Self-Organization and the Generation of Noise in Financial Markets

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Abstract

A dynamic model of financial markets with learning is demonstrated to produce a self-organized system that displays critical behavior. The price contains private information that traders learn to extract and employ to forecast future value. Since the price reflects the beliefs of the traders, the learning process is self referencing. As the market learns to correctly extract information from the price, the market deemphasizes private information. Despite the convergence of the model towards the parameters producing efficiency, pricing deviations remain constant due to the increased sensitivity of the price to small errors in information extraction produced by the model's own convergence.

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1. Introduction

When Grossman and Stiglitz (1980) find that no equilibrium exists able to produce efficient markets under rational expectations, they create one by introducing a second source of noise that hampers information extraction from the price. The equilibrium is created at the expense of market efficiency. Finding the conditions sufficient to create equilibrium is, of course, the traditional thrust of economic research. This paper explores a dynamic model of information extraction in a financial market that, like the initial Grossman-Stiglitz environment, lacks an equilibrium. The result is a self-organizing system that exhibits critical behavior. The point of attraction is market efficiency in which the price reveals the private information of the informed traders but in simulation the price fails to converge in concert with the other parameters.

A fully revealing efficient price is impossible in a Rational Expectations Equilibrium. Grossman and Stiglitz (1980) (hereafter GS) reach this conclusion based on a market in which fundamental information is costly to obtain. They find that traders who are informed with the private information have an impact on the price while uninformed traders use the price to costlessly extract the private information. The GS paradox is that a "long run" equilibrium does not exist. No trader wants to bear the cost of informing the market without reward.[1] Subsequent papers reach the same conclusion after relaxing the assumption of full rationality, for example by replacing rationality with dynamic learning on the part of the uninformed traders. This paper further generalizes GS by examining dynamic populations in concert with a dynamic learning process.

Among those papers extending GS is Bray (1982) introduced learning through repeated realizations of the GS single period model. Bray provides support for the GS assumption of full

[1] Muendler (2005) finds conditions under which an individual trader's can receive compensation for informing the remainder of the market.

rationality by finding that learning converges to the rational expectations equilibrium for any fixed population proportion of uninformed to informed traders. Hussman (1992) and Timmermann (1996) also examine learning in a financial market setting. Like Bray, they have a static population of traders, but with a multi-period asset paying a dividend that follows an AR(1) process.

Routledge (1999) examines a dynamic population in a noisy GS model. With two types of traders, a dynamic population process is introduced by allowing traders to switch strategies based on imitating more successful traders they encounter. Thus, the single act of imitation embodies the two dynamic processes of learning and population evolution. Modeled with a random supply of the risky asset, the exogenous noise ensures the existence of a stable REE. The process generally converges to this REE.

Bak, Tang, and Weisenfeld (1988) introduces the notion of self-organized criticality (SOC) by describing what has become the canonical example of a system that attains SOC, the sand pile. The slope of the sand pile increases as grains of sand are added. At the critical slope an additional grain of sand causes an avalanche which locally reduces the slope and temporarily returns the system to stability. The size of the avalanche is determined by local conditions. The distribution of the avalanche size follows a power law. The self-organization refers to the natural convergence of the system to the attracting critical slope. The criticality refers to the chaotic behavior of the avalanches near the critical slope. A number of papers have applied the notion of SOC to economic settings, including Bak *et al* (1992), Bak, Paczuski, and Shubik (1997), Berg *et al* (2001), Challet, Marsili and Zhang (2000), and Challet and Marsili (2002). The latter four describe financial markets as SOC.

Relative to Routledge (1999), Goldbaum (2005) separates the learning process from the population process in an examination of a market in which dividends are driven by a random walk. The model produces true Self-Organized Criticality with two market states, one stable and one unstable, separated by an attracting critical division of the population. The market presented in this paper also contains a phase transition, but it is the convergence towards the critical value, not the actual switching between phases, that makes the market behavior interesting. The convergence properties under learning as discussed by Marcet and Sargent (1989a, 1989b) are keenly relevant.

The paper proceeds as follows. Section 2 introduces the model of the market. The population is divided into those who are informed with private fundamental information and those who attempt to extract information from the price. The nature of the private information is developed along with the trading strategy of those who obtain it. The method by which traders learn to extract information from the price is also described in this section. Section 3 presents analysis of the model under learning and introduces the dynamic population process. Simulation results characterize the model's asymptotic behavior, displaying the critical behavior produced by the self-organizing aspect of the model. The model developed in Section 4 removes the artificial population division, allowing all traders to both receive private information and extract information from the price. This model is also examined through simulation. Conclusions are drawn in Section 5 of the paper.

2. Model

The market setting is similar to Goldbaum (2005) with the modification that the dividend follows an AR(1) process rather than a random walk. This small change produces substantially different market environment. The following provides a basic description of the environment and solutions based on the stationary dividend process.

2.1 The market

A large but finite number of agents, indexed by i = 1, ..., N, trade a risky asset and a risk-free bond. The risk-free bond, with a price of one, pays R. The risky asset is purchased at the market determined price, p_t, in period t. In t+1, it pays a stochastic dividend d_{t+1}, and sells for the market determined price p_{t+1}. The market participants are aware that the stochastic dividend is based on an AR(1) process centered around the commonly known d_0:

$$d_t = d_0 + \eta_t ,$$

$$\text{with } \eta_t = \phi\eta_{t-1} + \varepsilon_t ,\ \varepsilon_t \sim IIDN(0, \sigma_\varepsilon^2) . \tag{1}$$

Let $z_t = p_t + d_t$ and $\theta_{it} = 1/\gamma\sigma_{it}^2$ with $\sigma_{it}^2 = \mathrm{var}_{it}(z_{t+1})$ indicating the conditional variance. The parametner γ is the coefficient of absolute risk aversion. In each period, each myopic trader maximizes a negative exponential utility function on one period ahead wealth conditional on his individual information set (to be developed below). This produces the demand for the risky asset,

$$q_{it}(p_t) = \frac{E_{it}(p_{t+1} + d_{t+1} - Rp_t)}{\gamma Var_{it}(p_{t+1} + d_{t+1})} = \theta_{it}(E_{it}(z_{t+1}) - Rp_t) . \tag{2}$$

Assume K strategies for estimating payoffs, z_{t+1}. In a Walrasian equilibrium, the market price equates supply and demand for the asset. Supply is fixed to avoid the exogenous introduction of noise. For convenience, set fixed net supply of the risky asset to zero. Let N_k be the total number of traders employing information I^k. Let q_t^k be the *per capita* demand for the risky security among group k traders, $q_t^k(p_t) = \frac{1}{N_k}\sum_{i=1}^{N_k} q_{it}(p_t)$. Let n^k, $0 \le n^k \le 1$ be the proportion of the trader population employing strategy k, $\sum n^k = 1$. The price p_t clears the market by solving

$$0 = \sum_{k=1}^{K} n^k q_t^k(p_t)\,. \tag{3}$$

2.2 Information

2.2.1 The Fundamental Trader

The estimate of the future payoff is in the nature of Hellwig (1980). Trader i's private research indicates the fundamental value of the risky security captured by a signal based on d_{t+1} that is subject to idiosyncratic error. From the signal, the traders is able to extract a noisy indicator of the dividend deviation from the long run mean,

$$s_{it} = d_0 + \eta_{t+1} + e_{it}\text{, with } e_{it} \sim IIDN(0, \sigma_e^2)\,. \tag{4}$$

A linear projection of η_{t+1} onto the information set produces the fundamental investor's mean squared error minimizing forecast

$$E_i(\eta_{t+1} \mid \eta_t, s_{it}) = (1-\beta)\phi\eta_t + \beta s_{i,t} \tag{5}$$

where the weight β is known based on the traders' knowledge of the dividend and information processes,

$$\beta \equiv \mathrm{cov}(\eta_{t+1}, s_{it}) / \mathrm{var}(s_{it}) = \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_e^2}\,.$$

The "fundamental" price prevails in a market populated exclusively by fundamental investors. Derive the fundamental price by using the estimate (5) in (2),

$$p_t^F = p^F(\eta_t, \eta_{t+1}) = b_0 + b_1^F \eta_t + b_2^F \eta_{t+1} + \nu_t\,, \tag{6}$$

$\nu_t = \dfrac{\beta}{R-\phi}\dfrac{1}{N}\sum_i e_{it}$. Price is a function of the current private and public information.

