

Shape Homogeneity and Scale Heterogeneity of Downside Tail Risk*

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Abstract

We analyze the cross-sectional differences in the tail risk of equity returns and identify the drivers of tail risk. We provide two statistical procedures to test the hypothesis of cross-sectional downside tail shape homogeneity. An empirical study of 230 US non-financial firms shows that between 2008 and 2011 the cross-sectional tail shape is homogeneous across equity returns. The heterogeneity in tail risk over this period can be entirely attributed to differences in scale. The differences in scales are driven by the following firm characteristics: market beta, size, book-to-market ratio, leverage and bid-ask spread.

Keywords: Extreme Value Theory, Hypothesis Testing, Tail Index, Tail Risk

JEL Classification: C12, G11, G12

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1 Introduction

Downside risk of financial assets plays an important role and gives rise to concern for investors and risk managers. For example, in the banking industry the popularity of regulating the downside risk, measured by Value-at-Risk (VaR), has emerged through the regulatory accords such as Basel II and III. The unfolding of the recent financial crisis further raises the necessity of assessing the risks of potential extreme losses. From a risk management viewpoint, this calls for a comprehensive understanding of tail risk amongst regulators. From a corporate finance viewpoint, by managing tail risk, firms can attract long-term investors that are risk averse; thus, enhancing their access to capital markets.

This paper analyzes downside tail risk in stock returns across firms. First, we test the tail shape equivalence hypothesis. That is, we test whether the tail shapes in the downside tail distribution are cross-sectionally homogeneous. Second, we provide empirical evidence that when homogeneity of the tail shape holds across stock returns, the cross-sectional heterogeneity in downside tail risk is determined by differences in the scale. Lastly, we analyze potential firm-level determinants of downside tail risk.

In classic asset pricing theory, stock returns are assumed to follow a Gaussian distribution. Recent empirical literature shows that when modeling the distribution of financial asset returns, particularly that of equity returns, the Gaussian distribution underestimates the probability of extreme losses. Instead, there is large consensus that the tail distribution of asset returns is heavy-tailed, see, e.g., Mandelbrot (1963) and Jansen and De Vries (1991). Heavy-tails refer to the fact that the tail region of the distribution function exhibits a power law decay, as opposed to the exponential decay of the Gaussian distribution. Mathematically, denote the return of a financial asset by R , with the distribution function $F(x) = \Pr(R \leq x)$. The distribution function F is heavy-tailed if its left tail can be approximated by a power law as

$$F(-x) = \Pr(R \leq -x) \sim Ax^{-\alpha}, \quad \text{as } x \rightarrow \infty, \quad (1.1)$$

where α is the tail shape parameter, commonly referred to as the *tail index*, and A indi-

cates the *scale* of the distribution. In contrast to the Gaussian distribution, heterogeneity in the downside tail risk of a heavy-tailed distribution is manifest via differences in the tail shape and scale of the distribution. Note that in the Gaussian setup, tail risk is driven entirely by the variance.

The VaR measures the magnitude of an extreme event at a given tail probability within a fixed time period. Define, $VaR(p)$ as the VaR of R with tail probability p as $\Pr(R > VaR(p)) = p$. In the heavy-tailed framework, $VaR(p)$ is jointly determined by the tail index and scale as follows,

$$VaR(p) \approx \left(\frac{A}{p}\right)^{\frac{1}{\alpha}}. \quad (1.2)$$

Under the safety-first preference, Moore et al. (2012) show that if investors are sufficiently risk averse, that is, they guard amply enough against large, low probability high losses, the tail indices among different stock returns in a common market must be equivalent. This is called the *tail index equivalence* hypothesis. The intuition behind the tail index equivalence is as follows. For an extremely low probability level p , the asset with the lower tail index will always have a higher VaR. In addition, the VaRs corresponding to different tail indices are diverse when the probability level p tends to zero. Consequently, the differences in VaRs cannot be compensated by heterogeneity in expected returns. Therefore, if the admissible probability level of failure of investors is sufficiently low, tail index equivalence must hold. If tail index equivalence holds, the expected return in the safety-first pricing model of Arzac and Bawa (1977) can compensate for heterogeneity in the scale parameter.

Prior theoretical literature typically assumed the tail index equivalence hypothesis without justification, see, e.g. Hyung and de Vries (2005), Hyung and de Vries (2007), Ibragimov and Walden (2008) and Zhou (2010). For the most part, robust empirical testing of tail index equivalence has been lacking in the literature. Weak empirical evidence of tail index equivalence can be found in Jansen and De Vries (1991). However, their test bears two immediate drawbacks. First, they test the null hypothesis that the tail indices of two assets, α_1 and α_2 , are jointly equal to a hypothetical value α , whereas in

application the null hypothesis to be tested is usually $H_0 : \alpha_1 = \alpha_2$. Second, they impose a maintained assumption that stock returns are cross-sectionally independent. This is not plausible when conducting the test for stocks traded in same market. A decade later, Jondeau and Rockinger (2003), using likelihood ratio tests, provide empirical evidence of tail index equivalence for stock returns in a common geographical location using data from 20 countries. Although, their tests allow for dependence across time, they as well do not take into account the possible cross-sectional dependence across stock returns. On the contrary, the cross-sectional dependence across stock returns is a consequence of well known asset pricing models, such as the Capital Asset Pricing Model (CAPM) and Arbitrage Pricing Model (APT). The dependence in these models stems from common factors such as the market factor. Therefore, we construct testing procedures to accommodate this dependence feature.

