The Dynamic Process of Tax Reform

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Abstract

The tax reform literature, pioneered by Guesnerie [1977], uses static models but implicitly views tax reform as a dynamic process, i.e., as a policy-maker implementing incremental reforms over time. This paper studies tax reform in a dynamic version of the Diamond-Mirrlees-Guesnerie model and focuses on a specific aspect of the dynamic process, namely, the implications for tax reform of agents leaving bequests. The main idea is that a tax reform in one period will affect bequests and therefore endowments, equilibrium, and welfare in subsequent periods. Thus, the process of tax reform cannot be analyzed as a sequence of static economies; instead, the economies are linked by bequests. The paper undertakes a tax reform analysis *a la* Guesnerie, but with an added focus on individual welfare improving reforms for each generation. Second-best Pareto optima are then characterized, and these conditions are compared to the static optimal tax formulae derived in the literature. In particular, the key Diamond-Mirrlees result that production efficiency is desirable at second-best optima no longer holds in the presence of (effective) restrictions on the taxation of private savings. Restrictions on government savings (including balanced budget restrictions), however, do not disturb the desirability of production efficiency. Finally, the main results of the paper are shown to be robust to certain political constraints on the tax reform process.

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1. Introduction

The relevance of optimal tax formulae, as developed by Diamond and Mirrlees [1971] among others, has been questioned on several grounds. In particular, the implementation of such formulae would likely require a major upheaval of the existing tax system, which policy-makers are unwilling or unable to do. In practice, actual changes are 'slow and piecemeal'. This has shifted the emphasis from 'tax design' to 'tax reform', see especially Guesnerie [1977, 1995]. The tax reform literature takes the existing tax system as given, and examines the conditions under which there exist (differential) changes in taxes that are equilibrium preserving and Pareto improving, i.e., feasible reforms that increase the welfare of every agent in the economy.[1]

Despite using a static model, it is implicit that Guesnerie [1977, 1995] views tax reform as a dynamic process, i.e., as a policy-maker implementing incremental reforms over time. The dynamic aspect is analyzed explicitly in Fogelman, Quinzii, and Guesnerie [1978], who focus on the standard technical questions raised by dynamic systems: existence and stability. The present paper, however, studies a specific aspect of the dynamic process, namely, the implications for tax reform of agents leaving bequests. The main idea is fairly straightforward: a tax reform in one period will affect bequests and therefore endowments, equilibrium, and welfare in subsequent periods. Thus, the process of tax reform cannot be analyzed as a sequence of static economies; instead, the economies are dynamically linked by bequests. Accordingly, the taxation of savings

[1] The optimal tax literature has also been criticized for its reliance on the existence of a social welfare function. Tax reform analysis does not require a social welfare function to be specified, although it can be helpful in some aspects.

plays an important role, and the government can transfer resources through time by running (temporary) budget deficits and surpluses. These policy instruments are not available to the government in static models, in which there are no savings and the government's budget must always be balanced.

The paper employs a simple three-period model (the finite horizon prevents the government from running a Ponzi scheme). If the government undertakes a policy reform in period 1 bequests are likely to change, and therefore endowments, equilibrium, and welfare in subsequent periods will be disturbed. Thus, the government may be obligated to undertake reforms in periods 2 and 3, at least to restore equilibrium. Similarly, if the government does not reform in period 1 but does so in period 2, equilibrium and welfare in period 3 will be disturbed via the bequest mechanism. The model is described in greater detail in Section 2.

Section 3 undertakes a tax reform analysis *a la* Guesnerie [1977, 1995]. That is, we address the central questions of tax reform analysis: (i) does there exist an equilibrium preserving reform that makes a particular generation better-off?, and the usual question: (ii) does there exist an equilibrium preserving reform that makes every agent (generation) better-off? This provides the basis for a characterization of second-best Pareto optima in Section 4, i.e., conditions under which there does not exist a feasible reform that makes every generation better-off are derived. Special attention is paid to the government's debt instruments and the taxation of savings, as these instruments are not available to the government in static models.

Section 5 examines the implications of some possible constraints on the policy reform process. The first type of constraint considered is restrictions on changes in some of the government's policy instruments — specifically, restrictions on the taxation of savings and the government's ability to issue debt (say because the legislature will not approve such changes). The second type of constraint considered is a political constraint — that the welfare of generation 1 must not be reduced in the process of implementing a policy reform (say by the need of the government to get re-elected). The main finding is that the key Diamond-Mirrlees result that production efficiency is desirable at second-best optima no longer holds in the presence of (effective) restrictions on the taxation of private savings. Restrictions on government savings (including period-by-period balanced budget restrictions), however, do not disturb the desirability of production efficiency. Regarding the political constraint, it is shown to have a surprisingly straightforward effect on the conditions for second-best optimality. Section 6 contains some concluding remarks, and summarizes the main results of the paper.

2. The Model

The model has three periods, with consumers living for one period. There is a single representative consumer in each period, so the terms 'consumer' and 'generation' are used interchangeably. The dynamic structure of the model is broadly similar to the finite horizon OLG model of Blackorby and Brett [1999], but the key feature of the present model is that each generation cares about its own welfare as well as that of the following generation. This altruism materializes as a bequest. There are three types of bequest. The first is a storable capital good which forms the basis for the capital stock and can be purchased from the firm. The second is a bond which can be purchased from the government. From the point of view of each generation, the capital good and bond are perfect substitutes, as both shift-out the budget constraint of the recipient generation. Competition between the government and firm for private savings ensures that the prices of the capital good and government bond are the same. The third type of bequest is a human capital good, or educational bequest, which increases the effective labor supply of the recipient generation and therefore tilts its budget constraint. Thus, depending upon preferences and prices, each generation can choose to leave a capital bequest which simply boosts the income of the recipient, and/or an educational bequest which boosts the recipient's effective labor supply. The details of the model are now described.

2.1. Consumers

Generation 1 chooses its net (of endowment) consumption vector of non-storable goods $x^1 \in \mathbb{R}^n$, its leisure time $l^1 \in [0, 1]$, its capital bequest $\kappa^1 := k^1 + b^1 \in \mathbb{R}_+$, and its educational bequest $e^1 \in \mathbb{R}_+$, to solve the following program:

$$V^1(\pi^1, \rho^1, \alpha^1, \omega^1, R) = \max U^1(x^1, l^1, \kappa^1, e^1)$$

$$\text{subject to: } \pi^1 x^1 + \rho^1 \kappa^1 + \alpha^1 e^1 \le \omega^1 (1 - l^1) + R \tag{2.1}$$

where superscripts indicate the time period and corresponding generation, $V(\cdot)$ is the indirect utility function, $\pi \in \mathbb{R}^n_+$ is the consumer price vector corresponding to the net consumption vector x, $\rho \in \mathbb{R}_+$ is the consumer price of the capital good k and government bond b (which is the mechanism for leaving capital bequests), $\alpha \in \mathbb{R}_+$ is the consumer price of the educational good e (which is the mechanism for leaving educational bequests), $\omega \in \mathbb{R}_+$ is the wage (or consumer price of leisure), where the endowment of leisure is normalized to unity so that $(1 - l^1)$ is labor supplied, $R \in \mathbb{R}$ is a lump-sum tax, which is restricted to be the same for each generation,[2] and $U(\cdot)$ is the direct utility function. The restrictions $\kappa^1 \in \mathbb{R}_+$ and $e^1 \in \mathbb{R}_+$ ensure that generation 1 cannot die in debt. However, the purchase of government bonds can be positive or negative, depending upon whether the government is running a budget deficit or surplus.

The inclusion of bequests in generation 1's direct utility function is simpler (and arguably more realistic) than the usual assumption made regarding bequests in OLG models — that each generation knows the utility functions of the following generations and acts as if maximizing an infinite horizon utility function.[3] Program (2.1) simply says that generation 1 divides its income between purchases of non-storable goods for itself and bequests for generation 2 to maximize a utility function which represents generation 1's preferences over its own consumption and (in some sense) that of generation 2. In this framework, bequests provide a positive externality to the recipient generation, as they boost the recipient's income and indirect utility (see below).

Similarly, generation 2 chooses its net consumption vector $x^2 \in \mathbb{R}^n$, its leisure time $l^2 \in [0, 1]$, its capital bequest $\kappa^2 \in \mathbb{R}_+$, and its educational bequest $e^2 \in \mathbb{R}_+$, to solve the following program:

$$V^2(\pi^2, \rho^2, \alpha^2, \omega^2, R, \sigma^2, \kappa^1, e^1) = \max U^2(x^2, l^2, \kappa^2, e^2)$$

[2] R is sometimes called a poll tax or demogrant. The restriction that R be the same for each generation makes the model second-best, as the government does not have access to personalized lump-sum transfers.
[3] See chapter 3 in Blanchard and Fischer [1989]. However, Hu [1979] for example also includes bequests in the direct utility function.

$$\text{subject to: } \pi^2 x^2 + \rho^2\kappa^2 + \alpha^2 e^2 \leq \omega^2(1-l^2)(1+e^1) + R + \sigma^2\kappa^1 \qquad (2.2)$$

where $(1-l^2)(1+e^1)$ is generation 2's effective labor supply, and $\sigma^2 \in \mathbb{R}_+$ is the price generation 2 receives from selling the capital bequest it obtains from generation 1 to the firm (see below) and government at the beginning of the second period.

Generation 3 does not leave a bequest, as it is the last generation. Alternatively, generation 3 can be thought of as a purely non-altruistic generation. It chooses its net consumption vector $x^3 \in \mathbb{R}^n$ and leisure time $l^3 \in [0, 1]$ to solve the following program:

$$V^3(\pi^3, \omega^3, R, \sigma^3, \kappa^2, e^2) = \max U^3(x^3, l^3)$$

$$\text{subject to: } \pi^3 x^3 \leq \omega^3(1-l^3)(1+e^2) + R + \sigma^3\kappa^2 \qquad (2.3)$$

Note that generation 1 can be thought of as the 'start-up' generation, as it does not receive a bequest from the (non-existent) previous generation. Similarly, generation 3 is the 'shut-down' generation, as it does not leave a bequest. On the other hand generation 2, which both receives and leaves bequests, is of the usual type studied in infinite horizon models. Also note that it is implicit in the above formulation that the government taxes away all pure profits, as the consumers have no profit income. The assumption of a 100 percent tax on profits, or constant returns to scale in production which implies zero maximal profits, is standard in the optimal tax and tax reform literatures.

2.2. Prices and Taxes

Consumer prices are disconnected from producer prices by a complete system of specific taxes. That is:

$$\pi^1 = p^1 + t_x^1 \qquad \pi^2 = p^2 + t_x^2 \qquad \pi^3 = p^3 + t_x^3$$

$$\rho^1 = r^1 + t_k^1 \qquad \rho^2 = r^2 + t_k^2$$

$$\alpha^1 = a^1 + t_e^1 \qquad \alpha^2 = a^2 + t_e^2 \tag{2.4}$$

$$\omega^1 = w^1 + t_l^1 \qquad \omega^2 = w^2 + t_l^2 \qquad \omega^3 = w^3 + t_l^3$$

$$\sigma^2 = s^2 + \tau^2 \qquad \sigma^3 = s^3 + \tau^3$$

where $p \in \mathbb{R}_+^n$ is the producer price vector corresponding to x, $r \in \mathbb{R}_+$ is the price the firm receives from selling k, $a \in \mathbb{R}_+$ is the price the firm receives from selling e, $w \in \mathbb{R}_+$ is the price the firm pays for hiring effective labor, and $s \in \mathbb{R}_+$ is the price the firm pays for purchases of k. Taxes (or subsidies) are simply the difference between consumer prices and producer prices.