Advancing (6) one period, substituting it into the demand (2) and using the market clearing condition (3), the price coefficients solve to $b_0 = d_0/(R-1)$, $b_1^F = (1-\beta)\phi/(R-\phi)$ and $b_2^F = \beta/(R-\phi)$. Henceforth, the fact that p_t^F (and the general market clearing price, p_t) are functions of η_t and η_{t+1} will be implicit, captured by the subscript t. Future discussions of the fixed point price refer to the pricing equation at which the coefficients take on their fixed point values, not that there is a single value for the price at the fixed point.

For large N, the impact of the idiosyncratic signal noise on the price is negligible. Assume a sufficiently large N such that the ν_t term can be dropped.[2]

For $\sigma_e^2 \to \infty$, $p_t^F \to p_t^0 = \dfrac{d_0}{R-1} + \dfrac{\phi}{R-\phi}\eta_t$. As the private signal becomes increasingly noisy, the price converges to reflect just the public information contained in d_t, i.e. the price is Semistrong-form efficient according to the Fama (1970) definitions of efficiency. For $\sigma_e^2 = 0$, $p_t^F = p_t^{EM} = \dfrac{d_0}{R-1} + \dfrac{1}{R-\phi}\eta_{t+1}$. When traders receive a perfect noise-free signal on the next period's dividend, then the price fully reflects the d_{t+1} based value producing a Strong-form efficient price. Between the two extremes, p_t^F is not efficient, reflecting a weighted combination of d_t and d_{t+1} based information.

Fundamental traders rely on (6) in forming demand. Plug (6) back into (2) to solve for the average demand of the group of fundamental traders,

$$q_t^F = \theta_t^F\left(\frac{Rd_0}{R-1} + \frac{R}{R-\phi}((1-\beta)\phi\eta_t + \beta\eta_{t+1}) - Rp_t\right). \tag{7}$$

[2] Formally, ν_t is o(1).

2.2.2 Regression traders

The regression traders model the relationship between the payoff, $z_{t+1} = p_{t+1} + d_{t+1}$, and current market observables. The traders appropriately estimate

$$z_t = c_0 + c_1 p_{t-1} + c_2 \eta_{t-1} + \zeta_t \,. \tag{8}$$

The traders employ least-squares learning to update the parameters of their model. The learning process is self-referential with an endogenous state variable, p_{t-1}, included as a regressor. The value of c_0 is exogenous to beliefs and can be derived analytically by the traders given their knowledge of the dividend process. Thus traders impose the correct value of c_0. Let $x_t' = [1\ p_t\ \eta_t]$. The regression traders update the coefficients, $\mathbf{c}_t = [c_0\ c_{1t}\ c_{2t}]$, using the standard recursive updating algorithm for least-squares learning of Marcet and Sargent (1989a, 1989b):

$$\mathbf{c}_t = \mathbf{c}_{t-1} + (Q_t^{-1} x_{t-2}(z_{t-1} - \mathbf{c}_{t-1} x_{t-2}))'/t \,, \tag{9}$$

$$Q_t = Q_{t-1} + (x_{t-1} x_{t-1}' - Q_{t-1})/t \,,$$

given ($\mathbf{c}_0$, Q_0). The regression traders all rely on the same public information, and thus all employ the same forecast, $E(z_{t+1} \mid x_t)$. *Per capita* demand among regression traders is thus

$$q_t^R = \theta_t^R(\mathbf{c}_t x_t - R p_t)\,. \tag{10}$$

2.3 Price Formation

With $K = 2$, let $n_t = n_t^F$, and thus $(1-n_t) = n_t^R$. From (3),

$$0 = n_t q_t^F + (1 - n_t) q_t^R \tag{11}$$

Use (7), (10), and (11) to solve for the market clearing price. A consistent price function takes the form

$$p_t = p_t(n_t, \mathbf{c}_t) = b_0 + b_1(n_t, \mathbf{c}_t)\eta_t + b_2(n_t, \mathbf{c}_t)\eta_{t+1} \tag{12}$$

with

$$b_0 = d_0/(R-1),$$

$$b_1(n_t,\mathbf{c}_t) = \Psi(n_t,\mathbf{c}_t)^{-1}(n_t\theta_t^F \tfrac{R}{R-\phi}(1-\beta)\phi + (1-n_t)c_{2t}\theta_t^R), \qquad (13)$$

$$b_2(n_t,\mathbf{c}_t) = \Psi(n_t,\mathbf{c}_t)^{-1}(n_t\theta_t^F \tfrac{R}{R-\phi}\beta),$$

$$\Psi(n_t,\mathbf{c}_t) = n_t R\theta_t^F + (1-n_t)(R-c_{1t})\theta_t^R.$$

3. Analysis and simulation

3.1 Learning under fixed n

As with Bray (1982), consider a learning process with $n_t = n$ to determine whether a REE exists and if so whether the system will converge such that the traders can correctly extract d_{t+1} from the price.

3.1.1 A fixed point solution

Three equations describe the dynamic processes under a fixed n. Equation (1) is the exogenous dividend process. Equation (12) is the endogenous price equation. The coefficients of (12) depend on the beliefs of the regression traders as captured by (8) that evolve according to (9) and upon the proportion of the market relying of fundamental analysis.

The fixed point for the learning process produces a fixed point pricing function $p_t^*(n)$ based on the solution, for $0 < n \le 1$:

$$b_1^*(n) = \frac{n\theta^F(1-\beta)\phi}{(R-\phi)(n\theta^F + (1-n)\theta^R)},\ b_2^*(n) = \frac{n\theta^F\beta + (1-n)\theta^R}{(R-\phi)(n\theta^F + (1-n)\theta^R)}, \qquad (14)$$

$$c_1^*(n) = \frac{R}{(R-\phi)b_2^*(n)} = \frac{R(n\theta^F + (1-n)\theta^R)}{n\beta\theta^F + (1-n)\theta^R}, \qquad (15)$$

$$c_2^*(n) = \tfrac{\phi}{R-\phi}(R - c_1^*(n)) = -\frac{n(1-\beta)\phi\theta^F}{(R-\phi)(n\beta\theta^F + (1-n)\theta^R)},$$

$$\sigma_F^*(n)^2 = ((1-\beta)(R/(R-\phi))^2 + b_2^*(n)^2)\sigma_\varepsilon^2,\ \text{and} \qquad (16)$$

$$\sigma_R^*(n)^2 = b_2^*(n)^2 \sigma_\varepsilon^2,$$

and for $n = 0$ the regression captured by (8) is undefined since p_{t-1} is a linear function of η_{t-1}. Note that for $n \to 0$ $c_1^*(n) \to R$, $c_2^*(n) \to 0$ and for $n = 0$, $c_1^*(0) = R$, $c_2^*(0) = 0$ is a consistent solution producing $b_1^*(0) = \phi/(R\text{-}\phi)$ and $b_2^*(0) = 0$.

