Profit-Maximizing Sale of a Discrete Public Good: linear equilibria in the private-information subscription game

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January 11, 2008

Abstract

We analyze a symmetric Bayesian game in which two players individually contribute to fund a discrete public good; contributions are refunded if they do not meet a threshold set by the seller of the good. We provide a general characterization of symmetric equilibrium strategies that are continuous and nonconstant over the set of values for which the good has a positive chance of provision. Piecewise-linear strategies are our special focus. We characterize the distributions of players' private values that can support a continuous piecewise-linear symmetric equilibrium, and we calculate such equilibria for these distributions. Allowing the seller to charge a nonrefundable entry fee before players make their private contributions, we show these piecewise-linear equilibria can maximize the seller's expected profit over *all* incentive compatible selling mechanisms.

JEL Codes: H41, D61, D82
Keywords: discrete public good, subscription game, Revelation Principle

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DRAFT: Comments welcome

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1 Introduction

A *discrete* public good is either provided or not—quantity is not otherwise variable. The subscription game is a voluntary provision mechanism for such goods: Individuals privately contribute money; if contributions suffice to fund the good provision occurs, otherwise contributions are refunded. An example is a homeowners' association collecting pledges to fund a new swimming pool. If enough money is pledged, it will then be collected and the pool built; otherwise, no pool is built and no money is paid.

This game has been widely analyzed in the literature. The complete-information subscription game is well-understood, both for static and for dynamic problems (e.g. Admati and Perry, 1991; and Marx and Matthews, 2000). The welfare properties of some equilibria of the subscription game are encouraging. Indeed, even in a static framework, fully efficient equilibria exist. In the more realistic case where valuations for the public good are private information, however, results are neither as clear nor as promising, even restricting attention to static, two-player problems (cf. Alboth *et al.*, 2001; Barbieri and Malueg, 2007; Laussel and Palfrey, 2003; and Menezes *et al.*, 2001). From a positive point of view, the common message of these private-information papers is that a profusion of equilibria exists and analytical characterizations are very difficult. Therefore, these papers mostly focus on a uniform distribution of agents' valutions. From the normative point of view, the main focus of the previous analyses is efficiency. Menezes *et al.* (2001) show that the subscription game is classically inefficient. Laussel and Palfrey (2003) and Barbieri and Malueg (2007) use the less demanding notion of interim incentive efficiency.[1] Even for two-player subscription game where valuations uniformly distributed on [0,1], results are mixed. Laussel and Palfrey (2003) show that if the cost the cost of production exceeds 1 then there exist incentive efficient equilibria. In contrast, Barbieri and Malueg (2007) show that when the cost of provision is less than 1, interim incentive inefficient equilibria do not exist.

The papers mentioned above provide many examples of real-world situations that can be interpreted as subscription games. These observations are at odds with the unsatisfactory efficiency properties of the subscription game, if we believe private information is an important characteristic of the real-world situation under consideration. In this paper we provide a justification for the use of the subscription game that is not based on efficiency grounds, but on profit maximization. In other words, we ask the following natural question: Would a profit-maximizing producer of a public good ever choose the subscription game as an optimal selling mechanism? We show the answer is Yes.

A necessary preliminary step in our analysis is to sharpen the existing equilibrium characterization results. As in literature previously mentioned, we study a two-player subscription game in which players' values for the public good are private information, characterized by a continuous distribution function. We

[1] Classical and interim incentive efficiency are discussed in Holmström and Myerson (1983).

provide a general characterization of symmetric equilibrium strategies that are continuous and nonconstant over the set of values for which there is a positive probability the good is provided. Beyond our general characterization, we are particularly interested in equilibrium strategies that are piecewise-linear and are also strictly increasing over some interval of possible values. We show that for piecewise-linear equilibria to exist, players' values must be distributed according to an exponential distribution that does not include zero in its support or a reverse power function distribution (of which the uniform distribution is a special case).[2] Cornelli (1996) adapted Myerson's (1982) method of optimal auction design to characterize an optimal incentive compatible mechanism for the sale of a discrete public good. Within our framework, we show that, for the exponential and reverse power function distributions, the piecewise-linear equilibria of the subscription game (possibly with the introduction of nonrefundable entry fees) achieve the expected profit of an optimal mechanism for the seller; and if such piecewise-linear equilibria do not exist, then the subscription game cannot be an optimal mechanism for the seller. When piecewise-linear equilibria exist, we completely describe the seller's optimal optimal combination of completion threshold and entry fee.

The rest of the paper is organized as follows. Section 2 defines "regular" equilibria and provides a general characterization result. In Section 3 we completely describe piecewise-linear equilibria. Section 4 contains our main result on the optimality of the subscription game and relates it to the existing literature. Section 5 concludes.

2 The subscription game

Two players, 1 and 2, simultaneously contribute any positive amount to the funding of a public good. Player i's value for the good is v_i, $i = 1, 2$. Values v_1 and v_2 are independently distributed random variables with cumulative distribution function (cdf) F, which has support $[\underline{v}, \bar{v}]$, where $0 \leq \underline{v} < \bar{v} \leq \infty$. A player's realized value is known only to that player. We suppose F is absolutely continuous, with density function f. The third actor in the model is the collector, who *ex ante* specifies a contribution threshold t and provides the public good if and only if contributions total at least t. The cost of the public good is c, known to the collector and both players.

In the terminology of Admati and Perry (1991), we consider the *subscription game*: players' contributions are refunded if they are insufficient to cover t; the collector retains contributions exceeding t. The collector and two players are risk neutral. If the good is provided, then the payoff to player i is $v_i -$ (player i's contribution). If the good is not provided, then the payoff to player i is 0.

[2]There are other distributions that also support such piecewise-linear equilibrium strategies, but they would differ from the two families named only over types where (i) a player has no chance of obtaining the good or (ii) a player is sure he will obtain the good. These alternative distributions must agree with the families named over the interval of values where the piecewise-linear strategy is strictly increasing.

The foregoing description is common knowledge among the players and the collector. We focus on symmetric equilibrium strategies (s, s) such that completion happens with *ex ante* positive probability. In this case, standard arguments show $t < 2\bar{v}$ and $s(\bar{v}) = \max_{v \in [\underline{v}, \bar{v}]} s(v) \leq t$.

At this point some additional notation is useful. Define $\hat{v}$ as the lowest type who contributes $s(\bar{v})$, and define $\widetilde{v}$ as the lowest type who contributes $t - s(\bar{v})$;[3] that is,

$$\hat{v} \equiv \inf\{v \mid s(v) = s(\bar{v})\} \qquad \text{(lowest type contributing the maximum)}$$

$$\widetilde{v} \equiv \inf\{v \mid s(v) = t - s(\bar{v})\}. \qquad \text{(lowest type with positive probability of provision of the good)}$$

Because the interim probability of completion for type $v \in [\underline{v}, \widetilde{v})$ is zero, little more can be said about the shape of s on $[\underline{v}, \widetilde{v})$. To avoid uninteresting complications, we set $s = 0$ in this range. Lemmas 1–3 in Laussel and Palfrey (2003) establish that on $[\widetilde{v}, \bar{v}]$ a symmetric equilibrium contribution function s is non-decreasing and almost everywhere differentiable. A first possibility is that s is a step function. If an appropriate boundary condition, later defined in (4), is satisfied, then the following strategy constitutes the "halvesies" equilibrium: $s(v) = 0$ for $v \in [\underline{v}, t/2)$ and $s(v) = t/2$ for $v \in [t/2, \bar{v}]$. Menezes *et al.* (2001) point out, for $t < \bar{v}$, the "all-or-nothing" equilibrium is sure to exist: $s(v) = 0$ for $v < v^*$ and $s(v) = t$ for $v \geq v^*$, where v^* solves $vF(v) = t$. As will be seen later, however, if the subscription game is to be a profit-maximizing incentive compatible mechanism, then strategies must be continuous and strictly increasing in the region where a player's chance of obtaining the good is strictly between 0 and 1. These considerations lead us to focus on "regular" equilibria.

Definition 1. A symmetric equilibrium (s, s) is called *regular* if s is continuous and nonconstant on $[\widetilde{v}, \bar{v}]$.

Note that neither the halvesies equilibrium nor the all-or-nothing equilibrium is regular. Barbieri and Malueg (2007) establish that in a regular equilibrium s must be strictly increasing on $[\widetilde{v}, \hat{v}]$; furthermore $\widetilde{v}$ and $\hat{v}$ are not at all arbitrary, but rather are jointly determined as part of the equilibrium. We now characterize regular equilibria.

