in the case of $\gamma^0 \in (0, \min\{1, \alpha\})$. The derivative of $(1 - \gamma^0)/(1 - g_\epsilon(t;\gamma^0))$ with respect to $\epsilon$ is given by

$$te^{-t(2\delta_*-\epsilon)/2} \frac{(\alpha - \gamma^0)\delta_*}{(\alpha - \gamma_*)(2\delta_* - \epsilon)} - (1 - e^{-t(2\delta_*-\epsilon)/2}) \frac{2(1 - \gamma^0)\delta_*}{(1 - \gamma_*)(2\delta_* - \epsilon)^2}.$$

Taking the derivative of the above expression with respect to $t$, we find that it has the same sign as

$$\frac{\gamma^0 - \gamma_*}{1 - \gamma_*} - \frac{\alpha - \gamma^0}{\alpha - \gamma_*} \delta_* t,$$

which is decreasing in $t$ because by assumption $\gamma^0 < \alpha$. Note that the above is negative at any $t$ if $\gamma^0 < \gamma_*$. As a result, for any initial belief $\gamma^0 < \gamma_*$, if $\epsilon' > \epsilon$, then starting at the same initial belief $\gamma^0$, the equilibrium time path of belief of the uninformed $g_{\epsilon'}(t;\gamma^0)$ for $\epsilon'$ always stays below the time path $g_\epsilon(t;\gamma^0)$. Since $x_\epsilon(t)$ increases with $\epsilon$ for all $\gamma(t) < \gamma_*$, we have that the hazard rate conditional on the state being $L$ or $R$, given by $x_\epsilon(t) + \epsilon$, increases in $\epsilon$ for all $t$. Further, for any $\gamma^0 \in (\gamma_*, \min\{1, \alpha\})$, the time path $g_\epsilon(t;\gamma^0)$ for $\epsilon$ is initially above the time path $g_{\epsilon'}(t;\gamma^0)$, and can cross the latter at most once. In this case, for the beginning of the game, the time path of beliefs $\gamma(t)$ decreases at a slower rate as $\epsilon$ increases. The deadline effects on the hazard rate $x_\epsilon(t) + \epsilon$, are ambiguous.

### 5.3. Optimal exogenous exit rate

Welfare analysis is straightforward in the case of $\gamma^0 \in [\min\{1, \alpha\}, 1)$. Both $V_\epsilon(\gamma^0)$ and $U_\epsilon(\gamma^0)$ are increasing in $\epsilon$, because a greater exogenous rate of exit reduces the expected duration of the equilibrium play without affecting the corresponding terminal payoffs.

For the case of $\gamma^0 \in (0, \min\{1, \alpha\})$, the payoff function for the uninformed $U_\epsilon(\gamma^0)$, given by $\gamma^0 \underline{\pi}_M + (1 - \gamma^0)\overline{\pi}_D$, is strictly decreasing in the initial belief $\gamma^0$, but an increase in $\epsilon$ has no effect on it. To analyze the payoff function $V_\epsilon(\gamma^0)$ given by (5.2), it is convenient to use the fact that $\lim_{\gamma \to 0} H(\gamma) = 0$ to write

$$H(\gamma) = \int_0^\gamma h(\nu) \, \mathrm{d}\nu,$$

where

$$h(\nu) = \frac{2(1 - \gamma_*)}{(1 - \nu)^2} \left( \frac{\alpha(1 - \nu)}{\alpha - \nu} \right)^{(2\delta_*+\epsilon)/(2\delta_*-\epsilon)}.$$

To see how $V_\epsilon(\gamma^0)$ depends on the initial belief $\gamma^0$, we take the derivative of $H(\gamma)(1-\gamma)/\gamma$ with respect to $\gamma$, which yields

$$\begin{aligned}\frac{1-\gamma}{\gamma}h(\gamma)-\frac{1}{\gamma^2}H(\gamma)&=-\frac{\alpha}{\gamma^2}\left(1-\left(\frac{\alpha(1-\gamma)}{\alpha-\gamma}\right)^{2\epsilon/(2\delta_*-\epsilon)}\left(\frac{2(1-\gamma_*)\gamma}{\alpha-\gamma}+1\right)\right)\\&=-\frac{\alpha}{\gamma^2}\int_0^\gamma 2(1-\gamma_*)\left(\frac{\alpha(1-\nu)}{\alpha-\nu}\right)^{2\epsilon/(2\delta_*-\epsilon)}\frac{\nu((\alpha-\gamma_*)+(1-\gamma_*))}{(\alpha-\nu)^2(1-\nu)}\mathrm{d}\nu,\end{aligned}$$

which is negative because $\alpha>\gamma_*$. Thus, the payoff function of the informed is strictly decreasing in the initial belief $\gamma^0$.

Next, observe that $\alpha$ becomes arbitrarily large as $\epsilon$ goes to 0. We have

$$\lim_{\epsilon\to0}h(\nu)=\frac{2(1-\gamma_*)}{1-\nu}.$$

Thus,

$$\lim_{\epsilon\to0}H(\gamma)=2(1-\gamma_*)\ln(1-\gamma).$$

Since

$$\lim_{\epsilon\to0}\frac{(\alpha-\gamma_*)(\overline{\pi}_M-\underline{\pi}_M)+(1-\gamma_*)(\overline{\pi}_F-\underline{\pi}_F)}{(\alpha-\gamma_*)+(1-\gamma_*)}=\overline{\pi}_M-\underline{\pi}_M,$$

we have $\lim_{\epsilon\to0}V_\epsilon(\gamma)=V(\gamma)$, the payoff function for the informed when $\epsilon=0$.

To study the welfare effects of an increase in $\epsilon$ on $V_\epsilon(\gamma^0)$, note that both $\pi_\epsilon$ and $H(\gamma)$ depend on $\epsilon$. In fact, $\pi_\epsilon$ is decreasing in $\alpha$ and thus increasing in $\epsilon$. Further, since $\lim_{\gamma\to0}H(\gamma)=0$, and thus

$$\lim_{\gamma\to0}\frac{H(\gamma)}{\gamma}=\lim_{\gamma\to0}h(\gamma)=2(1-\gamma_*),$$

and since we have already shown that $H(\gamma)(1-\gamma)/\gamma$ is decreasing in $\gamma$, a sufficient condition for $V_\epsilon(\gamma)$ to be decreasing in $\epsilon$ is that $H(\gamma)$ is increasing in $\alpha$. A sufficient condition for the latter is that $\ln h(\gamma)$ is increasing in $\alpha$, or

$$-\ln\left(\frac{\alpha(1-\gamma)}{\alpha-\gamma}\right)+\frac{(\alpha-1)\gamma}{\alpha(1-\gamma)}\frac{(\alpha-\gamma_*)+(1-\gamma_*)}{2(1-\gamma_*)}>0.$$

Since the above is equal to 0 at $\gamma=0$, it is sufficient if its derivative with respect to $\gamma$ is strictly positive. This derivative is given by

$$\left(\frac{\alpha-1}{\alpha-\gamma}\right)^2\left(\frac{1}{1-\gamma}-\frac{1}{2(1-\gamma_*)}\right).$$

Therefore, $V_\epsilon(\gamma^0)$ decreases with $\epsilon$ so long as $\gamma^0 > \gamma_*$.[10]

## 6. Deadline Penalties

In this section we modify the model in section 2 by introducing an additional payoff loss $\lambda > 0$ for the two players when they fail to reach an agreement. This changes the no-delay game with $T = 0$ without affecting the differential equation for the evolution of the equilibrium belief of the uninformed. We view this modification as a robustness exercise, because it eliminates the continuum of equilibria in the no-delay game at $\gamma = \gamma_*$. Our main result that the optimal deadline is positive for an intermediate range of initial conflict turns out to be robust to this modification.

