Imposing Monotonicity Nonparametrically in First-Price Auctions *

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April 14, 2008

Abstract

Monotonicity of the equilibrium bidding strategy is a key property of structural auction models. Traditional nonparametric estimators provide a flexible means of uncovering salient features of auction data, but do not formally impose the monotonicity assumption that is inherent in the models during estimation. Here, we develop a nonparametric estimator which imposes the monotonicity assumption. We accomplish this by employing the constraint weighted bootstrapping theory developed in the statistics literature. The finite sample performance of our estimator is examined using simulated data, experimental data, as well as a naturally occurring data set composed of thousands of bids from Canadian timber auctions.

Keywords: Constrained Weighted Bootstrap, Bandwidth, Equilibrium Bidding Strategy

JEL Classification No: C12, C14, D44

*The authors have benefited greatly from comments made by Han Hong and Harry Paarsch and by participants in seminars at the University of Nevada, Reno and University Luiss, as well as from participants at NY Camp Econometrics III and the Conference on Auctions and Games held at Virginia Polytechnic Institute. All code used in the paper is available from the authors upon request. The usual disclaimer applies.

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1 Introduction

Nonparametric kernel methods are becoming increasingly popular tools for econometricians as researchers have begun to gravitate toward such methods when there is little prior information concerning the proper functional form and/or variable distributions (see, e.g., Li and Racine 2007). Frequently, this results from the use of statistical tests that formally reject the parametric model, but provide no evidence as to the direction in which to search for the correct parametric specification. Nonparametric method allow one to relax functional form assumptions of an unknown model, and instead find the best specification given the data.

These methods, while increasingly popular across all areas of econometrics, are commonly criticized on two grounds: computational complexity and too much flexibility. With respect to flexibility, in many contexts, researchers often have some – but incomplete – information about the proper specification. However, it is difficult to incorporate such structure into traditional nonparametric methods. For example, economic theory often places restrictions on the relationship between variables (such as homogeneity) that are not guaranteed to be hold in nonparametric estimation. Thus, researchers are left to choose between using a parametric approach, where the underlying specification may be mis-specified, but the restrictions from economic theory are easy to impose, and a nonparametric approach, where the functional form will not be mis-specified, but may not conform to economic theory. In this paper, we develop a new estimator to give researchers a third choice: impose restrictions from economic theory in a relatively straightforward manner within a nonparametric framework.

The motivation for our estimators stems from the analysis of a first-price auction. In a parametric setting, a researcher typically specifies *a priori* the distribution of values, and then derives the corresponding equilibrium bidding strategy (see, e.g., Paarsch 1992). This yields a specific bid density from which to conduct likelihood analysis. Due to the *a priori* specification of the distribution of values, the derivation of equilibrium bidding strategies has all of the important theoretical restrictions, such as monotonicity, imposed prior to estimation. In such settings, monotonicity naturally carries over to the likelihood function, but at the expense of having to specify the distribution of values.

If one decides to eschew *a priori* specifications of the value distribution, nonparametric methods to recover equilibrium bidding strategies, such as the estimator proposed in Guerre, Perrigne, and Vuong (2000; GPV hereafter), exist. However, existing methods, such as GPV, offer no guarantee that the estimated equilibrium bidding strategy will be monotonic. Thus, in the absence of prior information about the correct distribution for private values, researchers are forced to choose between a *potentially* incorrect distribution, but imposing monotonicity, or being agnostic about the true distribution, but ending up with a *potentially* non-monotonic equilibrium bidding strategy.

In this paper, we develop a nonparametric estimator that imposes the monotonicity assumption inherent in the theory of first-price auctions. Our approach provides a generalization of the estimator derived in GPV for first-price auctions within the independent private value paradigm (IPVP). Importantly, our estimator simplifies to the GPV estimator when the *estimated* inverse equilibrium bidding strategy is monotonic. With respect to the computational complexity of the nonparametric methods mentioned above, our estimator utilizes readily available quadratic programming routines and should pose few difficulties in implementation relative to the unconstrained estimator of GPV.

There are a number of reasons why the need for a generalization of the GPV estimator could arise in empirical applications. For example, optimization errors or irrationality in bidding would generate observed bidding strategies that deviate from that assumed by theory. There is an extensive literature in experimental auctions that document such deviations in the most simple of settings. Whether these results generalize to the field remains an ongoing empirical debate (see Levitt and List, 2007), but the pervasiveness of such deviations suggests the potential importance of our approach. Similarly the existence of omitted auction heterogeneities or uncaptured asymmetries across bidders - such as those generated by the operation of a cartel (e.g., see Bajari, 2001; Bajari and Ye, 2003) - may generate departures from equilibrium bidding strategies that are unaccounted for in the standard GPV framework.

To preview the discussion to follow, we note that the standard kernel density estimator used in GPV relies exclusively on the bandwidth. That is, the bandwidth represents the sole instrument by which the shape of the density being estimated can be manipulated. This is important since it puts researchers in a bind: we show that one can guarantee monotonicity by using a sufficiently large bandwidth, but that such guarantee may come at the expense of deviating from the 'optimal' bandwidth (and thus oversmoothing).[1] Moreover, since GPV advocate trimming the data based on the chosen bandwidth, using a larger bandwidth in order to guarantee monotonicity results in excessive trimming for a given sample size.

In contrast to the GPV estimator, our proposed estimator introduces a second instrument to manipulate the kernel density (distribution), thereby allowing constraints, such as monotonicity, to be imposed without deviating from the 'optimal' bandwidth. Our estimator relies on the constraint weighted bootstrapping theory developed in Hall, Huang, Gifford, and Gijbels (2001) and Hall and Huang (2001). To our knowledge, this theory has not been formally applied in economics, let alone the structural auction literature.

Our estimator should hold interest not only to researchers interested in auctions but also those who employ structural approaches to recover the primitives of economic models in other areas. For example, the ability to constrain nonparametric estimators should prove an indispensable tool for those who wish to use monotone comparative statics (see, e.g., Athey 2001, 2002) to recover structural parameters in stochastic optimization problems or games of incomplete information. More generally, economic theory often imposes a particular structure on models being estimated - i.e., curvature conditions in applied demand or production analysis. As conforming to such structural assumptions is a necessary condition for utilizing nonparametric methods in any structural analysis, it is thus important to augment existing

[1]In this respect, our approach formalizes sentiments expressed in Athey and Haile (2005) that bandwidth selection, specifically data-driven methods, are important to our understanding of structural auction models.

nonparametric approaches to allow the researcher to impose such fundamental behavioral assumptions.

Second, policymakers ought to take note of the findings presented herein as several of our simulations suggest that monotonicity hinges critically on the bandwidth selected. As a result, since bandwidth selection is under the direct control of the researcher, one could fictitiously display a monotonic relationship using existing nonparametric methods through manipulation of the bandwidth. This can be troubling, however, if such estimations are used to inform policymakers concerning the choice of auction format/design.

Third, our estimator should be of interest to econometricians interested in constrained nonparametric methods. Beginning with Gallant (1981) and Matzkin (1994), econometricians have developed methods to impose economic constraints while still pursuing nonparametric avenues. Our estimator adds to recent work that examines the imposition of curvature conditions in nonparametric settings (Beresteanu 2004; Chak, Madras and Smith 2005; Chernozhukov, Fernandez-Val, and Galichon 2007). In addition, the constraint weighted bootstrapping approach advocated here is similar in spirt to the empirical likelihood methods developed in Owen (1988) and the information theoretic approaches to GMM presented in Imbens, Spady, and Johnson (1998).[2] The constraint weighted bootstrapping methods can be viewed as imposing another level of constraints in the optimization of the empirical likelihood as both methods invoke power-divergence statistics.

Finally, in addition to the papers discussed above, a substantial amount of attention has been paid to the issue of monotonicity by statisticians. The longest standing method for imposing monotonicity has been isotonic regression (Mukerjee 1988; Mammen 1991). Ramsay (1988) suggested using constrained splines to smooth a monotonic function while Mammen, Marron, Turlach, and Wand (2001) suggested using projection based techniques to perform constrained smoothing. Related to our work is the method of Hall and Huang (2001), which uses constraint weighted bootstrapping to create a smooth, monotone regression function. Recently, Dette, Neumeyer, and Pilz (2006) suggest using rearrangement (proposed in economics by Chernozhukov et al. 2007) to construct a smooth, monotone regression function. In a Bayesian context, McCausland (2008) has shown how to construct estimators that obey curvature restrictions, such as monotonicity. Whether it be econometricians or statisticians, much attention has been paid to the construction of regression functions that are monotone and nonparametric. However, very little attention has been paid to the monotonic construction of survival functions, which underlies the structural auction estimator detailed in this paper, outside of the statistics and biostatistics literature. Thus, the method proposed here attempts to fill this void.