The solution in (14) - (16) is not a reduced form solution $b_2^*(n)$ and the conditional variance terms remain dependent.[3] The fixed point solutions based on the typical exogenous parameter values used in the simulations are plotted in Figure 1 as functions of n.

[Figure 1 about here]

At the fixed point, the price is fully revealing of the private information d_{t+1}, meaning that a the regression traders correctly deduce the relationship between price and payoff and trade as if d_{t+1} is known. Despite fully revealing d_{t+1} for all values of n, $p_t^*(n)$ only approaches p_t^{EM} as $n \to 0$. In general

$$p_t - p_t^{EM} = (b_{1t} + \phi b_{2t} - \phi/(R-\phi))\eta_t + (\phi b_{2t} - \phi/(R-\phi))\varepsilon_{t+1}. \qquad (17)$$

Since $b_1^*(n) + \phi b_2^*(n) = \phi/(R-\phi)\,\forall\, n$, the first term of (17) is equal to zero for all $p_t^*(n)$. The second term only converges to zero as $n \to 0$. With $p_t^*(0) = p_t^0$ there is no value of n at which $p_t^*(n) = p_t^{EM}$.

3.1.2 Performance

Let the measure of profits earned by each information source be the excess return realized for the risky asset multiplied by the group average demand:

$$\pi_t^k = q_t^k(p_{t+1} + d_{t+1} - Rp_t),\ k = F, R. \qquad (18)$$

[3] Solving for $b_2^*(n)$ involves solving a cubic polynomial. The solution is available, but pages long.

Based on the fixed point solution, the modeler with knowledge of n can compute

$$\left.\begin{aligned} E(\pi^F) &= -\theta^{F\,2}\theta^R\left(\frac{R}{R-\phi}\right)^2\left(\frac{n(1-n)(1-\beta)^2}{(n\theta^F+(1-n)\theta^R)^2}\right)\sigma_\varepsilon^2 \\ E(\pi^R) &= \theta^{F\,2}\theta^R\left(\frac{R}{R-\phi}\right)^2\left(\frac{n(1-\beta)}{n\theta^F+(1-n)\theta^R}\right)^2\sigma_\varepsilon^2 \end{aligned}\right\} \text{for } 0 < n \leq 1 \text{ and} \tag{19}$$

$$E(\pi^F) = \beta/(1\text{-}\beta),\ E(\pi^R) = 0 \text{ at } n = 0. \tag{20}$$

Thus, for all values of $n > 0$, the fixed point expected profits are positive for the regression traders and weakly negative for fundamental traders. A discrete jump in profits occurs at $n = 0$, reflecting the benefits to even noisy information on η_{t+1}, when the price reflects only the public time t information.

3.2 Evolution in the population

Allow for a dynamic population proportion. The Replicator Dynamic produces an evolutionary dynamic population in which the dominant strategy attracts converts from the inferior strategy.[4] The two choice version of the more general K choice replicator dynamic model found in Branch and McGough (2003) results in the transition equation

$$n_{t+1} = \begin{cases} n_t + r(\pi_t^{Fe} - \pi_t^{Re})(1-n_t) \text{ for } \pi_t^{Fe} \geq \pi_t^{Re} \\ n_t + r(\pi_t^{Fe} - \pi_t^{Re})n_t \text{ for } \pi_t^{Fe} < \pi_t^{Re} \end{cases} \tag{21}$$

where π_t^{Fe} and π_t^{Re} indicate the traders' performance measures of fundamental and market-based approaches, respectively. These are updated according to the process

$$\pi_t^{ke} = \pi_{t-1}^{ke} + (\pi_{t-1}^{k} - \pi_{t-1}^{ke})/t\,, \tag{22}$$

$\pi_0^{ke} = 0$, $k = F, R$.

A number of different functional forms for r exist in the literature. The simulations that

[4] This is consistent with how GS envisions evolution in the population. "If the [expected utility of the informed] is greater than the [expected utility of the uninformed], some individuals switch from being uninformed to being informed (and conversely). An overall equilibrium requires the two to have the same expected utility." *p*394.

follow are based upon

$$r(x) = \tanh(\delta x / 2)), \tag{23}$$

a choice that ensures $0 < n_t < 1$ for bounded $\pi_t^{Fe} - \pi_t^{Re}$. The parameter δ determines how responsive the population is to differences in expected profits. As a result, by construction, the discontinuity of $n = 0$ will never be realized in the simulation.

From (19), given $\mathbf{c}_t = \mathbf{c}^*(n_t)$ there is no value of n_t that produced equal profits, thus there is no joint fixed point to both the learning and population processes. Without the assumption $\mathbf{c}_t = \mathbf{c}^*(n_t)$, the learning and the population processes must evolve together. In Figure 2, the 3-dimentianal phase space in n_t, c_{1t}, and c_{2t} has been collapsed to two dimensions by setting c_{2t} to be consistent with the c_{1t} parameter as indicated by (15). In the phase space plotted in Figure 2, the learning process updates the regression coefficients and thus moves the model vertically in the phase space. The population process creates an evolution in n_t and thus moves the model horizontally. The curve labeled "c_1^*" plots the n_t dependent fixed point values $c_1^*(n_t)$ that represent the correct c_1 for the given n_t (the same curve labeled "c_t^*" in the second row of Figure 1).

The curves $c_1^-(n)$ and $c_1^+(n)$ are the two outer boundaries between which the regression traders' parameters are sufficiently accurate to produce positive expected profits. Along $c_1^+(n)$ the regression traders over-react to price innovations. Expected profits are zero because over-reaction forces $p_t(\mathbf{c}_t^+, n) = p_t^{EM}$ so that $E(p_{t+1} + d_{t+1} - Rp_t)$ in (18) is equal to zero. Along $c_1^-(n)$ the regression traders under-react to price innovations producing $p_t(\mathbf{c}_t^-, n) = p_t^F$ and thus $q_t^F = q_t^R = 0$ in (18). Above $c_1^+(n)$ and below $c_1^-(n)$ the regression traders' beliefs are sufficiently in error to produce expected profits for the fundamental traders. At $c_{1t} = \overline{c}_1$, the regression traders' strong positive response to a price increases creates an upward sloping demand curve. The curve thus represents an upper bound on the value of c_{1t}.

Analysis suggests a convergence path between the curves c_1 and c_1^+ that, under the RD, leads to the point $n_t = 0$ and $c_{1t} = R$.

[Figure 2 about here]

3.3 Simulations

Simulations are necessary to determine convergence properties when allowing for the interaction between the learning and population processes. Figures 3 through 6 display the typical evolution of endogenous parameters produced by $\delta = 0.01$ and $\delta = 1$. Each frame plots the time progression of one of the endogenous parameters of the model. Across the top row are plotted the price parameters b_{1t} and ϕb_{2t}. The second row contains the regression coefficients from the regression traders' extraction, c_{1t} and c_{2t}. The bottom row presents the difference $p_t - p_t^{EM}$ and the population parameter n_t. The time series are plotted in green. The solid black lines show the asymptotic values that reflect the market under full efficiency. The red line in the c_{1t} and c_{2t} frames plots the respective component of $\mathbf{c}_t^*(n_t)$, the correct coefficient values for the regression traders given the n_t value.