In this section we assume that $\lambda$ satisfies

Assumption 2. $\lambda < \frac{1}{2}(\overline{\pi}_M - \underline{\pi}_M)$.

In words, the deadline disagreement penalty is small enough that if the state is known to be $M$, the uninformed type still prefers the disagreement outcome of flipping a coin and paying the penalty $\lambda$ to conceding to the other side and avoiding the penalty. The role of this assumption in the analysis will become clear in the ensuing analysis.[11]

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[10] It is possible to identify a sufficient condition such that for small values of $\gamma^0$, the payoff function $V_\epsilon(\gamma^0)$ is increasing in $\epsilon$. To see this, note that $\pi_\epsilon$ is independent of $\epsilon$ if $\overline{\pi}_M - \underline{\pi}_M = \overline{\pi}_F - \underline{\pi}_F$. Since $h(\gamma)$ is point-wise decreasing in $\alpha$ for $\gamma < 2\gamma_* - 1$, there exist small values of $\gamma^0$ such that $H(\gamma)$ is decreasing in $\alpha$, implying $V_\epsilon(\gamma^0)$ is increasing in $\epsilon$.

[11] When Assumption 2 is violated, our equilibrium does not exist, because the equilibrium payoff $U^0(\gamma)$ for the uninformed from the no-delay game is too low relative to the payoff $U(\gamma)$ pinned down by atomless randomization. The equilibrium can be restored by allowing an additional atom in the concession rate at the deadline. Consider the equilibrium behavior at the deadline for $\gamma < \gamma^-$ where the uninformed type concedes with probability 1, and imagine that "an instant" before the deadline arrives the uninformed concedes with probability $y$. The payoff from following this strategy, by conceding, is

$$\gamma y U_r + \gamma(1-y)\underline{\pi}_M + (1-\gamma)\overline{\pi}_D,$$

while the payoff from persisting for this instant before the deadline and then conceding is

$$\gamma y \overline{\pi}_M + \frac{1}{2}\gamma(1-y)(\underline{\pi}_M + \overline{\pi}_M - 2\lambda) + (1-\gamma)\overline{\pi}_D.$$

For any value of $\lambda$, there is a $y < 1$ such that the uninformed is indifferent between the above two strategies. Further, for any value of $y$, the above payoff is at least as large as $U(\gamma)$, which we can use to replace the original no-delay payoff $U^0(\gamma)$ in constructing the boundary $B_\lambda$.

### 6.1. Equilibrium

First, we show that there is a unique equilibrium in the game without delay ($T = 0$). Fix a belief $\gamma$ that the state is $M$ for an uninformed player. Suppose that the opposing uninformed type concedes with probability $y$. The expected payoff to the uninformed player from conceding is

$$\gamma y\Big(\frac{\overline{\pi}_M + \underline{\pi}_M}{2} - \lambda\Big) + \gamma(1-y)\underline{\pi}_M + (1-\gamma)\overline{\pi}_D.$$

The expected payoff from persisting is

$$\gamma y\overline{\pi}_M + \gamma(1-y)\Big(\frac{\overline{\pi}_M + \underline{\pi}_M}{2} - \lambda\Big) + (1-\gamma)\Big(\frac{\overline{\pi}_D + \underline{\pi}_D}{2} - \lambda\Big).$$

The difference between the two payoffs is

$$-\gamma\Big(\frac{\overline{\pi}_M - \underline{\pi}_M}{2} - \lambda\Big) + (1-\gamma)\Big(\frac{\overline{\pi}_D - \underline{\pi}_D}{2} + \lambda\Big) - 2\gamma\lambda y.$$

The above is strictly decreasing in $y$, and therefore there is a unique equilibrium for any $\gamma$, given as follows. If

$$\gamma \leq \gamma_- \equiv \frac{\overline{\pi}_D - \underline{\pi}_D + 2\lambda}{\overline{\pi}_D - \underline{\pi}_D + \overline{\pi}_M - \underline{\pi}_M + 4\lambda},$$

then the difference in payoffs is always non-negative, and thus the unique equilibrium is $y = 1$; if

$$\gamma \geq \gamma_+ \equiv \frac{\overline{\pi}_D - \underline{\pi}_D + 2\lambda}{\overline{\pi}_D - \underline{\pi}_D + \overline{\pi}_M - \underline{\pi}_M},$$

the difference in payoffs is always non-positive and thus the unique equilibrium is $y = 0$; and if $\gamma \in (\gamma_-, \gamma_+)$, the unique equilibrium $y$ is given by

$$Y(\gamma) = \frac{\overline{\pi}_D - \underline{\pi}_D + 2\lambda}{4\lambda\gamma} - \frac{\overline{\pi}_D - \underline{\pi}_D + \overline{\pi}_M - \underline{\pi}_M}{4\lambda}. \tag{6.1}$$

By Assumption 2, we have $\gamma_+ \leq 1$. The continuum of equilibria when $\lambda = 0$ at $\gamma = \gamma_*$, with the uninformed conceding with probability between 0 and 1, is replaced by a set of

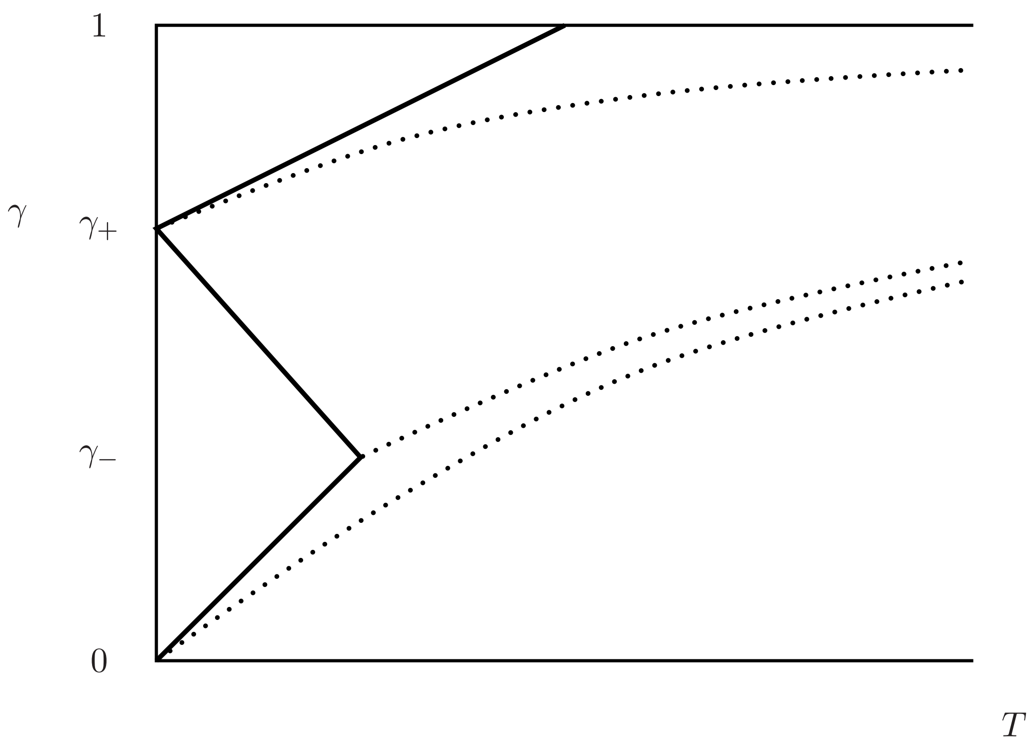


Figure 2

equilibria each corresponding to the uninformed with a belief $\gamma \in [\gamma_-, \gamma_+]$ conceding with some probability between 0 and 1.