The rest of the paper is laid out as follows. In Section 2 we briefly review the economics behind the first-price auction setup within the IPVP. Section 3 discusses the GPV nonparametric estimator for this auction style-paradigm pair. In Section 4 we describe how to implement the constraint weighted bootstrapping theory developed in the statistics literature to create a generalized estimator that imposes monotonicity. We provide several applications of the estimation method in Section 5 using simulated

[2]We thank Han Hong for suggesting this interesting connection between our work and the empirical likelihood approach.

data, experimental data, and a real world data set. In Section 6 we emphasize the usefulness of this style of nonparametric estimation beyond structural auctions and indicate several lines of possible future research.

2 Theoretical Background

Within the IPVP each player knows her value of the product to be auctioned, but no other player's value. Players values are assumed to be independent draws from $F(v)$, which is taken as common knowledge. Players select their bidding strategy to maximize their expected payout, given by $\pi^e(\cdot)$. This leads to the following maximization problem:

$$\max_b \quad \pi^e(b) = (v - b)F(\sigma_n)^{n-1}, \tag{1}$$

where $\sigma_n = \beta_n^{-1}(b)$ denotes the inverse of the bid function, $\beta_n(v)$, used by the player, v is her value and b is her corresponding bid when there are n total participants in the auction. The first order condition is given by:

$$-F(\sigma_n)^{n-1} + (n-1)(v - \beta_n)F(\sigma_n)^{n-2}f(\sigma_n)\sigma_n' = 0. \tag{2}$$

The assumption that the bid function is monotonic allows one to use $d\sigma_n/db = 1/\beta_n'(v)$ to simplify the above first order condition. We also note that by symmetry of the bidders, $\beta(v) = b$. These features allow us to simplify our solution as

$$\beta_n' + \frac{(n-1)f(v)}{F(v)}\beta_n = \frac{(n-1)vg(v)}{F(v)}, \tag{3}$$

which is a linear differential equation with solution, assuming the absence of a reserve price[3],

$$\beta_n(v) = v - \frac{\int_{\underline{v}}^{v} F(u)^{n-1}du}{F(v)^{n-1}}, \tag{4}$$

where $\underline{v}$ represents the left end of the support of the value distribution. If we allow for a reserve price, then our differential equation would have solution

$$\beta_n(v) = v - \frac{\int_{r}^{v} F(u)^{n-1}du}{F(v)^{n-1}} \quad \text{where} \quad r \leq v. \tag{5}$$

Note that the only difference between Equations (4) and (5) are the limits of integration, assuming that all potential bidders place bids. In essence the reserve price acts as a boundary condition in exactly the same way that $\underline{v}$ does in the no reserve setting. Paarsch and Hong (2006) provide a more detailed

[3] A reserve price is such that all submitted bids must be greater than this price.

description of this derivation and the IPVP in general.

3 Nonparametric Estimation in First Price Auctions

In a seminal paper on the identification and structural nonparametric estimation of a first-price auction, GPV provide a natural setting in which to think about the distribution of valuations within the IPVP in a nonparametric framework. Their analysis spurred (perhaps started) the growth of nonparametric structural estimation of auctions across paradigms, including affiliated private values (Li, Perrigne, and Vuong 2002) and conditionally independent private information (Krasnokutskaya 2004; Li, Perrigne, and Vuong 2000). Here, we describe their method under the situation of no reserve price.

The structural equilibrium bidding strategy derived in GPV is given as

$$v_i = b_i + \frac{G(b_i)}{(N-1)g(b_i)} = \xi(b_i, N, G), \tag{6}$$

where v_i and b_i are the value and bid for agent i, respectively. $G(b_i)$ is the cdf of the bid density and $g(b_i)$ is the bid density. Only the bid vector is observed by the econometrician. The functional forms of $G(\cdot)$ and $g(\cdot)$ must be assumed or estimated; then, the values, v_i, can be estimated along with the corresponding cdf and pdf, $F(\cdot)$ and $f(\cdot)$, respectively.

The nonparametric estimation approach given in GPV is as follows:

1. Estimate $g(b)$ using kernel methods.

$$\widehat{g}(b) = \frac{1}{nTh}\sum_{i=1}^{n}\sum_{t=1}^{T} K\left(\frac{b-b_{it}}{h}\right), \tag{7}$$

   where b is now indexed by both i and t. Here, t represents a particular auction, thus we are pooling bids from multiple auctions with the same number of bidders to increase the sample size. The bandwidth, h, depends on the sample size and converges to zero as T goes to ∞. The standard bias-variance tradeoff exists when considering how large or small to set the bandwidth. $K(\cdot)$ is a kernel function which is chosen to satisfy several unrestrictive conditions.

2. Estimate $G(b)$ using the empirical cdf

$$\widehat{G}(b) = \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} \mathbf{1}\{b_{it} \le b\}, \tag{8}$$

   where $\mathbf{1}\{A\}$ is the indicator that the event A is true.

3. Construct $\widehat{v}_{it} = \widehat{\xi}(b_{it}, n, G)$ using the above estimates to recover the values. Employ the truncation strategy of GPV (page 531, equation 6).

4. Estimate the density and distribution of values, $f(\widehat{v}_{it})$ and $F(\widehat{v}_{it})$, using equations (7) and (8) above, with b_{it} and b replaced with $\widehat{v}_{it}$ and v, respectively.

The above discussion omits two important details. First, the GPV estimator needs to trim the sample near the boundaries of the pseudo-values. Kernel density estimators are well known to be inconsistent near the edge of the support of the variable of interest. This contaminates the second stage recovery of the distribution of values. GPV propose trimming observations that are within one bandwidth of $\underline{b}$ and $\bar{b}$, the upper and lower bounds of the support for bids. This yields a consistent estimator on the interior of $\mathcal{I} = [\underline{b}, \bar{b}]$. Second, GPV show that $\xi(b_i, N, G)$ is strictly increasing for all $b_{it} \in \mathcal{I}$.[4] Their nonparametric approach, however, does not formally impose this condition in the estimation. To fill this void, the following section proposes a method to impose the monotonicity condition.

4 Monotonic Nonparametric Estimation in First-Price Auctions

4.1 Motivation

Prior to presenting our new estimator, it is worthwhile to step back and ask how fragile is the monotonic feature of the bid-value relationship?. If the GPV estimators yields a bid-value relationship that is monotonic in most cases, there may be little to no value added from the development of a new estimator.

To address this question, we perform simulations to examine whether, and to what extent, monotonicity is violated using the GPV estimator. We use the bandwidth and kernel employed in GPV, namely the triweight kernel, $K(u) = (35/32)(1-u^2)^3\mathbf{1}(|u| \leq 1)$, with bandwidth $1.06\widehat{\sigma}_b(NT)^{-1/5}$. Of first note is that this bandwidth actually leads to undersmoothing, as $(4/3)^{1/5}$ is the aysmptotically optimal scale factor for the Gaussian kernel, not the triweight kernel. The asymptotically optimal scale factor for the triweight kernel is $2.978 * (4/3)^{1/5} = 3.154$. This scaling factor comes from the theory on canonical kernels found in Marron and Nolan (1989). Essentially, to guarantee that the same degree of smoothing is present when different kernels are used, the bandwidth must be adjusted by a specific factor, in our case 2.978.[5] For completeness, we perform our simulations using both bandwidths.

To judge the monotonicity of the equilibrium bidding strategy, we determine if the sign of $\widehat{\xi}'(b_i, N, G)$ is negative on $\mathcal{I}$. We allow G to come from the gamma, exponential, log-normal, normal, Weibull, and a specific mixture of normals, all of which provide a theoretically monotonic equilibrium bidding strategy. We also vary the truncation points of the draws from these densities to correspond to assumption A2 in GPV. We will see that this truncation point also plays an important role in whether we find a high proportion of draws displaying monotonicity.

We take draws using five bidders and vary the number of auctions, T, so that our full sample sizes, nT, are 50, 100, or 1000. We vary the truncation points so that they contain 80, 90, or 95% of the data. Furthermore, as mentioned above, we use the bandwidth suggested in GPV, as well as the theoretically

[4] This is condition $C2$ of Theorem 1 in GPV.
[5] See equation 2.5 in Marron and Nolan (1989).

consistent rule-of-thumb bandwidth. For each scenario, we take 1000 draws and determine the proportion of those draws that display a monotonic bid-value relationship. For parsimony, we employ a fixed point design using 100 equally spaced points, ranging from the left to the right truncation points. Figure 1 displays the six equilibrium bidding strategies, as well as the parameters used to generate the data.

