Threshold Integrated Moving Average Models

(Does size matter? May be so)

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0. Outline

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  - Impulse Response Function
- Estimation
- Hypothesis Testing
- Application to Stock Prices (Hasbrouck (93) model on measuring price efficiency)
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I. Introduction

Q: Who Is the KING of the Time Series Models?

The Random Walk

$$y_t = y_{t-1} + \epsilon_t$$

- All the Kings have problems
- In the Random Walk ALL the shocks are Permanent and/or at every t there is always a permanent shock

$$\frac{\partial y_{t+k}}{\partial \epsilon_t} \neq 0 \text{ for } k \longrightarrow \infty$$

Solutions

- Some Standard Solutions: Permanent and Transitory decompositions
  - Uncorrelated Unobserved components (UC-0)
  - Perfect Correlated UC (Beveridge and Nelson (1981))
  - Correlated Unobserved components (UC-R)
- Some Problems:
  - Some of these decompositions may not always exist
  - In some decompositions all the shocks are permanent
  - They need identification assumptions that are not testable
  - At every t there is always a permanent shock

- Some Recent Solutions
  - Stochastic Permanent Breaks (Engle and Smith 1999)

$$y_t = m_t + \epsilon_t$$
$$m_t = m_{t-1} + q_{t-1}\epsilon_{t-1}$$
$$q_t = q(\epsilon_t)$$

    * The magnitude of the persistence depends on the size of the shock
    * Problem: All the shocks are permanent

  - TAR and TMA (Wecker (1981), Goojier (1998) and Guay and Scaillet (2003)) models

    * The magnitude of the persistence is regime-dependent
    * Problems: All the shocks are whether permanent or transitory

What are we looking for?

A type of Random Walk model with

- Two types of shocks:
  - ⋆ Permanent and Transitory
- Shock's identification assumptions that are
  - ⋆ Testable
  - ⋆ Coming from economic common sense

What is our proposal?

TIMA Models

II. TIMA Models

Threshold Integrated Moving Average Model (TIMA):

$$(1-L)y_t = x_t = \begin{cases} \epsilon_t - \theta_1 \epsilon_{t-1} & \text{if } |z_{t-1}| > r \\ \epsilon_t - \theta_2 \epsilon_{t-1} & \text{if } |z_{t-1}| \leq r \end{cases}$$

or

$$(1-L)y_t = x_t = \epsilon_t - \theta(z_{t-1})\epsilon_{t-1} \qquad (1)$$

with

$$\theta(z_t) = \begin{cases} \theta_1 & \text{if } |z_t| > r \\ \theta_2 & \text{if } |z_t| \leq r \end{cases}$$

Two types of TIMA models according to whether the threshold variable is observable or not:

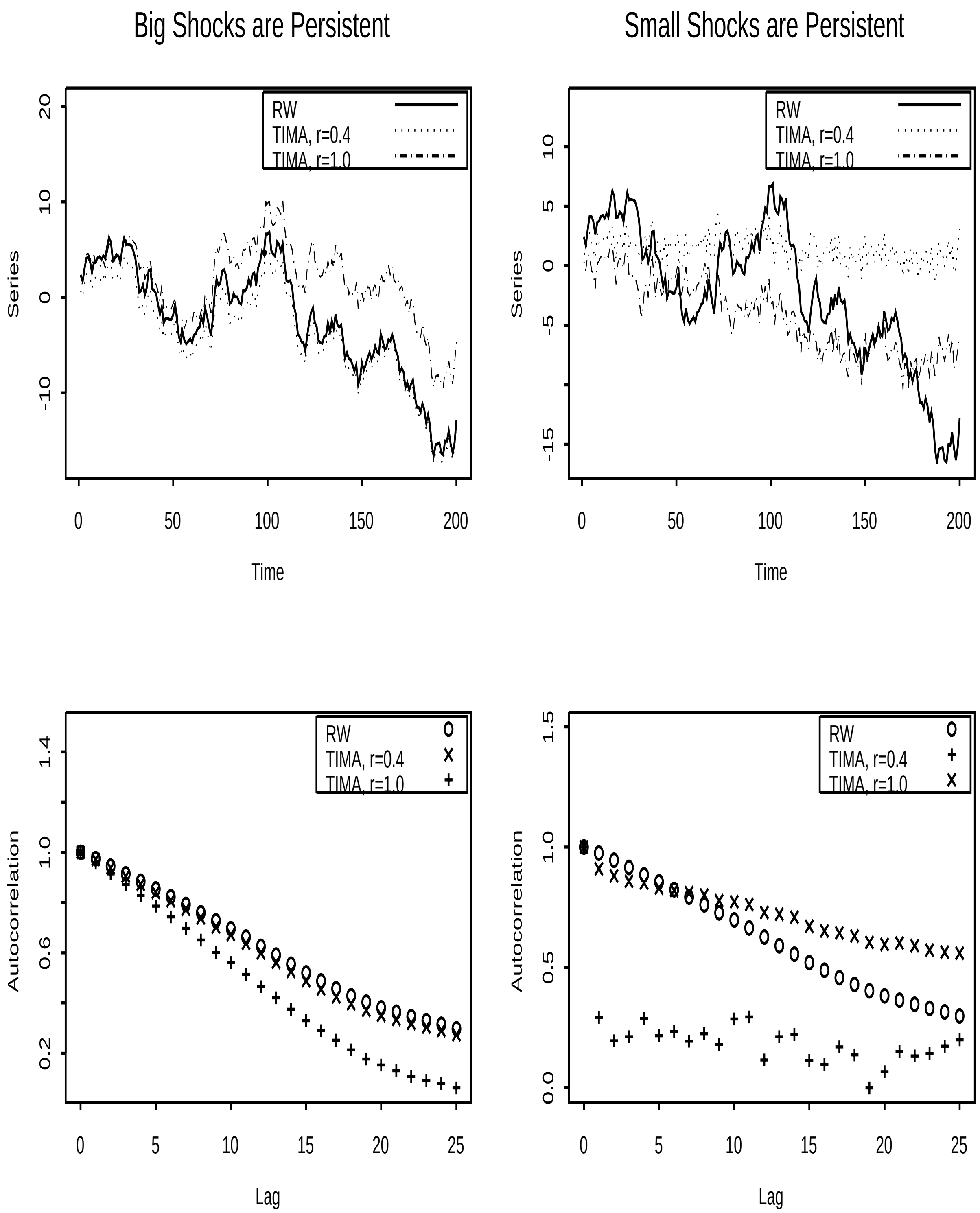
- **Observable TIMA**
  - Example:

$$z_t = (1 - L)y_t$$

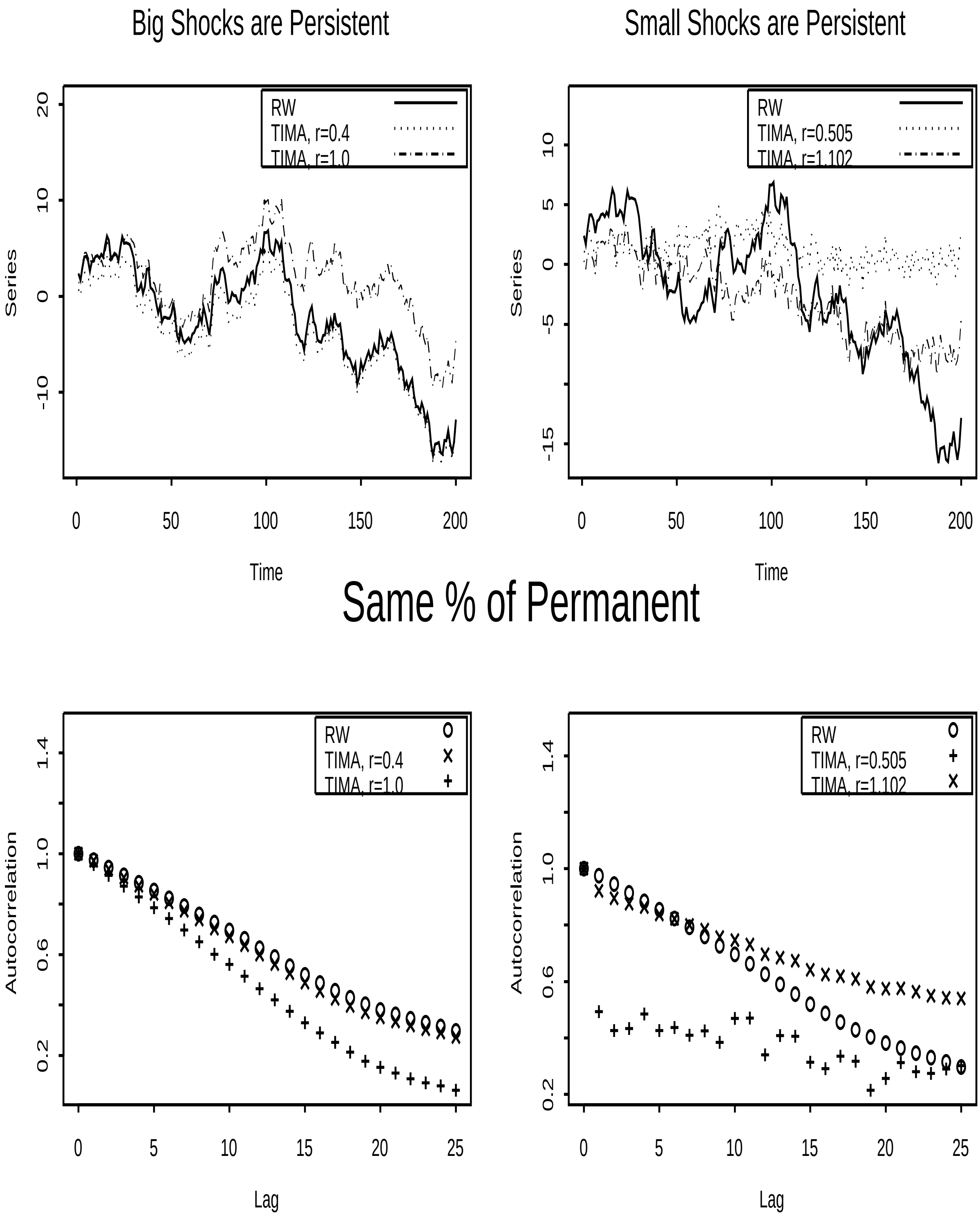
- **Unobservable TIMA**
  - Example: TIMA-shock