2.3. Production

The supply side of the economy consists of a single, aggregate, profit maximizing firm whose technology changes over time, making this formulation consistent with any assumptions regarding capital depreciation or technical progress. The firm's profit functions are given by $\psi_1(p^1, r^1, a^1, w^1)$, $\psi_2(p^2, r^2, a^2, w^2, s^2)$, and $\psi_3(p^3, w^3, s^3)$. Application of Hotelling's Theorem to these profit functions yields the firm's supply and demand functions:

$$x_1 = x_1(p^1, r^1, a^1, w^1) \qquad x_2 = x_2(p^2, r^2, a^2, w^2, s^2) \qquad x_3 = x_3(p^3, w^3, s^3)$$

$$k_1 = k_1(p^1, r^1, a^1, w^1) \qquad k_2 = k_2(p^2, r^2, a^2, w^2, s^2)$$

$$e_1 = e_1(p^1, r^1, a^1, w^1) \qquad e_2 = e_2(p^2, r^2, a^2, w^2, s^2) \tag{2.5}$$

$$L_1 = L_1(p^1, r^1, a^1, w^1) \qquad L_2 = L_2(p^2, r^2, a^2, w^2, s^2) \qquad L_3 = L_3(p^3, w^3, s^3)$$

$$K_2 = K_2(p^2, r^2, a^2, w^2, s^2) \qquad K_3 = K_3(p^3, w^3, s^3)$$

where the upper-case L and K are used to denote the firm's demands for effective labor and capital, i.e., it purchases capital from the consumers at the beginning of periods 2 and 3. The firm does not purchase any capital in period 1, as there is no supply by consumers, nor does it produce any capital in period 3, as there is no demand by consumers. Note that subscript notation indicates supply or demand by the firm.

2.4. General Competitive Equilibrium

Equilibrium is obtained if and only if:

Period 1:

$$x^1(\pi^1, \rho^1, \alpha^1, \omega^1, R) - x_1(p^1, r^1, a^1, w^1) \leq 0^{(n)} \tag{2.6}$$

$$k^1(\pi^1, \rho^1, \alpha^1, \omega^1, R) - k_1(p^1, r^1, a^1, w^1) \leq 0 \tag{2.7}$$

$$b^1(\pi^1, \rho^1, \alpha^1, \omega^1, R) - b_1 \leq 0 \tag{2.8}$$

$$e^1(\pi^1, \rho^1, \alpha^1, \omega^1, R) - e_1(p^1, r^1, a^1, w^1) \leq 0 \tag{2.9}$$

$$L_1(p^1, r^1, a^1, w^1) - [1 - l^1(\pi^1, \rho^1, \alpha^1, \omega^1, R)] \leq 0 \tag{2.10}$$

Period 2:

$$x^2(\pi^2, \rho^2, \alpha^2, \omega^2, R, \sigma^2, \kappa^1, e^1) - x_2(p^2, r^2, a^2, w^2, s^2) \leq 0^{(n)} \tag{2.11}$$

$$k^2(\pi^2, \rho^2, \alpha^2, \omega^2, R, \sigma^2, \kappa^1, e^1) - k_2(p^2, r^2, a^2, w^2, s^2) \leq 0 \tag{2.12}$$

$$b^2(\pi^2, \rho^2, \alpha^2, \omega^2, R, \sigma^2, \kappa^1, e^1) - b_2 \leq 0 \tag{2.13}$$

$$e^2(\pi^2, \rho^2, \alpha^2, \omega^2, R, \sigma^2, \kappa^1, e^1) - e_2(p^2, r^2, a^2, w^2, s^2) \leq 0 \tag{2.14}$$

$$L_2(p^2, r^2, a^2, w^2, s^2) - [1 - l^2(\cdot)][1 + e^1(\cdot)] \leq 0 \tag{2.15}$$

$$K_2(p^2, r^2, a^2, w^2, s^2) - k^1(\pi^1, \rho^1, \alpha^1, \omega^1, R) \leq 0 \tag{2.16}$$

$$B_2 - b^1(\pi^1, \rho^1, \alpha^1, \omega^1, R) \leq 0 \quad (2.17)$$

Period 3:

$$x^3(\pi^3, \omega^3, R, \sigma^3, \kappa^2, e^2) - x_3(p^3, w^3, s^3) \leq 0^{(n)} \quad (2.18)$$

$$L_3(p^3, w^3, s^3) - [1 - l^3(\cdot)][1 + e^2(\cdot)] \leq 0 \quad (2.19)$$

$$K_3(p^3, w^3, s^3) - k^2(\pi^2, \rho^2, \alpha^2, \omega^2, R, \sigma^2, \kappa^1, e^1) \leq 0 \quad (2.20)$$

$$B_3 - b^2(\pi^2, \rho^2, \alpha^2, \omega^2, R, \sigma^2, \kappa^1, e^1) \leq 0 \quad (2.21)$$

where b_1 and b_2 denote the supply of bonds by the government in periods 1 and 2, and B_2 and B_3 denote the government's demand (repurchases) of bonds in periods 2 and 3. The model's finite horizon prevents the government from engaging in some sort of Ponzi game, with Walras' Law ensuring that the government's budget is balanced over the three periods. Note that the periods are linked by the demand for and supply of capital, government bonds, and the educational good (human capital). For analytical purposes, it is assumed that the status quo is a 'tight' equilibrium, i.e., equations (2.6) − (2.21) all hold with equality.

3. Characterizing Reform Possibilities

Apart from the fixed 100 percent tax on profits, the government's policy instruments are the taxes on commodities, bequests, and labor, as well as the bonds and demogrant. As the government does not supply nor demand commodities, it uses its instruments only to redistribute income (or to pick equilibria in the sense of the Second Theorem of Welfare Economics), not to finance government services.

A *policy reform process* can be defined as a vector $\mathrm{d}P := \langle \mathrm{d}P^1, \mathrm{d}P^2, \mathrm{d}P^3, \mathrm{d}R\rangle$ where:

$$\mathrm{d}P^1 := \langle \mathrm{d}p^1,\ \mathrm{d}t_x^1,\ \mathrm{d}r^1,\ \mathrm{d}t_k^1,\ \mathrm{d}a^1,\ \mathrm{d}t_e^1,\ \mathrm{d}w^1,\ \mathrm{d}t_l^1,\ \mathrm{d}b_1\rangle$$

corresponds to changes in the government's first-period instruments, and similarly:

$$\mathrm{d}P^2 := \langle \mathrm{d}p^2,\ \mathrm{d}t_x^2,\ \mathrm{d}r^2,\ \mathrm{d}t_k^2,\ \mathrm{d}a^2,\ \mathrm{d}t_e^2,\ \mathrm{d}w^2,\ \mathrm{d}t_l^2,\ \mathrm{d}s^2,\ \mathrm{d}\tau^2,\ \mathrm{d}b_2,\ \mathrm{d}B_2\rangle$$

$$\mathrm{d}P^3 := \langle \mathrm{d}p^3,\ \mathrm{d}t_x^3,\ \mathrm{d}w^3,\ \mathrm{d}t_l^3,\ \mathrm{d}s^3,\ \mathrm{d}\tau^3,\ \mathrm{d}B_3\rangle$$

correspond to changes in the government's second- and third-period instruments.

A reform $\mathrm{d}P$ satisfies the equilibrium conditions (2.6) − (2.21) if and only if $\nabla Z\mathrm{d}P \leq 0^{(3n+13)}$, where ∇Z is a Jacobian matrix of excess-demand derivatives and is defined explicitly in the Appendix. The welfare of generation 1 increases if and only if $\nabla V^1\mathrm{d}P > 0$, where ∇V^1 is essentially the gradient of generation 1's indirect utility function (see the Appendix for details). Analogously, the welfares of generations 2 and 3 increase if and only if $\nabla V^2\mathrm{d}P > 0$ and $\nabla V^3\mathrm{d}P > 0$.

3.1. Equilibrium Preserving and Welfare Improving Reforms

Equilibrium preserving and welfare improving reforms are now characterized *a la* Guesnerie [1977: Proposition 4], Guesnerie [1995: Theorem 1, p.145], and more recently Murty and Russell [2003: Theorem 1]. Specifically, we want to know if there exists a feasible reform that makes a particular generation better-off, and if there exists a feasible reform that makes every generation better-off (which is the usual question addressed in the tax reform literature). The answer is given by Proposition 1 and its corollaries.

Proposition 1:

Let $\Gamma \subseteq \mathbb{R}^{6n+23}$ be the cone generated by taking non-negative linear combinations of the rows of ∇Z. If $\nabla V^i \in \Gamma$, then there does not exist an equilibrium preserving policy reform process that makes generation i better-off. Conversely, if $\nabla V^i \notin \Gamma$, then there exists an equilibrium preserving policy reform process that makes generation i better-off.

Proof:

By Farkas' Theorem (see Appendix), there exists a $\mu \in \mathbb{R}_+^{3n+13}$ such that $\mu\nabla Z = \nabla V^i$, or there exists a dP such that $\nabla Z\mathrm{d}P \leq 0^{(3n+13)}$ and $\nabla V^i\mathrm{d}P > 0$. Next, note that $\nabla V^i \in \Gamma$ implies that there must exist a $\mu \in \mathbb{R}_+^{3n+13}$ such that $\mu\nabla Z = \nabla V^i$. Conversely, $\nabla V^i \notin \Gamma$ implies that there cannot exist a $\mu \in \mathbb{R}_+^{3n+13}$ such that $\mu\nabla Z = \nabla V^i$. ■

Following the existing tax reform literature, we are particularly interested in the possibility of equilibrium preserving and Pareto improving reforms. In the present context, these are those (ideal) reforms which are feasible and improve the welfare of every generation.

Corollary 1.1:

If $\nabla V^1 \in \Gamma$ or $\nabla V^2 \in \Gamma$ or $\nabla V^3 \in \Gamma$, then there does not exist an equilibrium preserving and Pareto improving policy reform process. That is, the economy is at a local second-best optimum.