The data plotted in Figures 3 is from the first 100,000 periods of a simulation based on $\delta = 0.01$ after dropping the first 500 observations. The parameter δ from (23) determines the rate of adjustment in the population towards the superior performing choice. The smooth progression of n_t is the result of a population reluctant to switch approaches. The slow moving target makes it easy for the regression parameters to keep pace with the changing market environment.

The declining cycles seen in Figure 4 (and more pronounced in Figure 6) results from the evolution in n_t outpacing the evolution in the regression coefficients. The higher $\delta = 1$ setting produces greater responsiveness in the trader population to differences in performance. Profits earned by the accuracy of the regression equation cause n_t to drop, but at a rate faster than the coefficients can keep pace. When n_t becomes too low for the contemporaneous regression parameters, the pricing error allows the fundamental traders to profit, temporarily reversing the progress in n_t.

For both $\delta = 0.01$ and $\delta = 1$ the system appears to be in the process of convergence towards $n_t = 0$. Figures 5 and 6 displays the parameter values from the same simulation during the $t = 1.5\text{x}10^6$ to $t = 2.0\text{x}10^6$ interval. The most striking characteristic is that despite the continued convergence of the population and learning process towards greater efficient markets, the price and the price coefficients do not seem to reflect this advancement.

The self-organization of the market is the interplay between learning, profits, and the population. Early in the simulation the regression coefficients quickly converged to reflect the concurrent value of n_t. In (17), errors in beliefs by the regression traders cause the first term to deviate from zero, but for large n_t, the second term dominates, becoming the determining factor for the failure of market efficiency. The accuracy of regression produces $p_t \approx p_t^*(n_t)$, but the regression traders market impact remains low so that $p_t^*(n_t)$ remains distant from p_t^{EM}.

The accuracy of the regression trader beliefs produce superior performance, driving n_t towards zero and $p_t^*(n_t) \rightarrow p_t^{EM}$. For small values of n_t, $\phi b_2^*(n) - \phi/(R-\phi)$ is approximately zero and yet the simulation produces a price consistently different from p_t^{EM}. The errors in $b_{1t} \neq b_1^*$ and $b_{2t} \neq b_2^*$ caused by $\mathbf{c}_t \neq \mathbf{c}^*(n_t)$ mean neither term of (17) is zero. At this stage further convergence in n_t towards zero does not noticeably improve the market efficiency, but it would be reasonable to think that continued learning by the regression traders should.

The moving target for $\mathbf{c}_t \to \mathbf{c}^*(n_t)$ produced by the progression of $\mathbf{c}_t \to \mathbf{c}^*(n_t)$ is a hinderance to convergence, but convergence occurs nonetheless. This endogenously produced error is not the underlying cause of the failure to observe $p_t \text{->} p_t^{EM}$. Instead, the decreasing n_t produces instability by increasing the need for accuracy on the part of the regression traders. From (14), assuming $\mathbf{c}_t = \mathbf{c}^*(n)$, $n_t \to 0$ produces $b_1^*(n) \to 0$ and $\phi b_2^* \to \phi/(R-\phi)$. Thus, as $n_t \to 0$, $p_t^*(n_t) \to p_t^{EM}$. Without the assumption of $\mathbf{c}_t = \mathbf{c}^*(n_t)$, an examination of (13) reveals that both the numerator and denominator of both $b_1(n_t, \mathbf{c}_t)$ and $b_2(n_t, \mathbf{c}_t)$ converge to zero as both $\mathbf{c}_t \to \mathbf{c}^*(n_t)$ and $n_t \to 0$; $b_1(n_t, \mathbf{c}_t) \to 0/0$ and $b_2(n_t, \mathbf{c}_t) \to 0/0$. The closer the true market is to efficiency, the greater the impact of a small error in the regression trader model on the price. The market is self-organized to remain inefficient even as $\mathbf{c}_t \to \mathbf{c}^*(n_t)$ and $n_t \to 0$.

Consider again the phase space in Figure 2. Recall that along $c_1^+(n)$, $p_t = p_t^{EM}$. The convergence of $c_1^+(n) - c_1^* \to 0$ is a reflection of (17), that the price produced by the correct beliefs of the regression traders produces the convergence $p_t \to p_t^{EM}$. Recall as well that along $c_1^-(n)$, $p_t = p_t^0$. The narrowing of the distance between $c_1^-(n)$ and $c_1^+(n)$ as $n_t \to 0$ indicates that the pricing mechanism's tolerance for error in the regression trader's beliefs narrows as the market becomes increasingly reliant on the regression traders to set the price. Small errors by the regression traders when n_t is near zero produced substantial pricing error.

4. Complete Information

4.1 Learning

In this section traders are allowed to employ both the public price and the private signal in determining a forecast of the value of the risky security. Traders are modeled as learning agents individually estimating the relationship

$$z_t = c_0 + c_{1it} p_{t-1} + c_{2it} \eta_{t-1} + c_{3it} s_{it} + \varsigma_{it} \tag{24}$$

et $\mathbf{c}_{it}$ = [c_0 c_{1it} c_{2it} c_{3it}] be the estimated individual regression coefficients, where x_t’ = [1 p_t η_t $(\eta_{t+1} + e_{i,t})$]. Individual demand is

$$q_{it} = (\mathbf{c}_{it} x_t - R p_t) / \gamma \sigma^2_{it} . \tag{25}$$

The equilibrium price solution that sets $\sum_i q_{it}$ = 0 takes the same linear structure as in (12),

$$p_t = b_0 + b_1(\mathbf{c}_t)\eta_t + b_2(\mathbf{c}_t)\eta_{t+1} , \tag{26}$$

but with

$$b_1(\mathbf{c}_t) = \Psi_t^{-1} \sum_i \theta_{it} c_{2it} \text{ , } b_2(\mathbf{c}_t) = \Psi_t^{-1} \sum_i \theta_{it} c_{3it} \text{ ,} \tag{27}$$

$$\Psi_t = \sum_i \theta_{it} (R - c_{1it}) .$$

All traders share the same time consistent beliefs at the fixed point. The fixed point for the learning process produces the regression coefficients

$$c_1^* = R \text{ , } c_2^* = c_3^* = 0 \text{ ,}$$

while the price and the price coefficients are indeterminate at the fixed point,

$$b_1^* = c_2^* / (R - c_1^*) \text{ , } b_2^* = c_3^* / (R - c_1^*) .$$

As the system approaches the fixed point, traders are able to rely heavily on the price for information indicating the value of d_{t+1}. They place almost no weight on their noisy private signal since the price is an almost perfect indicator of d_{t+1}.

At the fixed point, both price and private information are removed from the agents’ demand function and demand for the risky asset becomes zero. The No Trade solution is attained. The price is removed because the traders believe the market has converged to the point at which p_t

fully reveals d_{t+1}, removing the ability of the price to reveal profitable trades. At the same time, the noisy private information is completely dominated by p_t, and thus receives zero weight. In a GS type paradox, the reality of the fixed point is just the opposite of the traders' belief. The price contains no information, both because no one trades based on the d_{t+1} information and because any price clears the market.

4.2 Simulations

Simulations produce market behavior much like that produced by the Replicator Dynamics. The typical progression can be seen in Figures 7 and 8. The positive value of the coefficient c_{3t} indicates that the traders choose to rely on the fundamental signal. The convergence of c_{3t} towards zero suggests that the traders are able to reduce their reliance on the private signal as the price becomes an increasingly reliant indicator of d_{t+1}. The greater reliance on the price is reflected in the increasing value of c_{1t} towards its fixed point value of R. As previously obtained, the convergence of the regression coefficients towards the efficient markets values does not produce convergence in p_t towards p_t^{EM}.