$$z_t = \epsilon_t$$

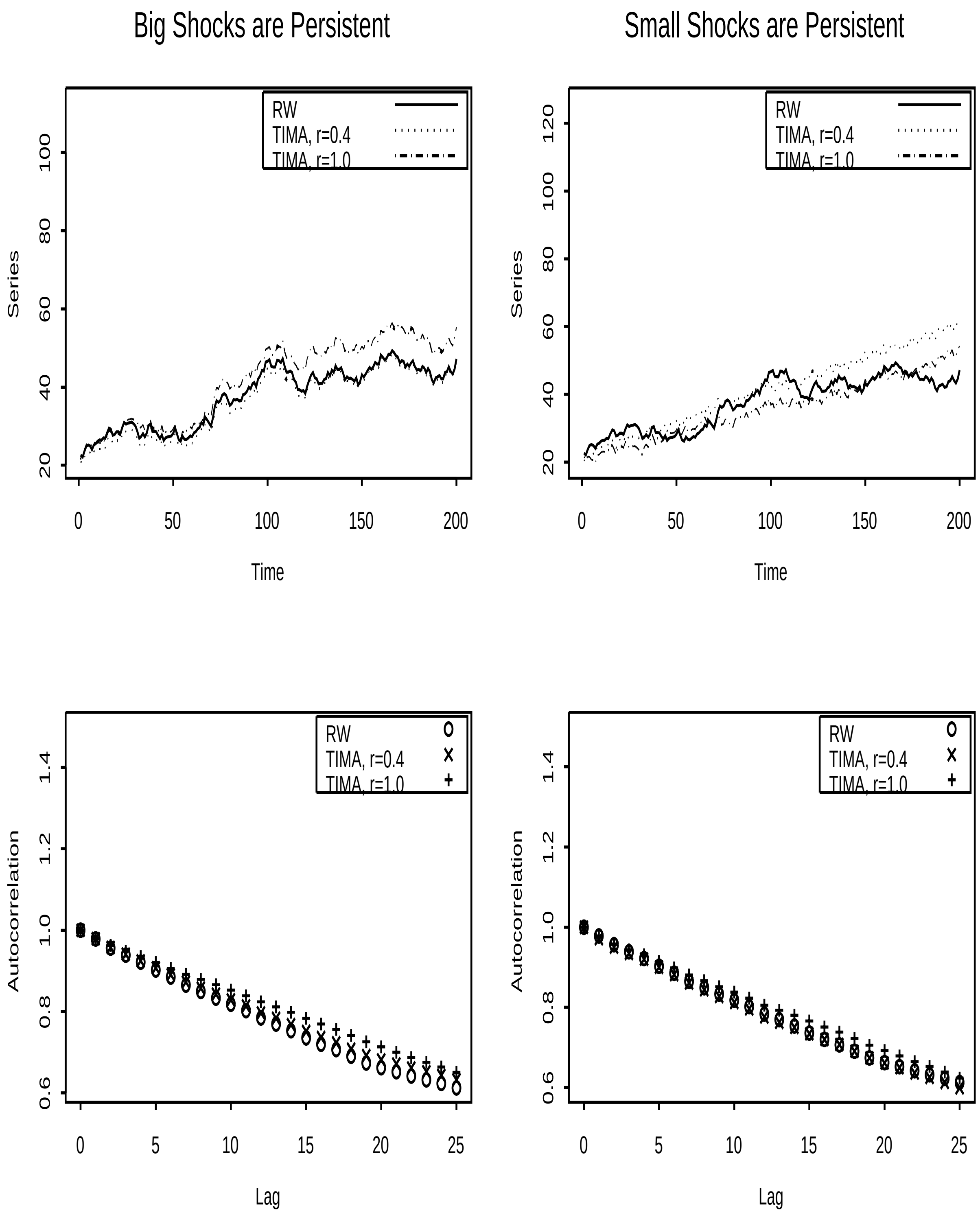
Some Graphical Examples



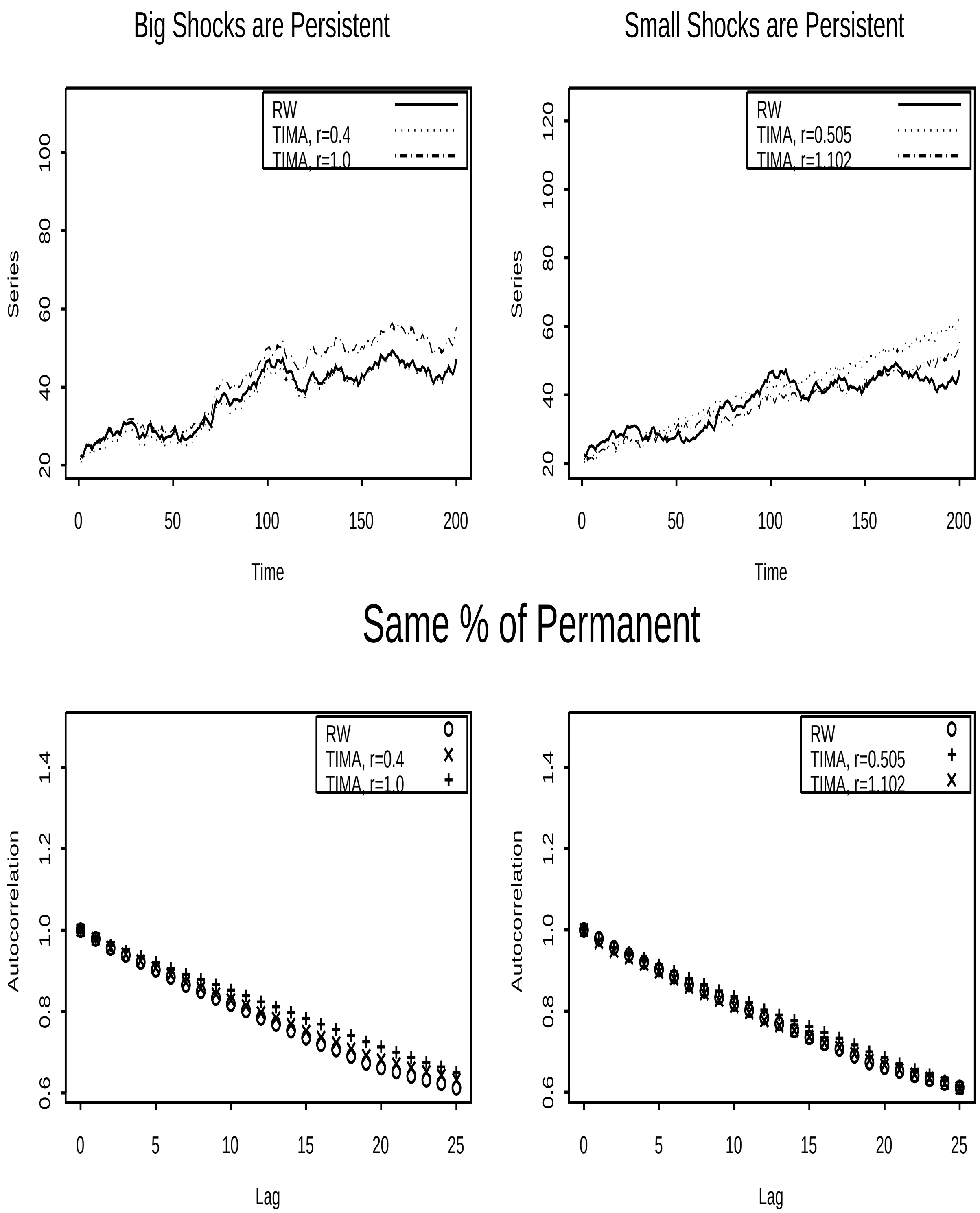
Some Graphical Examples (cont)



Some Graphical Examples (cont)



Some Graphical Examples (cont)



III. Assumptions

General Assumptions

G.0 ε_t iid $(0, \sigma_\varepsilon)$, with density $\infty > f_\varepsilon(\varepsilon_t) > 0$ $\forall \varepsilon_t$ and $\|\varepsilon_t\|_{2\gamma} < \infty$ for some $\gamma > 2$

G.1 $\left|\theta_1^0 - \theta_2^0\right| = \partial^0 > 0$

G.2 z_t is strictly stationary and $\alpha - mixing$ of size $-a$, with $a > 2/3$

G.3 $0 < \underline{p} \leq P\left(|z_t| < r / \Im_{t-1}\right) \leq \overline{p} < 1$, with $r \in (0, \overline{r})$

Interpretative Assumptions

I.0 In each threshold regime the shocks are m.d.s $E\left[\varepsilon_{t+j} 1\left(|z_{t+j}| > r\right) / \Im_{t-1}\right] = 0$ for $j \geq 0$

I.1 z_t is not Granger caused in mean by ε_t, in the sense of $E\left(z_t / \varepsilon_t, \Im_{t-1}\right) = E\left(z_t / \Im_{t-1}\right)$

Assumptions for Observable TIMAs

Invertibility

A.0 $E\left(\theta^2\left(z_t\right)/\Im_{t-1}\right) < 1$

Consistency and Testing

A.1 Parameter space $\theta^0_{(1\times 3)} \in \Theta$ is defined by