Corollary 1.2:

Suppose $\Gamma \cap -\Gamma = \{0^{(6n+23)}\}$, i.e., Γ is pointed. Let $-P(\Gamma)$ be the negative polar cone of Γ. If $-P(\Gamma) \subseteq -\Gamma$ and $\nabla V^1 \in -\Gamma$ and $\nabla V^2 \in -\Gamma$ and $\nabla V^3 \in -\Gamma$, then there exists *an equilibrium preserving and Pareto improving policy reform process. That is, the economy is not at a local second-best optimum.*

Corollary 1.3:

Suppose $\Gamma \cap -\Gamma = \{0^{(6n+23)}\}$, *i.e.,* Γ *is pointed. Let* $-P(\Gamma)$ *be the negative polar cone of* Γ. *If* $-\Gamma \subseteq -P(\Gamma)$ *and* $\nabla V^1 \in -P(\Gamma)$ *and* $\nabla V^2 \in -P(\Gamma)$ *and* $\nabla V^3 \in -P(\Gamma)$, *then there exists an equilibrium preserving and Pareto improving policy reform process. That is, the economy is not at a local second-best optimum.*

Corollary 1.4:

If $\nabla V^1 \notin \Gamma$ *and* $\nabla V^2 \notin \Gamma$ *and* $\nabla V^3 \notin \Gamma$, *and at least one of these vectors is an element of the complement of* $\Gamma \cup -P(\Gamma) \cup -\Gamma$ *(relative to* $\mathbb{R}^{6n+23}$*), then there may or may not exist an equilibrium preserving and Pareto improving policy reform process.*

Some intuition for Proposition 1 and its corollaries can be obtained by reference to Figure 1. The top panel of Figure 1 illustrates an example in which there exists an equilibrium preserving reform that makes generation 1 better-off, but generations 2 and 3 are made worse-off. A reform dP is equilibrium preserving if it forms an obtuse angle with all the rows of ∇Z. Letting Γ denote the cone generated by taking non-negative linear combinations of the rows of ∇Z, the set of equilibrium preserving reforms is therefore $-P(\Gamma)$, i.e., the negative polar cone of Γ. A reform dP makes generation i better-off if dP forms a strictly acute angle with the gradient of generation i's indirect utility function ∇V^i. In the top panel of Figure 1, the shaded area is the set of equilibrium preserving reforms which make generation 1 better-off. However, all equilibrium preserving reforms make generations 2 and 3 worse-off, as they all form obtuse angles with ∇V^2 and ∇V^3.

Further insight into Proposition 1 can be obtained by considering the hypothetical problem of choosing the policy instruments P to maximize the welfare of some generation i subject to the constraints (2.6) – (2.21). Suppose $\tilde{P}$ solves this problem, then by the Kuhn-Tucker Theorem there exists a $\tilde{\mu} \in \mathbb{R}_{+}^{3n+13}$ such that:

$$\tilde{\mu}\nabla Z = \nabla V^i \qquad \text{(KT)}$$

where $\tilde{\mu}$ is a vector of Kuhn-Tucker multipliers. If $\nabla V^i \notin \Gamma$, then there cannot exist a vector $\tilde{\mu} \in \mathbb{R}_{+}^{3n+13}$ such that (KT) is satisfied. That is, the status quo equilibrium does not 'look like' the policy instruments have already been chosen to maximize the welfare of generation i. In this case, there must exist a feasible reform which makes generation i better-off. Conversely, if there does not exist a feasible reform which makes generation i better-off, then (KT) is satisfied and the status quo equilibrium 'looks like' the policy instruments have already been chosen to maximize the welfare of generation i.

In order for there to exist an equilibrium preserving and Pareto improving reform, the vectors ∇V^1, ∇V^2, and ∇V^3 must be 'close enough' in the sense made precise by Corollaries 1.1 – 1.4. The bottom panel of Figure 1 illustrates an example in which there exists an equilibrium preserving and Pareto improving reform. In this example ∇V^1, ∇V^2, and ∇V^3 are all elements of $-\Gamma$, which is a sufficient condition for there to exist an equilibrium preserving and Pareto improving reform (refer Corollary 1.2).

The usefulness of Proposition 1 and its corollaries is that it forms the basis for the characterization of second-best Pareto optima (see below), and it provides an empirically implementable methodology for identifying the possibility of welfare improving reforms. That is, the criteria of Proposition 1 and its corollaries can be checked empirically by constructing the cone Γ and with knowledge of the consumers' consumption vectors. The construction of Γ requires empirical estimates (or assumptions) of various excess-demand derivatives, while survey data are available which provide information on household consumption. In sum, the information requirements of Proposition 1 and its corollaries can be met. Examples of empirical applications of tax reform theory, following the work of Guesnerie [1977], are King [1983] and Ahmad and Stern [1984].

4. Second-Best Pareto Optima

As noted by Guesnerie [1977] and Weymark [1979], necessary conditions for the non-existence of an equilibrium preserving and Pareto improving reform are, in fact, necessary conditions for the initial position to be Pareto optimal. In other words, these conditions characterize second-best optima as in the optimal tax literature, and are derived in the following proposition.

Proposition 2:

Let ∇V *be the* $3 \times (6n + 23)$ *matrix formed by the consumers' vectors* ∇V^1, ∇V^2, *and* ∇V^3. *If there does not exist a policy reform process* $\mathrm{d}P$ *such that*:

$$\nabla Z \mathrm{d}P \leq 0^{(3n+13)}$$

$$\nabla V \mathrm{d}P \gg 0^{(3)}$$

That is, if there does not exist an equilibrium preserving and Pareto improving policy reform process, then the economy is at a local second-best optimum and there exist two vectors of multipliers $\lambda > 0^{(3)}$ *and* $\mu \geq 0^{(3n+13)}$ *such that*:

$$\lambda \nabla V = \mu \nabla Z \tag{4.1}$$

Proof:

Follows from Motzkin's Theorem of the Alternative (see Appendix). ■

The existence of the multipliers guaranteed by Proposition 2 is not very informative in itself. However, each multiplier λ_i can be interpreted as the *social marginal utility of income* of generation i, and the multipliers μ are the *social shadow prices* of the $3n$ + 13 commodities. That is, if the government could choose its policy instruments to maximize a social welfare function $W(V^1, V^2, V^3)$ subject to (2.6) – (2.21), then λ_i is the implied welfare weight of generation i, specifically:

$$\lambda_i = \frac{\partial W(\cdot)}{\partial V^i(\cdot)} \frac{\partial V^i(\cdot)}{\partial I^i}$$

where $\partial V^i(\cdot) / \partial I^i$ is generation i's marginal utility of income, and μ is the vector of multipliers attached to the $3n$ + 13 constraints in (2.6) – (2.21). In other words, if there does not exist an equilibrium preserving and Pareto improving policy reform process, the economy 'looks like' the government has already chosen its policy instruments to maximize a social welfare function.

Expanding (4.1) yields the first-order (necessary) conditions for second-best optimality. The system (4.1) contains 29 equations which correspond to the government's $6n$ + 23 policy instruments. Each equation in (4.1) has a standard interpretation in the optimal tax literature: at an optimum, the *marginal social benefit* of a change in any instrument is equal to the *marginal social cost* of that change.[4] For example, a tax decrease which reduces the consumer price of some good will boost consumption of that good and welfare. This is the benefit. The cost is that the increase in demand must be met by an increase in supply by transferring resources from other sectors of the economy. Some characteristics of the second-best Pareto optima are discussed in the following corollaries. (Proofs, where necessary, are relegated to the Appendix.)

Corollary 2.1:

At all second-best Pareto optima production efficiency holds. That is, social shadow prices are proportional to producer prices in each period.

Corollary 2.2:

At all second-best Pareto optima, the multipliers attached to the bond market equations (2.8), (2.13), (2.17), *and* (2.21) *are all equal to zero. That is, differential changes in the government's debt instruments* b_1, b_2, B_2, *and* B_3 *have no social value.*

The term 'production efficiency' has been used in various guises in the optimal tax and tax reform literatures. Diamond and Mirrlees [1971] say production is efficient if technical rates of substitution in private and public sector production are equal. This is the usual definition of production efficiency. In Guesnerie [1977, 1995: Chapter 3] an equilibrium is temporarily (production) inefficient if it involves excess supply of some goods, i.e., 'non-tight' equilibria. Blackorby and Brett [2000], see also Guesnerie [1995: Chapter 4], define an equilibrium to be production efficient if producer prices are

[4] See, for example, Diamond and Mirrlees [1971: Equations 19 and 66], Dixit [1979: Equations 11 and 12],

proportional to social shadow prices. This is equivalent to the Diamond-Mirrlees definition if there is a public producer, because social shadow prices are the relevant prices for public production decisions. It is in this latter sense that this paper refers to production efficiency.

As noted by Guesnerie [1995: p.192], it is generally difficult to assess the relationship between social values and market prices (or more generally with technological conditions prevailing in the economy) in a second-best model with consumer prices disconnected from producer prices. However, Corollary 2.1 shows that social shadow prices are proportional to producer prices at all second-best optima. This reinforces the key Diamond-Mirrlees result (but in a dynamic model) that production efficiency is desirable at second-best Pareto optima, despite the second-best nature of the model.[5]

Corollary 2.2 states that, at all second-best Pareto optima, differential changes in the government's debt instruments have no social value. This follows from the fact that the government has complete and direct control over its bond issues. If changes in its debt instruments can increase social welfare, then the economy cannot be at an optimum.

Corollary 2.3:

Guesnerie [1995: Theorem 8, p.178], and Blackorby and Brett [2000: p.305].

[5] Recall the Lipsey and Lancaster [1956] general theorem of the second best, which asserts that if a distortion prevents the attainment of one of the conditions for first-best Pareto optimality, then the other conditions, although attainable, are generally no longer desirable. The distortion in the present context is the commodity taxes, which prevents marginal rates of substitution in consumption from being equated to marginal rates of transformation in production.

At all second-best Pareto optima, the marginal social benefits of increasing the taxes t_x^1, t_x^2, *and* t_x^3 *on first-, second-, and third-period consumption are given by, respectively*:

$$-\lambda_1 x^1 + \lambda_2 \left(\omega^2(1-l^2)\nabla_{\pi^1} e^1 + \sigma^2 \nabla_{\pi^1} \kappa^1\right) + \lambda_3 \left(\omega^3(1-l^3)\nabla_{\pi^1} e^2 + \sigma^3 \nabla_{\pi^1} \kappa^2\right)$$

$$-\lambda_2 x^2 + \lambda_3 \left(\omega^3(1-l^3)\nabla_{\pi^2} e^2 + \sigma^3 \nabla_{\pi^2} \kappa^2\right)$$

$$-\lambda_3 x^3$$

Therefore, given sufficient substitutability between bequests and the consumption goods, the marginal social benefits of increasing t_x^1 *and* t_x^2 *may be positive.*

In static models, the marginal social benefit of increasing consumption taxes is necessarily negative. In the present dynamic model, however, generations 1 and 2 may substitute bequests for their own consumption, boosting the incomes and welfare of the recipient generations. Thus, it is possible that the marginal social benefit of increasing consumption taxes is positive, especially in regions of the Pareto frontier in which the implied welfare weight of the saving (i.e., bequesting) generation is low relative to the recipient's implied welfare weight.