Proposition 1 (Characterization of a regular symmetric equilibrium)**.** *Suppose* $s : [\underline{v}, \bar{v}] \to \mathbf{R}_+$ *has the following properties, where* $\widetilde{v}$ *and* $\hat{v}$ *are defined above:*

a. $\underline{v} \leq \widetilde{v} < \hat{v} \leq \bar{v}$*;*

b. $s(v) = 0$ *for any* $v < \widetilde{v}$*;* s *is continuous and nondecreasing on* $[\widetilde{v}, \bar{v}]$*;*

[3] To see that $\widetilde{v}$ exists, note first that some type must contribute at least $t - s(\bar{v})$ otherwise completion never occurs. Note as well that not all types can contribute more than $t - s(\bar{v})$. If this were the case, then $\bar{v}$ has a profitable deviation to $s(\bar{v}) - \varepsilon$, for ε small enough, because contributions in excess of t do not generate any benefit for the contributors.

c. $s(\widetilde{v}) + s(\bar{v}) = t$*; and*

d. s *is strictly increasing and differentiable on* $(\widetilde{v}, \hat{v})$*.*

Define $G : [\widetilde{v}, \hat{v}] \to [\widetilde{v}, \bar{v}]$ *by* $s(v) + s(G(v)) = t$*. Then* (s, s) *is a regular symmetric equilibrium to the subscription game with threshold* t *if and only if* s *and* G *satisfy the following system of equations:*

$$s'(v)(1 - F(G(v))) + (v - s(v))f(G(v))G'(v) = 0 \quad \forall v \in (\widetilde{v}, \hat{v}), \tag{1}$$

$$G'(v) = -\frac{s'(v)}{s'(G(v))} \quad \forall v \in (\widetilde{v}, \hat{v}), \tag{2}$$

$$0 \leq s(v) \leq v \quad \forall v \in [\widetilde{v}, \bar{v}]; \tag{3}$$

$$\textit{and further, if } \widetilde{v} > \underline{v}, \textit{ then } s(\widetilde{v}) = \widetilde{v} \textit{ and } (\bar{v} - s(\hat{v}))(1 - F(G(\hat{v}))) \geq \bar{v} - t. \tag{4}$$

Conditions *a*–*d* describe a candidate strategy for a regular symmetric equilibrium. Condition *a* rules out the possibility that s is constant on $[\widetilde{v}, \bar{v}]$ (the halvesies equilibrium satisfies *b*–*d* because $\widetilde{v} = \hat{v} = t/2$ so the interval $(\widetilde{v}, \hat{v})$ is empty). Since s is strictly increasing on $(\widetilde{v}, \hat{v})$, the definition of G implies that, in equilibrium, type $v \in (\widetilde{v}, \hat{v})$ will see the threshold for completion reached if and only if the other player's type is at least as large as $G(v)$. Menezes *et al.* (2001) showed that a continuous equilibrium to the subscription game must satisfy (1) and (2). The value of Proposition 1 is that it further provides, with the addition of (3) and (4), conditions sufficient to characterize an equilibrium. The reader may be curious why the inequality in (4) applies only in the case where $\widetilde{v} > \underline{v}$. This condition says that a player with the highest value, $\bar{v}$, should not strictly benefit by contributing t (thereby ensuring the good is provided) rather than the specified level $s(\hat{v})$. This condition is not included when $\widetilde{v} = \underline{v}$ because in that case it is automatically satisfied. To see this, observe that if $\widetilde{v} = \underline{v}$, then $s(\underline{v}) + s(\hat{v}) = t$, so that a player contributing $s(\hat{v})$ is assured the good will be provided—hence, there is no profitable deviation to the (possibly) larger contribution t.

The characterization in Proposition 1 can be used to refine our understanding of regular equilibria. Using some algebraic manipulations and the identity $G(G(v)) = v$, we rewrite conditions (1) and (2) as

$$s'(v) = h(v)(G(v) - s(G(v))) \tag{5}$$

and

$$G'(v) = -\frac{h(v)}{v - s(v)} \frac{G(v) - s(G(v))}{h(G(v))} \tag{6}$$

where $h(y) \equiv f(y)/(1 - F(y))$ is the hazard rate function associated with F. A straightforward but tedious application of the contraction mapping theorem demonstrates that, if $\underline{v} > 0$, a regular equilibrium is sure to exist for t sufficiently small. More interesting is the following proposition showing that $s(\widetilde{v}) = \widetilde{v}$ and $\hat{v} < \bar{v}$ are incompatible. (The proof, relying on formulation (5) and (6), is in the Appendix.)

Proposition 2. *Suppose on $[\underline{v}, \bar{v})$ the density f is continuous and strictly positive. If (s, s) is a regular equilibrium with $s(\widetilde{v}) = \widetilde{v}$, then $\hat{v} = \bar{v}$.*

Barbieri and Malueg (2007) show that if $t = c < \bar{v}$ then an equilibrium to the subscription game can be interim incentive efficient only if $\hat{v} < \bar{v}$. For the particular case in which values are uniformly distributed over $[0, 1]$, Barbieri and Malueg (2007) show that all equilibria are interim incentive *inefficient.* Proposition 2 extends this negative result to all continuous distributions on $[0, 1]$ that are strictly positive at $v = 0$. The reason for this is that an incentive efficient equilibrium must be continuous, but for reasons discussed in Barbieri and Malueg (2007) such an equilibrium cannot have an atom at zero. Consequently, it must be that if (s, s) is a continuous equilibirum, then $\widetilde{v} = 0$ and $s(0) = 0$. Proposition 2 then implies that $\hat{v} = 1$, so there is no flat spot at the top. Hence, (s, s) cannot be incentive efficient.

3 Piecewise-linear equilibria

The equilibrium characterization of the previous section is fairly general. Even if step-function equilibria are excluded by requiring s to be regular, a vast multiplicity of equilibria results because the restrictions on the initial conditions $\widetilde{v}$ and $s(\widetilde{v})$ in Proposition 1 are weak. Indeed, $\widetilde{v}$ may be larger than $\underline{v}$ and, even when $\widetilde{v}$ equals $\underline{v}$, $s(\widetilde{v})$ is not uniquely determined except when $\underline{v}$ is zero. Moreover, in contrast to equilibrium derivation in the contribution game[4] (see Barbieri and Malueg, *forthcoming*), to describe all regular equilibria it is not sufficient to truncate and translate a baseline functional form. For example, as shown by Laussel and Palfrey (2003), for a given threshold t, different initial conditions yield different functional forms for different equilibria, even when valuations are uniformly distributed. Little progress seems possible without further simplifications. For this reason, we next turn to a special family of regular symmetric equilibria, those that are piecewise-linear. In particular, for $v \in [\widetilde{v}, \hat{v}]$ strategy s will be linear and strictly increasing, while on $[\hat{v}, \bar{v}]$ it is constant. For their eminent tractability, linear strategies have often been sought as a first step in the analysis of complex situations, such as double auctions. More importantly, as we shall see, with some exceptions, in our framework regular symmetric piecewise-linear equilibria have another desirable feature: they maximize the collector's profits.

[4] The contribution game is identical to the subscription game except contributions are never refunded.

3.1 Admissible distribution functions

When s is linear, equations (1) and (2) simplify considerably. The following lemmas characterize the class of distribution functions for which a regular symmetric piecewise-linear equilibrium strategy s can exist. For these distributions, the subsequent propositions show exactly when such equilibria exist, and in these cases they specify equilibrium strategies.

Lemma 1. *A regular symmetric piecewise-linear equilibrium exists only if the inverse hazard rate* $(1-F)/f$ *is linear on* $[\widetilde{v}, \hat{v}]$.

Proof. If s is linear on $[\widetilde{v}, \hat{v}]$, that is, $s(v) = \alpha v + \beta$, then the definition of G and the boundary condition $G(\widetilde{v}) = \hat{v}$ imply $G(v) = \hat{v} + \widetilde{v} - v$, so we can simplify (1) as $\alpha(1 - F(G(v))) - (v - (\alpha v + \beta))f(G(v)) = 0$, which upon rearrangement becomes

$$\frac{1 - F(G(v))}{f(G(v))} = \left(\frac{1-\alpha}{\alpha}\right) v - \frac{\beta}{\alpha}.$$

Using $G(G(v)) = v$, we obtain

$$\frac{1 - F(v)}{f(v)} = \left(\frac{1-\alpha}{\alpha}\right)(\hat{v} + \widetilde{v} - v) - \frac{\beta}{\alpha}, \tag{7}$$

so that on $[\widetilde{v}, \hat{v}]$ the inverse hazard rate is indeed linear. □

Lemma 1 restricts F only on $[\widetilde{v}, \hat{v}]$, the interval where (1) holds (cf. footnote 2). If we require linearity of the inverse hazard rate over the whole interval $[\underline{v}, \bar{v}]$, simple integration delivers the following lemma, whose proof is in the Appendix.