5. Conclusion

The analysis explores the convergence of an asset market towards an Efficient Markets Equilibrium under two information and population environments. In one setting, the trader population is divided into those who trade based on fundamental information and those who rely on endogenous market information. The proportions are endogenous based on past performance. In the second, this division is removed, allowing each trader to employ both fundamental and market information to forecast payoffs. By construction, the fundamental information is noisy. When working, the price aggregates across individual beliefs filtering out the idiosyncratic noise thus making the price a potentially superior indicator of value.

Both version of the model develop to produce the same outcome. The ability of the price to reveal the underlying fundamental information makes it a useful source of information that increases in importance as the market evolves. When trader population is divided, superior performance by the market based information leads to declining use of fundamental information as traders switch. When traders make use of both information sources, the co-evolution of the market and of learning leads the traders to become increasingly dependent on the market information and place negligible (but still positive) weight on the private signal.

In both versions, increased use of the market information allows the endogenous parameters of the model to converge toward those producing an Efficient Market. Nonetheless, the implied convergence in the market price towards full efficiency fails to materialize. Though the market is constructed to preclude the existence of an equilibrium with an efficient price, this absence does not appear to be the driving force behind the failure to achieve market efficiency. The failure to produce an efficient price is the result of the interaction between the accuracy of the regression trader model and the extent to which the market relies on this model to form price. As the population of traders decreases its dependence on the fundamental information the tolerance for error in the perceived model also declines. Thus, as the traders become increasingly accurate in their extraction of information from the price, the market demand for accuracy increases as well. The result is that despite the convergence in accuracy, the distribution of the pricing error remains stable through out the simulation.

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Figure 1 Fixed point values of endogenous parameters based on *n*

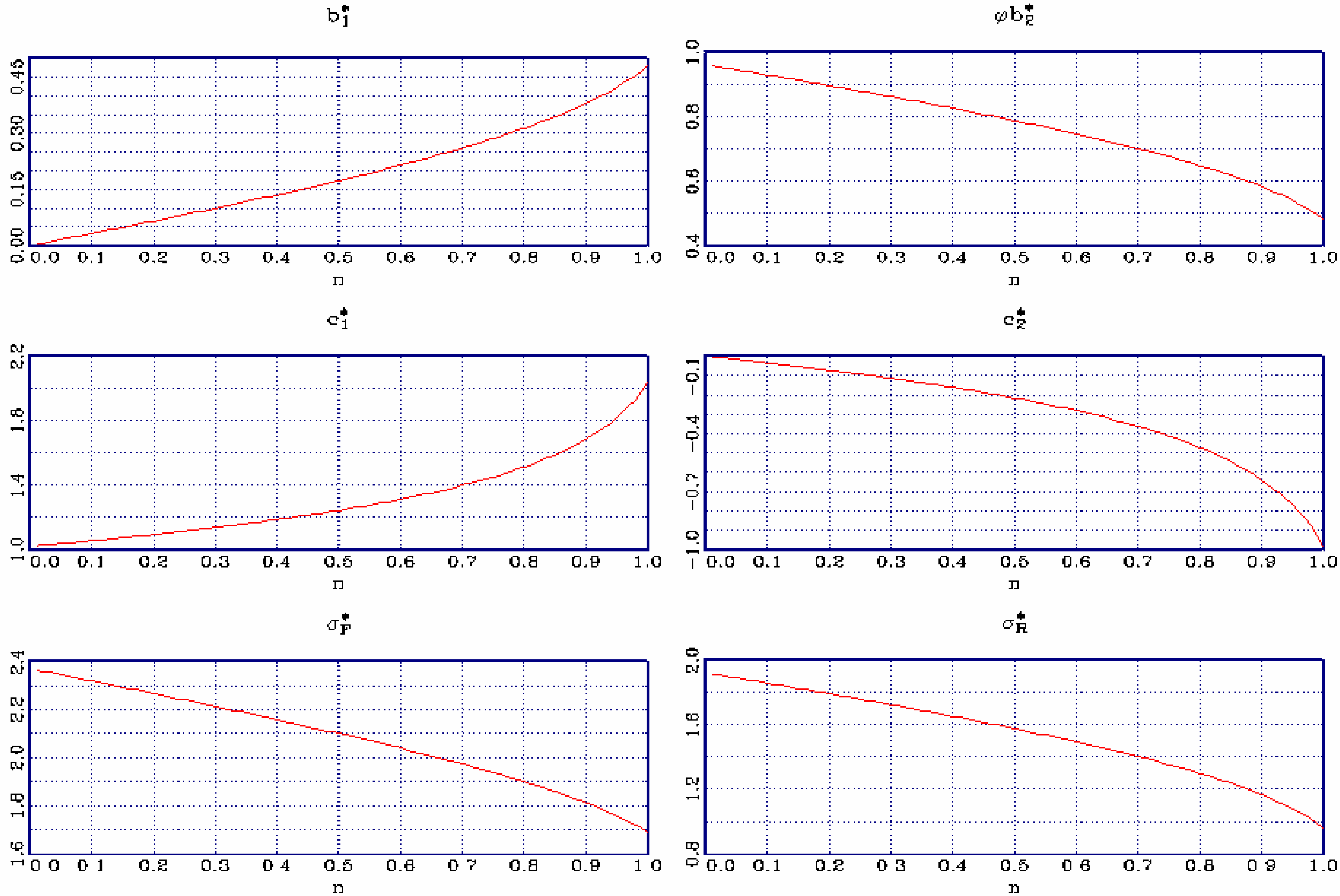

Figure 2: Phase Space in n_t and c_t^1.