We allow for tail dependence across stock returns by imposing a multivariate Extreme Value Theory (EVT) setup and construct two empirical tests: the *Minmax* and *Benchmark* statistical tests. To show the power of our testing procedures, we run extensive simulations. Then we apply both tests to a collection of 230 US non-financial firms from January 2000 to the end of 2011. We do not reject the tail index equivalence hypothesis from 2008 to 2011. In the periods where tail index equivalence holds, we apply a similar set of empirical tests to test for *tail risk equivalence*. This is equivalent to a joint test of tail index and scale homogeneity. Both tests reject tail risk equivalence over this period. This result provides weak evidence of tail scale heterogeneity.

To further detect tail scale heterogeneity, we analyze whether the tail scales are differentiated across firms with different firm-level characteristics. If heterogeneity in tail risk tail scale is solely due to estimation error, then we should not find any relation between the tail scale and firm-level characteristics. Through a series of cross-sectional regressions, we find that the book-to-market ratio, bid-ask spread and market beta are positive and significantly related to the scale parameter, while size and leverage are negative and significantly related to the scale. Hence, heterogeneity in the scale parameter is explained by firm-level characteristics. We thus confirm that the tail scale is cross-sectionally het-

erogeneous. In addition, we also highlight the potential drivers of downside tail risk.

These findings are comparable to the literature on the firm-level drivers of cross-sectional expected stock returns. The general intuition is that investors are compensated for higher risk taking with a premium in the expected returns. For our selection of the potential cross-sectional drivers of tail risk we refer to the following literature: Banz (1981), Fama and French (1992), Bhandari (1988), Basu (1977), Datar et al. (1998) and Amihud and Mendelson (1986).

The rest of the paper is organized as follows. In Section 2, we establish statistical methods to test tail index equivalence under cross-sectional dependency. A simulation study shows the performance of our testing procedure. We apply these two tests to 230 equities traded in US market in Section 3. In Section 4, we establish statistical methods on testing tail risk equivalence. In Section 5, we further analyze the drivers of tail risk at the firm level for those periods in which we fail to reject the tail index equivalence hypothesis. Section 6 concludes.

2 Statistical Tests: Tail Index Equivalence

2.1 The setup of the tests

We construct two statistical tests to determine whether the tail indices of different stock return series are equivalent under cross-sectional dependence. A first attempt on testing such a hypothesis has been conducted under a bivariate setup in Jansen and de Vries (1991) by assuming independence across stock returns. We begin by reviewing their procedure.

First, the tail indices of loss returns of stocks are estimated by the so-called Hill estimator as follows. Denote the return series of two stocks as $R_{i,t}$, $i = 1, 2$ and $t = 1, 2, \dots, n$. Denote the loss return as $X_{i,t} = -R_{i,t}$. By ranking all $\{X_{i,t}\}_{t=1}^n$ as $X_{i,(n)} \geq X_{i,(n-1)} \geq \dots \geq X_{i,(1)}$, the tail index α_i for stock return i is estimated by

$$\frac{1}{\hat{\alpha}_i} = \frac{1}{k} \sum_{j=1}^k \log X_{i,(n-j+1)} - \log X_{i,(n-k)},$$

where $k := k(n)$ is an intermediate sequence such that $k \rightarrow \infty$ and $k/n \rightarrow 0$ as $n \rightarrow \infty$. This estimator is proposed in Hill (1975). Mason (1982) proves the consistency of the Hill estimator under mild conditions. The asymptotic normality of the estimator had been studied in various authors, see e.g. Hall (1982), Davis and Resnick (1984), Haeusler and Teugels (1985). They find that

$$\sqrt{k} \left(\frac{\hat{\alpha}_i}{\alpha_i} - 1 \right) \xrightarrow{d} N(0, 1),$$

where $N(0, 1)$ refers to a standard normally distributed random variable. Later ? shows that the Hill estimator retains its properties for stationary data.

Next, in Jansen and de Vries (1991) test the null hypothesis $H_0 : \alpha_1 = \alpha_2 = \alpha$ for some given α . By assuming that the two stock return series are independent, the two estimators $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are independent. Hence, under the null that $\alpha_1 = \alpha_2 = \alpha$,

$$k_1 \left(\frac{\hat{\alpha}_1}{\alpha} - 1 \right)^2 + k_2 \left(\frac{\hat{\alpha}_2}{\alpha} - 1 \right)^2 \xrightarrow{d} \chi^2(2),$$

where k_i refers to the number of high order statistics used in estimation for α_i , and $\chi^2(2)$ refers to a random variable following the χ^2 distribution with degree of freedom 2.