$\Theta = [-1+\delta, 1-\delta] \times [-1+\delta, 1+\delta'] \times (0, \overline{r}]$
s.t $E\left(\theta^4\left(z_t\right)/\Im_{t-1}\right) \leq \overline{\lambda} < 1, \quad \forall\theta \in \Theta$
with δ and $\delta' > 0$

A.2 Conditional moment bound
$\max_{\{z',z\}}$
$E\left(\left|\varepsilon_t\right|^{2\gamma} \Big/ z' \leq |z_t| \leq z\right) \leq \overline{\sigma}^{2\gamma}_{\varepsilon/z} < \infty,$
with $\gamma \geq 1$, z' and z two different values of z_t

A.3 Full rank condition
$\min_{\{z',z\}}$
$E\left(\left|\varepsilon_t\right|^{2} \Big/ z' \leq |z_t| \leq z\right) \geq \underline{\sigma}^{2}_{\varepsilon/z} > 0$, and
$mv \leq E\left(1\left(r < |z_t| < r+v\right)/\Im_{t-1}\right) \leq Mv,$
with $m > 0$, $v > 0$, $M < \infty$, and z' and z two different values of z_t

Assumptions for TIMA-shock

Invertibility

A.4 $\lambda_1 = [\partial r f^m + E(|\theta(\varepsilon_{t-1})|)] < 1$, and $|\theta_1| < 1$; where $\partial = |\theta_1 - \theta_2|$, and $f^m = \max_e (f_\varepsilon(-r+e) + f_\varepsilon(r+e))$

Consistency and Testing

A.5 The parameter space $\theta^0 \in \Theta$ is defined by $\Theta = [-1+\delta, 1-\delta] \times [-1+\delta, 1] \times (0, \overline{r}]$

s.t.