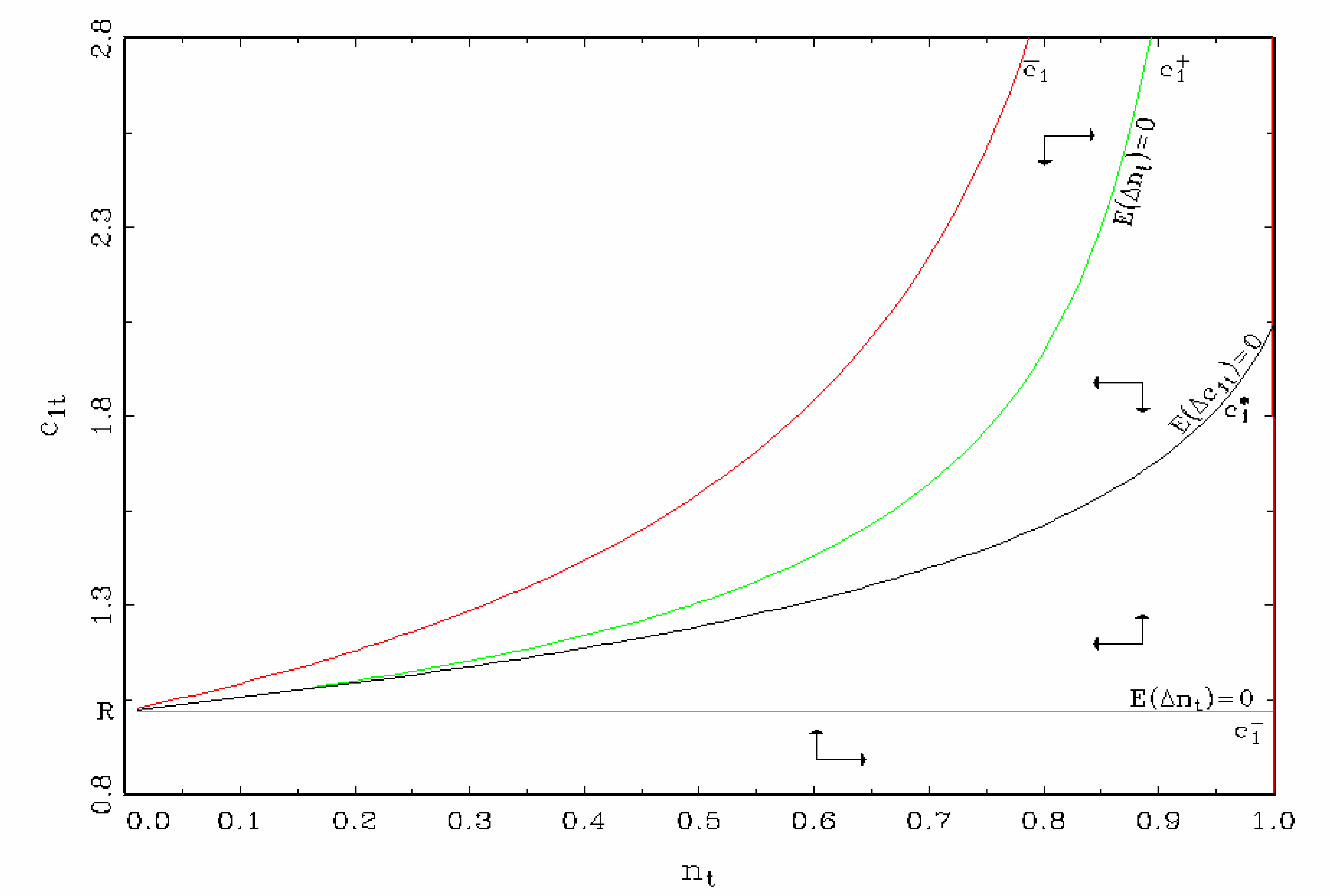

Figure 3: Evolution of endogenous parameters. Market 4: RD dynamics, δ = 0.01, *t* = 500 – 100,500

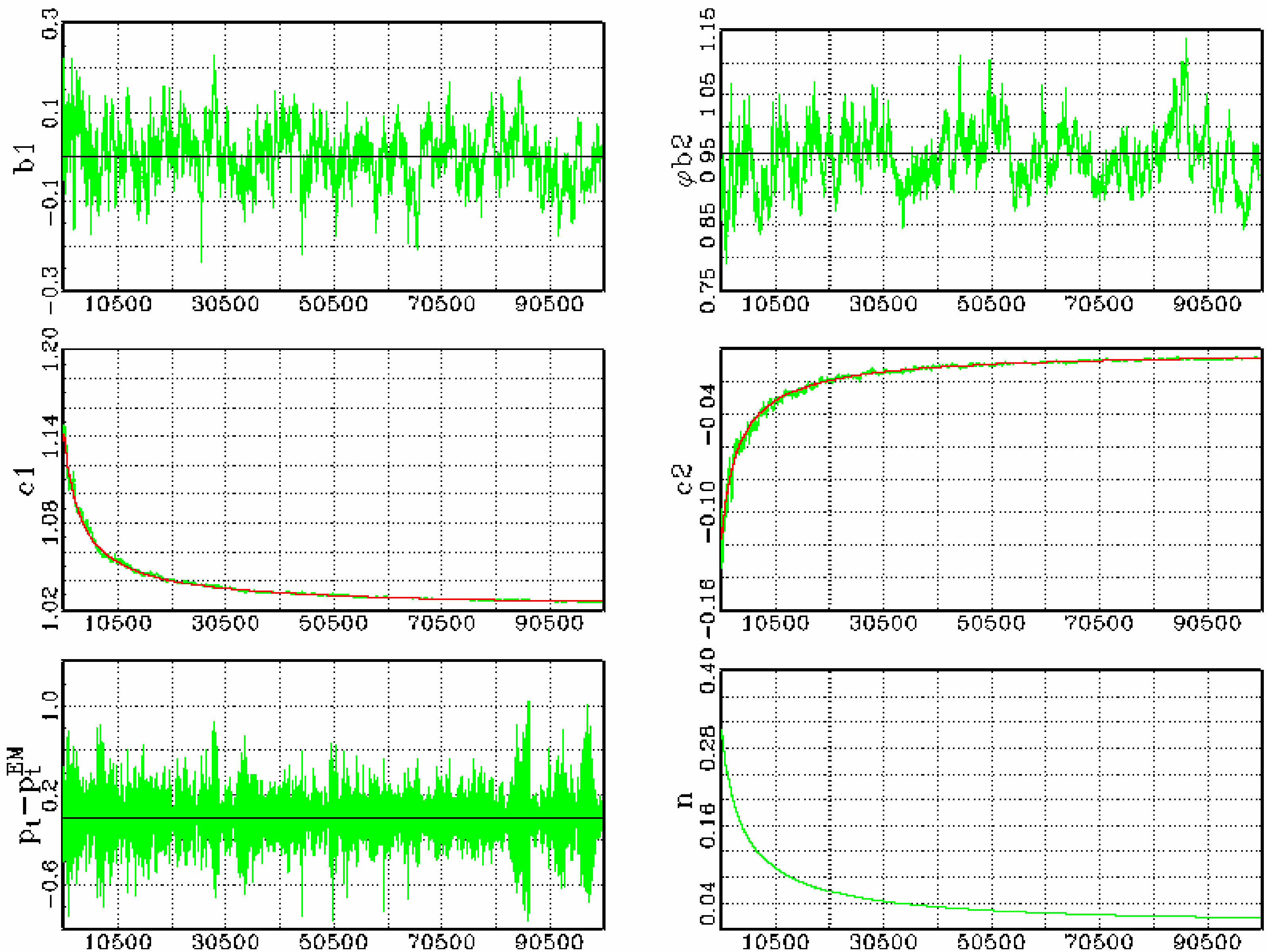

Figure 4: Evolution of endogenous parameters. Market 5: RD dynamics, δ = 1, t = 500 – 100,500.