The Jansen and de Vries test bears two immediate drawbacks. First, a hypothetical value of α in the null hypothesis has to be specified ex ante, whereas in an application a more suitable null hypothesis to be tested is $H_0 : \alpha_1 = \alpha_2$. Second, the maintained assumption that stock returns are independent is not plausible when conducting the test for stocks traded in same market over the same time period. The first drawback can be overcome by considering alternative testing statistics, while the second drawback calls for modeling the dependence structure.

We employ multivariate EVT to model the dependence structure across stock returns. Suppose $(X_{1,t}, X_{2,t})$, $t = 1, 2, \dots, n$ are observations from a bivariate distribution, indicated by random vector (X_1, X_2) . The EVT model in a bivariate setup is as follows. For

any $(x_1, x_2) \in [0, 1] \times [0, 1] / \{(0, 0)\}$,

$$\lim_{p \rightarrow 0} \frac{1}{p} \Pr(X_1 > VaR_1(px_1), X_2 > VaR_2(px_2)) = R(x_1, x_2),$$

where $VaR_i(p)$ refers to the VaR of X_i with tail probability p and $R(x_1, x_2)$ is a positive function defined on $[0, 1] \times [0, 1] / \{(0, 0)\}$. Such a model only imposes an assumption on the tail dependence between X_1 and X_2 without restricting the dependence at moderate level. This is sufficient for our purpose. We remark that all existing asset pricing models such as CAPM or other factor models satisfy such a dependence assumption provided that the pricing factors are heavy-tailed.

The following Proposition provides the joint limit distribution of the two Hill estimates $\hat{\alpha}_1$ and $\hat{\alpha}_2$, which is the theoretical foundation for constructing a valid test. The proof is postponed to the appendix.

Proposition 2.1 *Under a proper second order condition and choice of k sequence such that as $n \rightarrow \infty$,*

$$\sqrt{k} \left(\frac{\hat{\alpha}_i}{\alpha_i} - 1 \right) \xrightarrow{d} N(0, 1),$$

we have that,

$$\sqrt{k} \begin{pmatrix} \frac{\hat{\alpha}_1}{\alpha_1} - 1 \\ \frac{\hat{\alpha}_2}{\alpha_2} - 1 \end{pmatrix} \xrightarrow{d} N \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & R(1, 1) \\ R(1, 1) & 1 \end{pmatrix} \right).$$

Here $R(1, 1)$ is a usual tail dependence measure lying between 0 and 1, sometimes denoted by τ in literature (see De Jonghe (2010)). Multivariate EVT provides a consistent estimator for the correlation coefficient $\tau = R(1, 1)$ as follows.

$$\hat{\tau} = \frac{1}{k} \sum_{t=1}^n 1_{X_{1,t} > X_{1,(n-k)}} \text{ and } 1_{X_{2,t} > X_{2,(n-k)}}.$$

With the estimated covariance matrix, it is possible to simulate the joint limit distribution of the two Hill estimators.

Since we tend to compare the tail indices among a group of stocks, we extend the above

result to a broader context as follows. Denote the loss return for stock i and the market as $X_{i,t} = -R_{i,t}$, $i = 1, 2, \dots, d$ and $X_{M,t} = -R_{M,t}$, respectively, where $t = 1, 2, \dots, n$. Denote their tail indices as $\alpha_1, \dots, \alpha_d, \alpha_M$, with corresponding Hill estimators indicated by $\hat{\alpha}_1, \dots, \hat{\alpha}_d, \hat{\alpha}_M$. Similar to the bivariate case, one can prove that with proper second order condition and choice of the k sequence, as $n \rightarrow \infty$,

$$\sqrt{k} \left(\frac{\hat{\alpha}_1}{\alpha_1} - 1, \dots, \frac{\hat{\alpha}_d}{\alpha_d} - 1, \frac{\hat{\alpha}_M}{\alpha_M} - 1 \right)^T \xrightarrow{d} N(\mathbf{0}^T, \Sigma),$$

where $\mathbf{0}$ is a $(d+1) \times 1$ vector with zero component, and $\Sigma = (\tau_{i,j})_{(d+1) \times (d+1)}$ with $\tau_{i,j}$ indicating the $R(1,1)$ measure between stock i and j , $i, j = 1, \dots, d, M$.

We first construct a test for the null hypothesis $H_0 : \alpha_1 = \alpha_2 = \dots = \alpha_d$. Consider the following test statistic

$$T_1 = \frac{\max_{1 \leq i \leq d} \hat{\alpha}_i - \min_{1 \leq i \leq d} \hat{\alpha}_i}{\bar{\alpha}}, \quad \text{where } \bar{\alpha} = \frac{1}{d} \sum_{i=1}^d \hat{\alpha}_i.$$

Under the null hypothesis that $\alpha_1 = \dots = \alpha_d$, from the joint limit distribution of the Hill estimates, it follows that as $n \rightarrow \infty$, $\sqrt{k}T_1$ converges a limit distribution as $\max_{i=1}^d N_i - \min_{i=1}^d N_i$, where $(N_1, \dots, N_d)^T$ follows a d -dimensional normal distribution with mean zero and covariance matrix Σ_1 , which is a submatrix of Σ consisting of the first d rows and d columns.