Lemma 2. *If the inverse hazard rate of* F *is linear on* $[\underline{v}, \bar{v}]$, *then* F *is either*

$$F(v) = 1 - e^{-r(v-\underline{v})}, \text{ with } r > 0 \text{ and } \bar{v} = +\infty \tag{8}$$

or

$$F(v) = 1 - \left(\frac{\bar{v} - v}{\bar{v} - \underline{v}}\right)^r, \text{ with } r > 0 \text{ and } \bar{v} < \infty. \tag{9}$$

The cdf given by (8) is simply an exponential distribution with support $[\underline{v}, \infty)$; the cdf given by (9) we refer to as a "reverse power function" distribution.[5]

[5]To understand this terminology, consider the cdf on $[0, 1]$ given by $\Phi(v) = v^r$, where $r > 0$. This is often called a power function distribution, with density $\varphi(v) = rv^{r-1}$. For the distribution F in (9) with support $[0, 1]$, the variable $1 - v$ has the density function φ; that is, $f(v) = \varphi(1 - v)$.

The following propositions, with proofs in the Appendix, apply Proposition 1 to characterize regular piecewise-linear symmetric equilibria for the distributions of Lemma 2. The next proposition shows for exponential distributions that regular piecewise-linear equilibria are particularly simple.

Proposition 3 (Equilibria for exponential distributions)**.** *Suppose F is defined on $[\underline{v}, \infty)$ by $F(v) = 1 - e^{-r(v-\underline{v})}$, where $r > 0$. A regular piecewise-linear symmetric equilibrium exists if and only if $\underline{v} \geq 1/r$ and $t \geq 2(\underline{v} - 1/r)$. In this case, $\widetilde{v} = \underline{v}$ and the regular piecewise-linear symmetric equilibrium strategy is given by*

$$s(v) = \begin{cases} s(\hat{v}) & \text{if } \hat{v} < v \leq \bar{v} \\ v - \dfrac{1}{r} & \text{if } \underline{v} \leq v \leq \hat{v}, \end{cases} \tag{10}$$

where $\hat{v} = t - \underline{v} + 2/r$.

Note that by Proposition 3 a piecewise-linear equilibrium exists for the exponential distribution only if $\underline{v} > 0$, showing the frequent assumption that the support of players' values begins at 0 is not without loss of generality.

For the reverse power function distribution, the description of equilibria is slightly more involved. For clarity, we distinguish between the high-threshold case in Proposition 4 and the low-threshold case in Proposition 5.

Proposition 4 (Equilibria for reverse power function distributions, high-threshold case)**.** *Suppose F is defined on $[\underline{v}, \bar{v}]$ by $F(v) = 1 - \left(\frac{\bar{v}-v}{\bar{v}-\underline{v}}\right)^r$, where $r > 0$ and $0 \leq \underline{v} < \bar{v} < \infty$, and assume*

$$t > \frac{(2+r)\underline{v} + r\bar{v}}{1+r}. \tag{11}$$

A regular piecewise-linear symmetric equilibrium exists if and only if $t < 2\bar{v}$. The regular piecewise-linear symmetric equilibrium strategy is given by

$$s(v) = \begin{cases} \dfrac{r}{1+r} v + \dfrac{\widetilde{v}}{1+r} & \text{if } \widetilde{v} \leq v \leq \bar{v} \\ 0 & \text{if } \underline{v} \leq v < \widetilde{v}, \end{cases} \tag{12}$$

where $\widetilde{v} = \frac{t(1+r) - r\bar{v}}{2+r} \in (\underline{v}, \bar{v})$.

According to Proposition 4, when the threshold is high and players use piecewise-linear strategies, only when a player's value is sufficiently large does he have a chance of receiving the good, and even then is not assured of enjoying it. In contrast, the next proposition shows that for a low threshold, when using piecewise-linear strategies all players have a chance of enjoying the good, and those with sufficiently high values are assured they will. Thus, the low-cost case is similar to that of the exponential distribution.

Proposition 5 (Equilibria for reverse power function distributions, low-threshold case)**.** *Suppose F is defined on $[\underline{v}, \bar{v}]$ by $F(v) = 1 - \left(\frac{\bar{v}-v}{\bar{v}-\underline{v}}\right)^r$, where $r > 0$ and $0 \leq \underline{v} < \bar{v} < \infty$, and assume*

$$t \leq \frac{(2+r)\underline{v} + r\bar{v}}{1+r}. \tag{13}$$

A regular piecewise-linear symmetric equilibrium exists if and only if

$$t \geq \frac{r}{1+r}(\bar{v} - \underline{v}(2+r)),$$

and

$$t \geq -\frac{2}{1+r}(\bar{v} - \underline{v}(2+r)).$$

The regular piecewise-linear symmetric equilibrium strategy is given by

$$s(v) = \begin{cases} s(\hat{v}) & \text{if } \hat{v} < v \leq \bar{v} \\ \dfrac{r}{1+r}v - \dfrac{\bar{v} - \hat{v} - \underline{v}}{1+r} & \text{if } \underline{v} \leq v \leq \hat{v}, \end{cases} \tag{14}$$

where $\hat{v} = \frac{t(1+r)+2\bar{v}}{2+r} - \underline{v} \in (\underline{v}, \bar{v})$.

Though there typically exist multiple equilibria in the subscription game, we have focused on the piecewise-linear regular equilibria as especially attractive. However, it may be that in some cases they do not exist. Nevertheless, we next show that when they do exist they can maximize the seller's expected profit over all incentive compatible mechanisms. Additionally, for the exponential and reverse power function distributions, we show that when such equilibria do not exist the subscription game is not a profit-maximizing mechanism.

4 Profit-maximizing sale of a discrete public good: an application of linear equilibrium strategies

We begin this section by allowing the collector the choice of any incentive compatible and individually rational mechanism. We will then show how, when regular piecewise-linear equilibria exist, a slight modification of the subscription game implements the profit-maximizing allocation.

4.1 The profit-maximizing direct mechanism

The seller's profit-maximization problem can be solved adapting Myerson's (1982) optimal auction design. By the Revelation Principle it suffices to consider only direct mechanisms that are feasible: both incentive compatible and individually rational. Direct mechanisms are triples of functions (p, x_1, x_2) defined on $[\underline{v}, \bar{v}] \times [\underline{v}, \bar{v}]$. Players simultaneously report values v_1' and v_2'; then $x_i(v_1', v_2')$ is player i's payment to the seller and $p(v_1', v_2') \in [0, 1]$ is the probability the good is provided. The only difference with Myerson (1982) is that p is not indexed by players: we are dealing with a pure public good. Cornelli (1996) analyzed this problem, finding the following.

Lemma 3 (Cornelli, 1996)**.** *If $w(v) \equiv v - \frac{1-F(v)}{f(v)}$ is increasing on $[\underline{v}, \bar{v}]$, then a profit-maximizing feasible direct mechanism (p, x_1, x_2) has*

$$p(v_1, v_2) = \begin{cases} 1 & \text{if } w(v_1) + w(v_2) \geq c \\ 0 & \text{if } w(v_1) + w(v_2) < c. \end{cases} \tag{15}$$

Next we connect a player's conditional expected payment to the function p in a feasible mechanism. Assuming truthful revelation by player 2, player 1's expected payoff when reporting $\hat{v}_1$ is

$$U_1(\hat{v}_1|v_1) = \mathrm{E}[\, v_1 p\,(\hat{v}_1, v_2) - x_1(\hat{v}_1, v_2)\,] = v_1 P_1(\hat{v}_1) - X_1(\hat{v}_1),$$

where

$$P_1(\hat{v}_1) = \int_{\underline{v}}^{\bar{v}} p\,(\hat{v}_1, v_2)\, dF(v_2) \qquad \text{and} \qquad X_1(\hat{v}_1) = \int_{\underline{v}}^{\bar{v}} x_1\,(\hat{v}_1, v_2)\, dF(v_2).$$

Analogous formulas apply for player 2. In the truth-telling equilibrium, conditional on v_i, $P_i(v_i)$ is player i's perceived probability the good will be provided and $X_i(v_i)$ is his expected payment to the seller. Let $U_i^*(v_i) \equiv U_i(v_i \,|\, v_i)$ be player i's payoff in the truth-telling equilibrium of a feasible direct mechanism. Incentive compatibility requires

$$0 = \left.\frac{\partial U_i(\hat{v}_i|v_i)}{\partial \hat{v}_i}\right|_{\hat{v}_i = v_i} = v_i P_i'(v_i) - X_i'(v_i),$$

$$X_i'(v_i) = v_i P_i'(v_i). \tag{16}$$

The Envelope Theorem gives $dU_i^*(v_i)/dv_i = P_i(v_i)$, implying U_i^* is nondecreasing. Therefore, an incentive compatible mechanism is individually rational if and only if $U_i^*(\underline{v}) \geq 0$. As Myerson (1982) and Cornelli (1996) show, profit maximization requires this constraint to bind. The profit-maximizing conditional expected payment can be found by integrating (16) with the boundary condition ensuring $U_i^*(\underline{v}) = 0$. Expected cost is c times the probability the good is provided.