Table 1 displays the likelihood of estimating a monotonic equilibrium strategy for the given distributions mentioned above. The results show the sensitivity of monotonicity according to the size of the bandwidth, the sample size, and the level of truncation. One immediately notices that the proportion of estimated monotonic draws decreases as the truncation moves from 80 to 95%. This reflects the importance, or impact, of tail observations on the appearance of the estimator. As expected, we notice that the bandwidth is critical in uncovering a monotonic equilibrium strategy since switching from the Gaussian kernel's scale factor to the triweight's scale factor results in a more than sizeable shift in the proportion of samples that yield monotonic relationships. Indeed, below we will find that there exists a bandwidth such that the equilibrium bidding strategy is always monotonic for a given sample size. The sample size also has an intuitive effect on the probability of observing a monotonic equilibrium bidding strategy; the probability of observing a monotonic relationship is an increasing function of the sample size. This is to be expected since increasing the number of observations within a range fills in the spaces between points that exist in smaller samples.

These spaces add to the likelihood of estimating a nonmonotonic equilibrium bidding strategy regardless of the bandwidth used. Mathematically, we define the event that a sample contains a data point that is at least one bandwidth away from its nearest point as $C(\varepsilon)$. the probability of the event, $Pr(C(\varepsilon))$, is nonzero for a given bandwidth, sample size and truncation point. From the simulations above we see that as we increase the bandwidth or sample size, $Pr(C(\varepsilon)) \downarrow 0$. As the truncation point is increased, $Pr(C(\varepsilon)) \uparrow 0$.

This exercise, while heuristic in nature, does provide some insight into the ability of the GPV estimator to naturally provide an estimated monotonic equilibrium bidding strategy. The first striking feature is the tight relationship between monotonicity and the selected bandwidth. This is actually discouraging since bandwidth selection is arbitrary in this setting. While a researcher can select the asymptotically optimal bandwidth given the sample size as well as the kernel, it is at best 'asymptotically' optimal. Also, cross-validated bandwidth selection has remained largely unexplored in nonparametric auction estimation. This is interesting, as it is possible to select an automatically determined bandwidth to smooth the bid density optimally, but it is unknown whether this is the appropriate amount of smoothing for the equilibrium bidding strategy. Moreover, no selection criteria can be employed for bandwidth selection in this setting since the values, v, are unobserved, making it difficult, if not impossible, to formulate a criteria that leads to optimal smoothing of the equilibrium bidding strategy. This leads naturally to an estimator that nonparametrically constrains the GPV estimator to be monotonic, regardless of the bandwidth, thus removing the incentive to oversmooth in order to provide theoretically consistent results.

Formally, our claim above is that if $\mathcal{I}$ is a compact interval, then for all sufficiently large h, $\widehat{\xi}'(\cdot|h) > 0$ on $int(\mathcal{I})$, guaranteeing monotonicity of the equilibrium bidding strategy. To justify this argument, we assume that $K(\cdot)$ has two continuous derivatives in a small neighborhood of the origin, with $K(0) = 0$ and $K'(0) = 0$. We note that when $h \to \infty$, the following two relations hold:

$$K\left(\frac{b - b_{it}}{h}\right) \approx K(0) + \left(\frac{b - b_{it}}{h}\right) K'(0) + o_p(h^{-2}),$$

and

$$K'\left(\frac{b - b_{it}}{h}\right) \approx h^{-1}K'(0) + \left(\frac{b - b_{it}}{h^2}\right) K''(0) + o_p(h^{-3}),$$

which together imply

$$\widehat{g}(b|h) \approx h^{-1}K(0) + o_p(h^{-1}),$$

and

$$\widehat{g}'(b|h) \approx h^{-3}K''(0) \cdot (nT)^{-1} \sum_{i=1}^{n} \sum_{t=1}^{T} (b - b_{it}) + o_p(h^{-3}),$$

uniformly over $b \in \mathcal{I}$. From this it follows that for sufficiently large h, $\widehat{g}(b)^2 > |\widehat{g}'(b)| \; \forall \; b \in \mathcal{I}$. This shows that the numerator of the derivative of the GPV estimator (the denominator is always positive) will be positive everywhere given a sufficiently large bandwidth.

4.2 Monotone Estimation of the Bid Function

Rather than resorting to an arbitrarily large bandwidth or to parametric methods when the GPV estimator does not produce a monotonic result, we instead show how a modified version of the GPV estimator can be constrained to be monotonically increasing. The method we employ is known as constraint weighted bootstrapping in the statistics literature. It is becoming a common technique to impose monotonicity when estimating survival functions. We note that $\xi(\cdot)$ is similar to a survival function. Our approach is as follows:

1. Estimate $g(b)$ as
$$\widehat{g}(b|p) = \frac{1}{h} \sum_{i=1}^{n} \sum_{t=1}^{T} p_{it} K\left(\frac{b - b_{it}}{h}\right), \tag{9}$$
where the p_{it} are observation-specific weights. Note, the GPV estimator is a special case of our estimator where each weight is set equal to $1/nT$.

2. Estimate $G(b)$ as
$$\widehat{G}(b|p) = \int_{-\infty}^{b} \widehat{g}(u|p)du. \tag{10}$$
Notice that we are not constructing the cdf of the bids using the empirical distribution function. To ensure that our cdf corresponds to the pdf estimated in equation (9), we need to integrate the pdf as opposed to simply calculating the cdf with the empirical distribution function. This step is

not done in GPV, nor is it a common approach in studies that use both a cdf and a pdf in their estimation.[6] One may think that the reason for this is twofold. First, the asymptotic arguments are most likely easier to prove given widely known properties of the empirical distribution estimator. Second, the empirical distribution estimator is easier to construct than an integral of an estimated probability density.

3. Construct $\widehat{v}_{it} = \widehat{\xi}(b_{it}, n, G|p)$ using the above estimates to recover the values. Employ the truncation strategy of GPV (page 531, equation 6).

4. Estimate the density and distribution of values, $f(\widehat{v}_{it})$ and $F(\widehat{v}_{it})$ using equations (7) and (8) above, with b_{it} and b replaced with $\widehat{v}_{it}$ and v, respectively. Note here that because we have a two-step estimator, the recovery of the density and distribution of the values does not need to have any constraint weights incorporated into their estimation. this is because monotonicity has been imposed on the estimator of the bid function, which is used to create the pseudo-values, and the resulting pseud-values can then be treated as they were by GPV.

The crucial feature of our estimator is that the weights, p_{it} are selected to ensure that the values are monotonically increasing in the bids. To select the vector of weights, we choose $p = \{p_{11}, p_{12}, \ldots, p_{1T}, \ldots, p_{nT}\}$ to minimize a distance metric subject to the constraint that $\widehat{\xi}'(b_{it}, n, G|p) \geq 0$ on $\mathcal{I}$. If one desired to impose strict monotonicity, $\widehat{\xi}'(b_{it}, n, G|p) > 0$ on $\mathcal{I}$, the user needs to pick some small number δ such that $\widehat{\xi}'(b_{it}, n, G|p) > \delta$ on $\mathcal{I}$ so that this becomes computationally feasible.[7] We also impose the regularity conditions $p_{it} \geq 0 \ \forall i, t$ and $\sum_{i=1}^{n} \sum_{t=1}^{T} p_{it} = 1$. These conditions make the weights act as though they are drawn from a density and will prove useful when making comparisons to the uniform weights, $1/nT$, used in GPV. For simplicity, we choose to impose our nonnegativity constraint on

$$\hat{\xi}_1 = n\widehat{g}(b|p)^2 - \widehat{G}(b|p)\widehat{g}'(b|p). \tag{11}$$

Given that $\hat{\xi}'/\hat{\xi}_1$ is always nonnegative, this implies that both have the same sign.

Our distance metric is the power divergence measure introduced in Cressie and Read (1984) and proposed in Hall, Huang, Gifford, and Gijbels (2001) for monotone estimation of a hazard rate.[8] The power divergence measure is

$$D_\rho(p) = \frac{1}{\rho(1-\rho)}[nT - \sum_{i=1}^{n}\sum_{t=1}^{T}(nTp_{it})^\rho], \quad -\infty < \rho < \infty. \tag{12}$$

where $\rho \neq 0, 1$. One needs to take limits for $\rho = 0$ or 1. They are given as

$$D_0(p) = -\sum_{i=1}^{n}\sum_{t=1}^{T}\log(nTp_{it}); \qquad D_1(p) = \sum_{i=1}^{n}\sum_{t=1}^{T} p_{it}\log(nTp_{it}), \tag{13}$$

[6]See Martins-Filho and Yao (2008) for a recent example that does.
[7]This is also suggested in Hall and Huang (2001).
[8]It is also used in Hall and Huang (2001) for nonparametric monotone estimation of a regression function.

If one uses $\rho = 0.5$, this corresponds to Hellinger distance. Note, for all ρ we have $D_\rho(p) \geq 0\ \forall p$ and $D_\rho(p) = 0$ if and only if $p_{it} = 1/nT\ \forall i, t$. This suggests that departures from uniformity of the weights will correspond to a positive divergence measure, indicating the presence of regions of non-monotonicity.