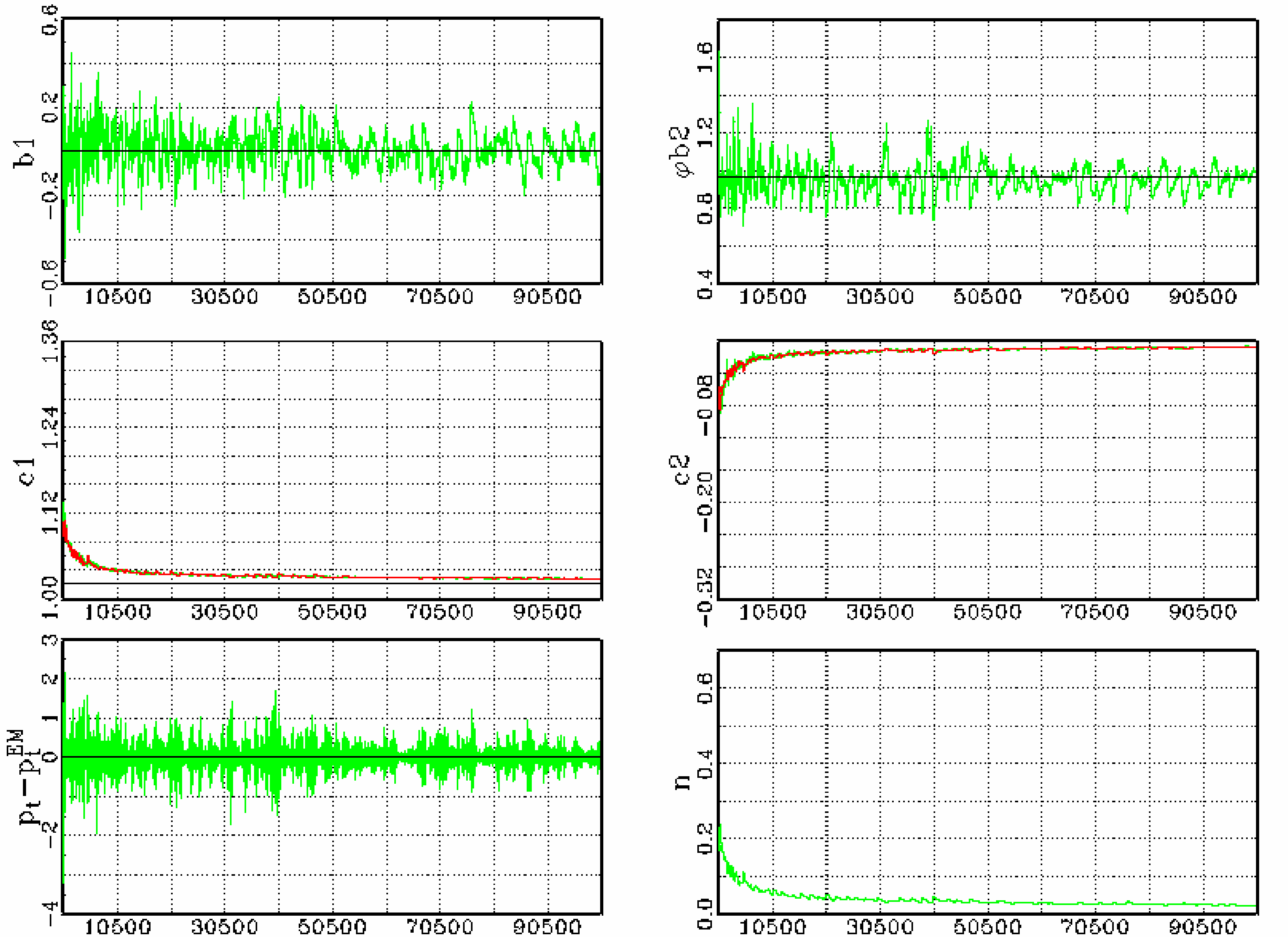

Figure 5: Evolution of endogenous parameters. Market 4: RD dynamics, $\delta = 0.01$, $t = 1.5 - 2.0\text{x}10^6$

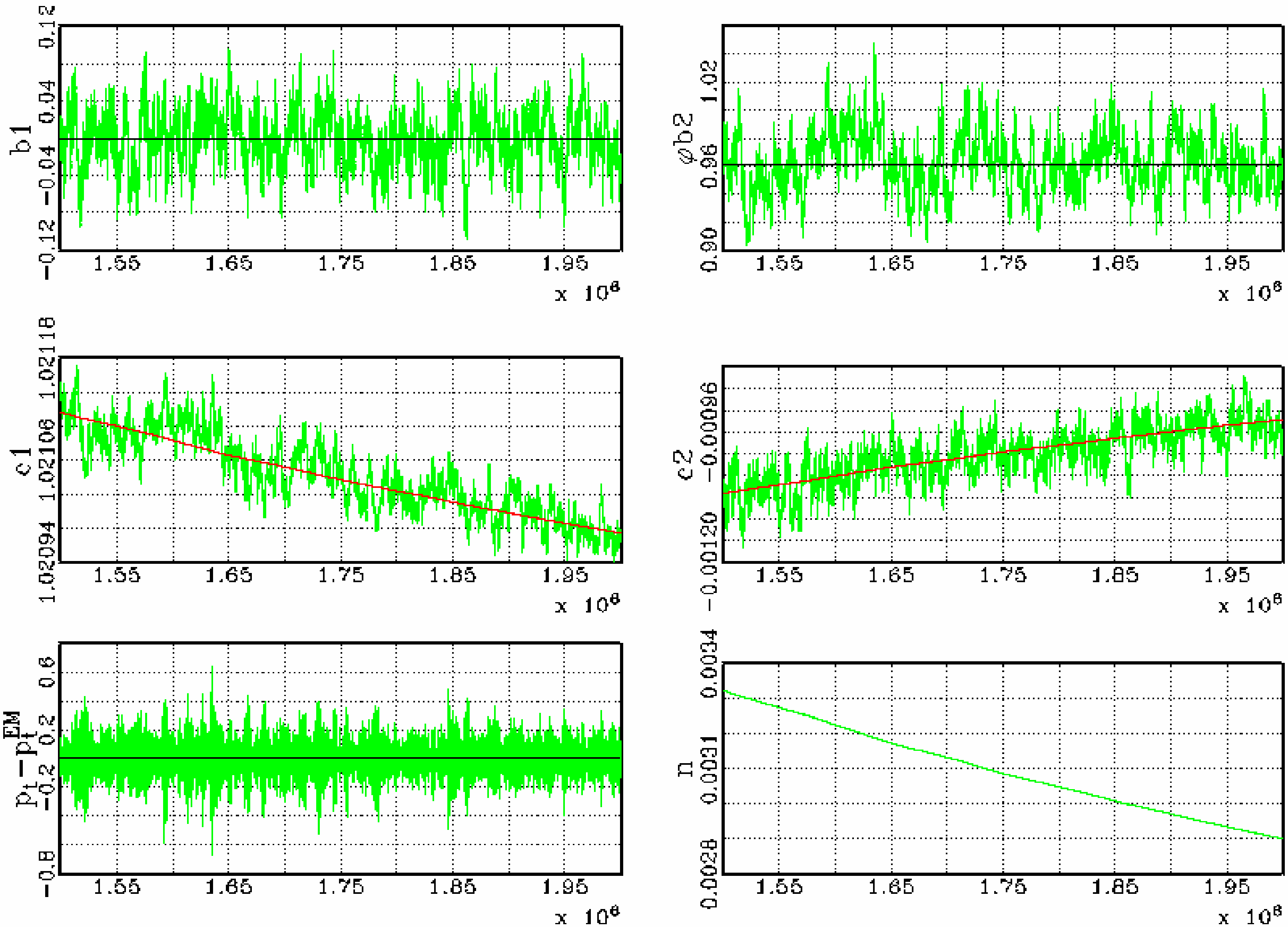

Figure 6: Evolution of endogenous parameters. Market 5: RD dynamics, $\delta = 1$, $t = 1.5 - 2.0\text{x}10^6$.