Since $\lambda$ changes the equilibrium continuation upon reaching the deadline, we need to redefine the boundary in the $T$-$\gamma$ space that separates the region where on equilibrium path the uninformed type persists and the region where the uninformed concedes with a positive flow rate. As in the case of $\lambda = 0$, let $s_\lambda(T)$ be the lowest initial belief of the uninformed below $\gamma_-$ such that it is an equilibrium for the uninformed to persist until the deadline followed by the uninformed types switching to concede, given by

$$s_\lambda(T) = \frac{2\delta T}{\overline{\pi}_M - \underline{\pi}_M - 2\lambda}.$$

In Figure 2, this is given by the line segment starting at $\gamma = 0$ and ending at $\gamma = \gamma_-$. Note that $s_\lambda(T)$ becomes steeper as $\lambda$ increases, and coincides with the vertical axis when Assumption 2 is binding. It follows that when Assumption 2 holds, we have $s_\lambda(T) > n(T)$ for all $T$. Next, let $p_\lambda(T)$ be the be lowest initial belief above $\gamma_+$ of the uninformed for which it is an equilibrium to persist for the entire duration of the game, given by

$$p_\lambda(T) = \frac{\overline{\pi}_D - \underline{\pi}_D + 2\lambda + 2\delta T}{\overline{\pi}_D - \underline{\pi}_D + \overline{\pi}_M - \underline{\pi}_M}.$$

In Figure 2, this is given by the line segment starting at $\gamma = \gamma_+$ and ending at $\gamma = 1$. Finally, instead of the interval of $[0, T_*]$ at $\gamma = \gamma_*$, we have a decreasing linear function $T(\gamma)$ representing the longest deadline for any initial belief $\gamma$ between $\gamma_-$ and $\gamma_+$ for the uninformed such that it is an equilibrium persist until the deadline followed by conceding with probability $Y(\gamma)$ as given in (6.1). This function is given by

$$T(\gamma) = \frac{\gamma_+ - \gamma}{8\lambda\delta}(\overline{\pi}_M - \underline{\pi}_M + \overline{\pi}_D - \underline{\pi}_D)(\overline{\pi}_M - \underline{\pi}_M - 2\lambda). \tag{6.2}$$

In Figure 2, it is represented by the line segment connecting $\gamma = \gamma_-$ and $\gamma = \gamma_+$. The new boundary, boldfaced and piece-wise linear in Figure 2, is given by

$$B_\lambda(\gamma) = \begin{cases} s_\lambda^{-1}(\gamma) & \text{if } \gamma \le \gamma_-, \\ T(\gamma) & \text{if } \gamma \in (\gamma_-, \gamma_+), \\ p_\lambda^{-1}(\gamma) & \text{if } \gamma \ge \gamma_+. \end{cases}$$

To describe the equilibrium for any initial belief $\gamma^0$ and deadline $T$, note that there are two cases if $T > B_\lambda(\gamma^0)$: either $\gamma^0 > n(T)$, so that there is a unique time $J_\lambda(\gamma^0) < T$ satisfying

$$T - J_\lambda(\gamma^0) = B_\lambda(g(J_\lambda(\gamma^0); \gamma^0)),$$

or $\gamma^0 \le n(T)$, which is equivalent to $Q(\gamma^0) \le T$. In words, either when the updated belief $g(t; \gamma^0)$ of the uninformed reaches the boundary $B_\lambda$ there is time left before the deadline, or the updated belief reaches 0 before the deadline expires. We can now state the equilibrium using $B_\lambda$.

Proposition 4. *Suppose that $T < \infty$ and $\lambda > 0$, and Assumption 1 holds. For any initial belief $\gamma^0 \in [0, 1)$, there is an equilibrium in which the informed always persists, and the strategy of the uninformed is: (i) if $\gamma^0 \in [0, n(T)]$, concede with a flow rate $\delta_*/g(t; \gamma^0)$ and concede with probability 1 at $t = Q(\gamma^0)$; (ii) if $\gamma^0 > n(T)$ and $T > B_\lambda(\gamma^0)$, concede with a flow rate $\delta_*/g(t; \gamma^0)$ for $t < J_\lambda(\gamma^0)$, and persist for all $t \in [J_\lambda(\gamma^0), T)$,* *followed by conceding with probability 1 at* $t = T$ *if* $g(J_\lambda(\gamma^0);\gamma^0) \leq \gamma_-$, *with probability* $Y(g(J_\lambda(\gamma^0);\gamma^0))$ *if* $g(J_\lambda(\gamma^0);\gamma^0) \in (\gamma_-, \gamma_+)$, *and with probability 0 if* $g(J_\lambda(\gamma^0);\gamma^0) \geq \gamma_+$; *(iii) if* $T \leq B_\lambda(\gamma^0)$, *persist for all* $t \in [0, T)$, *followed by conceding with probability 1 at* $t = T$ *if* $\gamma^0 \leq \gamma_-$, *with probability* $Y(\gamma^0)$ *if* $\gamma^0 \in (\gamma_-, \gamma_+)$, *and with probability 0 if* $\gamma^0 \geq \gamma_+$.

The proof of the above proposition follows the same steps as in the proof of Proposition 2, so we sketch the main steps. The proof of case (i) is identical to the no deadlines case, because by construction $Q(\gamma^0) < T$ for all $\gamma^0 \in [0, n(T)]$.

For case (iii), it suffices to verify that the informed types do not want to deviate to conceding. The equilibrium payoff is

$$\tilde{Y}(\gamma^0)\overline{\pi}_F + (1 - \tilde{Y}(\gamma^0))\Big(\frac{\overline{\pi}_F + \underline{\pi}_F}{2} - \lambda\Big) - \delta T,$$

where $\tilde{Y}(\gamma^0) = Y(\gamma^0)$ for $\gamma^0 \in (\gamma_-, \gamma_+)$, equals 1 for $\gamma \leq \gamma_-$ and 0 for $\gamma \geq \gamma_+$. The payoff from conceding right away is $\underline{\pi}_F$. It is optimal for the informed to persist if

$$\tilde{Y}(\gamma^0)(\overline{\pi}_F - \underline{\pi}_F) + (1 - \tilde{Y}(\gamma^0))\Big(\frac{\overline{\pi}_F - \underline{\pi}_F}{2} - \lambda\Big) \geq \delta T.$$

By construction, the uninformed type weakly prefers persisting until the deadline followed by conceding with probability $\tilde{Y}(\gamma^0)$ to conceding immediately. Since $\tilde{Y}(\gamma^0) > 0$ for $\gamma^0 < \gamma_+$, the equilibrium condition of the uninformed implies that

$$\gamma^0\tilde{Y}(\gamma^0)\Big(\frac{\overline{\pi}_M + \underline{\pi}_M}{2} - \lambda\Big) + \gamma^0(1 - \tilde{Y}(\gamma^0))\underline{\pi}_M + (1 - \gamma^0)\overline{\pi}_D - \delta T \geq \gamma^0\underline{\pi}_M + (1 - \gamma^0)\overline{\pi}_D,$$

or

$$\gamma^0\tilde{Y}(\gamma^0)\Big(\frac{\overline{\pi}_M - \underline{\pi}_M}{2} - \lambda\Big) \geq \delta T.$$

By Assumption 1 and Assumption 2, we have

$$\tilde{Y}(\gamma^0)(\overline{\pi}_F - \underline{\pi}_F) + (1 - \tilde{Y}(\gamma^0))\Big(\frac{\overline{\pi}_F - \underline{\pi}_F}{2} - \lambda\Big) > \frac{\overline{\pi}_F - \underline{\pi}_F}{2} - \lambda > \gamma^0\tilde{Y}(\gamma^0)\Big(\frac{\overline{\pi}_M - \underline{\pi}_M}{2} - \lambda\Big),$$

and thus the equilibrium condition of the informed is satisfied. For the case of $\gamma^0 \geq \gamma_+$ we have $\tilde{Y}(\gamma^0) = 0$, and the equilibrium condition of the uninformed is

$$\gamma^0\Big(\frac{\overline{\pi}_M + \underline{\pi}_M}{2} - \lambda\Big) + (1 - \gamma^0)\Big(\frac{\overline{\pi}_D + \underline{\pi}_D}{2} - \lambda\Big) - \delta T \geq \gamma^0\underline{\pi}_M + (1 - \gamma^0)\overline{\pi}_D,$$

which implies

$$\gamma^0\Big(\frac{\overline{\pi}_M - \underline{\pi}_M}{2} - \lambda\Big) > \delta T.$$

Thus, the equilibrium condition of the informed is satisfied under Assumption 1.