Regardless of the sampling distribution for the values of the players across auctions and the choice of $\mathcal{I}$, it is entirely plausible that $\widehat{\xi}_1$ will have a zero crossing on $\mathcal{I}$. An example can easily be constructed where a point lies both a bandwidth away from the boundary and from its nearest point. Label this event as $\mathcal{E}$. The probability of this event is strictly positive given minimal assumptions about the bid density. Fortunately, data sets that produce event $\mathcal{E}$ or a similar event, are pathological in nature. Even if one could not find a set of weights that guarantees a monotonic estimator, this should pose no problem. In fact, one can view this event as providing information about the true equilibrium bidding strategy or as evidence that other features of the auction are being ignored by the econometrician (Athey and Haile 2005).

While it may be argued that this procedure is entirely heuristic given the fact that many papers have confirmed monotonicity between bids and values [9], the ability to easily impose this condition when estimating models using auction data is important from an economic standpoint. Indeed, even if the weights are uniform, the researcher can be more confident that the equilibrium bidding strategy is monotonic than simply through visual inspection of the estimated surface. Additionally, while it may appear that monotonicity holds unconditionally between bids and values, the presence of covariates renders visual inspection useless in higher dimensions.

Theoretically, this estimator (ignoring truncation) is consistent following only minor modifications in the proof in Hall, Huang, Gifford, and Gijbels (2001). Given that the pseudo-value estimates are constructed identically to GPV, the theoretical properties of the value density estimator should follow directly. We do not consider asymptotic normality of this estimator and leave that for future research.

4.3 Extensions to Heterogeneous Auctions and Reserve Prices

Many auctions are characterized by differing numbers of bidders and/or use of reserve prices. We discuss a scenario encapsulating both to highlight the ease by which the constraint weighted bootstrapping estimation discussed above may be generalized. The GPV estimator of this auction setting relies on the following first order condition (written in terms of the actual bids):

$$v_i = b_i + \frac{1}{\mathcal{N} - 1} \left\{ \frac{G(b_i)}{g(b_i)} + \frac{F(r)}{(1 - F(r))g(b_i)} \right\}, \tag{14}$$

where $\mathcal{N}$ is the number of potential bidders and r is the reserve price of the auction. Within the IPVP, n_t, the number of actual bidders in auction t, has a binomial distribution with parameters $\mathcal{N}$ and $1 - F(r)$.

[9] See Figure 2 of Li, Perrigne, and Vuong (2000) for an example where $\widehat{\xi}(\cdot)$ is locally but not globally monotonic in OCS wildcat auctions.

A natural candidate estimator for $\mathcal{N}$ is

$$\widehat{\mathcal{N}} = \max_{t=1,\ldots,T} n_t. \tag{15}$$

Following Paarsch and Hong (2006, pg. 132), we estimate $1 - F(r)$ as $\bar{n}/\widehat{\mathcal{N}}$, where

$$\bar{n} = T^{-1} \sum_{t=1}^{T} n_t. \tag{16}$$

Before proceeding to estimation, we mention that the observed bids, b_i, must be transformed as $s_i = \sqrt{b_i - r}$ due to the proportionality between the actual bid density, $g(b)$, and $1/\sqrt{b-r}$ as b approaches r. This transformation prevents the density of bids to become unbounded near the reserve price. Using this transformation, we can write the first order condition in equation (14) as

$$v_i = s_i^2 + r + \frac{2s_i}{\mathcal{N}-1}\left\{\frac{G^*(s_i)}{g^*(s_i)} + \frac{F(r)}{(1-F(r))g^*(s_i)}\right\} = \xi(s_i, \mathcal{N}, G^*, r), \tag{17}$$

where $G^*(\cdot)$ and $g^*(\cdot)$ are the cdf and pdf, respectively, of the transformed bids. The GPV estimator follows by pooling bids across auctions and estimating the unobserved valuations as:

1. Estimate $\mathcal{N}$ and $1 - F(r)$ using the suggestions above.
2. Estimate $g^*(s)$ as
$$\widehat{g}^*(s|p) = \frac{1}{h} \sum_{t=1}^{T} \sum_{i=1}^{n_t} p_{it} K\left(\frac{s - s_{it}}{h}\right). \tag{18}$$
3. Estimate $G^*(b)$ as
$$\widehat{G}^*(s|p) = \int_{-\infty}^{s} \widehat{g}(u|p)du. \tag{19}$$
4. Construct $\widehat{v}_{it} = \widehat{\xi}(s_{it}, \mathcal{N}, G^*, r|p)$ using the above estimates to recover the values, following the truncation strategy of GPV on page 550.
5. Estimate the density and distribution of values, $f(\widehat{v}_{it})$ and $F(\widehat{v}_{it})$, using equations (7) and (8) above, with b_{it} and b replaced with $\widehat{v}_{it}$ and v, respectively.

To minimize the computational burden in selecting the weights, we choose to impose our nonnegativity constraint on

$$\widehat{\xi}_1 = 2s\widehat{\mathcal{N}}\widehat{g}^*(s|p)^2 + s\widehat{G}^*(s|p)\left(1 - 2\widehat{g}^{*\prime}(s|p)\right) + \widehat{F}(r)s(\widehat{g}^*(s|p) - 2)/(1 - \widehat{F}(r)). \tag{20}$$

We also maintain the regularity conditions $p_{it} \geq 0 \; \forall i, t$ and $\sum_{i=1}^{n} \sum_{t=1}^{T} p_{it} = 1$.

5 Applications

We use three sets of data to determine the performance of our estimator. We begin by examining simulated data where we know the exact form of the value distribution and create the corresponding equilibrium bids. We then move on to the assessment of experimental data to see if monotonicity arises in a laboratory setting where we are using bids submitted by participants, not the bids corresponding to the equilibrium bidding strategy. This is also useful because we know not only the value distribution, but the actual values, and therefore can more adequately address the criticisms of detecting monotonicity raised in Athey and Haile (2007) discussed above. Finally, we use data from timber auctions conducted in British Columbia to show the empirical relevance of our method.

5.1 Simulated Data

Our simulation experiments examine monotonic distributions that are theoretically consistent with an equilibrium bidding strategy. We consider $T = 100$ auctions, each having $n = 5$ bidders, which yields 500 observed bids. Our setup will involve 100 replications under each scenario. We choose the true distribution of private values to be either log-normal with parameters 0 and 1 or gamma with parameters 1 and 3. For the log-normal case, we follow the truncation strategy in GPV, discarding those value draws that are below 0.055 and above 2.5. Similarly, for the gamma distribution, we discard values that are below 0.0455 and above 4.982. For every replication we first draw nT values from the truncated distribution. We then compute the bids, b_{it}, using

$$b_{it} = v_{it} - \frac{1}{F(v_{it})^{n-1}} \int_{\underline{v}}^{v_{it}} F(u)^{n-1} du, \tag{21}$$

where $\underline{v}$ is the smallest value drawn from the truncated distribution for the given replication.

Using these generated data, we then employ our estimation procedure for each replication. We use (9) and (10) to estimate the density and distribution of the bids for a given set of weights. We employ the tri-weight kernel with the bandwidth used in GPV (Silverman's rule-of-thumb for a normal kernel). The weights are determined using $\rho = 0$, 0.5, and 1 and are found using the sequential quadratic programming routine SQPSolve in the programming language GAUSS 8.0. While our problem is not a quadratic programming problem, this type of solver uses a modified quadratic program to find the step length for moving in the direction of a minimum. Each iteration takes somewhat longer to run than those reported in GPV since, for any given replication, it is not guaranteed to generate a monotonic equilibrium bidding strategy. One problem that we encountered several times was that the program would not return feasible results. This was easily remedied, however, by changing the starting values.[10]

The simulation results are given in Figures 2 and 3. Panel (a) of each figure plots the true equilibrium

[10] Our starting values were selected at random from a uniform distribution and divided by the sum of the starting values to preserve the summation constraint.

bidding strategy along with the estimates from the GPV estimator and our estimator. The curves correspond to the 95^{th} percentile of the distance metric. Panel (b) of each figure depicts the envelope-curves of the weights after the constraints have been achieved. It is clear from our earlier simulations that the true data generating process provides a monotonic equilibrium bidding strategy. However, the finite sample results of the GPV estimator show regions where the derivative is negative. Our estimator corrects for these regions of non-monotonicity by changing the weights. In Panel (b) of each figure we see that the corresponding weights deviate from $1/nT$ in the bid region where the GPV estimator is non-monotonic.