4.2 Profit maximization in the subscription game

For implementation purposes, we slightly modify the standard subscription game. First, with binding commitment, the collector announces the threshold t and an individual's non-refundable entry fee f. Next, after observing the realization of their own private values, contributors independently and simultaneously decide whether to pay f. If either does not pay the entry fee, the game ends. If both pay f, the game moves to the third stage, in which contributors play the standard subscription game: players simultaneously contribute any amount; these contributions are refunded if they total less than t but otherwise are retained by the collector. Only in the latter case does the collector provide the good, incurring cost c.

The next propositions show that, when regular piecewise-linear regular equilibria exist, the collector can choose t and f so that his profit in the modified subscription game coincides with his profit in the optimal direct mechanism described in Section 4.1. In turn, we will reference the necessary and sufficient conditions in Propositions 3, 4, and 5. Given threshold t, we will take the corresponding equilibria described in Propositions 3, 4, and 5 to be the continuation equilibria after players pay the entry fee f.

We first consider the exponential distribution in (8). Because $w(v) = v - 1/r$ is increasing in v, Lemma 3 applies. Straightforward calculations show that when $c \leq 2(\underline{v} - 1/r)$ the collector optimally provides the public good with probability one, and charges $2\underline{v}$. This can be easily accomplished in our modified subscription game by setting the entry fee $f = \underline{v}$ and the threshold $t = 0$. For higher costs, the following proposition shows how the collector obtains the same profit as in the optimal direct mechanism.

Proposition 6 (Profit maximization: exponential distributions)**.** *Suppose F is defined on $[\underline{v}, \infty)$ by $F(v) = 1 - e^{-r(v-\underline{v})}$, where $r > 0$ and $\underline{v} \geq 1/r$. Assume further that $c > 2(\underline{v} - 1/r)$. The collector can obtain the same profit as in the optimal direct mechanism in Section 4.1 by setting the contribution threshold at $t = c$ and the nonrefundable entry fee at $f = e^{-2-cr+2\underline{v}r}/r$. The continuation equilibrium is described in* (10)*.*

Our proof uses Myerson's (1982) Revenue Equivalence Theorem. For given c, if t is chosen so the provision region in the subscription game matches that in the corresponding optimal mechanism, then the probability a player obtains the good conditional on his value ($P_i(v_i)$ in the notation of Section 4.1) is the same in both settings.[6] By Myerson's result, each player's expected payments will then be the same in both regimes if the player with value $\underline{v}$ earns the same conditional expected payoff (namely 0) in both settings. We set the

[6] It is now clear, too, why step-function equilibria cannot be profit-maximizing. The "southwest" boundary of the region of optimal provision is where $w(v_1) + w(v_2) = c$. If $w(\cdot)$ is strictly increasing, then this boundary will be strictly decreasing and continuous in $v_1 \times v_2$-space. Step-function equilibria give a boundary that is not strictly decreasing.

entry fee f to make this so, leaving players willing to pay f. Because provision regions coincide, expected costs are identical.

Proof of Proposition 6. As noted above, $w(v) = v - (1 - F(v))/f(v) = v - 1/r$, so the profit-maximizing direct mechanism provides the good exactly where $v_1 + v_2 \geq c + 2/r$.

Now consider the provision region for the regular equilibrium of Proposition 3. As given in (10), $s(v) = v - 1/r$ for all $v \leq \hat{v} = t - \underline{v} + 2/r$, and the good is provided if and only if $v_1 + v_2 \geq t + 2/r$. Therefore, to match the two provision regions, the subscription game threshold is set at $t = c$. If no entry fee were required, the payoff to the lowest type would be

$$(\underline{v} - s(\underline{v})) \Pr(v > \hat{v}) = \frac{1}{r} e^{-r(\hat{v} - \underline{v})} = \frac{1}{r} e^{-2+2r\underline{v}-rc}.$$

Therefore, to extract all surplus from the lowest-value type, the entry fee is set to $e^{-2+2r\underline{v}-rc}/r$. □

It is worth noting that Proposition 6 covers the case $2(\underline{v} - 1/r) < c \leq 2\underline{v}$, where a fully efficient equilibrium of the subscription game exists. Indeed, because $c \leq 2\underline{v}$, the contributors' asymmetric information is irrelevant for efficiency purposes: it is common knowledge that the good should always be provided and both contributors are willing to contribute $c/2$ regardless of their private information. Nonetheless, the collector finds it more profitable to restrict the probability of provision as described in Proposition 6. This behavior of the collector is completely analogous to a monopolist restricting the quantity supplied to increase the price charged, or to the imposition of a reserve price in optimal auctions.

While for the exponential distribution piecewise-linear regular equilibria do not exist when $\underline{v} < 1/r$, there may well exist nonlinear equilibria continuous on $[\widetilde{v}, \hat{v}]$. Nevertheless, no such equilibria of the (modified) subscription game yield the profit of the optimal mechanism. Thus, without restricting attention to linear strategies, the following proposition provides a converse to the previous result.

Proposition 7. *Suppose F is defined on $[\underline{v}, \infty)$ by $F(v) = 1 - e^{-r(v-\underline{v})}$, where $r > 0$ and $\underline{v} < 1/r$. For any continuation equilibrium of the subscription game, the collector cannot obtain the same profit as in the optimal direct mechanism in Section 4.1.*

Proof. Applying Lemma 3, we obtain that the provision region in the optimal mechanism is $v_1 + v_2 \geq c + 2/r$. If an equilibrium of the subscription game can match this provision region it must have $G'(v) = -1$, so $G(v) = k_1 - v$ for some constant k_1, where G is defined in Proposition 1. Equation (1) then reduces to

$$s'(v) = r(v - s(v));$$

a solution to this differential equation must have the form

$$s(v) = v - \frac{1}{r} + k_2\, e^{-rv},$$

for some constant k_2. When $k_2 = 0$ one obtains our earlier candidate for a linear-strategy equilibrium. However, because $\underline{v} < 1/r$, Proposition 3 implies this is not an equilibrium. When $k_2 \neq 0$, constancy of $s(v) + s(G(v))$ on $[\widetilde{v}, \hat{v}]$ implies that, for some constant k_3,

$$k_1 - \frac{2}{r} + k_2(e^{-rv} + e^{-r(k_1 - v)}) = k_3;$$

differentiation with respect to v implies $e^{-rv} = e^{-r(k_1-v)}$, which cannot be satisfied for an interval of v's. Hence, there is no equilibrium with $k_2 \neq 0$. □

Combining Propositions 6 and 7, the following corollary provides necessary and sufficient conditions for the subscription game to maximize profit when values have an exponential distribution.

Corollary 1. *Suppose F is defined on $[\underline{v}, \infty)$ by $F(v) = 1 - e^{-r(v-\underline{v})}$, where $r > 0$. Assume further that $c > 2(\underline{v} - 1/r)$. Through the modified subscription game the collector can obtain the same profit as in the optimal direct mechanism if and only if $\underline{v} \geq 1/r$.*

We now consider reverse power function distributions. Lemma 3 again applies, and straightforward calculations show that when $cr \leq 2((1+r)\underline{v} - \bar{v})$ (the low-cost case), the collector optimally provides the public good with probability one and charges $2\underline{v}$—there are many ways to implement this allocation via the subscription game. For $cr > 2((1+r)\underline{v} - \bar{v})$ it is convenient to distinguish two cases. We begin with the high-cost case. (The proofs of the remaining propositions follow the logic of those of Propositions 6 and 7 and can be found in the Appendix.)