The marginal social benefits of decreasing the taxes on labor income, however, are always positive as in static models, provided the bequests are normal goods (which is probably a safe assumption). That is:

Corollary 2.4:

At all second-best Pareto optima, the marginal social benefits of decreasing the taxes t_l^1, t_l^2, *and* t_l^3 *on first-, second-, and third-period labor income are given by, respectively*:

$$\lambda_1(1-l^1) + \lambda_2\left(\omega^2(1-l^2)\nabla_{\omega^1}e^1 + \sigma^2\nabla_{\omega^1}\kappa^1\right) + \lambda_3\left(\omega^3(1-l^3)\nabla_{\omega^1}e^2 + \sigma^3\nabla_{\omega^1}\kappa^2\right)$$

$$\lambda_2(1-l^2)(1+e^1) + \lambda_3\left(\omega^3(1-l^3)\nabla_{\omega^2}e^2 + \sigma^3\nabla_{\omega^2}\kappa^2\right)$$

$$\lambda_3(1-l^3)(1+e^2)$$

Therefore, if the bequests κ^1, e^1, κ^2, *and* e^2 *are all normal goods, then the marginal social benefits of decreasing the taxes on labor income are always positive.*

5. Constraints on the Policy Reform Process

It has thus far been assumed that the government can change all taxes and issue bonds at will. In reality, however, it is well known that no central authority controls all taxes, and typically the government must seek legislative approval for changes. This can potentially hinder the government's ability to implement reforms. It is also the case that the government may have to ensure that a reform does not reduce the welfare of the current generation, say by the need to get re-elected. This places a political constraint on the set of reforms. These issues are addressed in this section.

5.1. Restrictions on the Government's Policy Instruments

The taxation of capital has long been a controversial issue, and proposed increases in capital taxation may be blocked by the legislature for political reasons. Restrictions on the government's ability to issue debt, such as period-by-period balanced budget restrictions, have also been proposed. We explore the consequences of such restrictions on the policy reform process and the nature of second-best Pareto optima.

First, suppose the government is not permitted to increase the taxes on savings (i.e., purchases of capital and the educational good) by generations 1 and 2. That is, we impose the restrictions $\mathrm{d}t_k^1 \leq 0$, $\mathrm{d}t_e^1 \leq 0$, $\mathrm{d}t_k^2 \leq 0$, and $\mathrm{d}t_e^2 \leq 0$. By Motzkin's Theorem, if there is no solution dP to:

$$\nabla Z \mathrm{d}P \leq 0^{(3n+13)} \qquad \nabla V \mathrm{d}P \gg 0^{(3)} \qquad \begin{pmatrix} 0^{(2n+1)} & -1 & 0^{(4n+21)} \\ 0^{(2n+3)} & -1 & 0^{(4n+19)} \\ 0^{(4n+8)} & -1 & 0^{(2n+14)} \\ 0^{(4n+10)} & -1 & 0^{(2n+12)} \end{pmatrix} \mathrm{d}P \geq 0^{(4)}$$

That is, if there does not exist an equilibrium preserving and Pareto improving policy reform process (with the restrictions on savings taxation), then there exist vectors $\lambda > 0^{(3)}$, $\mu \geq 0^{(3n+13)}$, and $\theta \geq 0^{(4)}$ such that:

$$\lambda \nabla V + \theta \begin{pmatrix} 0^{(2n+1)} & -1 & 0^{(4n+21)} \\ 0^{(2n+3)} & -1 & 0^{(4n+19)} \\ 0^{(4n+8)} & -1 & 0^{(2n+14)} \\ 0^{(4n+10)} & -1 & 0^{(2n+12)} \end{pmatrix} = \mu \nabla Z \tag{5.1}$$

The system (5.1) is the same as the system (4.1) which characterizes unrestricted second-best optima, except the 4th, 6th, 13th, and 15th equations in (5.1) include the multipliers $\langle \theta_1, \theta_2, \theta_3, \theta_4 \rangle$ which correspond to the restrictions on savings taxation. Note that if any of the restrictions are binding, i.e., if there exists an equilibrium preserving and Pareto improving reform in the absence of the restrictions, then $\theta \neq 0^{(4)}$. This leads to the following proposition, which is proved in the Appendix.

Proposition 3:

Suppose the government is not permitted to increase the taxes on savings by generations 1 and 2. Then production efficiency holds in period 1 (*period 2*) *at all second-best Pareto optima if and only if the constraints on the taxation of savings in period 1* (*period 2*) *are not binding*. (*Note*: *production efficiency always remains desirable in period 3*.)

In other words, production efficiency is no longer a necessary condition for second-best optimality if the government cannot increase the taxes on savings. In regions of the Pareto frontier in which the constraints are not binding, production efficiency remains desirable. However, when one of the constraints is binding, the marginal social benefit of increasing the corresponding tax on savings is greater than the marginal social cost. In this case, the government would like to increase the tax to directly determine bequests and the supplies of capital and/or effective labor available to the firm. Instead, it must distort producer prices relative to social shadow prices as an indirect (and imperfect) means of control. Roughly speaking, this is the classic 'two wrongs make a right' result of second-best theory.

The second way that the government can influence intertemporal income transfers is by issuing bonds itself. However, history suggests that government deficits and debt levels can become excessive, imposing a burden on future generations. In response, some economists have argued in favor of various forms of fiscal constraint, the most extreme of which are balanced budget restrictions. The following result, which is proved in the Appendix, shows that such restrictions do not disturb production efficiency.

Proposition 4:

Production efficiency remains desirable at all second-best Pareto optima in the presence of constraints on the government's debt instruments, including period-by-period balanced budget restrictions, whether or not the constraints are binding.

From the point of view of production efficiency, then, restrictions on the government's ability to issue debt are preferable to restrictions on the government's ability to tax private savings. Even though the capital good and government bonds are perfect substitutes, purchases and sales of bonds are transactions between consumers and the government only; they do not involve the production sector. The producer is affected only indirectly by the prices it receives and pays for capital.

5.2. Social Welfare and Political Constraints

Until now attention has focused on the possibility of Pareto improving reforms, i.e., reforms which increase the welfare of every generation. In general, however, the Pareto criterion can be criticized as being much too demanding, especially if the status quo equilibrium involves a highly unequal distribution of income. Moreover, the set of feasible reforms which increase the welfare of *every* agent in the economy is likely to be small, if not empty. Therefore, the government may instead refer to a social welfare function $W(V^1, V^2, V^3)$. However, the government may have to ensure that a reform does not reduce the welfare of the current generation, say by the need to get re-elected. This places a political constraint on the set of social-welfare improving reforms.

First, we show in Proposition 5 that a characterization analogous to that in Proposition 1 goes through using the social-welfare criterion (excluding for the moment political constraints).

Proposition 5:

Let $\Gamma \subseteq \mathbb{R}^{6n+23}$ *be the cone generated by taking non-negative linear combinations of the rows of* ∇Z, *and let* ∇W *be the gradient of the social welfare function. If* $\nabla W \in \Gamma$, *then there does not exist an equilibrium preserving policy reform process that increases social welfare. Conversely, if* $\nabla W \notin \Gamma$, *then there exists an equilibrium preserving policy reform process that increases social welfare.*

Proof:

By Farkas' Theorem, there exists a $\mu \in \mathbb{R}_{+}^{3n+13}$ such that $\mu\nabla Z = \nabla W$, or there exists a dP such that $\nabla Z\mathrm{d}P \leq 0^{(3n+13)}$ and $\nabla W\mathrm{d}P > 0$. Next, note that $\nabla W \in \Gamma$ implies that there must exist a $\mu \in \mathbb{R}_{+}^{3n+13}$ such that $\mu\nabla Z = \nabla W$. Conversely, $\nabla W \notin \Gamma$ implies that there cannot exist a $\mu \in \mathbb{R}_{+}^{3n+13}$ such that $\mu\nabla Z = \nabla W$. ■

Corollary 5.1:

If $\nabla W \in \Gamma$ *the economy is at a local second-best optimum and there exists a vector of multipliers* $\mu \in \mathbb{R}_{+}^{3n+13}$ *such that*:

$$\nabla W = \mu\nabla Z \tag{5.2}$$

The gradient ∇W of the social welfare function $W(V^1, V^2, V^3)$ is defined explicitly in the Appendix. Note that the system of equations (5.2) is identical to the system (4.1). Thus, it makes no difference whether second-best optima are characterized using the Pareto criterion or the social-welfare criterion. Intuition for this finding is provided in the discussion of Proposition 6 below.

We can now examine the implications of political constraints on the policy reform process. Specifically, suppose the welfare of generation 1 cannot be reduced in the process of implementing a social-welfare improving reform. By Motzkin's Theorem, if there does not exist a policy reform process dP such that:

$$\nabla Z\mathrm{d}P \leq 0^{(3n+13)} \qquad \nabla W\mathrm{d}P > 0 \qquad \nabla V^1\mathrm{d}P \geq 0$$

then there exists $\gamma > 0$, $\bar{\mu} \geq 0^{(3n+13)}$, and $\bar{\theta} \geq 0$ such that:

$$\nabla W + \theta\nabla V^1 = \mu\nabla Z$$

where $\theta = \bar{\theta}/\gamma$ and $\mu = \bar{\mu}/\gamma$. If the political constraint is binding ($\theta \neq 0$) there exists a social-welfare improving reform, but all such reforms make generation 1 worse-off. The following result can now be stated.

Proposition 6:

Suppose the government is prevented from implementing social-welfare improving reforms which make generation 1 worse-off. Then second-best Pareto optima are still characterized by (4.1), *except* λ_1 *is replaced by* $\lambda_1 + \theta$.

That is, the political constraint simply acts to increase the implied welfare weight of generation 1 from λ_1 to $\lambda_1 + \theta$. (A proposition analogous to Proposition 6 would be obtained if some other generation were the politically favored generation.) The intuition can be explained by reference to Figure 2. Conditions for second-best optimality hold at *all* points on the Pareto frontier. The political constraint, requiring that the welfare of generation 1 not be reduced below its initial level (say $\bar{U}^1$), means that the economy is initially at a point on the Pareto frontier (say $\bar{E}$) which is relatively favorable to generation 1. In the absence of the political constraint, the government could implement a social-welfare improving reform and move the economy towards E^*.

Thus, the social-welfare approach and political constraints do not fundamentally alter the necessary conditions for second-best Pareto optimality obtained in Section 4. This implies that the results obtained earlier in the paper are robust to this alternative approach (except that the implied welfare weight of generation *i* is increased if the political constraint favoring generation *i* is binding).