5.2 Experimental Data

Our experimental data were originally collected by Dyer, Kagel and Levin (1989). Since the data have been used in Bajari and Hortaçsu (2005) and are discussed there as well, we provide only limited details. MBA students at the University of Houston participated in a series of first-price sealed bid auctions over the course of two hours. Subjects submitted contingent bids, based on the number of other bidders in the auction (either 2 or 5). However, we are treating the submitted bids (with either 3 or 6 bidders) as the actual bids for our purposes. Values were drawn from a $\mathcal{U}[0, 30]$ density. As in Bajari and Hortaçsu (2005), we drop the submitted bids for the first five auctions of a given run of the experiment. This leaves us with 23 auctions over three experimental runs. We have a total of 414 bids, regardless of the number of bidders, since we are ignoring the contingent bidding aspect of the experiment.

We use the same kernel (Gaussian) and bandwidth ($1.06\widehat{\sigma}_b(NT)^{-1/5}$) as in Bajari and Hortaçsu (2005). For the three bidder case, Figure 4 displays a slight non-monotonic portion of the observed bidding strategy, which in the uniform value case should be exactly linear.[11] This result is interesting since we learn that random samples from known value distributions could generate non-monotonic bidding strategies for a given bandwidth. We find here that with a known value distribution, but with humans deciding how to bid, we still have non-monotonic portions of the estimated equilibrium bidding strategy. This lends further credibility to our ability to constrain the estimator to be monotonic.

Figure 5 shows that the GPV estimator is monotonic by itself in the six bidder case. This also relates to Bajari and Hortaçsu's (2005) argument that the bids from the six bidder case provide more realistic estimates of the values than those bids from the three bidder case. Thus, there is weak evidence that as the number of bidders increases, the theoretical properties of the GPV estimator are being adhered to by the human subjects. Again, these results are contingent on the bandwidth that we selected; the conclusions drawn from these figures may change given a change in the bandwidth.

[11] We use $\rho = 0.5$ to determine the constraint weights.

5.3 Naturally Occurring Data

To further display the empirical merit of our techniques, we use a sub-sample of the data investigated in List, Millimet, and Price (2007). They collected data from sealed bid timber auctions in British Columbia, Canada. We use the three largest sub-samples of bids that correspond to homogenous samples. This provides us with 257 auctions with three bidders, 282 auctions with four bidders, and 165 auctions with five bidders, giving us 771, 968, and 825 total observations, respectively. For more details on the collection and description of the data, refer to List, Millimet, and Price (2007).

We construct the monotonically increasing equilibrium bidding strategies for each of the three sub-samples, and provide the corresponding value densities, using the truncation recommended in GPV. We use the same kernel and bandwidth as GPV to estimate both the equilibrium bidding strategy as well as the density of pseudo-values. The estimated equilibrium bidding strategies and the corresponding pseudo-value densities are presented in Figures 6 through 8.

For the three bidder auctions, we see that the equilibrium bidding strategy is non-monotonic using the GPV estimator. In particular, we note that there are two places of the estimated equilibrium bidding strategy that are non-monotonic. While visually hard to detect, they are easily tracked in the lower left panel which plots the constraint weights versus the uniform weights. With $\rho = 0.5$ we have $D_\rho = 0.709$. We are cautious to not imply that the agents in these auctions are irrational, however, as omitted covariates could be causing the observed non-monotonic portions of the equilibrium strategy (see Section 5.2 in Athey and Haile 2005 for more on this argument).

For the four bidder auctions displayed in Figure 7, we find that the equilibrium bidding strategy is again non-monotonic using the GPV estimator. This region of non-monotonicity is very slight and almost impossible to detect without resorting to Panel (c) of the figure, however. With $\rho = 0.5$, we have $D_\rho = 0.206$, which is also suggestive of the four bidder auction being closer to uniformity than the three bidder auction, consistent with the experimental findings above.

Finally, the five bidder auctions are analogous to those discussed in the four bidder case. Again, the region of non-monotonicity is virtually indistinguishable from the unconstrained GPV estimator. We do notice a slight region of non-monotonicity very near the upper boundary of our bids, which is conveyed through the plot of the constraint weights. As before, with $\rho = 0.5$ we have $D_\rho = 0.016$, implying that with more players, bidders are behaving closer to what theory predicts.

6 Conclusion

Nonparametric methods have become increasingly popular tools in econometrics given their flexibility. However, a shortcoming often pointed out is that they may be *too* flexible. Specifically, while the fully parametric model is unknown, some information is known and should be imposed in the nonparametric estimation. For example, in structural settings, often the researcher has some information *a priori*. Indeed, this is the case of structural estimation of first-price auctions, where monotonicity of the equilibrium bidding strategy is assumed to hold.

The result has been leaving researchers with an unappealing choice: impose too much structure by using potentially mis-specified parametric methods, but guarantee monotonicity, or impose too little structure by using nonparametric methods that fail to guarantee monotonicity. However, in this paper, we offer a third option: constrained nonparametric methods.

Specifically, we have extended a nonparametric method originally proposed for estimating a survival function that can accommodate theoretical restrictions, such as monotonicity, in structural auction models. We believe that this technique has value far beyond that of structural auctions. In fact, given the importance of monotone comparative statics in economics, (see, e.g., Athey 2001, 2002), these techniques should prove indispensable for the use of nonparametric estimation as a structural tool.

Our work has also discovered that errors in bidding within an experiment can lead to non-theoretical conclusions. The data collected in Dyer, Kagel and Levin (1989) revealed that bidding errors produced a small, non-monotonic portion of the equilibrium strategy when there were three bidders. We also showed that monotonicity, in an empirical setting, is directly linked to the bandwidth used. This lends merit to the argument that is well known throughout the statistics and econometrics literature that bandwidth selection is critical, regardless of the setting. Given that monotonicity is a theoretical restriction, bandwidth selection becomes even more important when holding steadfast to assumptions stemming from theory.

While we have laid out the framework for constrained nonparametric analysis of auctions, much remains to be done. Future research is needed to extend these methods both within the IPVP, as well as beyond. Within the IPVP, the methods need to be augmented to allow for auction heterogeneity in terms of risk aversion, covariates, and learning. Outside of the IPVP, these methods can be tailored to the Affiliate Private Value, Common Value, and Conditionally Independent Private Information paradigms that have been developed.

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Table 1: Likelihood of an Estimated Monotonic Equilibrium Bidding Strategy

	Truncation/Sample Size								
		80%			90%			95%	
	50	100	1000	50	100	1000	50	100	1000
Gamma									
$h = 1.06$	10.9	13.0	33.3	6.3	8.3	18.0	5.0	4.8	9.7
$h = 3.154$	98.1	99.0	100.0	95.8	97.3	100.0	92.3	95.1	99.6
Exponential									
$h = 1.06$	11.5	14.1	32.8	7.0	8.0	19.3	4.5	5.0	8.0
$h = 3.154$	97.8	99.0	99.9	95.2	97.1	99.9	91.9	93.7	99.6
Log-Normal									
$h = 1.06$	11.7	12.6	27.1	4.0	4.4	8.4	1.7	1.8	2.1
$h = 3.154$	97.6	98.9	100.0	90.4	93.6	99.5	81.2	85.0	96.4
Normal									
$h = 1.06$	35.4	49.9	98.5	36.1	55.0	98.2	35.3	53.6	97.5
$h = 3.154$	99.1	99.7	100.0	99.2	99.7	100.0	99.4	99.9	100.0
Weibull									
$h = 1.06$	43.3	62.2	98.1	41.7	60.5	97.1	40.4	59.2	96.4
$h = 3.154$	100.0	100.0	100.0	100.0	100.0	100.0	99.8	100.0	100.0
Mixture Normal									
$h = 1.06$	9.2	9.3	12.2	9.8	10.6	20.1	14.2	12.2	27.0
$h = 3.154$	92.4	93.9	99.1	94.2	95.6	99.8	95.6	96.5	100.0