Given the estimates of each bivariate $\tau_{i,j}$, we can simulate the random vector $(N_1, \dots, N_d)^T$ and consequently obtain the simulated limit distribution of $\sqrt{k}T_1$. The null hypothesis is rejected when a high value of T_1 is observed. More specifically, if $\sqrt{k}T_1$ is above a certain threshold determined from the simulated limit distribution. For details on a fast simulation procedure for the limit distribution, see Appendix 7.2. Since the test is based on comparing the minimum and the maximum of the estimated tail indices, we call this the *Minmax Tail Shape* test.

Next, we construct a test on the null hypothesis $H_0 : \alpha_1 = \alpha_2 = \dots = \alpha_d = \alpha_M$. This, of course, can be accomplished via the *Minmax Tail Shape* test by regarding the market return as the $(d+1)$ -th stock. Nevertheless, we can treat the market return as

a specific benchmark and construct a different test. Consider the following test statistic

$$T_2 = \sum_{i=1}^d \left(\frac{\hat{\alpha}_i}{\hat{\alpha}_M} - 1 \right)^2.$$

Similar to the discussion for the first test, under the null hypothesis that $\alpha_1 = \dots = \alpha_d = \alpha_M$, from the joint limit distribution of the Hill estimators, it follows that as $n \rightarrow \infty$, $\sqrt{k}T_2$ converges to a limit distribution as $\sum_{i=1}^d (N_i - N_M)^2$, where $(N_1, \dots, N_d, N_M)^T$ follows a d -dimensional normal distribution with mean zero and covariance matrix Σ . Similarly, we reject the null if $\sqrt{k}T_2$ is above a certain threshold determined from the simulated limit distribution. Since this test compares the estimated tail indices of the stocks to that of the benchmark, we call it the *Benchmark Tail Shape* test.

2.2 Simulations

Before applying the statistical tests to real data, we run a series of simulations to verify the performance of the testing procedures. We simulate data from two different samples and present the test results regarding each sample.

We use the Student-t distribution in all simulations¹. We simulate data with a built-in dependence structure in two different ways:

Sample 1. In the first sample, we simulate random vectors $(Y_{1,t}, \dots, Y_{d,t})$ with common marginal tail index and differentiated scales.

1. Generate $(d+1)n$ independent and identically distributed (i.i.d.) random variables $\{X_{i,t}\}$, $i = 1, \dots, d+1$, $t = 1, \dots, n$. Each $X_{i,t}$ follows the standard Student-t distribution with degrees of freedom α .
2. We use $\{X_{d+1,t}\}$, $t = 1, \dots, n$ as a common factor to generate a dependence structure as follows:

$$Y_{i,t} = \begin{cases} \lambda X_{d+1,t} + (1 - \lambda)X_{i,t}, & i = 1, \dots, d/2, t = 1, \dots, n, \\ \lambda X_{d+1,t} + (1 - \lambda) * 2X_{i,t}, & i = d/2 + 1, \dots, d, t = 1, \dots, n, \end{cases}$$

¹The Student-t distribution falls into the class of heavy-tailed distributions. Furthermore, the tail index associated with a given Student-t distribution is equal to its corresponding degree of freedom.

where $\lambda \in [0, 1]$ determines the degree of dependence across the observations².

Then, all $\{Y_{1,t}, \dots, Y_{d,t}\}$ share the same tail index α and are dependent for $\lambda > 0$. Note that their scales are cross-sectionally different. For instance, the ratio of the scale of $Y_{1,t}$ to that of $Y_{d/2+1,t}$ is $\frac{\lambda^\alpha + (1-\lambda)^\alpha}{\lambda^\alpha + [2(1-\lambda)]^\alpha}$.

Sample 2. In the second sample, we simulate random vectors $(Z_{1,t}, \dots, Z_{d,t})$ with differentiated tail indices.

1. Generate $\frac{dn}{2}$ i.i.d. random variables $\{X_{i,t}\}$, $i = 1, \dots, d/2$, $t = 1, \dots, n$. Each $X_{i,t}$ follows the standard Student-t distribution with a degree of freedom α_1 .
2. Generate $(\frac{d}{2} + 1)n$ i.i.d. random variables $\{X_{i,t}\}$, $i = d/2 + 1, \dots, d + 1$, $t = 1, \dots, n$. Each $X_{i,t}$ follows the standard Student-t distribution with a degree of freedom α_2 .
3. We again use $\{X_{d+1,t}\}$, $t = 1, \dots, n$ as a common factor to generate a dependence structure as follows.

$$Z_{i,t} = \lambda X_{d+1,t} + (1 - \lambda) X_{i,t}, \quad i = 1, \dots, d, \quad t = 1, \dots, n,$$

where $\lambda \in [0, 1]$.