Proposition 8 (Profit maximization: reverse power distributions, high cost)**.** *Suppose F is defined on $[\underline{v}, \bar{v}]$ by $F(v) = 1 - \left(\frac{\bar{v}-v}{\bar{v}-\underline{v}}\right)^r$, where $r > 0$ and $0 \leq \underline{v} < \bar{v} < \infty$. Assume further that c satisfies $\underline{v}(1+r) + (r-1)\bar{v} \leq cr < 2\bar{v}r$. The collector can obtain the same profit as in the optimal direct mechanism in Section 4.1 by setting the contribution threshold at*

$$t = \frac{cr(2+r) + 2\bar{v}}{(1+r)^2}$$

and the nonrefundable entry fee at $f = 0$. The continuation equilibrium is described in (12)*.*

Observe that in the high-cost case, the unmodified subscription game is optimal—an entry fee need not be charged. This is because types near $\underline{v}$ have no chance of obtaining the good and therefore earn zero surplus. For lower costs of production the seller can be expected to set a lower threshold; once the type-$\underline{v}$ contributor has a chance of obtaining the good, an entrance fee must be added to ensure this contributor is left with no surplus. Nevertheless, turning to the medium-cost case, we note that the functional form of the optimal threshold is the same as in the high-cost case.

Proposition 9 (Profit maximization: reverse power distributions, medium cost)**.** *Suppose F is defined on $[\underline{v},\bar{v}]$ by $F(v) = 1 - \left(\frac{\bar{v}-v}{\bar{v}-\underline{v}}\right)^r$, where $r > 0$ and $0 \leq \underline{v} < \bar{v} < \infty$. Assume further that c satisfies $2((1+r)\underline{v} - \bar{v}) < cr \leq \underline{v}(1+r) + (r-1)\bar{v}$. If either*

a. $\bar{v} - \underline{v}(2+r) \leq 0$; or

b. $\bar{v} - \underline{v}(2+r) > 0$ and $cr \geq \bar{v}(r-1) - \underline{v}r(1+r)$,

then the collector can obtain the same profit as in the optimal direct mechanism in Section 4.1 by setting the contribution threshold at

$$t = \frac{cr(2+r) + 2\bar{v}}{(1+r)^2}$$

and the nonrefundable entry fee at

$$f = \frac{\left(\bar{v}\,(r-1) + \underline{v}\,(1+r) - cr\right)^{r+1}}{(1+r)^{r+2}\,(\bar{v}-\underline{v})^r}.$$

The continuation equilibrium is described in (14)*.*

Our analysis has implications for the study by Alboth *et al.* (2001), who consider a seller of a discrete public good facing two contributors whose individual values are independently and uniformly distributed over $[0,1]$. The seller's cost of production is zero. Setting a threshold t that total contributions must reach before the good is provided, the seller expects to earn positive revenue in the Bayesian equilibrium of the subscription game. Given $t < 2$, Alboth *et al.* derive the unique symmetric equilibrium in which a player's strategy is a strictly increasing, continuously differentiable function of his own value, having range $[0,t]$. (Thus, *every* equilibrium they derive has $\widetilde{v} = 0$ and $\hat{v} = 1$.) Varying threshold t they seek the corresponding equilibrium maximizing expected revenue; revenue is so maximized at $t = 1/2$, achieving value $1/3$; the associated equilibrium strategy is $s(v) = v/2$ for all $v \in [0,1]$. Their analysis, however, leaves unanswered three important questions:

1. Do other subscription game equilibria yield greater revenue?

2. Do other selling mechanisms yield greater revenue?

3. If $c > 0$ can any of Alboth *et al.*'s strictly increasing equilibria be optimal for the seller?

Our analysis implies the answer to all three questions is No. Propositions 8 covers the case where values are uniformly distributed over $[0,1]$ ($\underline{v}=0$, $\bar{v}=1$, $r=1$). If $c=0$ (the case of Alboth *et al.*), then the subscription game yields the profit of the optimal mechanism when the threshold is set at $t=1/2$, the entry fee is $f=0$, and players use strategy $s(v)=v/2$. Thus, not only is the equilibrium Alboth *et al.* derive profit-maximizing in the subscription game, no other mechanism yields greater profit. For $t\neq 1/2$ the equilibria Alboth *et al.* derive are highly nonlinear and apparently give provision regions without the linear boundary found in an optimal mechanism, and therefore would appear not to maximize the seller's profit over all incentive compatible mechanisms.[7] To see this is indeed the case, note that in the equilibria of Alboth *et al.*, where strategies are strictly increasing throughout, all types in $(0,1]$ have a strictly positive probability of obtaining the good. However, for $c\in(0,2)$, the optimal mechanism specifies that players with value $v\in[0,c/2)$ have zero chance of obtaining the good.[8] Thus, the provision regions of the optimal mechanism and the equilibria of Alboth *et al.* do not match, implying by the Revenue Equivalence Theorem that the revenues are not equal.

Analogous to Proposition 7, our final proposition provides for the reverse power function distribution the same kind of converse result to the optimality of linear strategies: if linear strategies do not exist, then the (modified) subscription game cannot maximize the collector's profits.

Proposition 10. *Suppose F is defined on $[\underline{v},\bar{v}]$ by $F(v)=1-\left(\frac{\bar{v}-v}{\bar{v}-\underline{v}}\right)^r$, where $r>0$ and $0\leq\underline{v}<\bar{v}<\infty$. Assume further that $2((1+r)\underline{v}-\bar{v})\leq cr\leq\underline{v}(1+r)+(r-1)\bar{v}$. If $\bar{v}-\underline{v}(2+r)>0$ and $cr<\bar{v}(r-1)-\underline{v}r(1+r)$, then, for any continuation equilibrium of the subscription game, the collector cannot obtain the same profit as in the optimal direct mechanism in Section 4.1.*

Combining Propositions 8–10, the following corollary provides necessary and sufficient conditions for the subscription game to maximize profit when values have a reverse power function distribution.

Corollary 2. *Suppose F is defined on $[\underline{v},\bar{v}]$ by $F(v)=1-\left(\frac{\bar{v}-v}{\bar{v}-\underline{v}}\right)^r$, where $r>0$ and $0\leq\underline{v}<\bar{v}<\infty$. Assume further that $2((1+r)\underline{v}-\bar{v})\leq cr\leq 2r\bar{v}$. Through the modified subscription game the collector can obtain the same profit as in the optimal direct mechanism if and only if $\bar{v}\leq(2+r)\underline{v}$ or $cr\geq\bar{v}(r-1)-\underline{v}r(1+r)$.*

5 Conclusion

In the symmetric subscription game, piecewise-linear regular equilibria exist for the exponential and reverse power distributions of players' values. Moreover, if a regular symmetric equilibrium strategy is piecewise

[7]Similarly, with a uniform distribution of values, the contribution game cannot be optimal (even when $c=0$, as in Alboth *et al.*) as the continuous equilibria have a provision region bounded by a hyperbola (Barbieri and Malueg, *forthcoming*).

[8]To see this, recall that for values uniformly distributed over $[0,1]$, the optimal mechanism provides the good if and only if $v_1+v_2\geq 1+c/2$.

linear, then the distribution of players' types must agree with an exponential or reverse power function distribution, at least over the interval $[\widetilde{v}, \hat{v}]$. For the exponential and reverse power function distributions we have characterized exactly when piecewise-linear regular equilibria exist, and we have shown such equilibria can be profit-maximizing for the seller. Moreover, for these distributions only piecewise-linear equilibria can maximize the seller's profit over all incentive compatible mechanisms. It remains an open question whether, for other distributions of players' types, nonlinear equilibria of the subscription game (possibly with entry fees) can be optimal for the seller.

Appendix

Proof of Proposition 1. Necessity: Note that on $[\widetilde{v}, \hat{v}]$ the pairs $(v, G(v))$ represent the equilibrium completion frontier in $v_1 \times v_2$-space. The payoff to a player with value $v \in [\widetilde{v}, \hat{v}]$ contributing $s(v_a) \in [s(\widetilde{v}), s(\hat{v})]$ is $U(s(v_a)|v) = (v - s(v_a))(1 - F(G(v_a)))$; since $s(v)$ is optimal it must satisfy the following first-order optimization condition

$$s'(v)(1 - F(G(v))) + (v - s(v))f(G(v))G'(v) = 0,$$

which is (1). Differentiating the defining equation for G, we obtain $s'(v) + s'(G(v))G'(v) = 0$, which can be rearranged to give (2). Since contributions and payoffs in equilibrium must be positive, we need $0 \leq s(v) \leq v$, for $v \in [\widetilde{v}, \bar{v}]$.