6. Concluding Comments

The existing tax reform literature implicitly views tax reform as a dynamic process, but it uses static models. Examples of this literature include: Guesnerie [1977, 1995], Diewert [1978], Dixit [1979], Weymark [1979], Blackorby and Brett [2000], and Murty and Russell [2003]. This paper has examined tax reform in a dynamic version of the Diamond-Mirrlees-Guesnerie model (which is the classic model used in optimal tax and tax reform) whereby the periods (or economies) are dynamically linked by bequests. The main contributions of the paper are:

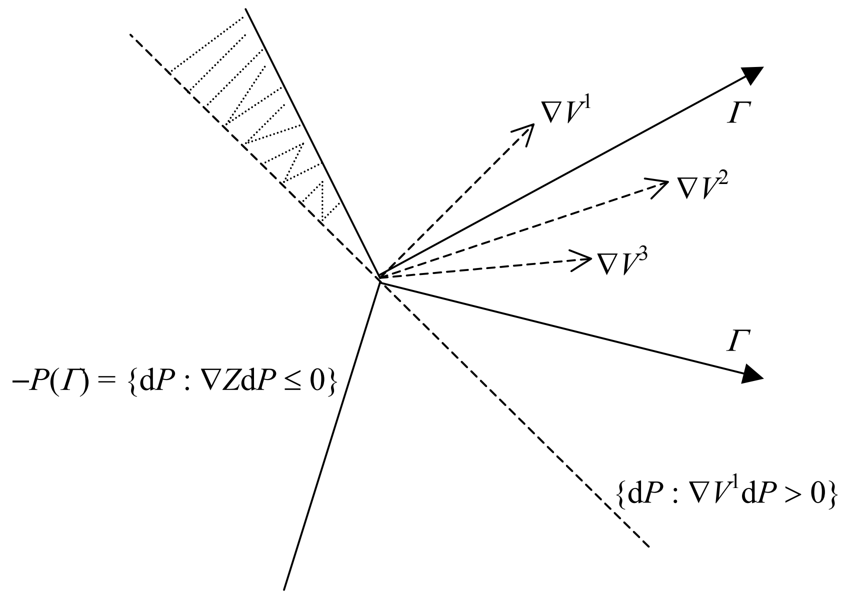
- the development of a simple dynamic model which makes it possible to analyze tax reform as a dynamic process (as it should be);
- the development of a tax reform methodology which makes it possible to derive the conditions under which each individual generation (or agent) can and cannot be made better-off, in addition to the usual Pareto improving conditions;

- second-best Pareto optima have been characterized as in the optimal tax literature, with special attention paid to intertemporal conditions and how these compare to static optimal tax formulae;
- production efficiency is shown to be desirable at all second-best Pareto optima, reinforcing the key Diamond-Mirrlees result (but in a dynamic model), but not if there are restrictions on the taxation of private savings. Restrictions on government savings, however, do not disturb the desirability of production efficiency; and
- it makes no difference whether second-best optima are characterized using the Pareto criterion or the social-welfare criterion, and political constraints which favor some generation simply increase the implied welfare weight of that generation (without disturbing fundamentally the conditions for second-best optimality).

FIGURE 1

Stylized Geometric Description of Proposition 1 and its Corollaries

E.g. 1: *There exists an equilibrium preserving reform that makes generation 1 better-off* (*shaded area*), *but generations 2 and 3 are made worse-off*.



E.g. 2: *There exists an equilibrium preserving and Pareto improving reform*.

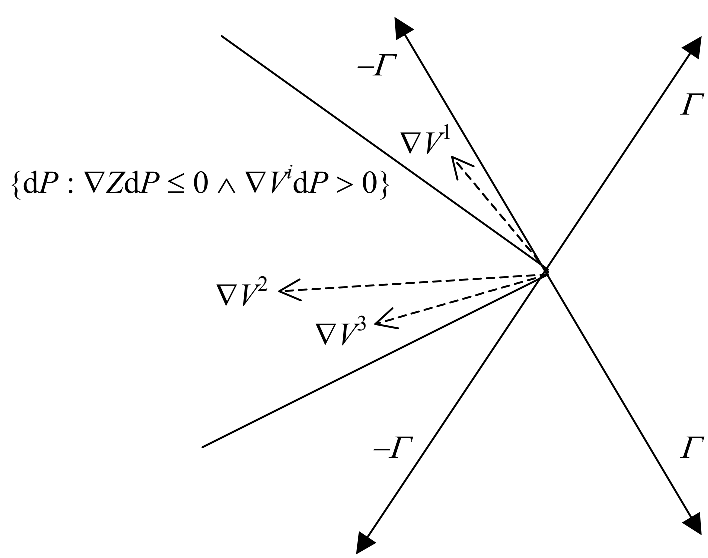
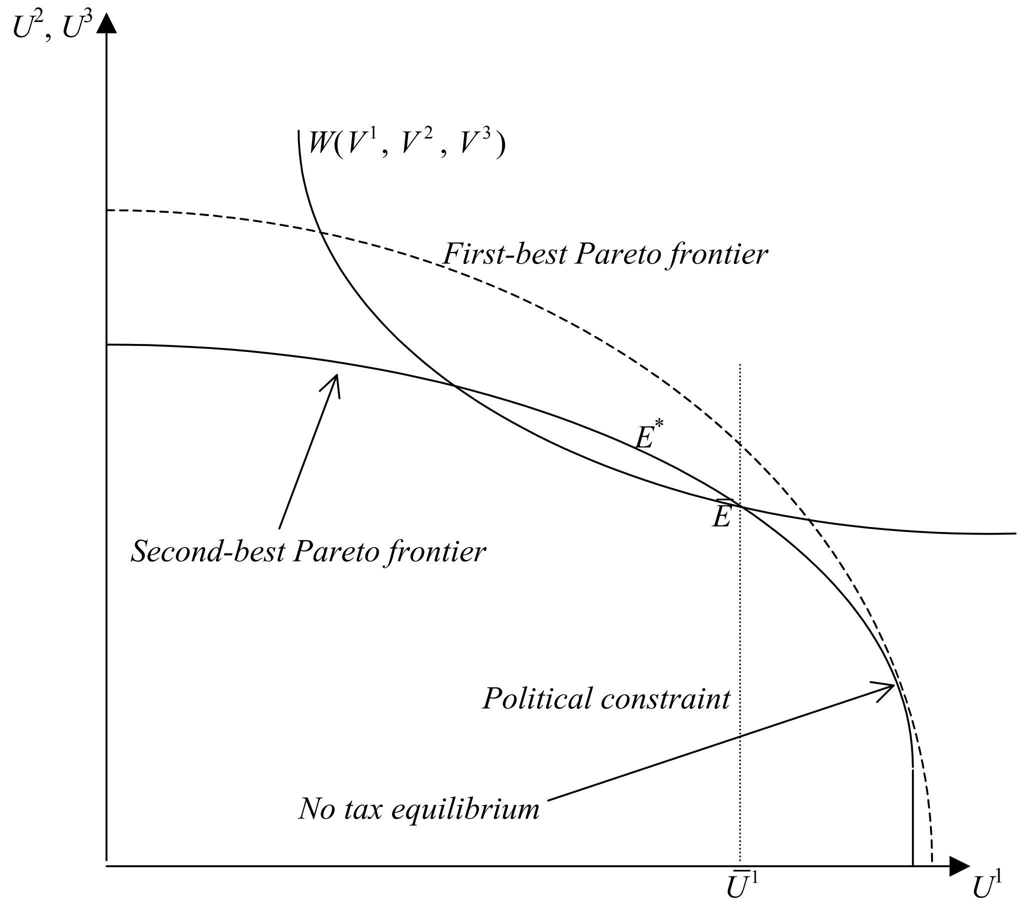


FIGURE 2

Social Welfare and Political Constraints



Appendix

(I) ***Farkas' Theorem.***

Let **A** be an $a \times n$ matrix and let **b** be an n-dimensional vector. Then either

$$\mathbf{Ax} \le 0^{(a)} \qquad \mathbf{bx} > 0$$

has a solution **x**, where **x** is an n-dimensional vector, or

$$\mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda} = \mathbf{b}$$

has a solution $\lambda \ge 0^{(a)}$, where λ is an a-dimensional vector, but never both.

Motzkin's Theorem of the Alternative.

Let **A**, **B**, and **C** be $a \times n$, $b \times n$, and $c \times n$ matrices. Then either

$$\mathbf{Ax} \gg 0^{(a)} \qquad \mathbf{Bx} \ge 0^{(b)} \qquad \mathbf{Cx} = 0^{(c)}$$

has a solution **x**, where **x** is an n-dimensional vector, or

$$\mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}_1 + \mathbf{B}^{\mathrm{T}}\boldsymbol{\lambda}_2 + \mathbf{C}^{\mathrm{T}}\boldsymbol{\lambda}_3 = 0^{(n)} \qquad \boldsymbol{\lambda}_1 > 0^{(a)},\ \boldsymbol{\lambda}_2 \ge 0^{(b)},\ \boldsymbol{\lambda}_3 \text{ unrestricted}$$

has a solution $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$, and $\boldsymbol{\lambda}_3$, where $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$, and $\boldsymbol{\lambda}_3$ are a-dimensional, b-dimensional, and c-dimensional vectors, but never both.

(II) ***Proof of Corollary 2.1.***

Subtracting the first equation in (4.1) from the second yields:

$$\langle \mu_1, \ldots, \mu_n \rangle \nabla_{p^1} x_1 + \mu_{n+1} \nabla_{p^1} k_1 + \mu_{n+3} \nabla_{p^1} e_1 - \mu_{n+4} \nabla_{p^1} L_1 = 0^{(n)} \tag{A.1}$$

Subtracting the third equation in (4.1) from the fourth yields:

$$\langle \mu_1, \ldots, \mu_n \rangle \nabla_{r^1} x_1 + \mu_{n+1} \nabla_{r^1} k_1 + \mu_{n+3} \nabla_{r^1} e_1 - \mu_{n+4} \nabla_{r^1} L_1 = 0 \tag{A.2}$$

Subtracting the fifth equation in (4.1) from the sixth yields:

$$\langle \mu_1, \ldots, \mu_n \rangle \nabla_{a^1} x_1 + \mu_{n+1} \nabla_{a^1} k_1 + \mu_{n+3} \nabla_{a^1} e_1 - \mu_{n+4} \nabla_{a^1} L_1 = 0 \tag{A.3}$$

Subtracting the seventh equation in (4.1) from the eighth yields:

$$\langle \mu_1, ..., \mu_n \rangle \nabla_{w^1} x_1 + \mu_{n+1} \nabla_{w^1} k_1 + \mu_{n+3} \nabla_{w^1} e_1 - \mu_{n+4} \nabla_{w^1} L_1 = 0 \tag{A.4}$$

Combining (A.1) – (A.4) yields:

$$\langle \mu_1, ..., \mu_n, \mu_{n+1}, \mu_{n+3}, \mu_{n+4} \rangle \nabla_{\hat{p}^1} \hat{x}_1 = 0^{(n+3)} \tag{A.5}$$

where $\hat{x}_1$ combines the supply of x_1, k_1, e_1, and L_1, and $\hat{p}^1$ combines the producer prices p^1, r^1, a^1, and w^1.

Next, it follows from homogeneity of degree zero of $x_1(p^1, r^1, a^1, w^1)$, $k_1(p^1, r^1, a^1, w^1)$, $e_1(p^1, r^1, a^1, w^1)$, and $L_1(p^1, r^1, a^1, w^1)$ in producer prices that:

$$\nabla_{\hat{p}^1} \hat{x}_1 \langle p_1^1, ..., p_n^1, r^1, a^1, w^1 \rangle = 0^{(n+3)} \tag{A.6}$$

It now follows from (A.5), (A.6), and symmetry of the Hessian of the profit function $\psi_1(p^1, r^1, a^1, w^1)$ that $\langle \mu_1, ..., \mu_n, \mu_{n+1}, \mu_{n+3}, \mu_{n+4} \rangle = \eta \langle p_1^1, ..., p_n^1, r^1, a^1, w^1 \rangle$ for some scalar $\eta > 0$. That is, first-period social shadow prices are proportional to first-period producer prices. In a similar manner, it can be shown that second- and third-period social shadow prices are proportional to second- and third-period producer prices. ■

(III) *Proof of Proposition 3*.