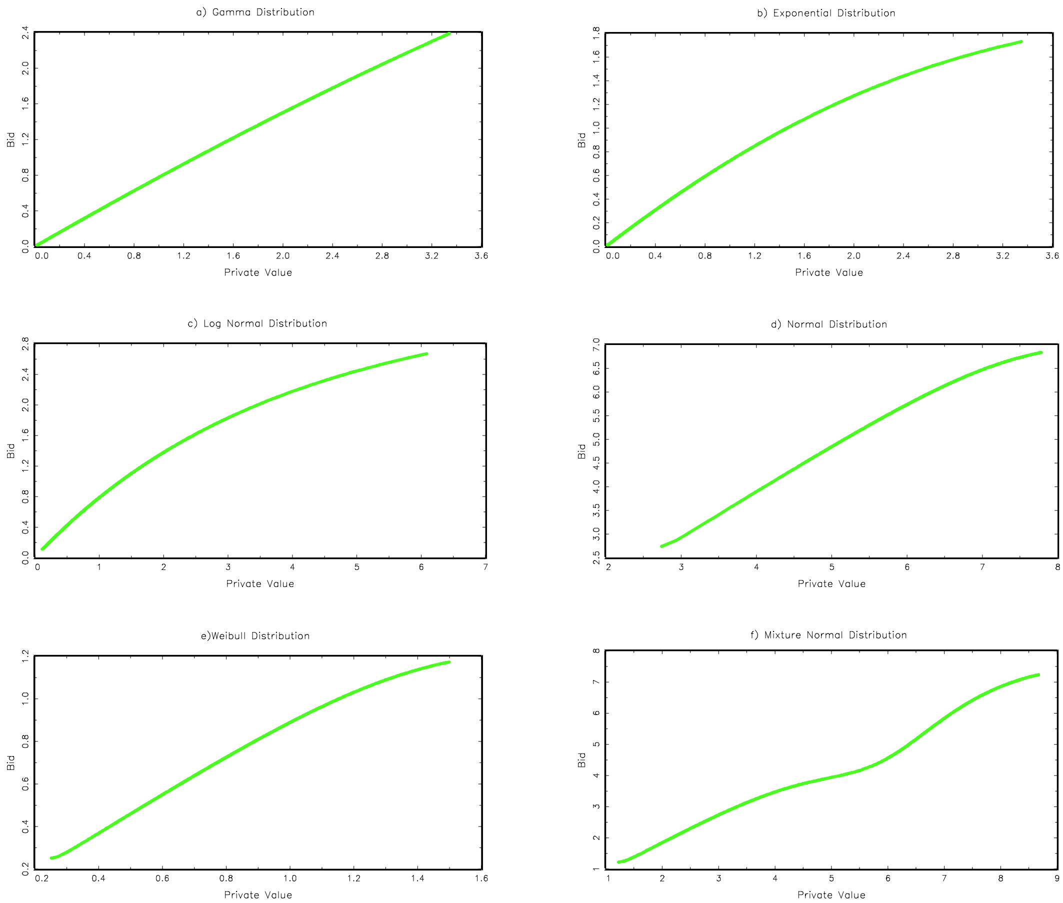

Figure 1: Panel a shows the true equilibrium strategy when $G \sim$ Gamma(3,1). Panel b is for Exponential(1) and panel c is Log-Normal(1,0). Panel d is the true equilibrium strategy for a Normal (6,1) and panel e represents a Weibull(3,1). Panel f represents an equal mixture of Normal(7.2,1) and Normal(3,1).

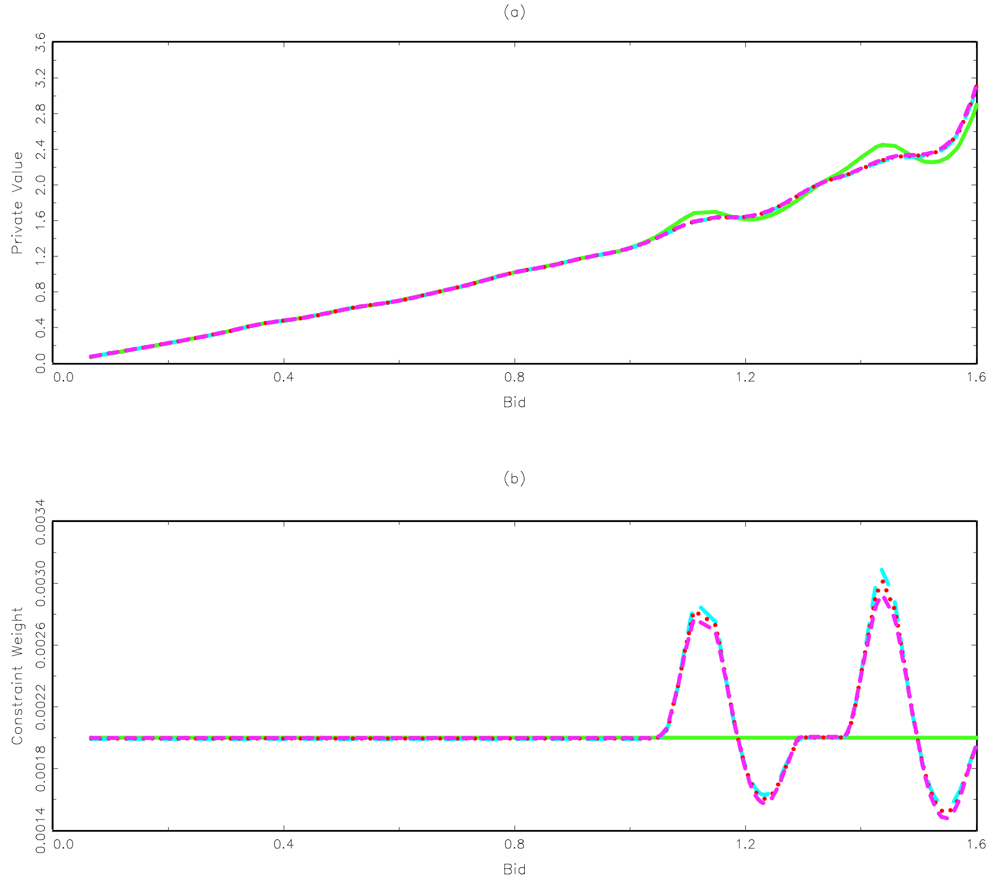

Figure 2: Monotonization for data simulated from a truncated log-normal distribution. Panel (a) represents the 95th percentile of $D_0(\hat{p}) = 1.811407$, the long-dashed line, the GPV estimator, the solid line, as well as the constrained GPV estimator for the same dataset with $\rho = 0.5$, $D_{0.5}(\hat{p}) = 1.846482$ represented by the short-dashed line, and with $\rho = 1$, $D_1(\hat{p}) = 0.003746$ represented by the dotted line. Panel (b) depicts the envelope curves of the values of $\hat{p}$ after the monotonicity constraint had been achieved with $\rho = 0, 0.5,$ and 1, again with the respective line types.

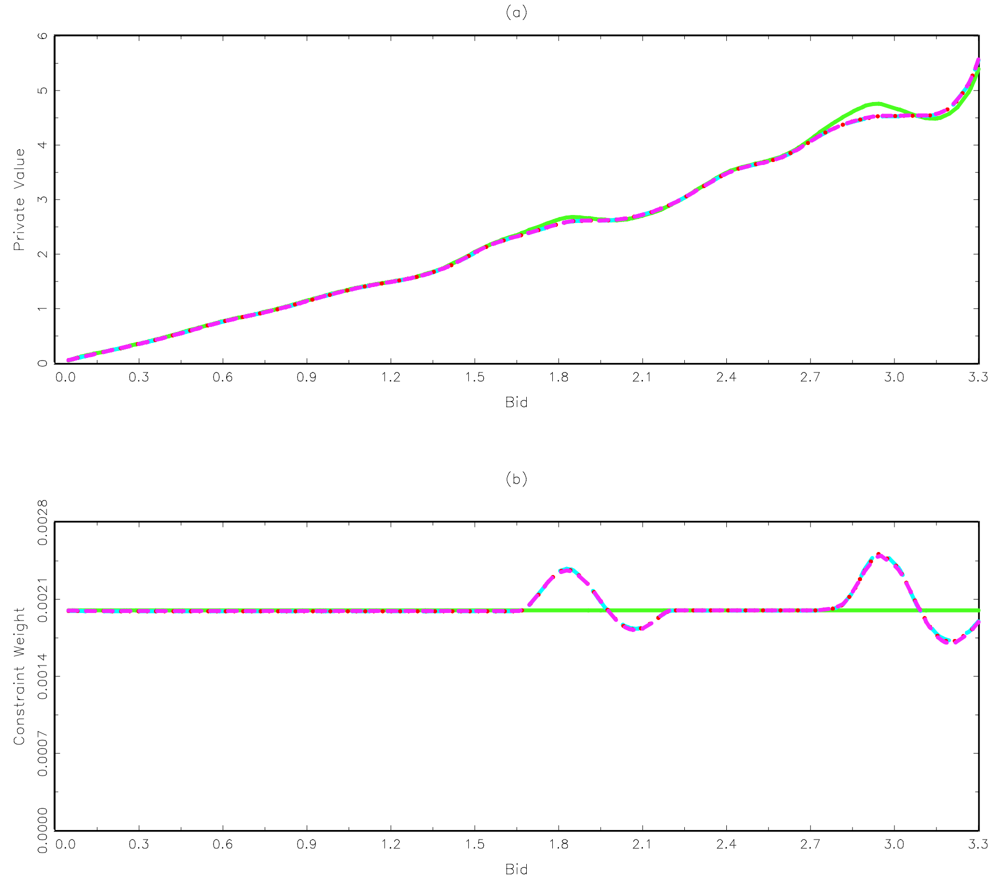

Figure 3: Monotonization for data simulated from a truncated gamma distribution. Panel (a) represents the 95th percentile of $D_0(\hat{p}) = 0.615096$, the long-dashed line, the GPV estimator, the solid line, as well as the constrained GPV estimator for the same dataset with $\rho = 0.5$, $D_{0.5}(\hat{p}) = 0.622522$ represented by the short-dashed line, and with $\rho = 1$, $D_1(\hat{p}) = 0.001259$ represented by the dotted line. Panel (b) depicts the envelope curves of the values of $\hat{p}$ after the monotonicity constraint had been achieved with $\rho = 0, 0.5,$ and 1, again with the respective line types.