If $\widetilde{v} > \underline{v}$, then two additional conditions must be satisfied. First, it is necessary that

$$s(\widetilde{v}) = \widetilde{v}, \tag{17}$$

for if $s(\widetilde{v}) < \widetilde{v}$ then types in $[\underline{v}, \widetilde{v})$ sufficiently near $\widetilde{v}$ could profitably deviate to a contribution of $s(\widetilde{v})$. Second, it must be that

$$(\bar{v} - s(\hat{v}))(1 - F(G(\hat{v}))) \geq \bar{v} - t, \tag{18}$$

so that the player with the highest possible value, $\bar{v}$, cannot, by ensuring the project is provided, strictly increase his payoff with a contribution of t rather than $s(\hat{v})$.

Sufficiency: Now we show that conditions (1)–(4) suffice to characterize a regular symmetric equilibrium. To show that the first-order condition (1) actually identifies a best response, first observe that for any announcement $v_a \in (\widetilde{v}, \hat{v})$,

$$\frac{dU(s(v_a)|v)}{dv_a} = -s'(v_a)(1 - F(G(v_a))) - (v - s(v_a))f(G(v_a))G'(v_a)$$

$$= (v_a - v) f(G(v_a)) G'(v_a),$$

where the second equality follows from (1) evaluated at $v = v_a$. Because $G'(v_a) \leq 0$, we have

$$\frac{dU(s(v_a)|v)}{dv_a} \gtreqless 0 \text{ for } v_a \lesseqgtr v,$$

implying that $U(s(v_a)|v)$ is quasiconcave in $v_a \in (\widetilde{v}, \hat{v})$, achieving a maximum at $s(v_a) = s(v)$. By continuity of s (in v) and U (in s and v), s is also optimal for $v = \widetilde{v}, \hat{v}$. Thus, we have shown that over the interval of contributions $[s(\widetilde{v}), s(\hat{v})]$, $s(v)$ is optimal for all $v \in [\widetilde{v}, \hat{v}]$. It remains to show $s(v)$ is globally optimal for all $v \in [\widetilde{v}, \hat{v}]$. A contribution less than $s(\widetilde{v})$ implies the good will not be provided and the corresponding payoff is 0; hence, given (3), such a contribution is not strictly better than $s(v)$. A contribution in the interval $(s(\hat{v}), t)$ does not increase the probability of provision beyond contributing $s(\hat{v})$ as all types $v < \widetilde{v}$ contribute 0. Therefore, the only other deviation from $s(v)$ to consider is t, which would ensure the good is provided. However, we see that, for any $v \in [\widetilde{v}, \hat{v}]$,

$$\begin{aligned}
U(s(v)|v) &\geq U(s(\hat{v})|v) && (s(v) \text{ is optimal in } [0, s(\hat{v})]) \\
&= (v - s(\hat{v}))(1 - F(G(\hat{v}))) \\
&= (v - \bar{v})(1 - F(G(\hat{v}))) + (\bar{v} - s(\hat{v}))(1 - F(G(\hat{v}))) \\
&\geq (v - \bar{v})(1 - F(G(\hat{v}))) + (\bar{v} - t) && (\text{by condition (4)}) \\
&= (v - t) + (\bar{v} - v) F(G(\hat{v})) \\
&\geq v - t,
\end{aligned}$$

showing a deviation to t is not profitable.

Turning attention to types in $(\hat{v}, \bar{v}]$, we show $s(v) = s(\hat{v})$ is optimal. There are two possibilities. The first is $\widetilde{v} = \underline{v}$. Here, because $s(\hat{v}) + s(\widetilde{v}) = s(\bar{v}) + s(\underline{v}) = t$ and s is nondecreasing, a contribution $s(\hat{v})$ is enough to ensure provision of the good, so any larger contribution is dominated by $s(\hat{v})$. The second possibility is $\widetilde{v} > \underline{v}$. In this case, condition (4) is sufficient to discourage a deviation to a contribution equal to t by type $\bar{v}$, and to any contribution in $(s(\hat{v}), t)$ because $s(v) = 0$ for $v \in [\underline{v}, \widetilde{v})$. The argument of the previous paragraph applies here, too, to show that (4) is sufficient to ensure no type $v \in (\hat{v}, \bar{v}]$ has a strictly profitable deviation of t or larger.

As for types in $[\underline{v}, \widetilde{v})$, they can exist only when $\widetilde{v} > \underline{v}$. In this case, $s(\widetilde{v}) = \widetilde{v}$ by condition (4) so types in $[\underline{v}, \widetilde{v})$ find it optimal to contribute any amount less than $s(\widetilde{v})$, in order to avoid triggering completion with some positive probability and thereby realizing a strictly negative payoff. Therefore, we can set their contributions to zero. This establishes that $s(v)$ is optimal for all v. □

Proof of Proposition 2. By contradiction assume that, for some $t > 0$, (s, s) is a regular equilibrium with $s(\widetilde{v}) = \widetilde{v}$ and $\hat{v} < \bar{v}$. Then, it must satisfy the conditions in Proposition 1. Define $G : [\widetilde{v}, \hat{v}] \to [\widetilde{v}, \hat{v}]$ as in Proposition 1. Then G is strictly decreasing and continuous, with $G(\widetilde{v}) = \hat{v}$ and $G(\widetilde{v}) = \hat{v}$. Define v_m as the type that contributes $t/2$: $v_m \equiv s^{-1}(t/2)$; note that in a regular equilibrium v_m exists and $v_m \in (\widetilde{v}, \hat{v})$. Fix any $v \in [\widetilde{v}, v_m]$ and observe that $G(v) \in [v_m, \hat{v}]$. Define m as

$$m \equiv \min_{v \in [v_m, \hat{v}]} \frac{v - s(v)}{h(v)}.$$

Note that m is well-defined because s and h are continuous and, for $v \in [v_m, \hat{v}]$, $h(v)$ is bounded away from zero because of the assumption $\hat{v} < \bar{v}$. Moreover, since $U(s(v)|v) = U^*(v)$ is strictly increasing in v and $s(\widetilde{v}) = \widetilde{v}$, it follows that $s(v) < v$ for $v \in [v_m, \hat{v}]$. Therefore we see that $m > 0$. Because $f(\underline{v}) > 0$ and f is continuous on $[\underline{v}, \bar{v})$, there exists $v_0 \in (\underline{v}, v_m]$ and $\underline{f} > 0$ such that $f(v) \geq \underline{f}$ for all $v \in [\underline{v}, v_0]$. The bounds m and $\underline{f}$ are used next.

On the one hand, since G is bounded, for $v \in [\widetilde{v}, v_0]$ we have

$$G(v) - G(v_m) = G(v) - v_m \leq \hat{v}. \tag{19}$$

On the other hand, for $v \in (\widetilde{v}, v_0]$ we have

$$\begin{aligned}
G(v) - G(v_m) &= \int_{v_m}^{v} G'(y)\, dy \\
&= \int_{v}^{v_m} \frac{h(y)}{y - s(y)} \frac{G(y) - s(G(y))}{h(G(y))} && \text{(by (6))} \\
&\geq m \int_{v}^{v_m} \frac{h(y)}{y - s(y)}\, dy && \text{(definition of } m\text{)} \\
&\geq m \int_{v}^{v_m} \frac{f(y)}{y - s(y)}\, dy \\
&\geq m\underline{f} \int_{v}^{v_0} \frac{1}{y - s(y)}\, dy && (v_0 \leq v_m\text{; definition of } \underline{f}) \\
&\geq m\underline{f} \int_{v}^{v_0} \frac{1}{y - \widetilde{v}}\, dy \\
&= m\underline{f} \log\left(\frac{v_0 - \widetilde{v}}{v - \widetilde{v}}\right) \to \infty \quad \text{as } v \downarrow \widetilde{v},
\end{aligned}$$

showing (19) is violated for v close to $\widetilde{v}$. This contradiction implies (s, s) cannot be a regular equilibrium. □

Proof of Lemma 2. Consider the equation for a linear inverse hazard rate

$$\frac{1 - F(v)}{f(v)} = \gamma v + \delta,$$

which can be restated as

$$\frac{-f(v)}{1 - F(v)} = -\frac{1}{\gamma v + \delta}. \tag{20}$$

If $\gamma = 0$, then $\delta > 0$ and integration of (20) yields

$$\log(1 - F(v)) = -\frac{1}{\delta} v + \lambda,$$

where λ is a constant of integration. The condition $F(\underline{v}) = 0$ implies $\lambda = \underline{v}/\delta$, so

$$F(v) = 1 - e^{-r(v - \underline{v})},$$

where $r = 1/\delta > 0$; and the condition $\lim_{v \to \bar{v}} F(v) = 1$ implies $\bar{v} = +\infty$.