For case (ii), note that the continuation play is in equilibrium for any $\gamma^0$ at $t = J_\lambda(\gamma^0)$ by case (iii). The continuation payoff for the uninformed $\mathcal{U}(J_s(\gamma^0))$ is thus equal to $U(g(J_\lambda(\gamma^0);\gamma^0))$ by the definition of $J_\lambda(\gamma^0)$. It follows that the strategy of the uninformed is an equilibrium for $t < J_\lambda(\gamma^0)$. For the informed, the equilibrium continuation payoff at $t = J_\lambda(\gamma^0)$ is given in case (ii) above. The equilibrium payoff function $V_\lambda(\gamma)$ at any $\gamma = g(t;\gamma^0)$ for $t < J_\lambda(\gamma^0)$ is given by (3.6) with the boundary condition that

$$\begin{aligned} V_\lambda(g(J_\lambda(\gamma^0);\gamma^0)) = \tilde{Y}(g(J_\lambda(\gamma^0);\gamma^0))\overline{\pi}_F + (1-\tilde{Y}(g(J_\lambda(\gamma^0);\gamma^0)))\Big(\frac{\overline{\pi}_F + \underline{\pi}_F}{2} - \lambda\Big) \\ - \delta(T - J_\lambda(\gamma^0)). \end{aligned} \tag{6.3}$$

The claim that it is optimal for the informed to persist at all $t < J_\lambda(\gamma^0)$ follows from the identical arguments in the proof of case (ii) of Proposition 2.

## 6.2. Deadline effects

Fix an initial belief $\gamma^0$. If $T < B_\lambda(\gamma^0)$ the equilibrium expected payoffs for both the informed and the uninformed are strictly decreasing in $T$. Since the two players disagree until the deadline, lengthening $T$ increases the loss from delay without changing the equilibrium behavior upon reaching the deadline.

Fix $\gamma^0$ and $T$ such that $T \geq B_\lambda(\gamma^0)$. Since the uninformed type initially concedes at positive flow rate, the expected payoff of the uninformed types is pinned down by the expected payoff from concession, which equals $U(\gamma^0)$ and does not change with $T$. The deadline effects are entirely captured by changes in the expected payoff of the informed types.

As in the case with no deadline penalties, the equilibrium payoff of the informed $V_\lambda(\gamma^0)$ varies one-to-one with a constant $C_\lambda$ in the solution (3.6) to the same differential equation (3.4) as in the case of $\lambda = 0$. Further, the deadline effect of an increase in $T$ on the equilibrium payoff $V(\gamma^0)$ of the informed for the initial belief $\gamma^0$ is the same as its effect on the constant $C_\lambda$. The presence of the deadline penalty $\lambda$ changes only the boundary condition that determines $C_\lambda$ through equation (4.1). In particular, if $\gamma^0 \in [0, n(T)]$, then $C_\lambda$ is such that $V_\lambda(0) = \overline{\pi}_F$. Otherwise, $C_\lambda$ is such that $V_\lambda(g(J_\lambda(\gamma^0); \gamma^0))$ is as given in (6.3).

Parallel to the case of $\lambda = 0$, for any initial belief $\gamma^0 \geq \gamma_-$, let $S_\lambda(\gamma^0)$ be the deadline such that when the belief of the uninformed as determined by $g(t; \gamma^0)$ reaches $\gamma_-$ the time remaining is $T(\gamma_-)$. That is,

$$g(S_\lambda(\gamma^0) - T(\gamma_-); \gamma^0) = \gamma_-.$$

Next, for any deadline $T$, let $k_\lambda(T)$ be the initial belief such that the updated belief reaches $\gamma_+$ at $t = T$. That is,

$$g(T; k_\lambda(T)) = \gamma_+.$$

Note that as in the case of $\lambda = 0$, we have $k_\lambda(T) \leq p_\lambda(T)$ with equality only at $T = 0$ when $k_\lambda(0) = p_\lambda(0) = \gamma_+$.

We claim that the deadline effects on the informed are: non-existent, if $\gamma^0 \in [0, n(T))$; negative, if $T \leq T(\gamma_-)$ and $\gamma^0 \in [n(T), s(T))$, or if $T > S_\lambda(\gamma^0)$ and $\gamma^0 \geq n(T)$; positive, if $\gamma^0 \in [\gamma_-, \gamma_+]$ and $T \in [T(\gamma^0), S_\lambda(\gamma^0)]$, or if $\gamma^0 > \gamma_+$ and $T \in [k_\lambda^{-1}(\gamma^0), S_\lambda(\gamma^0)]$; and positive, if $\gamma^0 \in [k_\lambda(T), p_\lambda(T)]$. Equivalently, our claim is that in the region of $T > B_\lambda(\gamma^0)$ in the $T$-$\gamma^0$ space where the uninformed type initially concedes at a positive flow rate, the deadline effect is nil if the updated belief never hits the boundary $B_\lambda(\gamma^0)$; negative if it hits the boundary in the segment $s_\lambda$; and positive if it hits the boundary either in the segment $T(\gamma)$ or in the segment $p_\lambda$.

The case where the updated belief never hits the boundary is identical to the case of $\gamma^0 \in [0, n(T)]$ for $\lambda = 0$. There is no deadline effect.

The case where the updated belief hits $s_\lambda$ is identical to the corresponding case of $\gamma^0 \in (n(T), s(T))$ when $\lambda = 0$, because the deadline penalty is never incurred on the equilibrium path. The deadline effect is negative.

Next, consider the case where the updated belief hits $p_\lambda$ at time $t = J_\lambda(\gamma^0)$. The boundary condition given by (6.3) with $\tilde{Y}(g(J_\lambda(\gamma^0); \gamma^0)) = 0$, with

$$g(J_\lambda(\gamma^0); \gamma^0) = \frac{\overline{\pi}_D - \underline{\pi}_D + 2\lambda + 2\delta(T - J_\lambda(\gamma^0))}{\overline{\pi}_D - \underline{\pi}_D + \overline{\pi}_M - \underline{\pi}_M} = 1 - (1 - \gamma^0)e^{\delta_* J_\lambda(\gamma^0)}.$$

Thus, the boundary condition can be written as

$$V_\lambda(g(J_\lambda(\gamma^0);\gamma^0)) = \frac{\overline{\pi}_F + \underline{\pi}_F}{2} - \frac{1}{2}(g(J_\lambda(\gamma^0);\gamma^0)(\overline{\pi}_D - \underline{\pi}_D + \overline{\pi}_M - \underline{\pi}_M) - (\overline{\pi}_D - \underline{\pi}_D)),$$

which is identical to (4.2), with $\lambda$ disappearing. The result of the argument then follows unchanged from the argument given for the case of $\gamma^0 \in [k(T), p(T))$ when $\lambda = 0$, and from the fact that $\gamma_+$ increases with $\lambda$.