When $\alpha_1 > \alpha_2$ and $\lambda > 0$, $\{Z_{1,t}, \dots, Z_{d,t}\}$ have the same tail index α_2 , but different scales³. When $\alpha_1 < \alpha_2$, $\{Z_{1,t}, \dots, Z_{d,t}\}$ have different tail indices. The tail index of $\{Z_{j,t}\}$ is α_1 for $j = 1, \dots, d/2$ and α_2 for $j = d/2 + 1, \dots, d$.

Both samples are generated with dimension $d = 100$ and number of observations $n = 1000$. Then we apply the *Minmax Tail Shape* and *Benchmark Tail Shape* tests outlined in Subsection 2.1 to the simulated observations. In the case of the *Benchmark Tail Shape* test, the variable $X_{101,t}$ is taken as the “benchmark” with tail index α_M . For each simulated data set, we run 11 different simulations with different degrees of dependence, e.g. $\lambda = 0, 0.05, \dots, 0.45, 0.5$.

²A non-zero value of λ creates tail dependence between two vectors of observations $(Y_{i,t}, Y_{j,t})$. The tail dependence measure, τ , is an increasing function of λ , for any given α . When $\lambda = 0$ we have $\tau = 0$.

³Zhou (2010) shows that in a linear combination of two heavy-tailed random variables, the heavier-tailed random variable (the lower tail index) determines the first-order tail index of the linear combination.

For each fixed value of λ , we generate 1000 simulations. In each simulation, we calculate a test statistic and corresponding p-value, pv_{al_j} , $j = 1, \dots, 1000$. The p-value is determined by comparing the test statistic with the simulated limit distribution from Appendix 7.2. Then, we calculate the percentage of simulations in which the p-value falls below the 5% level in each of the two tests by

$$\phi = \left(\frac{1}{1000} \sum_{j=1}^{1000} 1_{pv_{al_j} \leq 0.05} \right) \cdot 100\%.$$

An acceptable test should produce $\phi = 5\%$ if the null hypothesis is valid. If the null hypothesis is not valid, a high value of ϕ shows a high power of the test.

Table 1 reports the percentage of simulations in which the p-value falls below the 5% level. We first use two values of the tail index, $\alpha = 3$ and 4, to generate data as in Sample 1. For this sample, the null hypothesis holds. We observe that both tests are valid as the ϕ values are around 5%. Nevertheless, with a moderate tail dependence, i.e. a high value of λ , both of the *Minmax Tail Shape* and *Benchmark Tail Shape* tests reject the null hypothesis in less than 5% of the simulations. This indicates that both tests do not tend to have a high probability of type I error. For low λ , i.e. low tail dependence, the *Minmax Tail Shape* test performs better, because the *Benchmark Tail Shape* test rejects in more than 5% of the simulations.

Then, we generate data as in Sample 2 using the following pairings: $\{\alpha_1, \alpha_2\} = \{3, 4\}$, $\{4, 3\}$. For the pair $\{\alpha_1, \alpha_2\} = \{3, 4\}$, the null hypothesis does not hold. In the simulation, both tests show some power in rejecting the false null hypothesis, though not at a high level. As the dependence level increases, i.e. λ increases, the power declines, especially in the case of the *Benchmark Tail Shape* test. This can be explained by the fact that if all simulated data load heavily on the common factor, the theoretical differences in the tail indices, stemming from the idiosyncratic factor, is diminished. For the pair $\{\alpha_1, \alpha_2\} = \{4, 3\}$, the null hypothesis on tail index equivalence holds. In this case, the *Benchmark Tail Shape* test performs well by reporting a percentage of rejection close to 5%. On the other hand, the *Minmax Tail Shape* test tends to over reject the true null

hypothesis. This problem is particularly severe for a low λ value. This is due to the fact that a low loading on the common factor is not sufficient for generating data that reflect the theoretical equivalence of the tail indices.

An important factor contributing to the apparent low power in rejecting a false hypothesis is the small difference between α_1 and α_2 above. To enlarge the difference, we also generate data from Sample 2 with $\{\alpha_1, \alpha_2\} = \{3, 5\}$ and $\{5, 3\}$. The general feature of the performance of the two tests remains, but the performance is improved. For the case that $\{\alpha_1, \alpha_2\} = \{3, 5\}$, the rejection power increases considerably for both tests. For the case that $\{\alpha_1, \alpha_2\} = \{5, 3\}$, at least for the *Benchmark Tail Shape* test, the precision of the type I error is improved as the numbers get closer to 5%.

In summary, as in the case for most statistical tests, a local alternative destroys the power of a test. But when the tail index differences increase, the power property improves. From an economic point of view, the low power in the case that the tail indices are close is much less of a concern. In such a case, the properties of the two equity returns will be also similar as to make an arbitrage unprofitable.