$\lambda_1^* = [\partial r f^{m,*} + \lambda_2^*(\theta_1, \theta_2, r)] < 1 \quad \forall \theta \in \Theta$,

with $f^{m,*} = 4 \max_e f_\varepsilon(e)$,

$\lambda_2^*(\theta_1, \theta_2, r) = |\theta_1|(1 - \overline{p}(r)) + |\theta_2| \overline{p}(r)$,

$\sup_k P(|\varepsilon_t + k| < r) \leq \overline{p}(r)$,

and $|\theta_1| < 1$

III. Properties

INVERTIBILITY

- We focus on $(1-L)y_t = x_t$
- We use the general invertibility definition introduced by Granger and Andersen (1978) improved by Hallin (1980)

Definition (Granger and Andersen): The process $x_t = g(x_{t-1}, \varepsilon_{t-1}, ..., x_{t-p}, \varepsilon_{t-p}) + \varepsilon_t$ will be invertible if

$$\lim_{t\to\infty} E\left(e_t^2\right) = 0$$

with

$$e_t = \varepsilon_t - \widehat{\varepsilon}_t = \varepsilon_t - (x_t - g(x_{t-1}, \widehat{\varepsilon}_{t-1}, ..., x_{t-p}, \widehat{\varepsilon}_{t-p}))$$

RESULTS

Theorem 1 (Observable TIMA) Given G.0 and A.0 the process $\{x_t\}$ is invertible.

Theorem 2 (TIMA-shock) Given G.0 and A.4 the process $\{x_t\}$ is invertible.

Persistence

Persistence: It is the effect of a shock, ε_t, at time t, in the future sample path of the series, $\{y_{t+k}\}_{k=0}^{\infty}$.

Permanent shock: If this effect in y_{t+k} does not vanish when $k \to \infty$.

To analyze this property we use the **General Impulse Response Function** (Koop, Pesaran and Potter(1996), and Potter (2000)) defined as:

$$GI\left(k, \varepsilon_t, w_{t-1}\right) =$$

$$E\left[y_{t+k}/\, \varepsilon_t, w_{t-1}\right] - E\left[y_{t+k}/\, w_{t-1}\right],$$

$$k = 0, 1, ...,$$

with w_{t-1} a particular history of $\Im_{t-1}$.

Some Results

Under the assumptions **I.0** or **I.1**, the TIMA model (1) has the following GI:

$$GI\,(k>0,\varepsilon_t,w_{t-1}) = [(1-\theta(z_t))]\,\varepsilon_t$$

Case: $|\theta_1|<1, \theta_2=1$

$$GI\,(k>0,\varepsilon_t,w_{t-1}) = \begin{cases} (1-\theta_1)\,\varepsilon_t & \texttt{if} \quad |z_t|>r \\ 0 \quad \texttt{if} \quad |z_t|\le r \end{cases}$$

Case: $\theta_1=0, \theta_2=1$

$$GI\,(k>0,\varepsilon_t,w_{t-1}) = \begin{cases} \varepsilon_t & \texttt{if} \quad |z_t|>r \\ 0 \quad \texttt{if} \quad |z_t|\le r \end{cases}$$

Some Simulations

TIMA Model:

$$(1-0.5L)(1-L)y_t = \begin{cases} \epsilon_t - 0.5\epsilon_{t-1} & \text{if } |\epsilon_{t-1}| > 0.6 \\ \epsilon_t - \epsilon_{t-1} & \text{if } |\epsilon_{t-1}| \leq 0.6 \end{cases}$$

TAR Model:

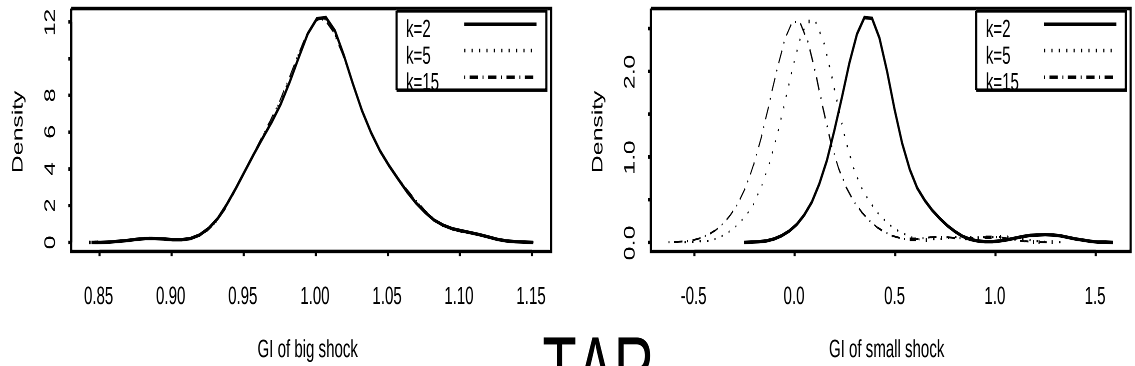
$$y_t = \begin{cases} y_{t-1} + \epsilon_t & \text{if } |\epsilon_{t-1}| > 0.6 \\ 0.5y_{t-1} + \epsilon_t & \text{if } |\epsilon_{t-1}| \leq 0.6 \end{cases}$$

IMA Model:

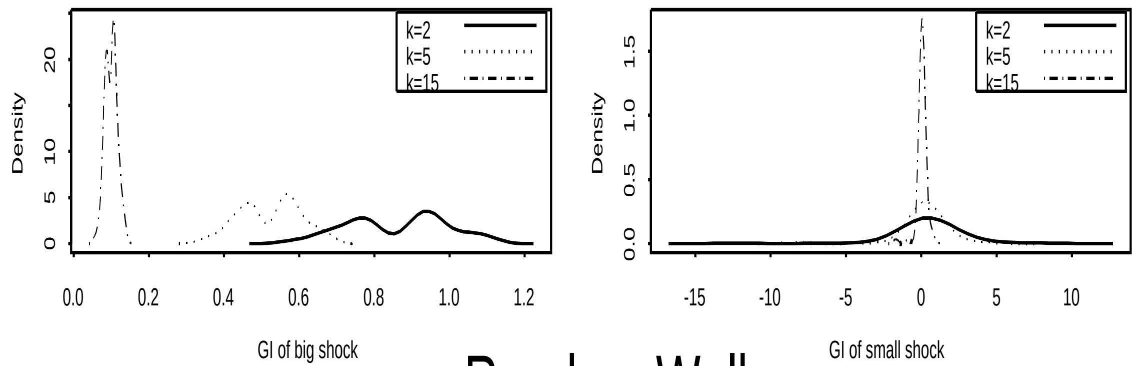
$$y_t = y_{t-1} + \epsilon_t - 0.2\epsilon_{t-1}$$