If $\gamma \neq 0$, integration of (20), along with the boundary conditions $F(\underline{v}) = 0$ and $F(\bar{v}) = 1$, yields

$$F(v) = 1 - \left(\frac{\bar{v} - v}{\bar{v} - \underline{v}}\right)^r,$$

for some $r \neq 0$. Since F must be increasing, we have $r > 0$. □

Proof of Proposition 3. The proof applies Proposition 1. If $s(v) = \alpha v + \beta$ on $[\widetilde{v}, \hat{v}]$, then $G(v) = (t - 2\beta)/\alpha - v$ so equation (2) is satisfied. If F is given in (8), then (as in (7)) equation (1) becomes

$$\frac{1}{r} = \frac{1 - \alpha}{\alpha}(\hat{v} + \widetilde{v} - v) - \frac{\beta}{\alpha} \qquad \forall v \in (\widetilde{v}, \hat{v}), \tag{21}$$

which is satisfied if and only if

$$\alpha = 1 \text{ and } \beta = -\frac{1}{r}. \tag{22}$$

Therefore, on $[\widetilde{v}, \hat{v}]$ a regular piecewise-linear equilibrium strategy must be $s(v) = \alpha v + \beta = v - \frac{1}{r}$; consequently, the necessary condition $s(\widetilde{v}) = \widetilde{v}$ if $\widetilde{v} > \underline{v}$ can never be satisfied, implying by (4) that $\widetilde{v} = \underline{v}$. Because contributions must be non-negative, we have

$$\underline{v} - \frac{1}{r} = \widetilde{v} - \frac{1}{r} \geq 0. \tag{23}$$

Finally, the condition $s(\widetilde{v}) + s(\hat{v}) = t$ implicitly defines $\hat{v}$ as

$$\hat{v} = t - \underline{v} + \frac{2}{r}.$$

The requirement that $\hat{v} \geq \underline{v}$ yields

$$t \geq 2\left(\underline{v} - \frac{1}{r}\right). \tag{24}$$

Conditions (23) and (24) are the necessary conditions stated in the proposition. If these conditions are satisfied, then s given in (10) satisfies a–d and (1)–(3) of Proposition 1; and with $\widetilde{v} = \underline{v}$, it follows that (s, s) is a symmetric equilibrium. □

Proof of Proposition 4. The proof applies Proposition 1. If $s(v) = \alpha v + \beta$ on $[\widetilde{v}, \hat{v}]$, then $G(v) = (t - 2\beta)/\alpha - v$ so equation (2) is satisfied. If F is given by (9), then (as in (7)) equation (1) becomes

$$\frac{1}{r}(\bar{v} - v) = (\hat{v} + \widetilde{v} - v)\,\frac{1-\alpha}{\alpha} - \frac{\beta}{\alpha} \qquad \forall v \in (\widetilde{v}, \hat{v}), \tag{25}$$

which is satisfied if and only if

$$\alpha = \frac{r}{1+r} \text{ and } \beta = -\frac{\bar{v} - (\hat{v} + \widetilde{v})}{1+r}. \tag{26}$$

Therefore, on $[\widetilde{v}, \hat{v}]$ a regular piecewise-linear equilibrium strategy must be

$$s(v) = \frac{r}{1+r}v - \frac{\bar{v} - (\hat{v} + \widetilde{v})}{1+r}. \tag{27}$$

By the definition of G, we must have

$$s(\widetilde{v}) + s(\hat{v}) = t,$$

or, using (27),

$$t = \left(\frac{2+r}{1+r}\right)(\widetilde{v} + \hat{v}) - \frac{2\bar{v}}{1+r}, \tag{28}$$

which can be rewritten as

$$\begin{aligned}
\widetilde{v} + \hat{v} &= \left(\frac{1+r}{2+r}\right)t + \frac{2\bar{v}}{2+r} \\
&> \frac{(2+r)\underline{v} + r\bar{v}}{2+r} + \frac{2\bar{v}}{2+r} && \text{(by (11))} \\
&= \bar{v} + \underline{v};
\end{aligned}$$

and, since $\hat{v} \leq \bar{v}$ we must have $\widetilde{v} > \underline{v}$. When $\widetilde{v} > \underline{v}$, the necessary condition $s(\widetilde{v}) = \widetilde{v}$ applies, and using (27) we obtain $\hat{v} = \bar{v}$. Therefore, $\widetilde{v}$ is implicitly determined by $s(\widetilde{v}) + s(\bar{v}) = t$, yielding

$$\widetilde{v} = \frac{t(1+r) - r\bar{v}}{2+r}.$$

In order for the last expression to belong to $(\underline{v}, \bar{v})$ it must be that

$$\frac{r\bar{v} + \underline{v}(2+r)}{1+r} < t < 2\bar{v},$$

thus completing the proof of necessity of the Proposition.

The foregoing shows that if there exists a symmetric piecewise-linear regular equilibrium (s, s), then on $[\widetilde{v}, \hat{v}]$ strategy s has the form in (27). Note that the strategy s in (27) satisfies conditions *a*–*d*. To show further that it constitutes an equilibrium we consider the case $\widetilde{v} > \underline{v}$, which is possible because $t \geq \frac{r\bar{v}+\underline{v}(2+r)}{1+r}$. Using (27), the necessary condition that $s(\widetilde{v}) = \widetilde{v}$ now implies $\hat{v} = \bar{v}$, so the candidate equilibrium strategy is given by (12). This strategy has properties *a*–*d* of Proposition 1 and satisfies the sufficient conditions (1)–(3). To complete the proof, it only remains to show that (4) is also satisfied. Because $t = s(\bar{v}) + s(\widetilde{v}) = s(\bar{v}) + \widetilde{v}$, the remaining condition in (4) can be rewritten as

$$(\bar{v} - (t - \widetilde{v}))(1 - F(\widetilde{v})) \geq \bar{v} - t,$$

which, after substituting for F and rearranging terms, becomes

$$\frac{\bar{v} - (t - \widetilde{v})}{\bar{v} - t} \left(\frac{\bar{v} - \widetilde{v}}{\bar{v} - \underline{v}} \right)^r \geq 1. \tag{29}$$

It now suffices to show (29) is satisfied. From the condition $t = \widetilde{v} + s(\bar{v})$, we obtain

$$\widetilde{v} = \frac{t(1+r) - r\bar{v}}{2+r}.$$

Consequently, we have

$$\bar{v} - (t - \widetilde{v}) = \frac{(2+r)(\bar{v} - t) + t(1+r) - r\bar{v}}{2+r} = \frac{2\bar{v} - t}{2+r} \tag{30}$$

and

$$\bar{v} - \widetilde{v} = \frac{(2+r)\bar{v} - t(1+r) + r\bar{v}}{2+r} = \frac{2\bar{v} - t}{2+r}(1+r); \tag{31}$$

so, to show that (29) is satisfied when $\widetilde{v} > \underline{v}$, we need to show

$$\frac{(2\bar{v}-t)^{r+1}}{(\bar{v}-t)} \frac{1}{2+r} \left(\frac{1}{\bar{v}-\underline{v}} \times \frac{1+r}{2+r}\right)^{r} \geq 1 \qquad \forall t > \frac{(2+r)\underline{v}+r\bar{v}}{1+r}. \tag{32}$$

Consider the function

$$\frac{(2\bar{v}-t)^{r+1}}{\bar{v}-t} \tag{33}$$

on the left-hand side of (32). Its derivative with respect to t is

$$\begin{aligned}
\frac{-(r+1)(2\bar{v}-t)^{r}(\bar{v}-t)+(2\bar{v}-t)^{r+1}}{(\bar{v}-t)^{2}} &= \frac{(2\bar{v}-t)^{r}}{(\bar{v}-t)^{2}}\left(-(r+1)(\bar{v}-t)+2\bar{v}-t\right) \\
&= \frac{(2\bar{v}-t)^{r}}{(\bar{v}-t)^{2}}\left(rt+(1-r)\bar{v}\right) \\
&\gtreqless 0 \iff t \gtreqless \frac{r-1}{r}\bar{v}.
\end{aligned}$$