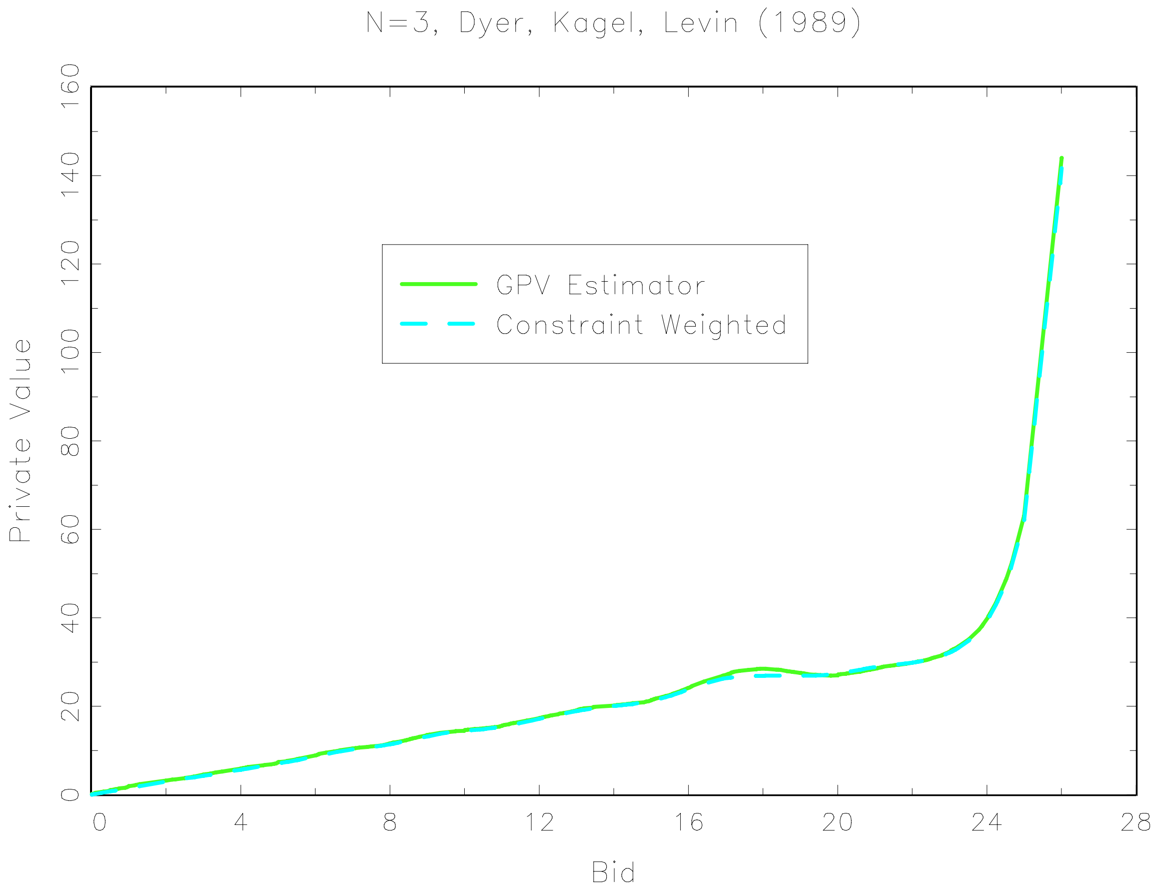

Figure 4: Equilibrium Bidding Strategy for 3 bidder experiment. The solid line represents the unconstrained GPV estimator while the dashed line is the monotonically constrained GPV estimator.

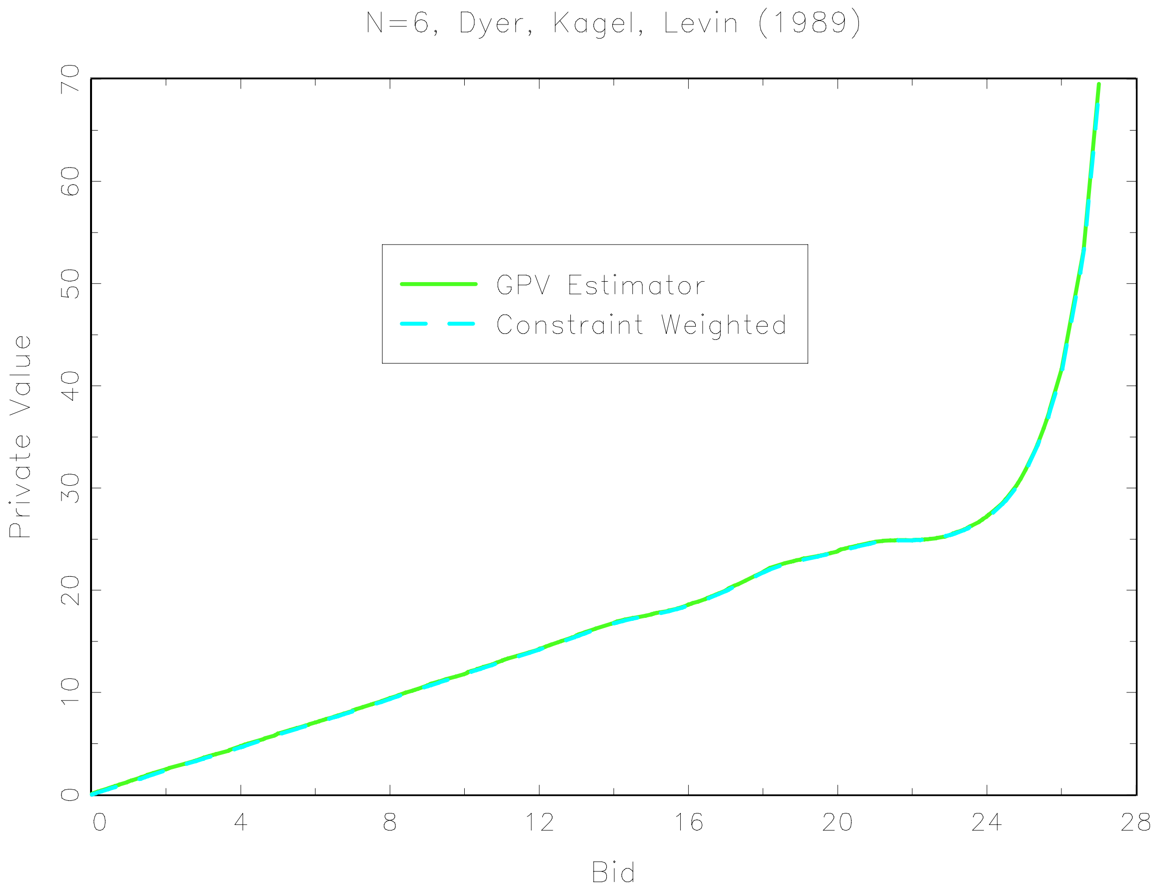

Figure 5: Equilibrium Bidding Strategy for 6 bidder experiment. The solid line represents the unconstrained GPV estimator while the dashed line is the monotonically constrained GPV estimator.