$$GI\left(k, \varepsilon_t, w_{t-1}\right)/\varepsilon_t$$

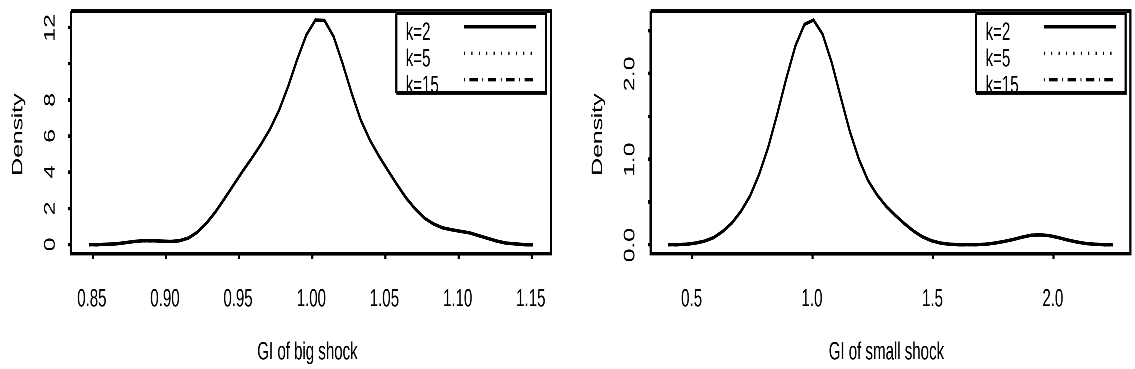
TIMA



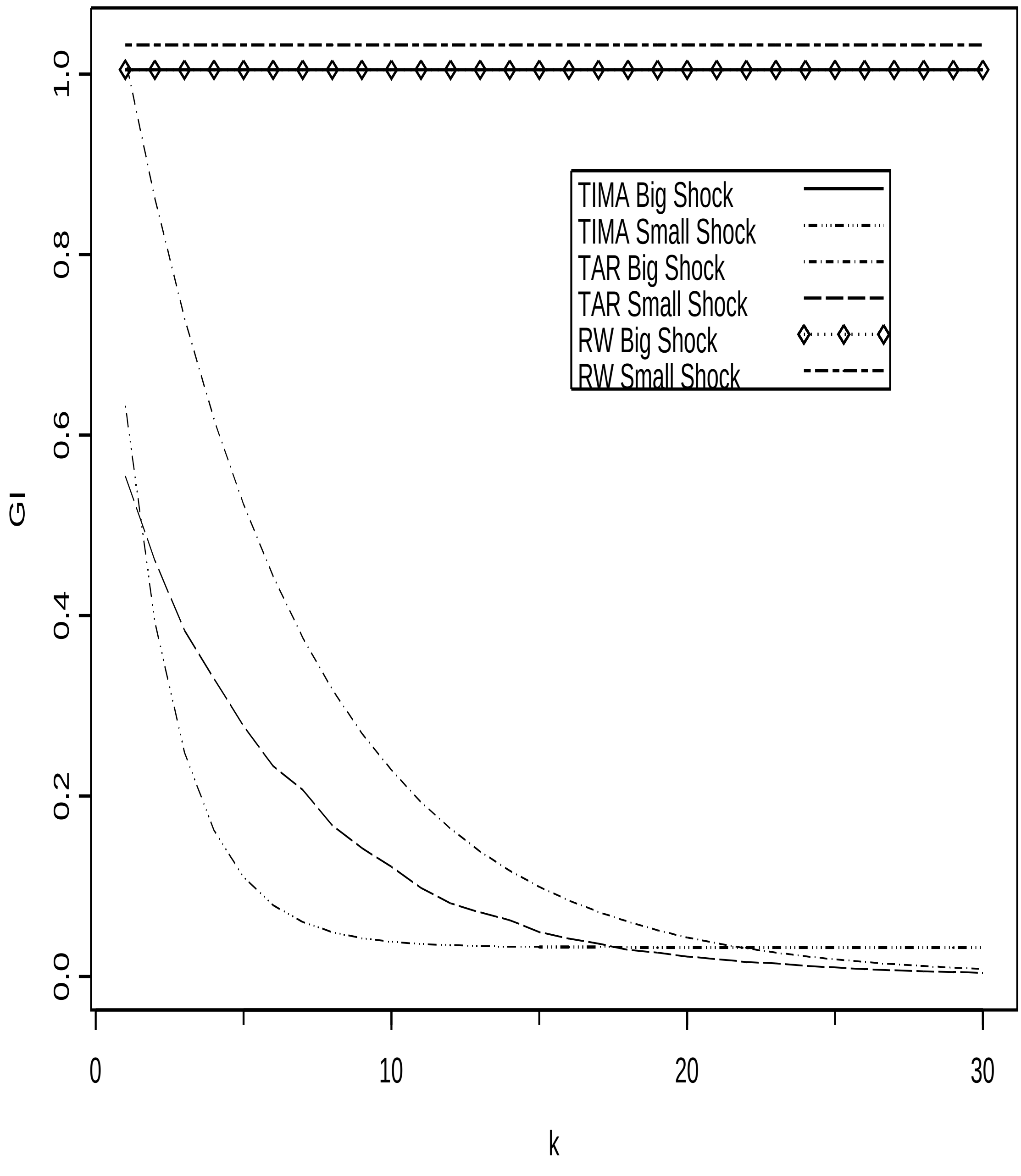
TAR



Random Walk



Mean($GI\left(k, \varepsilon_t, w_{t-1}\right)/\varepsilon_t$)



IV. Estimation

Method: Least Squares

Objective Function:

$$Q_T(\theta) = \sum_{t=1}^{T} e_t^2(\theta)$$

with $e_t = \theta(z_{t-1}) e_{t-1} + x_t$

Results for Observable TIMAs

Theorem 3 Under assumptions $G.0 - G.4$ and $A.1 - A.3$,

$\widehat{\theta}_{i,T} = \theta_i^0 + O_p\left(T^{-1/2}\right)$ and $\widehat{r}_T = r_0 + O_p\left(T^{-1}\right)$.

Results for TIMA-Shock

Theorem 4 Under assumptions $G.0 - G.1$ and $A.5$,

$\widehat{\theta}_{i,T} = \theta_i^0 + O_p\left(T^{-1}\right)$ and $\widehat{r}_T = r_0 + O_p\left(T^{-1}\right)$.

V. Inference

Results for Observable TIMA

Theorem 5 (r known) Under assumptions $G.0 - G.4$, $A.1 - A.3$, with r known, and $H\left(\theta^0\right)$ being a positive matrix,

$$T^{1/2}\left(\widehat{\theta} - \theta^0\right) \xrightarrow{d} N\left(0, 4H^{-1}\left(\theta^0\right)\Omega H^{-1}\left(\theta^0\right)\right).$$

Theorem 6 (r unknown) Under assumptions $G.0 - G.4$ and $A.1 - A.3$ with r estimated by LS, and $H\left(\theta^0\right)$ being a positive matrix,

$$T^{1/2}\left(\widehat{\theta}\left(\widehat{r}\right) - \theta^0\right) \xrightarrow{d} N\left(0, 4H^{-1}\left(\theta^0\right)\Omega H^{-1}\left(\theta^0\right)\right).$$

Results for TIMA Shock

Theorem 7 Under assumptions $G.0 - G.4$ and $A.5$, $z_t = \widehat{\varepsilon}_t$, with r estimated by LS and $H\left(\theta^0\right)$ being a positive matrix,

$$T^{1/2}\left(\widehat{\theta}\left(\widehat{\varepsilon}\right) - \theta^0\right) \xrightarrow{d} N\left(0, 4H^{-1}\left(\theta^0\right)\Omega H^{-1}\left(\theta^0\right)\right),$$

where $\widehat{\theta}\left(\widehat{\varepsilon}\right)$ is a second step estimator of θ^0.

Testing Strategy

Step 1: Test the Threshold Hypothesis $\theta_1 = \theta_2$

- Obtain the Test Statistic, with $R = (1, -1)$,

$$\mathrm{W}_T(r) = \left(R\widehat{\theta}_T(r)\right)' \left[RVar\left(\widehat{\theta}_T(r)\right)R'\right]^{-1}\left(R\widehat{\theta}_T(r)\right)$$

$$W_T = \sup_r W_T(r) \Longrightarrow \; Something$$

- Obtain the $p - values$ by Bootstrap methods

Step 2: Transitory Hypothesis $\theta_j = 1$ for some $j = 1, 2$

- Obtain the Test Statistic, with $R = (0, 1)$,

$$\mathrm{W}_T(\widehat{r}) = \left(R\widehat{\theta}_T(\widehat{r}) - 1\right)' \left[RVar\left(\widehat{\theta}_T(\widehat{r})\right)R'\right]^{-1}\left(R\widehat{\theta}_T(\widehat{r}) - 1\right)$$

$$W_T \xrightarrow{d} \chi^2(1)$$

Alternative Method for Testing the Threshold Hypothesis

- Adjust an IMA model to y_t:

$$(1-L)y_t = \widehat{\varepsilon}_t - \widehat{\theta}\widehat{\varepsilon}_{t-1}$$

- Regress $\widehat{\varepsilon}_t$ on $\widehat{\varepsilon}_{t-1}$, and obtain R^2_R
- Regress $\widehat{\varepsilon}_t$ on $\widehat{\varepsilon}_{t-1}$, $1(|z_{t-1}| < r)\widehat{\varepsilon}_{t-1}$, and obtain $R^2_{UR}(r)$
- Construct $W_T(r) = KF_T(r)$, where