For the remaining case of the updated belief hitting $T(\gamma)$, we rewrite the expression (4.1) for the constant $C_\lambda$ in the solution (3.6) to the differential equation (3.4) as

$$\begin{aligned} C_\lambda = -\overline{\pi}_F &- \frac{g(J_\lambda(\gamma^0);\gamma^0)}{1 - g(J_\lambda(\gamma^0);\gamma^0)}(\overline{\pi}_F - V_\lambda(g(J_\lambda(\gamma^0);\gamma^0))) \\ &+ \Big(\frac{1}{1 - g(J_\lambda(\gamma^0);\gamma^0)} + \ln(1 - g(J_\lambda(\gamma^0);\gamma^0))\Big)(\overline{\pi}_M - \underline{\pi}_M). \end{aligned}$$

From the expression (6.2) we have

$$g(J_\lambda(\gamma^0);\gamma^0) = \gamma_+ - \frac{8\lambda\delta(T - J_\lambda(\gamma^0))}{(\overline{\pi}_D - \underline{\pi}_D + \overline{\pi}_M - \underline{\pi}_M)(\overline{\pi}_M - \underline{\pi}_M - 2\lambda)} = 1 - (1-\gamma^0)e^{\delta_* J_\lambda(\gamma^0)}.$$

Note that the second equation above implies that $\partial J_\lambda(\gamma^0)/\partial T > 0$. The boundary value $V_\lambda(g(J_\lambda(\gamma^0);\gamma^0))$ is given by equation (6.3) with $\tilde{Y}(g(J_\lambda(\gamma^0);\gamma^0)) = Y(g(J_\lambda(\gamma^0);\gamma^0))$ for $g(J_\lambda(\gamma^0);\gamma^0) \in [\gamma_-, \gamma_+]$. Thus, $V_\lambda(g(J_\lambda(\gamma^0);\gamma^0))$ satisfies

$$\begin{aligned} \overline{\pi}_F - V_\lambda(g(J_\lambda(\gamma^0);\gamma^0)) = (1 - Y(g(J_\lambda(\gamma^0);\gamma^0)))&\frac{\overline{\pi}_F - \underline{\pi}_F + 2\lambda}{2} \\ + \frac{\gamma_+ - g(J_\lambda(\gamma^0);\gamma^0)}{8\lambda}&(\overline{\pi}_D - \underline{\pi}_D + \overline{\pi}_M - \underline{\pi}_M)(\overline{\pi}_M - \underline{\pi}_M - 2\lambda). \end{aligned}$$

It follows that $C_\lambda$ depends on $T$ only through $g(J_\lambda(\gamma^0);\gamma^0)$, and the deadline effect on the equilibrium payoff of the informed is positive if and only if the derivative of $C_\lambda$ with respect to $g(J_\lambda(\gamma^0);\gamma^0)$ is negative. The derivative of $C_\lambda$ with respect to $g(J_\lambda(\gamma^0);\gamma^0)$ is given by

$$\frac{g(J_\lambda(\gamma^0);\gamma^0)}{1 - g(J_\lambda(\gamma^0);\gamma^0)}\frac{\mathrm{d}V_\lambda(g(J_\lambda(\gamma^0);\gamma^0))}{\mathrm{d}g(J_\lambda(\gamma^0);\gamma^0)} - \frac{\overline{\pi}_F - V_\lambda(g(J_\lambda(\gamma^0);\gamma^0))}{(1 - g(J_\lambda(\gamma^0);\gamma^0))^2} + g(J_\lambda(\gamma^0);\gamma^0)(\overline{\pi}_M - \underline{\pi}_M),$$

where

$$\begin{aligned} \frac{\mathrm{d}V_\lambda(g(J_\lambda(\gamma^0);\gamma^0))}{\mathrm{d}g(J_\lambda(\gamma^0);\gamma^0)} = &-\frac{1}{8\lambda g^2(J_\lambda(\gamma^0);\gamma^0)}(\overline{\pi}_F - \underline{\pi}_F + 2\lambda)(\overline{\pi}_D - \underline{\pi}_D + 2\lambda) \\ &+ \frac{1}{8\lambda}(\overline{\pi}_D - \underline{\pi}_D + \overline{\pi}_M - \underline{\pi}_M)(\overline{\pi}_M - \underline{\pi}_M - 2\lambda). \end{aligned}$$

Thus, $\mathrm{d}C_\lambda/\mathrm{d}g(J_\lambda(\gamma^0);\gamma^0)$ has the same sign as

$$-(\overline{\pi}_F - \underline{\pi}_F + 2\lambda)(\overline{\pi}_M - \underline{\pi}_M + 2\lambda) + 8\lambda g(J_\lambda(\gamma^0);\gamma^0)(\overline{\pi}_M - \underline{\pi}_M)$$
$$+ (g(J_\lambda(\gamma^0);\gamma^0)(2 - g(J_\lambda(\gamma^0);\gamma^0)) - \gamma_+)(\overline{\pi}_D - \underline{\pi}_D + \overline{\pi}_M - \underline{\pi}_M)(\overline{\pi}_M - \underline{\pi}_M - 2\lambda).$$

The above is an increasing function of $g(J_\lambda(\gamma^0);\gamma^0)$, and at $g(J_\lambda(\gamma^0);\gamma^0) = \gamma_+$ is equal to

$$-((\overline{\pi}_F - \underline{\pi}_F) - \gamma_+(\overline{\pi}_M - \underline{\pi}_M))(\overline{\pi}_M - \underline{\pi}_M + 2\lambda),$$

which is negative by Assumption 1.

An immediate corollary of the above analysis of the deadline effects on welfare is that the optimal deadline is either 0, or $S_\lambda(\gamma^0)$.

### 6.3. Optimal deadline

Now we show that for any fixed $\lambda$ satisfying Assumption 2 with strict inequality, there exist thresholds $\underline{\gamma}_\lambda$ and $\overline{\gamma}_\lambda$, with $\gamma_- < \underline{\gamma}_\lambda < \gamma_+ < \overline{\gamma}_\lambda < 1$, such that the optimal deadline for any initial belief $\gamma^0$ of the uninformed is $S_\lambda(\gamma^0)$ if and only if $\gamma^0 \in (\underline{\gamma}_\lambda, \overline{\gamma}_\lambda)$.

The equilibrium payoff functions for the informed and uninformed in the no-delay game are given by

$$U_\lambda^0(\gamma^0) = \begin{cases} \frac{1}{2}\gamma^0(\overline{\pi}_M + \underline{\pi}_M - 2\lambda) + (1-\gamma^0)\overline{\pi}_D & \text{if } \gamma^0 \in [0, \gamma_-), \\ \gamma^0\underline{\pi}_M + (1-\gamma^0)\overline{\pi}_D + \frac{1}{2}\gamma^0 Y(\gamma^0)(\overline{\pi}_M - \underline{\pi}_M - 2\lambda) & \text{if } \gamma^0 \in [\gamma_-, \gamma_+], \\ \frac{1}{2}\gamma^0(\overline{\pi}_M + \underline{\pi}_M - 2\lambda) + \frac{1}{2}(1-\gamma^0)(\overline{\pi}_D + \underline{\pi}_D - 2\lambda) & \text{if } \gamma^0 \in (\gamma_+, 1); \end{cases}$$

and

$$V_\lambda^0(\gamma^0) = \begin{cases} \overline{\pi}_F & \text{if } \gamma^0 \in [0, \gamma_-), \\ Y(\gamma^0)\overline{\pi}_F + \frac{1}{2}(1 - Y(\gamma^0))(\overline{\pi}_F + \underline{\pi}_F - 2\lambda) & \text{if } \gamma^0 \in [\gamma_-, \gamma_+], \\ \frac{1}{2}(\overline{\pi}_F + \underline{\pi}_F) - \lambda & \text{if } \gamma^0 \in (\gamma_+, 1). \end{cases}$$