The proof is analogous to that of Corollary 2.1, except subtracting the third equation in (5.1) from the fourth yields:

$$\langle \mu_1, ..., \mu_n \rangle \nabla_{r^1} x_1 + \mu_{n+1} \nabla_{r^1} k_1 + \mu_{n+3} \nabla_{r^1} e_1 - \mu_{n+4} \nabla_{r^1} L_1 = \theta_1 \tag{A.2'}$$

and subtracting the fifth equation in (5.1) from the sixth yields:

$$\langle \mu_1, ..., \mu_n \rangle \nabla_{a^1} x_1 + \mu_{n+1} \nabla_{a^1} k_1 + \mu_{n+3} \nabla_{a^1} e_1 - \mu_{n+4} \nabla_{a^1} L_1 = \theta_2 \tag{A.3'}$$

The expression corresponding to (A.5) becomes:

$$\langle \mu_1, \ldots, \mu_n, \mu_{n+1}, \mu_{n+3}, \mu_{n+4} \rangle \nabla_{\hat{p}^1} \hat{x}_1 = \langle 0^{(n)}, \theta_1, \theta_2, 0 \rangle \tag{A.5'}$$

From (A.5') and (A.6) it follows that first-period social shadow prices are proportional to first-period producer prices if and only if $\theta_1 = \theta_2 = 0$, i.e., if and only if the constraints preventing increases in first-period savings taxation are not binding. In a similar manner, it can be shown that production efficiency holds in the second period if and only if $\theta_3 = \theta_4 = 0$, i.e., if and only if the constraints preventing increases in second-period savings taxation are not binding. ■

(IV) ***Proof of Proposition 4.***

Requiring that the government balance its budget period-by-period requires that the following restrictions be placed on its bond issues: $\mathrm{d}b_1 + b_1 = 0$, $\mathrm{d}b_2 + b_2 = 0$, $\mathrm{d}B_2 + B_2 = 0$, and $\mathrm{d}B_3 + B_3 = 0$. There exists an equilibrium preserving and Pareto improving reform with the government's budget balanced period-by-period if and only if there exists a solution dP to the following non-homogenous system:

$$\nabla Z \mathrm{d}P \leq 0^{(3n+13)} \qquad \nabla V \mathrm{d}P \gg 0^{(3)} \qquad \begin{pmatrix} 0^{(2n+6)} & 1 & 0^{(4n+16)} \\ 0^{(4n+15)} & 1 & 0^{(2n+7)} \\ 0^{(4n+16)} & 1 & 0^{(2n+6)} \\ 0^{(6n+21)} & 1 & 0 \end{pmatrix} \mathrm{d}P = \begin{pmatrix} -b_1 \\ -b_2 \\ -B_2 \\ -B_3 \end{pmatrix} \tag{A.7}$$

The non-homogenous system (A.7) has a solution dP if and only if the following homogeneous system (A.8) has a solution $\langle \mathrm{d}P, D\rangle$, where D is a dummy variable used to convert the non-homogenous system (A.7) into the homogeneous system (A.8).

$$\begin{pmatrix} \nabla Z & 0^{(3n+13)} \end{pmatrix} \begin{pmatrix} \mathrm{d}P \\ D \end{pmatrix} \leq 0^{(3n+13)} \qquad \begin{pmatrix} \nabla V & 0^{(3)} \\ 0^{(6n+23)} & 1 \end{pmatrix} \begin{pmatrix} \mathrm{d}P \\ D \end{pmatrix} \gg 0^{(4)}$$

$$\begin{pmatrix} 0^{(2n+6)} & 1 & 0^{(4n+16)} & b_1 \\ 0^{(4n+15)} & 1 & 0^{(2n+7)} & b_2 \\ 0^{(4n+16)} & 1 & 0^{(2n+6)} & B_2 \\ 0^{(6n+21)} & 1 & 0 & B_3 \end{pmatrix} \begin{pmatrix} \mathrm{d}P \\ D \end{pmatrix} = 0^{(4)} \tag{A.8}$$

By Motzkin's Theorem, if there is no solution $\langle \mathrm{d}P, D\rangle$ to (A.8), then there exist vectors $\langle \lambda, \xi\rangle > 0^{(4)}$, $\mu \geq 0^{(3n+13)}$, and θ unrestricted such that:

$$\begin{pmatrix} \lambda & \xi \end{pmatrix} \begin{pmatrix} \nabla V & 0^{(3)} \\ 0^{(6n+23)} & 1 \end{pmatrix} + \theta \begin{pmatrix} 0^{(2n+6)} & 1 & 0^{(4n+16)} & b_1 \\ 0^{(4n+15)} & 1 & 0^{(2n+7)} & b_2 \\ 0^{(4n+16)} & 1 & 0^{(2n+6)} & B_2 \\ 0^{(6n+21)} & 1 & 0 & B_3 \end{pmatrix} = \mu \begin{pmatrix} \nabla Z & 0^{(3n+13)} \end{pmatrix} \tag{A.9}$$

The only differences between (A.9) and the unrestricted conditions for second-best optimality contained in (4.1) are that θ_1, θ_2, θ_3, and θ_4 are added, respectively, to the 9th, 20th, 21st, and 28th equations in (4.1), and an additional condition:

$$\xi + \theta_1 b_1 + \theta_2 b_2 + \theta_3 B_2 + \theta_4 B_3 = 0$$

is introduced. However, none of these changes affect the equations required to prove production efficiency (see the proof of Corollary 2.1). Therefore, restrictions on the government's debt instruments do not affect the desirability of production efficiency. ■

(V) *Defining the Matrices and Vectors.*

The Jacobian matrix ∇Z is obtained by stacking the following matrices ∇Z^1, ∇Z^2, and ∇Z^3, which correspond to the first-, second-, and third-period market clearing equations.

$$
\nabla Z^1 := \begin{pmatrix}
\nabla_{\pi^1}x^1 - \nabla_{p^1}x_1 & \nabla_{\pi^1}x^1 & \nabla_{\rho^1}x^1 - \nabla_{r^1}x_1 & \nabla_{\rho^1}x^1 & \nabla_{\alpha^1}x^1 - \nabla_{a^1}x_1 & \nabla_{\alpha^1}x^1 \\
\nabla_{\pi^1}k^1 - \nabla_{p^1}k_1 & \nabla_{\pi^1}k^1 & \nabla_{\rho^1}k^1 - \nabla_{r^1}k_1 & \nabla_{\rho^1}k^1 & \nabla_{\alpha^1}k^1 - \nabla_{a^1}k_1 & \nabla_{\alpha^1}k^1 \\
\nabla_{\pi^1}b^1 & \nabla_{\pi^1}b^1 & \nabla_{\rho^1}b^1 & \nabla_{\rho^1}b^1 & \nabla_{\alpha^1}b^1 & \nabla_{\alpha^1}b^1 \\
\nabla_{\pi^1}e^1 - \nabla_{p^1}e_1 & \nabla_{\pi^1}e^1 & \nabla_{\rho^1}e^1 - \nabla_{r^1}e_1 & \nabla_{\rho^1}e^1 & \nabla_{\alpha^1}e^1 - \nabla_{a^1}e_1 & \nabla_{\alpha^1}e^1 \\
\nabla_{\pi^1}l^1 + \nabla_{p^1}L_1 & \nabla_{\pi^1}l^1 & \nabla_{\rho^1}l^1 + \nabla_{r^1}L_1 & \nabla_{\rho^1}l^1 & \nabla_{\alpha^1}l^1 + \nabla_{a^1}L_1 & \nabla_{\alpha^1}l^1
\end{pmatrix} \sim
$$

$$
\begin{pmatrix}
\nabla_{\omega^1}x^1 - \nabla_{w^1}x_1 & \nabla_{\omega^1}x^1 & 0^{(n)} & 0^{(n\times n)} & 0^{(n\times n)} & 0^{(n)} & 0^{(n)} & 0^{(n)} & 0^{(n)} & 0^{(n)} & 0^{(n)} \\
\nabla_{\omega^1}k^1 - \nabla_{w^1}k_1 & \nabla_{\omega^1}k^1 & 0 & 0^{(n)} & 0^{(n)} & 0 & 0 & 0 & 0 & 0 & 0 \\
\nabla_{\omega^1}b^1 & \nabla_{\omega^1}b^1 & -1 & 0^{(n)} & 0^{(n)} & 0 & 0 & 0 & 0 & 0 & 0 \\
\nabla_{\omega^1}e^1 - \nabla_{w^1}e_1 & \nabla_{\omega^1}e^1 & 0 & 0^{(n)} & 0^{(n)} & 0 & 0 & 0 & 0 & 0 & 0 \\
\nabla_{\omega^1}l^1 + \nabla_{w^1}L_1 & \nabla_{\omega^1}l^1 & 0 & 0^{(n)} & 0^{(n)} & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \sim
$$

$$
\begin{pmatrix}
0^{(n)} & 0^{(n)} & 0^{(n)} & 0^{(n)} & 0^{(n\times n)} & 0^{(n\times n)} & 0^{(n)} & 0^{(n)} & 0^{(n)} & 0^{(n)} & 0^{(n)} & \nabla_R x^1 \\
0 & 0 & 0 & 0 & 0^{(n)} & 0^{(n)} & 0 & 0 & 0 & 0 & 0 & \nabla_R k^1 \\
0 & 0 & 0 & 0 & 0^{(n)} & 0^{(n)} & 0 & 0 & 0 & 0 & 0 & \nabla_R b^1 \\
0 & 0 & 0 & 0 & 0^{(n)} & 0^{(n)} & 0 & 0 & 0 & 0 & 0 & \nabla_R e^1 \\
0 & 0 & 0 & 0 & 0^{(n)} & 0^{(n)} & 0 & 0 & 0 & 0 & 0 & \nabla_R l^1
\end{pmatrix}
$$