  $F_T(r) = \frac{(R^2_{UR}(r)-R^2_R(r))(T-K)}{(1-R^2_{UR}(r))K}$,

  and obtain the $p-value$ of $\sup_r(W_T(r))$ by bootstrap methods

Extra Issues

- **More Dynamics**

  Always in the AR part

- **Number of Regimes**

  "Estimation and Model Selection Based Inference in Single and Multiple Threshold Models" (Gonzalo and Pitarakis), *Journal of Econometrics* (2002)

- **Inference on the Threshold Parameter** r

  "Sample splitting and threshold estimation," (Caner and Hansen) *Econometrica*, (2000)
  "Subsampling Inference in Threshold Autoregressive Models" (Gonzalo and Wolf)(2003),(http://halweb.uc3m.es/jgonzalo/)

A Simulation Exercise

- **Case A (Observable TIMA)**: z_t *iid*

For Size, DGP_1 :

$$y_t = y_{t-1} + \varepsilon_t,$$

For Power, DGP_2 :

$$(1-L)y_t = \begin{cases} \epsilon_t & \text{if } |z_{t-1}| > 0.3 \\ \epsilon_t - \epsilon_{t-1} & \text{if } |z_{t-1}| \leq 0.3 \end{cases}$$

- **Case B (TIMA-Shock)**: $z_t = \varepsilon_t$

For Size, DGP_1 :

$$y_t = y_{t-1} + \varepsilon_t$$

For Power, DGP_2 :

$$(1-L)y_t = \begin{cases} \varepsilon_t & \text{if } |\varepsilon_{t-1}| > 0.3 \\ \varepsilon_t - \varepsilon_{t-1} & \text{if } |\varepsilon_{t-1}| \leq 0.3 \end{cases}$$

Results for Observable TIMA

H_0: $\theta_1 = \theta_2 = 0$ - No Threshold				
		Sup	Mean	Exp
T=200	$\alpha = 5\%$	0.045	0.035	0.035
	$\alpha = 10\%$	0.105	0.095	0.105
T=400	$\alpha = 5\%$	0.050	0.035	0.040
	$\alpha = 10\%$	0.090	0.070	0.090

H_a: $\theta_1 = 0, \theta_2 = 1, r = 0.3$				
		Sup	Mean	Exp
T=200	$\alpha = 5\%$	0.96	0.86	0.94
	$\alpha = 10\%$	0.98	0.91	0.97

Number of Monte Carlo replications, N=200

Number of Bootstrap replications, B=500

Results for TIMA Shock

H_0: $\theta_1 = \theta_2 = 0$				
		Sup	Mean	Exp
T=200	$\alpha = 5\%$	0.035	0.07	0.045
	$\alpha = 10\%$	0.075	0.105	0.100
T=400	$\alpha = 5\%$	0.065	0.075	0.06
	$\alpha = 10\%$	0.115	0.12	0.115

Power Table for TIMA Shock Model under construction !!!!

Number of Monte Carlo replications, N=200

Number of Bootstrap replications, B=500

Application: Stock Prices

We apply the **TIMA Shock** for measuring the deviations between actual transaction prices and implicit efficient prices, following a method developed by Hasbrouck (1993).

We decompose the transaction price into the sum of two components:

$$p_t = m_t + s_t$$

Efficient price: $m_t = m_{t-1} + w_t$

Pricing error: $s_t = \alpha w_t + \eta_t$

The pricing error can come from diverse microstructure effects and it has been described by Lo and MacKinlay (1988), Frama and French (1988) and Poterba and Summers (1988) among others.

The proposed **measure** of the deviation is the **dispersion of the transitory components,** σ_s.

Unfortunately without assumptions about s_t this measure, σ_s, is not identified.

The standard identification conditions are:

- $\alpha = 0$, $\eta_t \neq 0 \Rightarrow$ (UC-0)
- $\alpha > 0$, $\eta_t = 0 \Rightarrow$ (B-N)

We propose the following model:

$$m_t = m_{t-1} + 1(|w_t| > r)w_t$$

$$s_t = 1(|w_t| \leq r)w_t$$

From it is straightforward to obtain the following representation:

$$(1-L)p_t = \begin{cases} w_t & \text{if } |w_{t-1}| > r \\ w_t - w_{t-1} & \text{if } |w_{t-1}| \leq r \end{cases}$$

We test this model with S&P500 daily series.

Logarithm of S&P500, 2 January 1998 - 29 July 2003

Results

H_0: Linear model is REJECTED (p-value=0.061)

H_0: Unit root in the MA part of the small shocks regime is NOT REJECTED (p-value=0.46)

CONCLUSION: TIMA-shock model is NOT rejected. Therefore SIZE matters.

Estimated Model under linearity:

$$p_t - p_{t-1} = \widehat{\varepsilon}_t - 0.017\widehat{\varepsilon}_{t-1}$$

Estimated Model under TIMA-shock structure:

$$p_t - p_{t-1} = \widehat{\varepsilon}_t - 1(|\widehat{\varepsilon}_{t-1}| < 0.0043)\widehat{\varepsilon}_{t-1}$$

Proportion of Transitory Shocks: 0.27%.

CASE			
	$\alpha = 0$ $(UC - 0,$ $Watson(1986))$	$\eta_t = 0$ (BN)	TIMA
$\widehat{\sigma}_s \times 100$	0.181	0.024	0.126

VII. Conclusion

If you want to test whether SIZE matters or not for *permanent* and *transitory* issues, TIMA is the PERFECT model to use.