Under the deadline $T = S_\lambda(\gamma^0)$, the payoff to the uninformed is simply $U_\lambda^S(\gamma^0) = U(\gamma^0)$ as in (3.2). To compute the payoff to the informed, we use (3.6) and the boundary condition

$$V_\lambda(\gamma_-) = \overline{\pi}_F - \delta T(\gamma_-).$$

We can then write the constant in (4.1) as

$$C_\lambda = -\overline{\pi}_F - \frac{\gamma_-^2}{2(1-\gamma_-)}(\overline{\pi}_M - \underline{\pi}_M - 2\lambda) + \Big(\frac{1}{1-\gamma_-} + \ln(1-\gamma_-)\Big)(\overline{\pi}_M - \underline{\pi}_M), \quad (6.4)$$

and therefore

$$V_\lambda^S(\gamma^0) = \overline{\pi}_F - \frac{1-\gamma^0}{\gamma^0}\left(\ln\Big(\frac{1-\gamma^0}{1-\gamma_-}\Big) + \frac{\gamma^0-\gamma_-}{(1-\gamma^0)(1-\gamma_-)}\right)(\overline{\pi}_M - \underline{\pi}_M)$$
$$- \frac{1-\gamma^0}{\gamma^0}\frac{\gamma_-^2}{1-\gamma_-}\Big(\frac{\overline{\pi}_M-\underline{\pi}_M}{2} - \lambda\Big).$$

The difference in ex ante welfare $W_\lambda^S(\gamma^0) - W_\lambda^0(\gamma^0)$ is

$$\frac{1}{2-\gamma^0}(U_\lambda^S(\gamma^0) - U_\lambda^0(\gamma^0)) + \frac{1-\gamma^0}{2-\gamma^0}(V_\lambda^S(\gamma^0) - V_\lambda^0(\gamma^0)) \equiv \frac{1}{2(2-\gamma^0)}\Delta_\lambda(\gamma^0).$$

Since $Y(\gamma_-) = 1$, we have

$$\Delta_\lambda(\gamma_-) = -\gamma_-(\overline{\pi}_M - \underline{\pi}_M - 2\lambda) - \gamma_-(1-\gamma_-)(\overline{\pi}_M - \underline{\pi}_M - 2\lambda) < 0.$$

Since $Y(\gamma_+) = 0$, we have

$$\Delta_\lambda(\gamma_+) = (1-\gamma_+)(\overline{\pi}_F - \underline{\pi}_F - 2\lambda) - \frac{2(1-\gamma_+)^2}{\gamma_+}\frac{\gamma_-^2}{1-\gamma_-}(\overline{\pi}_M - \underline{\pi}_M - 2\lambda)$$
$$- \frac{2(1-\gamma_+)^2}{\gamma_+}\left(\ln\Big(\frac{1-\gamma_+}{1-\gamma_-}\Big) + \frac{\gamma_+-\gamma_-}{(1-\gamma_+)(1-\gamma_-)}\right)(\overline{\pi}_M - \underline{\pi}_M).$$

Using Assumption 1, we can show that

$$\Delta_\lambda(\gamma_+) \geq \frac{1-\gamma_+}{\overline{\pi}_M - \underline{\pi}_M + 2\lambda}\big((1-\gamma_-)(\overline{\pi}_M - \underline{\pi}_M - 2\lambda)^2 + 8(1-\gamma_+)\lambda(\overline{\pi}_M - \underline{\pi}_M)\big) > 0.$$

Thus, there exists a $\underline{\gamma}_\lambda \in (\gamma_-, \gamma_+)$ such that $\Delta_\lambda(\underline{\gamma}_\lambda) = 0$. Taking derivatives of $\Delta_\lambda(\gamma^0)$ with respect to $\gamma^0 \in (\gamma_-, \gamma_+)$ and evaluating at $\underline{\gamma}_\lambda$ using $\Delta_\lambda(\underline{\gamma}_\lambda) = 0$ yield

$$\frac{\gamma_+(1-\gamma_-)}{\underline{\gamma}_\lambda(\gamma_+-\gamma_-)}(\overline{\pi}_F - \underline{\pi}_F + 2\lambda) + \frac{\gamma_-(2\underline{\gamma}_\lambda - \gamma_+(1+\underline{\gamma}_\lambda))}{\underline{\gamma}_\lambda(1-\underline{\gamma}_\lambda)(\gamma_+-\gamma_-)}(\overline{\pi}_M - \underline{\pi}_M - 2\lambda) - 2(\overline{\pi}_M - \underline{\pi}_M)$$
$$> \frac{1-\gamma_-}{\gamma_+-\gamma_-}(\overline{\pi}_F - \underline{\pi}_F + 2\lambda) + \frac{2\gamma_- - \gamma_+(1+\gamma_-)}{(1-\gamma_-)(\gamma_+-\gamma_-)}(\overline{\pi}_M - \underline{\pi}_M - 2\lambda) - 2(\overline{\pi}_M - \underline{\pi}_M)$$
$$> \frac{(1-\gamma_-)\gamma_+}{\gamma_+-\gamma_-}(\overline{\pi}_M - \underline{\pi}_M + 2\lambda) + \frac{2\gamma_- - \gamma_+(1+\gamma_-)}{(1-\gamma_-)(\gamma_+-\gamma_-)}(\overline{\pi}_M - \underline{\pi}_M - 2\lambda) - 2(\overline{\pi}_M - \underline{\pi}_M),$$

where the first inequality follows because the first term in the expression is decreasing in $\underline{\gamma}_\lambda$ while the second term is increasing in $\underline{\gamma}_\lambda$, and the second inequality uses Assumption 1 and Assumption 2. The above can be shown to be equal to

$$\Big(\frac{\overline{\pi}_M-\underline{\pi}_M}{2} - \lambda\Big)\left(\frac{\overline{\pi}_D-\underline{\pi}_D}{\lambda}\left(\frac{\overline{\pi}_D-\underline{\pi}_D}{\overline{\pi}_M-\underline{\pi}_M+2\lambda} + \frac{3}{2}\right) + \frac{\overline{\pi}_M-\underline{\pi}_M-2\lambda}{\overline{\pi}_M-\underline{\pi}_M+2\lambda} + \frac{\overline{\pi}_M-\underline{\pi}_M}{\lambda} - 2\right),$$

which is positive by Assumption 2. As a result, $\underline{\gamma}_\lambda$ is unique, with $\Delta_\lambda(\gamma^0) > 0$ if $\gamma^0 \in (\underline{\gamma}_\lambda, \gamma_+)$, and the opposite holding if $\gamma^0 \in (\gamma_-, \underline{\gamma}_\lambda)$.