Because

$$\frac{(2+r)\underline{v}+r\bar{v}}{1+r} > \frac{r}{1+r}\bar{v} > \frac{r-1}{r}\bar{v},$$

it follows that (33) is strictly increasing in t for $t > \frac{(2+r)\underline{v}+r\bar{v}}{1+r}$. Therefore, for any $t > \frac{(2+r)\underline{v}+r\bar{v}}{1+r}$ we have

$$\begin{aligned}
\frac{(2\bar{v}-t)^{r+1}}{\bar{v}-t} &> \left.\frac{(2\bar{v}-t)^{r+1}}{\bar{v}-t}\right|_{t=\frac{(2+r)\underline{v}+r\bar{v}}{1+r}} \\
&= (2+r)\left(\frac{\bar{v}-\underline{v}}{\bar{v}-(2+r)\underline{v}}\right)\left(\frac{2+r}{1+r}\right)^{r}(\bar{v}-\underline{v})^{r}.
\end{aligned} \tag{34}$$

Using (34) we have

$$\begin{aligned}
\text{lhs of (32)} &\geq \left[(2+r)\left(\frac{\bar{v}-\underline{v}}{\bar{v}-(2+r)\underline{v}}\right)\left(\frac{2+r}{1+r}\right)^{r}(\bar{v}-\underline{v})^{r}\right]\frac{1}{(2+r)}\left(\frac{1}{\bar{v}-\underline{v}} \times \frac{1+r}{2+r}\right)^{r} \\
&= \frac{\bar{v}-\underline{v}}{\bar{v}-(2+r)\underline{v}} \\
&\geq \frac{\bar{v}-\underline{v}}{\bar{v}-\underline{v}} = 1,
\end{aligned}$$

establishing (32), thereby completing the proof. □

Proof of Proposition 5. We proceed exactly as in the proof of Proposition 4 to derive that $\widetilde{v} > \underline{v}$ implies

$$t > \frac{r\bar{v}+\underline{v}(2+r)}{1+r},$$

contradicting the definition of the low-threshold case. Therefore, in the low-threshold case, regular piecewise-linear symmetric equilibria must have $\widetilde{v} = \underline{v}$. In this case $\hat{v}$ is implicitly determined by $s(\underline{v}) + s(\hat{v}) = t$, which, using (27), yields after rearrangement

$$\hat{v} = \frac{t(1+r) + 2\bar{v}}{2+r} - \underline{v}.$$

Substituting this value into (27) yields

$$s(\underline{v}) = \frac{r}{1+r}\underline{v} - \frac{\bar{v}r - t(1+r)}{(2+r)(1+r)};$$

but the necessary condition $s(\underline{v}) \geq 0$ thus implies

$$t \geq \frac{r}{1+r}(\bar{v} - \underline{v}(2+r)).$$

At the same time, we must have

$$\underline{v} \leq \hat{v} = \frac{t(1+r) + 2\bar{v}}{2+r} - \underline{v};$$

thus implying

$$t \geq -\frac{2}{(1+r)}(\bar{v} - \underline{v}(2+r)),$$

and completing the proof of necessity of the Proposition.

The foregoing shows as well that the strategy (14) has properties *a*–*d* of Proposition 1 and satisfies the sufficient conditions (1)–(3). Since $\widetilde{v} = \underline{v}$, condition (4) is automatically satisfied. This establishes the sufficiency of Proposition 5. □

Proof of Proposition 8. Because

$$v - \frac{1 - F(v)}{f(v)} = \left(\frac{1+r}{r}\right)v - \frac{\bar{v}}{r},$$

Lemma 3 shows the profit-maximizing mechanism provides the good if and only if

$$c \leq \left(\frac{1+r}{r}\right)(v_1 + v_2) - \frac{2\bar{v}}{r},$$

which is equivalent to

$$v_1 + v_2 \geq \frac{2\bar{v} + cr}{1+r}. \tag{35}$$

Note how

$$t = \frac{cr(2+r) + 2\bar{v}}{(1+r)^2}$$

and $cr \geq \underline{v}(1+r)+(r-1)\bar{v}$ imply

$$t \geq \frac{(\underline{v}(1+r)+(r-1)\bar{v})(2+r)+2\bar{v}}{(1+r)^2} = \frac{\underline{v}(2+r)+\bar{v}r}{(1+r)},$$

the definition of high-threshold case in (11). Therefore, the strategy in (12) is part of a symmetric equilibrium where completion happens if and only if $s(v_1)+s(v_2) \geq t$, or

$$v_1+v_2 \geq \frac{2\bar{v}+t(1+r)}{2+r} = \frac{2\bar{v}+cr}{1+r},$$

using (12) and the choice of t in the Proposition, showing the provision region in this subscription-game equilibrium coincides with that in the optimal mechanism (see (35)). To conclude, note that in (12) we have $\widetilde{v} > \underline{v}$, so that $U_i^*(\underline{v}) = 0$. Therefore, $f = 0$ is optimal. □

Proof of Proposition 9. Note how the choice of t in the Proposition and $cr \leq \underline{v}(1+r)+(r-1)\bar{v}$ imply

$$t \leq \frac{\underline{v}(2+r)+\bar{v}r}{(1+r)},$$

the definition of low-threshold case in (13). Therefore, if the strategy in (14) is part of a symmetric equilibrium, proceeding as for the proof of Proposition 8, one may show that the completion region induced by (14) is the same specified in Lemma 3. The choice of the entry fee f delivers

$$0 = U_i^*(\underline{v}) = (\underline{v} - s(\underline{v}))(1-F(\hat{v})) - f,$$

using (14). Therefore, we just need to verify the additional conditions specified in Proposition 5. When $\bar{v} - \underline{v}(2+r) > 0$ we need

$$t \geq \frac{r}{1+r}(\bar{v} - \underline{v}(2+r)),$$

which, after substitution, reduces to

$$cr > \bar{v}(r-1) - \underline{v}r(1+r),$$

and this condition is required by the statement of this Proposition. When $\bar{v} - \underline{v}(2+r) < 0$ we need

$$t \geq -\frac{2}{1+r}(\bar{v} - \underline{v}(2+r)),$$

which, after substitution, yields

$$cr > 2\underline{v}\,(1+r) - 2\bar{v},$$

and this condition is equivalent to the assumption that the collector does not produce the good with probability one in the optimal mechanism: $cr > 2(1+r)\underline{v} - \bar{v}$. $\square$

Proof of Proposition 10. The proof proceeds with the same steps of the proof of Proposition 7, so $G(v) = k_1 - v$ for some constant k_1. Equation (1) now reduces to

$$s'(v) = \frac{r(v - s(v))}{\bar{v} - k_1 + v};$$

a nonconstant solution to this differential equation must have the form

$$s(v) = \frac{k_1 - \bar{v} + rv}{1+r} + k_2(k_1 - \bar{v} - v)^{-r},$$

for some constant k_2. With the same steps as in the proof of Proposition 7, one can establish that s cannot be an equilibrium when $k_2 \neq 0$, while the conditions in this Proposition imply that s cannot be an equilibrium when $k_2 = 0$, because $s(v)$ is not a valid linear equilibrium. $\square$

References

Admati, Anat R., and Motty Perry (1991), Joint projects without commitment, *Review of Economic Studies* 58, 259–276.

Alboth, Dirk, Anat Lerner, and Jonathan Shalev (2001), Profit maximizing in auctions of public goods, *Journal of Public Economic Theory* 3, 501–525.

Barbieri, Stefano, and David A. Malueg, 2007, Private provision of a discrete public good: continuous-strategy equilibria in the private-information subscription game, available at `http://ssrn.com/abstract=985118`.

Barbieri, Stefano, and David A. Malueg, Private provision of a discrete public good: efficient equilibria in the private-information contribution game, *forthcoming* in *Economic Theory*.

Cornelli, Francesca, 1996, Optimal selling procedures with fixed costs, *Journal of Economic Theory* 71, 1–30.

Holmström, Bengt, and Roger B. Myerson (1983), Efficient and durable decision rules with incomplete information, *Econometrica* 51: 1799–1820.

Laussel, Didier, and Thomas R. Palfrey (2003), Efficient equilibria in the voluntary contributions mechanism with private information, *Journal of Public Economic Theory* 5, 449–478.

Marx, Leslie M., and Steven A. Matthews (2000), Dynamic voluntary contribution to a public project, *Review of Economic Studies* 67: 327–358.

Menezes, Flavio M., Paulo K. Monteiro, and Akram Temimi (2001), Private provision of discrete public goods with incomplete information, *Journal of Mathematical Economics* 35, 493–514.