3 Testing the tail index equivalence

3.1 Data and Methodology

In this section we apply the statistical tests described in Section 2 to test for tail index equivalence among equities traded on a common market. We use daily equity return data of non-financial US companies listed on both the NYSE and the NASDAQ from 2000 to 2011. We split our sample into 9 overlapping periods with 4 years of data in each period (i.e. 2000-2003, 2001-2004, ..., 2008-2011). We select stocks that are traded with sufficient regularity as follows. Each stock selected must have non-zero returns on at least 80% of the trading days in each period, and were traded throughout the entire 11 year period. In this way, each stock, in each period, has a sufficient number of observations for the tail index estimation. Furthermore, the selected firms must have available firm-level accounting data as described in Subsection 5.2. This selection procedure results in 230

stocks. For the case of the *Benchmark test*, we use the returns from the S&P 500 index as the market return⁴.

For each of the 9 periods we estimate both the tail index and the scale parameter for each selected stock and the S&P 500 index. The summary statistics of the two parameters for each period can be found in the first five columns of Table 2 and Table 3. We observe that there is a huge dispersion in the scale parameter for each period. However, the inter-period tail index estimates are roughly between 2 and 4. Notice that the dispersion in the tail index estimates shrinks for the period of 2005-2008. Furthermore, columns 6, 7 of Table 2 report the $\hat{\tau}$ tail dependence estimates. The estimate is the average of the estimated τ value across all pairs, whereas $\hat{\tau}_M$ is the average of the estimated τ between equities and the market return. These results indicate that we cannot ignore the cross-sectional dependence when testing the tail index equivalence hypothesis across equities. Moreover, we observe that the tail dependence is initially low, but increases starting in the 2005-2008 period. From our simulation results in the previous subsection, our statistical tests can be applied to such a data set with a moderate level of tail dependence.

A technical issue that arises in the estimation procedure regards the choice of the intermediate sequence, k , for the Hill estimator. The theoretical conditions on k are not relevant for a finite sample analysis. Instead of taking an arbitrary k , a usual procedure is to calculate the tail index for different k values and draw a line plot of the estimates against the k values. With a low k value, the estimator exhibits a large variance, while for a high k value, since the estimation uses too many observations from the moderate level, it exhibits a bias. Therefore, k is chosen by picking the first stable part of the line plot starting from low k , which balances the trade-off between the variance and the bias. The estimates then follow from such a choice of k . Because k is chosen from a stable part of the line plot, a small variation of the k value does not change the estimated value. Thus, the estimates are not sensitive to the exact k value. Following such a procedure, we keep the percentage of (tail) observations used in the estimation constant at a level

⁴The return series for the 230 stocks and the market return from the S&P 500 index are collected from the CRSP data source from January 2000 to December 2011.

of 3.5% for each sample, i.e. $k/n = 3.5\%^5$.

From these estimates we apply the statistical testing procedures from Section 2.1 to test whether the tail indices are jointly equivalent in the cross-section, i.e.

$$H_0^1 : \alpha_1 = \dots = \alpha_d (= \alpha_M).$$

3.2 Results

The results of both tests for each period are reported in the last two columns of Table 2. Both tests reject the null hypothesis of tail index equivalence for the first 4 periods (i.e. 2000-2003, ..., 2003-2006). However, the null hypothesis is not rejected at the 95% confidence level for the last 3 periods (i.e. 2006-2009, ..., 2008-2011). For the periods 2004-2007 and 2005-2008, the *Minmax Tail Shape* test rejects the null hypothesis while the *Benchmark Tail Shape* test fails to reject the null hypothesis at the 95% confidence level. Note that the p value of the *Minmax Tail Shape* test in the period 2005-2008 is close to 5%. From these results, we conclude that the tail indices are cross-sectionally heterogeneous for the period starting from 2000 and ending in 2004. On the contrary, the tail indices are cross-sectionally equivalent for the period from 2008 to the end of 2011. The overlapping nature of the tests makes it difficult however to determine when the actual break date occurs.

To detect the break, we take a more granular approach by testing the tail index equivalence hypothesis using a one-month rolling window. In doing so, we are able to examine more precisely when tail index equivalence holds and when it fails. Figure 1 shows the results of the one month rolling window analysis over the entire sample period. The plots indicate the p -values of the two tests. From the *Benchmark Tail Shape* test, we see that we do not reject tail index equivalence for all dates after September 2008 at the 95% confidence level. The *Minmax Tail Shape* test provides a similar pattern.

The analysis is consistent with the theoretical prediction of Moore et al. (2012) that tail index equivalence should hold when investors have a strong downside risk concern,

⁵Plots are available upon request.

e.g. they hold a sufficiently low admissible probability of failure. The initiation of the financial crisis increased investor awareness of downside risk. Before the crisis a potential high admissible probability level permitted heterogeneity in the tail indices among stocks traded in a common market. Investor passiveness towards downside risk preceding the crisis, coupled with models that underestimated the probability of extreme downside returns (e.g. Gaussian distribution assumption), potentially created an environment where the downside risk of stocks were mis-priced in the market. On the contrary, after the financial crisis hit, investors became increasingly aware of the downside risk inherent in their portfolios and began to hold a lower admissible probability of failure to guard against it. This leads to the homogeneity in the tail indices.