At the other end, we have

$$\lim_{\gamma^0 \to 1} \Delta_\lambda(\gamma^0) = -(\overline{\pi}_M - \underline{\pi}_M - 2\lambda) < 0.$$

Thus, there exists a $\overline{\gamma}_\lambda \in (\gamma_+, 1)$ such that $\Delta_\lambda(\overline{\gamma}_\lambda) = 0$. The derivative of $\Delta_\lambda(\gamma^0)$ with respect to $\gamma^0 \in (\gamma_+, 1)$ is given by

$$\begin{aligned} &-(\overline{\pi}_F - \underline{\pi}_F + \overline{\pi}_D - \underline{\pi}_D + 2\lambda) - 3(\overline{\pi}_M - \underline{\pi}_M) + \frac{(1-(\gamma^0)^2)\gamma_-^2}{(\gamma^0)^2(1-\gamma_-)}(\overline{\pi}_M - \underline{\pi}_M - 2\lambda) \\ &+ \frac{2(1-(\gamma^0)^2)}{(\gamma^0)^2}\left(\ln\left(\frac{1-\gamma^0}{1-\gamma_-}\right) + \frac{\gamma^0 - \gamma_-}{(1-\gamma^0)(1-\gamma_-)}\right)(\overline{\pi}_M - \underline{\pi}_M). \end{aligned}$$

As in the case of $\lambda = 0$, the sum of the last two terms in the above expression is increasing in $\gamma^0$ and approaches $4(\overline{\pi}_M - \underline{\pi}_M)$ as $\gamma^0$ approaches 1. Thus,

$$\Delta'_\lambda(\gamma^0) < -(\overline{\pi}_F - \underline{\pi}_F + \overline{\pi}_D - \underline{\pi}_D + 2\lambda) + (\overline{\pi}_M - \underline{\pi}_M) < 0,$$

by Assumption 2. It follows that $\overline{\gamma}_\lambda$ is uniquely defined in $(\gamma_+, 1)$, and $\Delta_\lambda(\gamma^0) > 0$ for $\gamma^0 \in (\gamma_+, \overline{\gamma}_\lambda)$ and the opposite holds for $\gamma^0 \in (\overline{\gamma}_\lambda, 1)$.

### 6.4. Optimal deadline penalty

Now we show that $W^S_\lambda(\gamma^0)$ is increasing in the deadline penalty $\lambda$ for all initial belief $\gamma^0 > \gamma_-$. Since $U^S_\lambda(\gamma^0)$ is independent of $\lambda$, and since $V^S_\lambda(\gamma^0)$ depends on $\lambda$ only through $C_\lambda$ as given in (6.4), it suffices to show that $C_\lambda$ is an increasing function of $\lambda$. Taking derivatives, we have

$$\frac{\mathrm{d}C_\lambda}{\mathrm{d}\lambda} = \frac{\gamma_-^2}{1-\gamma_-} + \frac{\gamma_-}{2(1-\gamma_-)^2}\frac{\mathrm{d}\gamma_-}{\mathrm{d}\lambda}(4\lambda + \gamma_-(\overline{\pi}_M - \underline{\pi}_M - 2\lambda)).$$

The above has the same sign as

$$(\overline{\pi}_D - \underline{\pi}_D + 2\lambda)(\overline{\pi}_M - \underline{\pi}_M + 2\lambda) + ((\overline{\pi}_M - \underline{\pi}_M) - (\overline{\pi}_D - \underline{\pi}_D))\left(4\lambda + \gamma_-(\overline{\pi}_M - \underline{\pi}_M - 2\lambda)\right),$$

which is greater than

$$(\overline{\pi}_D - \underline{\pi}_D)(\overline{\pi}_M - \underline{\pi}_M - 2\lambda)(1 - \gamma_-) > 0$$

by Assumption 2. It follows that for any initial belief of the uninformed $\gamma^0 \in (\underline{\gamma}_\lambda, \overline{\gamma}_\lambda)$, an increase in the deadline penalty $\lambda$ raises the ex ante welfare under the optimal deadline $T = S_\lambda(\gamma^0)$.

The above analysis suggests that for intermediate values of initial beliefs $\gamma^0$ it may be optimal to allow the deadline penalty $\lambda$ to be as great as possible conditional on adopting the optimal deadline $S_\lambda(\gamma^0)$. The following assumption is sufficient for this observation to be valid:

Assumption 3. $\overline{\pi}_M - \underline{\pi}_M \leq \overline{\pi}_D - \underline{\pi}_D$.

Note that $\gamma_+$ is always increasing in $\lambda$, and $\gamma_-$ and $\gamma_+$ converge to the single value $\gamma_*$ when $\lambda$ approaches 0. Under Assumption 3, we can easily verify that $\gamma_-$ is decreasing in $\lambda$. It follows that $\gamma_* \in (\gamma_-, \gamma_+)$ for all $\lambda > 0$.

We claim that if the initial belief $\gamma^0$ is strictly above $\gamma_*$ but sufficiently close to it, then conditional on the optimal deadline $S_\lambda$, the optimal deadline penalty $\lambda$ is equal to $\frac{1}{2}(\overline{\pi}_M - \underline{\pi}_M)$.

To establish the above claim, we first show that in the no delay game ($T = 0$), the optimal deadline penalty $\lambda$ is 0 for any initial belief $\gamma^0$ strictly above $\gamma_*$ but sufficiently close to it. Note from the expressions of $U^0_\lambda$ and $V^0_\lambda$, for any such initial belief $\gamma^0$, $\lambda$ is small so that $\gamma^0 > \gamma_+$, then as $\lambda$ increases, both $V^0_\lambda$ and $U^0_\lambda$ decrease. If $\lambda$ is such that $\gamma^0 \leq \gamma_+$, we have

$$\begin{aligned}\frac{\partial W^0_\lambda(\gamma^0)}{\partial \lambda} &= \frac{\gamma^0}{2(2-\gamma^0)}\left(-2Y(\gamma^0) + (\overline{\pi}_M - \underline{\pi}_M - 2\lambda)\frac{\partial Y(\gamma^0)}{\partial \lambda}\right) \\ &\quad - \frac{1-\gamma^0}{2(2-\gamma^0)}\left(2(1-Y(\gamma^0)) - (\overline{\pi}_F - \underline{\pi}_F + 2\lambda)\frac{\partial Y(\gamma^0)}{\partial \lambda}\right).\end{aligned}$$

Using equation (6.1), we can show that $\partial W^0_\lambda(\gamma^0)/\partial\lambda$ has the same sign as

$$(\gamma^0(\overline{\pi}_M - \underline{\pi}_M) + (1-\gamma^0)(\overline{\pi}_F - \underline{\pi}_F))(\gamma^0(\overline{\pi}_M - \underline{\pi}_M) - (1-\gamma^0)(\overline{\pi}_D - \underline{\pi}_D)) - 4\lambda^2(1-2(1-\gamma^0)^2).$$

The first term in the above expression approaches 0 when $\gamma^0$ approaches $\gamma_*$ from above, while the second term is strictly negative for all $\gamma^0 > \gamma_*$, as $\gamma_* \geq \frac{1}{2}$ by Assumption 3. As a result, the optimal deadline penalty $\lambda$ is 0 conditional of no delay ($T = 0$) for any any initial belief $\gamma^0$ strictly above $\gamma_*$ but sufficiently close to it.