$$
\nabla Z^2 := \begin{pmatrix}
\nabla_{\pi^1}x^2 & \nabla_{\pi^1}x^2 & \nabla_{\rho^1}x^2 & \nabla_{\rho^1}x^2 \\
\nabla_{\pi^1}k^2 & \nabla_{\pi^1}k^2 & \nabla_{\rho^1}k^2 & \nabla_{\rho^1}k^2 \\
\nabla_{\pi^1}b^2 & \nabla_{\pi^1}b^2 & \nabla_{\rho^1}b^2 & \nabla_{\rho^1}b^2 \\
\nabla_{\pi^1}e^2 & \nabla_{\pi^1}e^2 & \nabla_{\rho^1}e^2 & \nabla_{\rho^1}e^2 \\
\nabla_{\pi^1}l^2(1+e^1) + & \nabla_{\pi^1}l^2(1+e^1) + & \nabla_{\rho^1}l^2(1+e^1) + & \nabla_{\rho^1}l^2(1+e^1) + \\
\nabla_{\pi^1}e^1(l^2-1) & \nabla_{\pi^1}e^1(l^2-1) & \nabla_{\rho^1}e^1(l^2-1) & \nabla_{\rho^1}e^1(l^2-1) \\
-\nabla_{\pi^1}k^1 & -\nabla_{\pi^1}k^1 & -\nabla_{\rho^1}k^1 & -\nabla_{\rho^1}k^1 \\
-\nabla_{\pi^1}b^1 & -\nabla_{\pi^1}b^1 & -\nabla_{\rho^1}b^1 & -\nabla_{\rho^1}b^1
\end{pmatrix} \sim
$$

$$\begin{pmatrix} \nabla_{\alpha^1}x^2 & \nabla_{\alpha^1}x^2 & \nabla_{\omega^1}x^2 & \nabla_{\omega^1}x^2 & 0^{(n)} \\ \nabla_{\alpha^1}k^2 & \nabla_{\alpha^1}k^2 & \nabla_{\omega^1}k^2 & \nabla_{\omega^1}k^2 & 0 \\ \nabla_{\alpha^1}b^2 & \nabla_{\alpha^1}b^2 & \nabla_{\omega^1}b^2 & \nabla_{\omega^1}b^2 & 0 \\ \nabla_{\alpha^1}e^2 & \nabla_{\alpha^1}e^2 & \nabla_{\omega^1}e^2 & \nabla_{\omega^1}e^2 & 0 \\ \begin{matrix}\nabla_{\alpha^1}l^2(1+e^1)+ \\ \nabla_{\alpha^1}e^1(l^2-1)\end{matrix} & \begin{matrix}\nabla_{\alpha^1}l^2(1+e^1)+ \\ \nabla_{\alpha^1}e^1(l^2-1)\end{matrix} & \begin{matrix}\nabla_{\omega^1}l^2(1+e^1)+ \\ \nabla_{\omega^1}e^1(l^2-1)\end{matrix} & \begin{matrix}\nabla_{\omega^1}l^2(1+e^1)+ \\ \nabla_{\omega^1}e^1(l^2-1)\end{matrix} & 0 \\ -\nabla_{\alpha^1}k^1 & -\nabla_{\alpha^1}k^1 & -\nabla_{\omega^1}k^1 & -\nabla_{\omega^1}k^1 & 0 \\ -\nabla_{\alpha^1}b^1 & -\nabla_{\alpha^1}b^1 & -\nabla_{\omega^1}b^1 & -\nabla_{\omega^1}b^1 & 0 \end{pmatrix} \sim$$

$$\begin{pmatrix} \nabla_{\pi^2}x^2-\nabla_{p^2}x_2 & \nabla_{\pi^2}x^2 & \nabla_{\rho^2}x^2-\nabla_{r^2}x_2 & \nabla_{\rho^2}x^2 & \nabla_{\alpha^2}x^2-\nabla_{a^2}x_2 \\ \nabla_{\pi^2}k^2-\nabla_{p^2}k_2 & \nabla_{\pi^2}k^2 & \nabla_{\rho^2}k^2-\nabla_{r^2}k_2 & \nabla_{\rho^2}k^2 & \nabla_{\alpha^2}k^2-\nabla_{a^2}k_2 \\ \nabla_{\pi^2}b^2 & \nabla_{\pi^2}b^2 & \nabla_{\rho^2}b^2 & \nabla_{\rho^2}b^2 & \nabla_{\alpha^2}b^2 \\ \nabla_{\pi^2}e^2-\nabla_{p^2}e_2 & \nabla_{\pi^2}e^2 & \nabla_{\rho^2}e^2-\nabla_{r^2}e_2 & \nabla_{\rho^2}e^2 & \nabla_{\alpha^2}e^2-\nabla_{a^2}e_2 \\ \begin{matrix}\nabla_{\pi^2}l^2(1+e^1) \\ +\nabla_{p^2}L_2\end{matrix} & \nabla_{\pi^2}l^2(1+e^1) & \begin{matrix}\nabla_{\rho^2}l^2(1+e^1) \\ +\nabla_{r^2}L_2\end{matrix} & \nabla_{\rho^2}l^2(1+e^1) & \begin{matrix}\nabla_{\alpha^2}l^2(1+e^1) \\ +\nabla_{a^2}L_2\end{matrix} \\ \nabla_{p^2}K_2 & 0^{(n)} & \nabla_{r^2}K_2 & 0 & \nabla_{a^2}K_2 \\ 0^{(n)} & 0^{(n)} & 0 & 0 & 0 \end{pmatrix} \sim$$

$$\begin{pmatrix} \nabla_{\alpha^2}x^2 & \nabla_{\omega^2}x^2-\nabla_{w^2}x_2 & \nabla_{\omega^2}x^2 & \nabla_{\sigma^2}x^2-\nabla_{s^2}x_2 & \nabla_{\sigma^2}x^2 \\ \nabla_{\alpha^2}k^2 & \nabla_{\omega^2}k^2-\nabla_{w^2}k_2 & \nabla_{\omega^2}k^2 & \nabla_{\sigma^2}k^2-\nabla_{s^2}k_2 & \nabla_{\sigma^2}k^2 \\ \nabla_{\alpha^2}b^2 & \nabla_{\omega^2}b^2 & \nabla_{\omega^2}b^2 & \nabla_{\sigma^2}b^2 & \nabla_{\sigma^2}b^2 \\ \nabla_{\alpha^2}e^2 & \nabla_{\omega^2}e^2-\nabla_{w^2}e_2 & \nabla_{\omega^2}e^2 & \nabla_{\sigma^2}e^2-\nabla_{s^2}e_2 & \nabla_{\sigma^2}e^2 \\ \nabla_{\alpha^2}l^2(1+e^1) & \begin{matrix}\nabla_{\omega^2}l^2(1+e^1) \\ +\nabla_{w^2}L_2\end{matrix} & \nabla_{\omega^2}l^2(1+e^1) & \begin{matrix}\nabla_{\sigma^2}l^2(1+e^1) \\ +\nabla_{s^2}L_2\end{matrix} & \nabla_{\sigma^2}l^2(1+e^1) \\ 0 & \nabla_{w^2}K_2 & 0 & \nabla_{s^2}K_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim$$

$$\begin{pmatrix} 0^{(n)} & 0^{(n)} & 0^{(n\times n)} & 0^{(n\times n)} & 0^{(n)} & 0^{(n)} & 0^{(n)} & 0^{(n)} & 0^{(n)} & \nabla_R x^2 \\ 0 & 0 & 0^{(n)} & 0^{(n)} & 0 & 0 & 0 & 0 & 0 & \nabla_R k^2 \\ -1 & 0 & 0^{(n)} & 0^{(n)} & 0 & 0 & 0 & 0 & 0 & \nabla_R b^2 \\ 0 & 0 & 0^{(n)} & 0^{(n)} & 0 & 0 & 0 & 0 & 0 & \nabla_R e^2 \\ 0 & 0 & 0^{(n)} & 0^{(n)} & 0 & 0 & 0 & 0 & 0 & \begin{matrix}\nabla_R e^1(l^2-1)+ \\ \nabla_R l^2(1+e^1)\end{matrix} \\ 0 & 0 & 0^{(n)} & 0^{(n)} & 0 & 0 & 0 & 0 & 0 & -\nabla_R k^1 \\ 0 & 1 & 0^{(n)} & 0^{(n)} & 0 & 0 & 0 & 0 & 0 & -\nabla_R b^1 \end{pmatrix}$$

$$\nabla Z^3 := \begin{pmatrix} \nabla_{\pi^1} x^3 & \nabla_{\pi^1} x^3 & \nabla_{\rho^1} x^3 & \nabla_{\rho^1} x^3 \\ \nabla_{\pi^1} l^3(1+e^2)+ & \nabla_{\pi^1} l^3(1+e^2)+ & \nabla_{\rho^1} l^3(1+e^2)+ & \nabla_{\rho^1} l^3(1+e^2)+ \\ \nabla_{\pi^1} e^2(l^3-1) & \nabla_{\pi^1} e^2(l^3-1) & \nabla_{\rho^1} e^2(l^3-1) & \nabla_{\rho^1} e^2(l^3-1) \\ -\nabla_{\pi^1} k^2 & -\nabla_{\pi^1} k^2 & -\nabla_{\rho^1} k^2 & -\nabla_{\rho^1} k^2 \\ -\nabla_{\pi^1} b^2 & -\nabla_{\pi^1} b^2 & -\nabla_{\rho^1} b^2 & -\nabla_{\rho^1} b^2 \end{pmatrix} \sim$$

$$\begin{pmatrix} \nabla_{\alpha^1} x^3 & \nabla_{\alpha^1} x^3 & \nabla_{\omega^1} x^3 & \nabla_{\omega^1} x^3 & 0^{(n)} \\ \nabla_{\alpha^1} l^3(1+e^2)+ & \nabla_{\alpha^1} l^3(1+e^2)+ & \nabla_{\omega^1} l^3(1+e^2)+ & \nabla_{\omega^1} l^3(1+e^2)+ & \\ \nabla_{\alpha^1} e^2(l^3-1) & \nabla_{\alpha^1} e^2(l^3-1) & \nabla_{\omega^1} e^2(l^3-1) & \nabla_{\omega^1} e^2(l^3-1) & 0 \\ -\nabla_{\alpha^1} k^2 & -\nabla_{\alpha^1} k^2 & -\nabla_{\omega^1} k^2 & -\nabla_{\omega^1} k^2 & 0 \\ -\nabla_{\alpha^1} b^2 & -\nabla_{\alpha^1} b^2 & -\nabla_{\omega^1} b^2 & -\nabla_{\omega^1} b^2 & 0 \end{pmatrix} \sim$$

$$\begin{pmatrix} \nabla_{\pi^2} x^3 & \nabla_{\pi^2} x^3 & \nabla_{\rho^2} x^3 & \nabla_{\rho^2} x^3 \\ \nabla_{\pi^2} l^3(1+e^2)+ & \nabla_{\pi^2} l^3(1+e^2)+ & \nabla_{\rho^2} l^3(1+e^2)+ & \nabla_{\rho^2} l^3(1+e^2)+ \\ \nabla_{\pi^2} e^2(l^3-1) & \nabla_{\pi^2} e^2(l^3-1) & \nabla_{\rho^2} e^2(l^3-1) & \nabla_{\rho^2} e^2(l^3-1) \\ -\nabla_{\pi^2} k^2 & -\nabla_{\pi^2} k^2 & -\nabla_{\rho^2} k^2 & -\nabla_{\rho^2} k^2 \\ -\nabla_{\pi^2} b^2 & -\nabla_{\pi^2} b^2 & -\nabla_{\rho^2} b^2 & -\nabla_{\rho^2} b^2 \end{pmatrix} \sim$$