When the tail index equivalence hypothesis holds, the downside tail risk of equity returns are characterized by the scale parameter of the tail distribution. In the remainder of the paper we focus on the periods where tail index equivalence is not rejected and analyze cross-sectional downside tail risk heterogeneity via the scale parameter.

4 Statistical Tests: Tail Risk Equivalence

When the tail index equivalence hypothesis holds, the theoretical model in Moore et al. (2012) describes how downside risk can be priced by safety-first investors. More specifically, investors are compensated with higher expected return for holding assets with higher downside risk, e.g. assets with a larger scale parameter. Given the heterogeneity in expected returns, heterogeneity in the scale parameters should be allowed. In this section, we provide evidence on scale heterogeneity by testing for *tail scale equivalence* for the periods where tail index equivalence. This is equivalent to testing for tail risk equivalence.

4.1 The setup of the tests

We construct statistical test on whether the tail indices and scales of different stock return series are jointly equivalent. The null hypothesis is that $H_0 : \alpha_1 = \alpha_2 = \dots =$

α_d and $A_1 = A_2 = \dots = A_d$.

We recall the notion that the loss return are $X_{i,t} = -R_{i,t}$, $i = 1, \dots, d$ and $t = 1, \dots, n$. Under the null hypothesis, from (1.2), we get that $\frac{VaR_1(p)}{VaR_2(p)} \rightarrow 1$ as $p \rightarrow 0$. This is the theoretical relation we aim to test. An estimator for $VaR_i(p)$ at the $p = \frac{k}{n}$ level is the $(k+1)$ -th high order statistic $X_{i,(n-k)}$. Due to the cross-sectional dependence across stock returns, the order statistics are also dependent.

We continue with the setup on the dependence issue as in Subsection 3.1. The following Proposition provides the joint limit distribution of the $d+1$ order statistics.

Proposition 4.1 *With proper choice of k sequence such that as $n \rightarrow \infty$,*

$$\sqrt{k} \left(\frac{X_{i,(n-k)}}{VaR_i(k/n)} - 1 \right) \xrightarrow{d} N \left(0, \frac{1}{\alpha_i^2} \right),$$

for all $i = 1, \dots, d, M$, we have that,

$$\sqrt{k} \left(\frac{X_{1,(n-k)}}{VaR_1(k/n)} - 1, \dots, \frac{X_{d,(n-k)}}{VaR_d(k/n)} - 1, \frac{X_{M,(n-k)}}{VaR_M(k/n)} - 1 \right)^T \xrightarrow{d} N(\mathbf{0}^T, \mathbf{B}\Sigma\mathbf{B}),$$

where $\mathbf{0}$ is a $(d+1) \times 1$ vector with zero component, $\mathbf{B} = \text{diag}\{1/\alpha_1, \dots, 1/\alpha_d, 1/\alpha_M\}$ and $\Sigma = (\tau_{i,j})_{(d+1) \times (d+1)}$ with $\tau_{i,j}$ indicating the $R(1,1)$ measure between stock i and j , $i, j = 1, \dots, d, M$.

From Proposition 4.1, the construction of the tests is analogous to the tests of tail index equivalence.

We first construct a *Minmax Tail Risk* test. We use $\bar{\alpha} = \frac{1}{d} \sum_{i=1}^d \hat{\alpha}_i$ as an estimator for the equivalent marginal tail index. Then, consider the following test statistic

$$T_1 = \bar{\alpha} \frac{\max_{1 \leq i \leq d} X_{i,(n-k)} - \min_{1 \leq i \leq d} X_{i,(n-k)}}{\bar{X}_{(n-k)}},$$

where $\bar{X}_{(n-k)} = \frac{1}{d} \sum_{i=1}^d X_{i,(n-k)}$. Under the null hypothesis, from the joint limit distribution of the order statistics, we get that as $n \rightarrow \infty$, $\sqrt{k}T_1$ converges to a limit distribution as $\max_{i=1}^d N_i - \min_{i=1}^d N_i$. Here $(N_1, \dots, N_d)^T$ is a random vector that follows a d -dimensional normal distribution with mean zero and covariance matrix Σ_1 , which is a

submatrix of Σ consisting of the first d rows and d columns.

To test the hypothesis including the market return, we construct a *Benchmark Tail Risk* test statistic as,

$$T_2 = \alpha_M^2 \sum_{i=1}^d \left(\frac{X_{i,(n-k)}}{X_{M,(n-k)}} - 1 \right)^2 .$$

Similar to the discussion for the first test, under the null hypothesis, it follows that as $n \rightarrow \infty$, kT_2 converges to a limit distribution as $\sum_{i=1}^d (N_i - N_M)^2$. The random vector $(N_1, \dots, N_d, N_M)^T$ follows a $(d + 1)$ -dimensional normal distribution with mean zero and covariance matrix Σ .