Fix any $\gamma^0 > \gamma_*$. Compare $U^0$ and $V^0$ (with $\lambda = 0$) to the corresponding payoff functions $U^S_{\overline{\lambda}}$ and $V^S_{\overline{\lambda}}$ for $T = S_\lambda(\gamma^0)$ and $\lambda = \frac{1}{2}(\overline{\pi}_M - \underline{\pi}_M)$. We have

$$U^S_{\overline{\lambda}}(\gamma^0) - U(\gamma^0) = -\gamma\frac{\overline{\pi}_M - \underline{\pi}_M}{2} + (1-\gamma\frac{\overline{\pi}_D - \underline{\pi}_D}{2},$$

which is equal to 0 when $\gamma^0 = \gamma_*$. On the other hand,

$$V^S_{\overline{\lambda}}(\gamma^0) = \overline{\pi}_F - \frac{1-\gamma^0}{\gamma^0}\left(\ln\left(\frac{1-\gamma^0}{1-\gamma_{**}}\right) + \frac{1}{1-\gamma^0} - \frac{1}{1-\gamma_{**}}\right)(\overline{\pi}_M - \underline{\pi}_M),$$

where $\gamma_{**}$ is the minimum of $\gamma_-$ when $\lambda = \frac{1}{2}(\overline{\pi}_M - \underline{\pi}_M)$, given by

$$\gamma_{**} = \frac{\overline{\pi}_D - \underline{\pi}_D + \overline{\pi}_M - \underline{\pi}_M}{\overline{\pi}_D - \underline{\pi}_D + 3(\overline{\pi}_M - \underline{\pi}_M)}.$$

Under Assumption 1, for $V^S_{\overline{\lambda}}(\gamma^0) > V^0(\gamma^0)$, it suffices if

$$\frac{2-\gamma^0}{2(1-\gamma^0)} - \frac{1}{1-\gamma_{**}} + \ln\left(\frac{1-\gamma^0}{1-\gamma_{**}}\right) < 0.$$

It is straightforward to verify that

$$\frac{2-\gamma^0}{2(1-\gamma^0)} < \frac{2-\gamma_*}{2(1-\gamma_*)} < \frac{1}{1-\gamma_{**}}.$$

Thus, for any any initial belief $\gamma^0$ strictly above $\gamma_*$ but sufficiently close to it, the ex ante payoff of the informed and uninformed is strictly greater when $T = S_\lambda$ and $\lambda = \frac{1}{2}(\overline{\pi}_M - \underline{\pi}_M)$ than when $T = 0$ and $\lambda = 0$. It follows that for such initial belief of the uninformed, the optimal deadline penalty is the maximum.

## 7. Concluding Remarks

This paper is an outgrowth of our earlier paper (Damiano, Li and Suen, 2008). In that paper we use a discrete time model with more restrictive preference assumptions to show that costly delay can improve strategic information aggregation and hence ex ante welfare in a variety of environments with regards to deadlines. See Damiano, Li and Suen (2008) for a more comprehensive list of references to related papers on dynamic games with asymmetric information. However, the discrete time framework is not suitable for studying the issue of optimal deadlines in strategic information aggregation, because an explicit characterization of equilibrium play is difficult to obtain.

There is a sizable literature on the "deadline effect" in war of attrition and bargaining games. Hendricks, Weiss and Wilson (1988) characterize mixed-strategy Nash equilibria of a continuous time, complete information war of attrition game, in which there is a mass point of concession at the deadline and no concession in a time interval proceeding it. Spier (1992) shows that in pretrial negotiations with incomplete information, the settlement probability is U-shaped. Ma and Manove (1993) find strategic delay in bargaining games with complete information, by assuming that there may be exogenous, random delay in offer transmission. As early offers are rejected and the deadline approaches, there is an increasing risk of missing the deadline and negotiation activities pick up. Also in a bargaining game with complete information, Fershtman and Seidmann (1993) introduce the assumption that, by rejecting an offer, players commit to not accepting poorer offers in the future. They show that when players are sufficiently patient, there is a unique subgame perfect equilibrium in which players wait until the deadline to reach an agreement. Ponsati (1995) studies a war of attrition game in which each player has private information about his payoff loss incurred by conceding to the opponent and must choose the timing of concession. She shows that there is a unique pure strategy equilibrium in which both players never concede before the deadline is reached if their payoff losses are sufficiently large. Sandholm and Vulkan (1999) consider a bargaining game in which two players make offers continuously and an agreement is reached as soon as the offers are compatible with each other. The only private information a player has is the deadline he faces. They show that the only equilibrium is each player persisting by demanding the whole pie until the deadline and then switching to concede everything to his opponent. Finally, Yildiz (2004) shows that when players in a bargaining game are overly optimistic about their bargaining power at the deadline, it is an equilibrium to persist until close to the deadline to reach an agreement. However, when there is uncertainty about when the deadline arrives, the deadline effect disappears.

In our model the positive welfare effects of extending the deadline are directly related to the deadline behavior of the uninformed stopping the concessions at some point and then conceding with a positive probability upon reaching the deadline. A longer deadline is beneficial for the informed in spite of the fact that the uninformed type persists for a longer period of time, because the latter concedes with a greater probability when the deadline is reached. The failure in inducing this deadline behavior is the reason that stochastic deadlines, or exogenous negotiation breakdowns, are ineffective in raising the ex ante welfare for the informed. However, an implicit assumption we have made in modeling stochastic deadlines is that exogenous breakdowns occur at a constant flow rate. We have not investigated either time-varying flow rates, or atoms in the flow rate. The latter case is perhaps more natural way of modeling stochastic deadlines, and is likely to generate some deadline behavior and positive deadline effects. A related point is that we have assumed throughout that the two parties incur payoff losses from delay at a constant flow rate. We may imagine that delay cost may exhibit atoms in the flow rate that correspond to temporary suspensions of the negotiation process. We expect such atoms to generate some kind of deadline behavior and positive deadline effects. All these issues are left for future research.

In our framework of negotiation with a finite deadline, we have shown that there is a boundary in the space of the belief of the uninformed and the time remaining to the deadline, which separates the region where the uninformed types concede at the same flow rate as when there is no deadline, and the region where the uninform types stop the concessions and the evolution of the belief until the deadline and then concede with the same probability as when there is no delay. Modifications to the no-delay game change the equilibrium play only through changing the shape of the boundary. Although we have chosen the most natural no-delay game in our setup, it would be interesting to decouple the no-delay game and the no-deadline game. One way of formalizing this decoupling is to keep the no-deadline game and model the no-delay payoffs in reduced forms. Doing so may provide more general insights about the deadline effects and the optimal deadlines than the present model.

Our repeated proposal game is symmetric, and we have restricted the attention to symmetric equilibria. This restriction is without loss of generality so long as the informed types persist throughout the game. Games with asymmetric preferences and delay costs are worth future research because asymmetry adds an interesting element to equilibrium dynamics of information aggregation. The restriction to equilibria in which the informed types always persist is natural in our setup because the informed types know what the mutually preferred choice is. Our Assumption 1, which states that the payoff loss from making the wrong choice is greater for the informed types than the payoff loss from conceding in the disagreement state for the uninformed, is shown to be sufficient for us to focus on equilibrium play of the uninformed and turn to the informed only for welfare analysis. In a more general setup, instead of having three states and one perfectly informed type, we may have one type better informed about the mutually preferred choice than the other type. This would be more challenging as there is no longer the dichotomy between strategic analysis and welfare analysis, but the present paper may still provide a good starting point.

Our result that the optimal deadline is positive and increasing for intermediate levels of initial conflicts hinges on two implicit assumptions about the game that may be questioned in practice. First, the two parties in the joint decision situation are assumed to be able to commit to a precise deadline at the start of the negotiation process. According to our characterization of equilibrium play, before the process reaches the critical point when the parties are supposed to become inactive until the deadline arrives, they have no incentive to renegotiate the deadline. However, as soon as the critical point is reached, they would want to jump to the end-game play immediately. Of course if such renegotiation of the deadline is anticipated the equilibrium play before this critical point would be changed. If we find a way to formalize this commitment issue, it is potentially interesting to investigate further. The other implicit assumption we have made is that the initial belief of the uninformed is common knowledge between the two parties when setting the deadline. We hasten to emphasize that our result that extending the deadline can have positive welfare effects is robust to slight perturbations to the initial belief of the uninformed. However, a perhaps more interesting issue is whether the two parties may find some way to communicate their knowledge about the initial degrees of conflict before jointly setting the deadline for negotiation. Such communication raises strategic issues that are worth further research in the future.

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