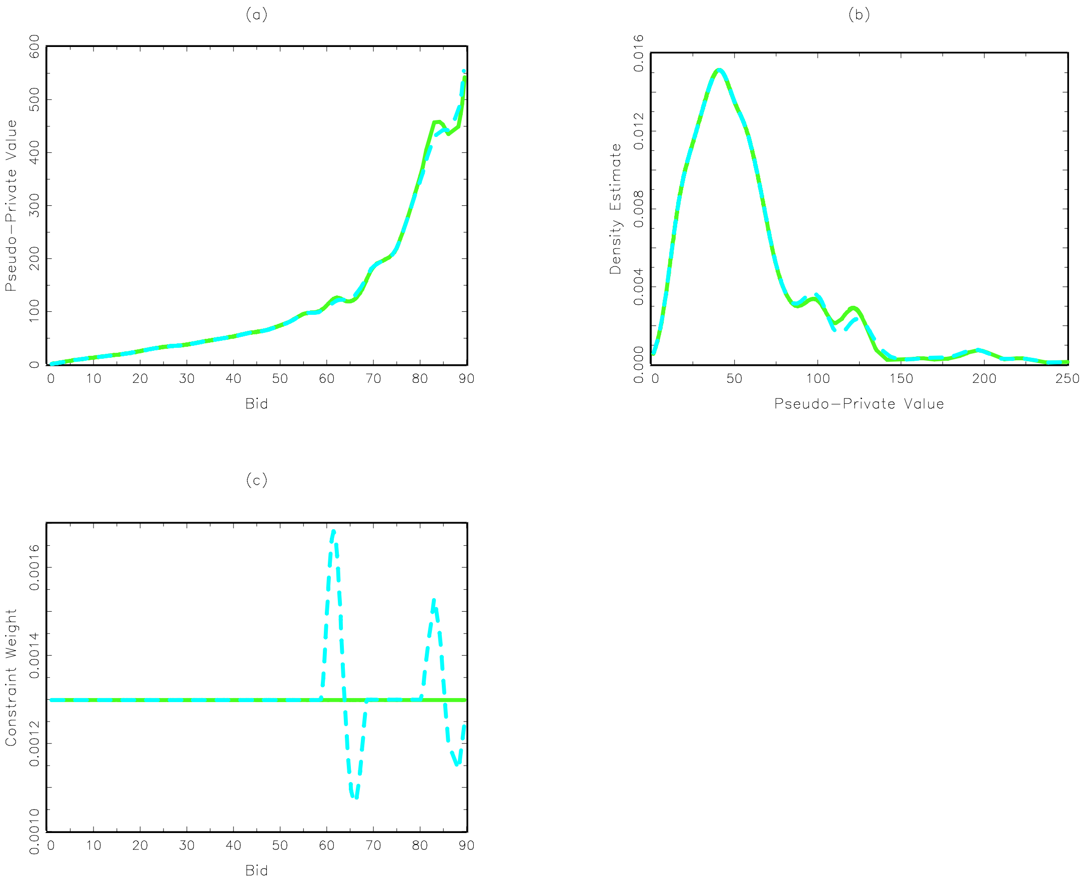

Figure 6: Monotonized Equilibrium Bidding Strategy for Timber Auctions with 3 bidders. In Panel (a) the solid line represents the unconstrained GPV estimator while the dashed line is the monotonically constrained GPV estimator. Panel (b) represents the corresponding value density while Panel (c) shows the bootstrap weights.

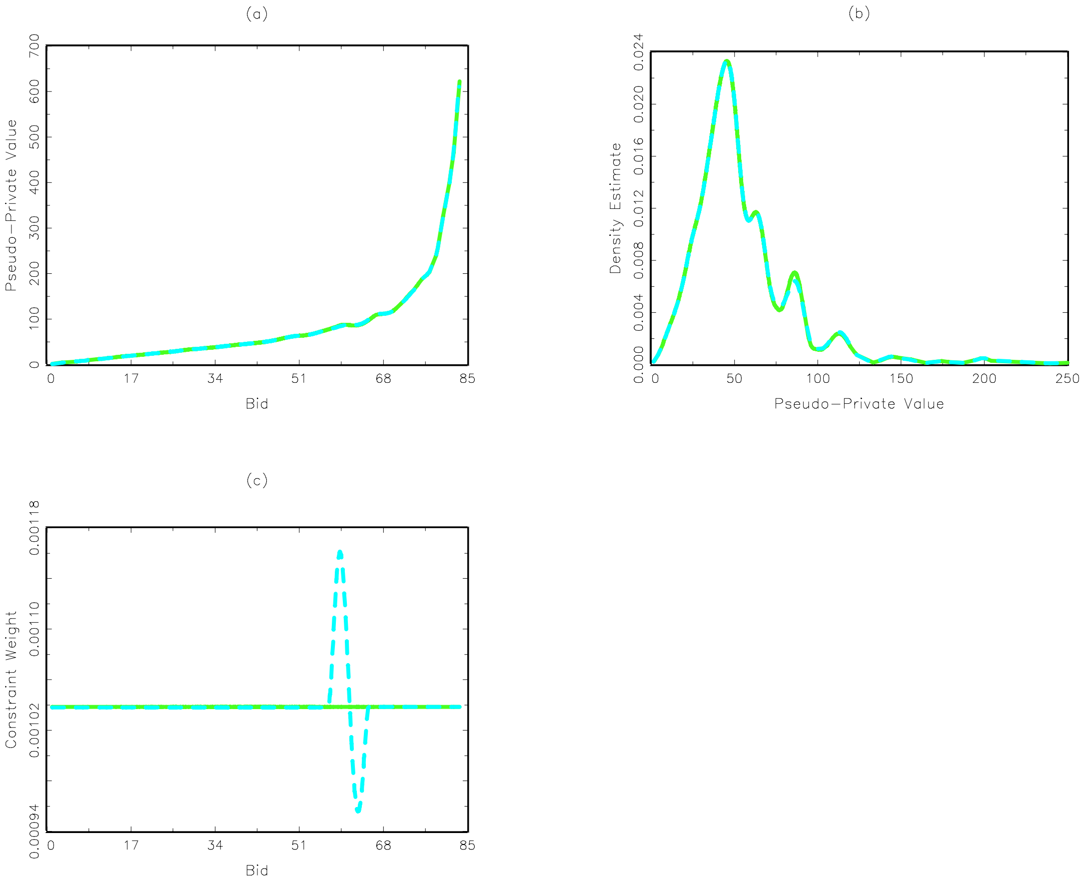

Figure 7: Monotonized Equilibrium Bidding Strategy for Timber Auctions with 4 bidders. In Panel (a) the solid line represents the unconstrained GPV estimator while the dashed line is the monotonically constrained GPV estimator. Panel (b) represents the corresponding value density while Panel (c) shows the bootstrap weights.

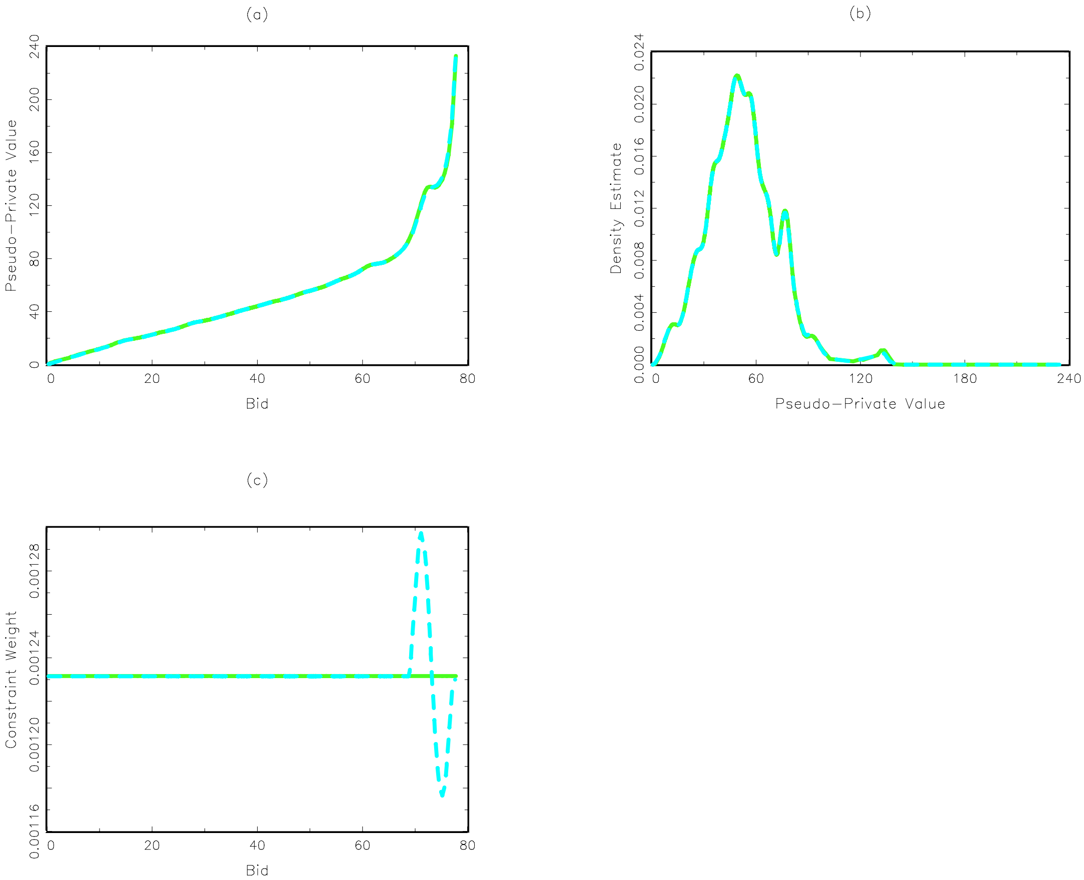

Figure 8: Monotonized Equilibrium Bidding Strategy for Timber Auctions with 5 bidders. In Panel (a) the solid line represents the unconstrained GPV estimator while the dashed line is the monotonically constrained GPV estimator. Panel (b) represents the corresponding value density while Panel (c) shows the bootstrap weights.