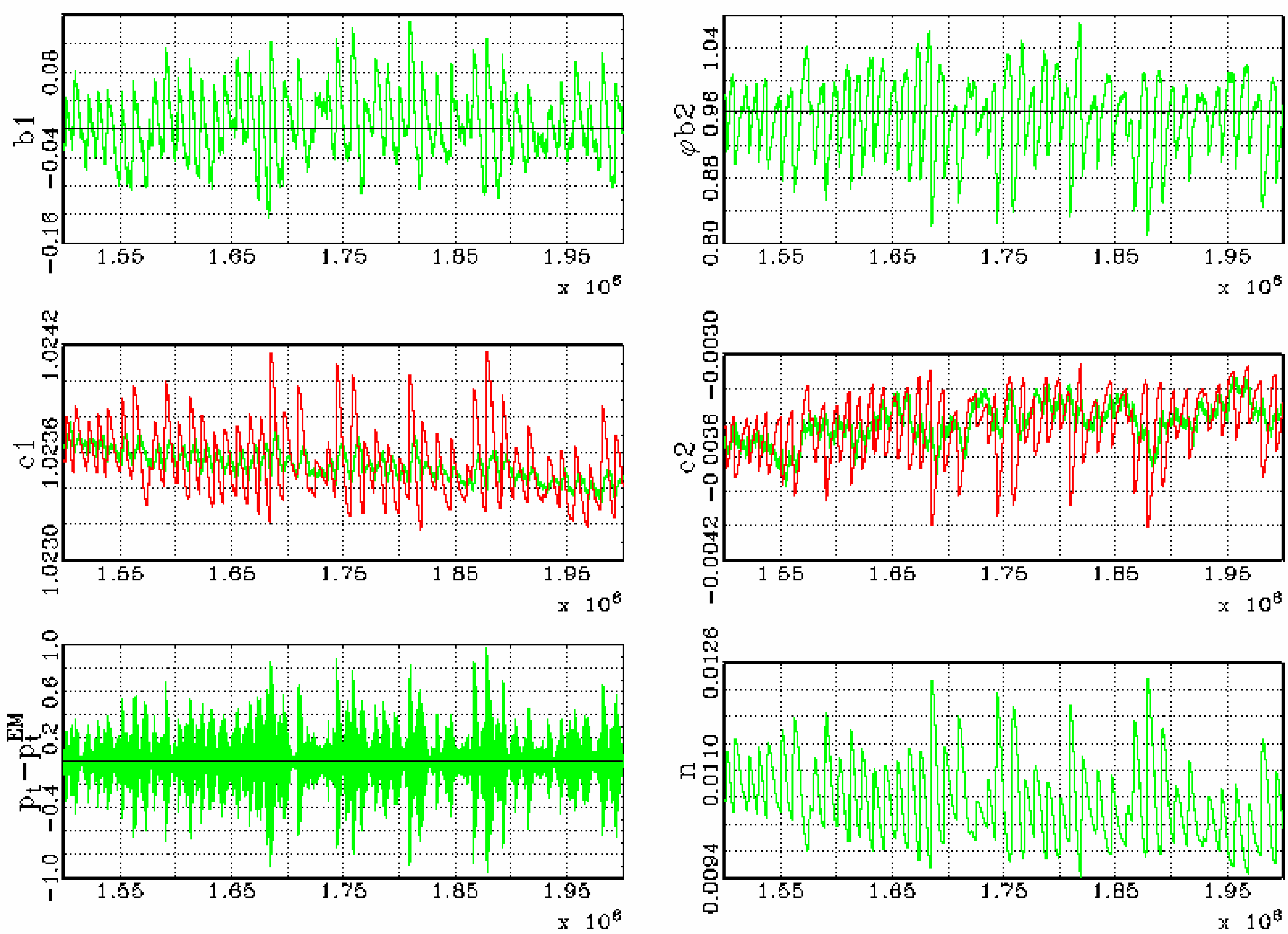

Figure 7: Evolution of endogenous parameters. Market 6: Complete Information, t = 500 – 100,500.

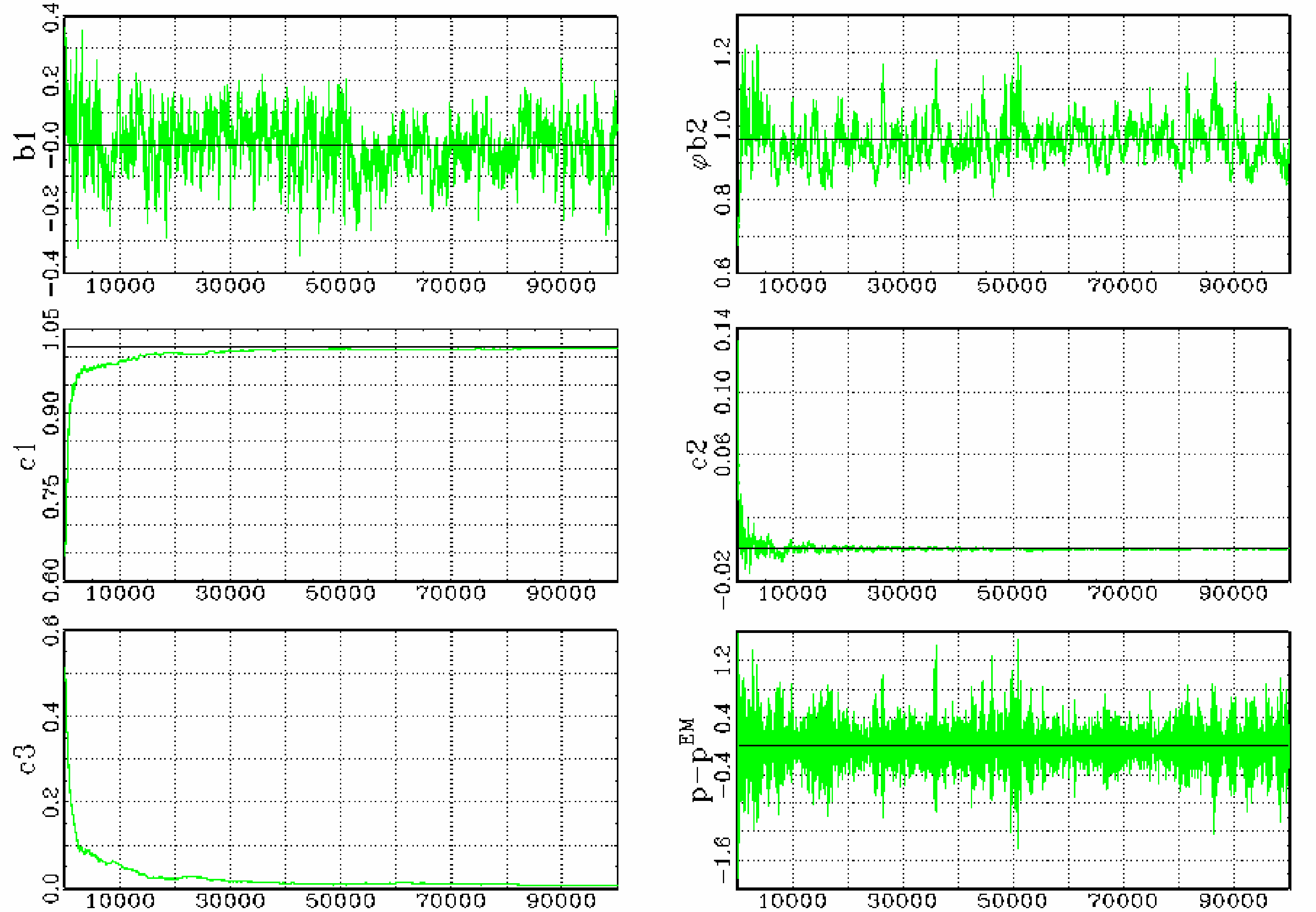

Figure 8: Evolution of endogenous parameters. Market 6: Complete Information, $t = 1.5 - 2.0\text{x}10^6$.