$$\begin{pmatrix} \nabla_{\alpha^2}x^3 & \nabla_{\alpha^2}x^3 & \nabla_{\omega^2}x^3 & \nabla_{\omega^2}x^3 \\ \nabla_{\alpha^2}l^3(1+e^2)+ & \nabla_{\alpha^2}l^3(1+e^2)+ & \nabla_{\omega^2}l^3(1+e^2)+ & \nabla_{\omega^2}l^3(1+e^2)+ \\ \nabla_{\alpha^2}e^2(l^3-1) & \nabla_{\alpha^2}e^2(l^3-1) & \nabla_{\omega^2}e^2(l^3-1) & \nabla_{\omega^2}e^2(l^3-1) \\ -\nabla_{\alpha^2}k^2 & -\nabla_{\alpha^2}k^2 & -\nabla_{\omega^2}k^2 & -\nabla_{\omega^2}k^2 \\ -\nabla_{\alpha^2}b^2 & -\nabla_{\alpha^2}b^2 & -\nabla_{\omega^2}b^2 & -\nabla_{\omega^2}b^2 \end{pmatrix} \sim$$

$$\begin{pmatrix} \nabla_{\sigma^2}x^3 & \nabla_{\sigma^2}x^3 & 0^{(n)} & 0^{(n)} & \nabla_{\pi^3}x^3-\nabla_{p^3}x_3 & \nabla_{\pi^3}x^3 \\ \nabla_{\sigma^2}e^2(l^3-1)+ \\ \nabla_{\sigma^2}l^3(1+e^2) & \nabla_{\sigma^2}e^2(l^3-1)+ \\ \nabla_{\sigma^2}l^3(1+e^2) & 0 & 0 & \nabla_{\pi^3}l^3(1+e^2) \\ +\nabla_{p^3}L_3 & \nabla_{\pi^3}l^3(1+e^2) \\ -\nabla_{\sigma^2}k^2 & -\nabla_{\sigma^2}k^2 & 0 & 0 & \nabla_{p^3}K_3 & 0^{(n)} \\ -\nabla_{\sigma^2}b^2 & -\nabla_{\sigma^2}b^2 & 0 & 0 & 0^{(n)} & 0^{(n)} \end{pmatrix} \sim$$

$$\begin{pmatrix} \nabla_{\omega^3}x^3-\nabla_{w^3}x_3 & \nabla_{\omega^3}x^3 & \nabla_{\sigma^3}x^3-\nabla_{s^3}x_3 & \nabla_{\sigma^3}x^3 \\ \nabla_{\omega^3}l^3(1+e^2) \\ +\nabla_{w^3}L_3 & \nabla_{\omega^3}l^3(1+e^2) & \nabla_{\sigma^3}l^3(1+e^2) \\ +\nabla_{s^3}L_3 & \nabla_{\sigma^3}l^3(1+e^2) \\ \nabla_{w^3}K_3 & 0 & \nabla_{s^3}K_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim$$

$$\begin{pmatrix} 0^{(n)} & \nabla_R x^3 \\ 0 & \nabla_R e^2(l^3-1)+ \\ & \nabla_R l^3(1+e^2) \\ 0 & -\nabla_R k^2 \\ 1 & -\nabla_R b^2 \end{pmatrix}$$

To obtain the consumers' vectors ∇V^1, ∇V^2, and ∇V^3, note that the welfare of generation 1 increases if and only if:

$$\mathrm{d}V^1(\pi^1,\rho^1,\alpha^1,\omega^1,R) = \nabla_{\pi^1}V^1(\cdot)\mathrm{d}p^1 + \nabla_{\pi^1}V^1(\cdot)\mathrm{d}t_x^1$$
$$+ \nabla_{\rho^1}V^1(\cdot)\mathrm{d}r^1 + \nabla_{\rho^1}V^1(\cdot)\mathrm{d}t_k^1 + \nabla_{\alpha^1}V^1(\cdot)\mathrm{d}a^1 + \nabla_{\alpha^1}V^1(\cdot)\mathrm{d}t_e^1$$
$$+ \nabla_{I^1}V^1(\cdot)\nabla_{\omega^1}I^1\mathrm{d}w^1 + \nabla_{I^1}V^1(\cdot)\nabla_{\omega^1}I^1\mathrm{d}t_l^1 + \nabla_{I^1}V^1(\cdot)\nabla_R I^1\mathrm{d}R > 0$$

where $I^1 := \omega^1(1-l^1)+R$ is generation 1's income. Equivalently, using Roy's Theorem, the welfare of generation 1 increases if and only if:

$$-x^1\mathrm{d}p^1 - x^1\mathrm{d}t_x^1 - \kappa^1\mathrm{d}r^1 - \kappa^1\mathrm{d}t_k^1 - e^1\mathrm{d}a^1 - e^1\mathrm{d}t_e^1 + (1-l^1)\mathrm{d}w^1 + (1-l^1)\mathrm{d}t_l^1 + \mathrm{d}R > 0$$

Thus, ∇V^1 is defined by:

$$\nabla V^1 := \langle -x^1, -x^1, -\kappa^1, -\kappa^1, -e^1, -e^1, (1-l^1), (1-l^1), 0^{(4n+16)}, 1\rangle$$

Similarly, ∇V^2 and ∇V^3 are defined by:

$$\nabla V^2 := \langle \beta^2\nabla_{\pi^1}V^2(\cdot), \beta^2\nabla_{\pi^1}V^2(\cdot), \beta^2\nabla_{\rho^1}V^2(\cdot), \beta^2\nabla_{\rho^1}V^2(\cdot), \beta^2\nabla_{\alpha^1}V^2(\cdot),$$

$$\beta^2\nabla_{\alpha^1}V^2(\cdot), \beta^2\nabla_{\omega^1}V^2(\cdot), \beta^2\nabla_{\omega^1}V^2(\cdot), 0, -x^2, -x^2, -\kappa^2, -\kappa^2, -e^2, -e^2,$$

$$(1-l^2)(1+e^1), (1-l^2)(1+e^1), \kappa^1, \kappa^1, 0^{(2n+7)}, 1\rangle$$

$$\nabla V^3 := \langle \beta^3\nabla_{\pi^1}V^3(\cdot), \beta^3\nabla_{\pi^1}V^3(\cdot), \beta^3\nabla_{\rho^1}V^3(\cdot), \beta^3\nabla_{\rho^1}V^3(\cdot), \beta^3\nabla_{\alpha^1}V^3(\cdot),$$

$$\beta^3\nabla_{\alpha^1}V^3(\cdot), \beta^3\nabla_{\omega^1}V^3(\cdot), \beta^3\nabla_{\omega^1}V^3(\cdot), 0, \beta^3\nabla_{\pi^2}V^3(\cdot), \beta^3\nabla_{\pi^2}V^3(\cdot), \beta^3\nabla_{\rho^2}V^3(\cdot),$$

$$\beta^3\nabla_{\rho^2}V^3(\cdot), \beta^3\nabla_{\alpha^2}V^3(\cdot), \beta^3\nabla_{\alpha^2}V^3(\cdot), \beta^3\nabla_{\omega^2}V^3(\cdot), \beta^3\nabla_{\omega^2}V^3(\cdot), \beta^3\nabla_{\sigma^2}V^3(\cdot),$$

$$\beta^3\nabla_{\sigma^2}V^3(\cdot), 0, 0, -x^3, -x^3, (1-l^3)(1+e^2), (1-l^3)(1+e^2), \kappa^2, \kappa^2, 0, 1\rangle$$

where $\beta^2 = \dfrac{1}{\partial V^2(\cdot)/\partial I^2}$ and $\beta^3 = \dfrac{1}{\partial V^3(\cdot)/\partial I^3}$.

The gradient of the social welfare function $W(V^1, V^2, V^3)$ is given by:

$$\nabla W := \langle -\lambda_1 x^1 + \lambda_2\beta^2\nabla_{\pi^1}V^2(\cdot) + \lambda_3\beta^3\nabla_{\pi^1}V^3(\cdot), -\lambda_1 x^1 + \lambda_2\beta^2\nabla_{\pi^1}V^2(\cdot) + \lambda_3\beta^3\nabla_{\pi^1}V^3(\cdot),$$

$$-\lambda_1\kappa^1 + \lambda_2\beta^2\nabla_{\rho^1}V^2(\cdot) + \lambda_3\beta^3\nabla_{\rho^1}V^3(\cdot), -\lambda_1\kappa^1 + \lambda_2\beta^2\nabla_{\rho^1}V^2(\cdot) + \lambda_3\beta^3\nabla_{\rho^1}V^3(\cdot),$$

$$-\lambda_1 e^1 + \lambda_2\beta^2\nabla_{\alpha^1}V^2(\cdot) + \lambda_3\beta^3\nabla_{\alpha^1}V^3(\cdot), -\lambda_1 e^1 + \lambda_2\beta^2\nabla_{\alpha^1}V^2(\cdot) + \lambda_3\beta^3\nabla_{\alpha^1}V^3(\cdot),$$

$\lambda_1(1-l^1)\ +\ \lambda_2\beta^2\nabla_{\omega^1}V^2(\cdot)\ +\ \lambda_3\beta^3\nabla_{\omega^1}V^3(\cdot),\ \lambda_1(1-l^1)\ +\ \lambda_2\beta^2\nabla_{\omega^1}V^2(\cdot)\ +$

$\lambda_3\beta^3\nabla_{\omega^1}V^3(\cdot),\ 0,\ -\lambda_2 x^2\ +\ \lambda_3\beta^3\nabla_{\pi^2}V^3(\cdot),\ -\lambda_2 x^2\ +\ \lambda_3\beta^3\nabla_{\pi^2}V^3(\cdot),\ -\lambda_2\kappa^2\ +$

$\lambda_3\beta^3\nabla_{\rho^2}V^3(\cdot),\ -\lambda_2\kappa^2\ +\ \lambda_3\beta^3\nabla_{\rho^2}V^3(\cdot),\ -\lambda_2 e^2\ +\ \lambda_3\beta^3\nabla_{\alpha^2}V^3(\cdot),\ -\lambda_2 e^2\ +$

$\lambda_3\beta^3\nabla_{\alpha^2}V^3(\cdot),\ \lambda_2(1-l^2)(1+e^1)\ +\ \lambda_3\beta^3\nabla_{\omega^2}V^3(\cdot),\ \lambda_2(1-l^2)(1+e^1)\ +$

$\lambda_3\beta^3\nabla_{\omega^2}V^3(\cdot),\ \lambda_2\kappa^1\ +\ \lambda_3\beta^3\nabla_{\sigma^2}V^3(\cdot),\ \lambda_2\kappa^1\ +\ \lambda_3\beta^3\nabla_{\sigma^2}V^3(\cdot),\ 0,\ 0,\ -\lambda_3 x^3,\ -\lambda_3 x^3,$

$\lambda_3(1-l^3)(1+e^2),\ \lambda_3(1-l^3)(1+e^2),\ \lambda_3\kappa^2,\ \lambda_3\kappa^2,\ 0,\ \lambda_1+\lambda_2+\lambda_3\rangle$

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