The limit distribution of the two tests are exactly the same as those testing whether the tail indices are equivalent. Thus, the simulation methods on the limit distributions are also identified.

4.2 Results

We apply both the *Minmax Tail Risk* and the *Benchmark Tail Risk* tests to test the tail risk equivalence hypothesis. The results of both tests are given in the last two columns in Table 3. Under both testing procedures we reject the null hypothesis of joint equivalence of both the tail index and the scale parameter, and therefore tail risk, at the 95% confidence level in the last four periods (2005-2008, ..., 2008-2011). We therefore conclude that the downside tail risks across stocks are heterogeneous in each period. By showing that downside tail risk of stocks have significant cross-sectional variation, coupled with the result of tail index equivalence in the last four periods, we provide evidence in support of cross-sectional heterogeneity in the scale parameter. We are aware of the fact that the evidence of tail scale heterogeneity is weak, because the conclusion is drawn from combining observations from two interrelated tests.

5 Determinants of Downside Tail Risk

5.1 Potential Determinants

In this section, we provide alternative evidence for the tail scale heterogeneity by investigating the cross-sectional determinants of the scale parameter from firm-level characteristics. If heterogeneity in the scale is due to estimation error, then variation in the scale parameter estimates should not be explained by difference in firm characteristics. On the other hand, if cross-sectional variation in the scales can be explained by variation in the firm characteristics then the heterogeneity cannot be attributed to estimation error.

We analyze the heterogeneity of downside tail risk in the four periods, 2005-2008, ..., 2008-2011, by identifying cross-sectional determinants of the scale parameter from firm-level characteristics. Since Moore et al. (2012) show that the scales should be priced in the expected returns when the tail index hypothesis holds, we conjecture that the potential determinants of the scale parameter should be similar to those of expected returns in the prevailing literature. Because expected returns have to compensate for scale heterogeneity, the chosen firm-level characteristics are the known drivers of cross-sectional variation in expected stock returns. These include size, book-to-market equity, leverage, earnings-to-price, share turnover and the bid-ask spread in addition to the market beta backed out from the CAPM model.

5.2 Data and Methodology

We collect annual firm-level accounting data for the same 230 non-financial firms used in Subsection 3.1 from the COMPUSTAT data source for the years 2004, ..., 2007. The sample periods are chosen to pre-date by one year the periods where we find evidence of tail index equivalence⁶. For each year in the sample period, we calculate firm characteristics for each firm. The size of a firm is calculated as the logarithm of its year-end market capitalization. The book-to-market ratio is calculated by taking the logarithm

⁶By taking firm-level characteristics from the year preceding the sample period of the scale estimates, we aim to analyze forward-looking drivers of tail risk.

of the ratio of year-end book value of equity and the year-end market value of equity. This identification is often related to the growth potential of a firm. A firm with a high book-to-market ratio is a “growth stock”, whereas a firm with a low book-to-market equity ratio is a “value stock”. Firm leverage is calculated as long-term outstanding debt divided by year-end value of book equity. The earnings-to-price ratio is calculated by the earnings per share divided by the share’s year-end closing price. It is often referred to as the “earnings yield”. A higher earnings-to-price ratio translates to a higher earnings yield. The variable that proxies for share liquidity is calculated as the logarithm of the ratio of total volume of common shares traded in the year and by the total number of common shares outstanding. The proxy for investor asymmetric information is calculated as the difference in the highest annual closing price and the lowest annual closing price of the stock divided by the average annual closing price of the stock. We refer to this variable as the bid-ask spread. The higher the bid-ask spread the greater the level of asymmetric information or uncertainty regarding the firm’s stock value. Finally, the market beta is taken from the following estimate

$$\beta_i = \frac{Cov\{R_i, R_m\}}{\sigma_m^2}$$

where R_i and R_m are daily return series of the firm’s stock and the market portfolio during the year and σ_m is the variance on the market return during the same year.

For each period consisting of 4 years, we match the estimated scale parameters with the annual accounting data recorded at the end of the year preceding the beginning of the period (e.g. the scale parameter estimated over the 2005-2008 period is matched with accounting data for 2004). The selection of the 230 firms used in Section 3.1 was conditioned ex-ante on having available firm characteristics with no missing observations over the period of analysis. Furthermore, because the fiscal year end date of firms may not always align with the calendar year end date, we only select firms with equivalent fiscal and calendar year end dates.

To analyze the potential firm level determinants that drive the differences in the scale parameter, we run 4 separate cross-sectional regressions. An estimate of the scale

parameter directly follows from an estimate of the tail index as $\hat{A} = \frac{k}{n} X_{n,n-k}^{\hat{\alpha}}$, where $k = k(n)$ is the same intermediate sequence used in the estimation of the tail index. Hall (1982) shows that the scale parameter is consistent and asymptotically normal distributed as,

$$\sqrt{k}(\log \frac{n}{k})^{-1}(\frac{\hat{A}}{A} - 1) \sim \mathcal{N}(0, 1). \quad (5.1)$$

Since we are under the assumption of tail index equivalence, we estimate the scale parameter by using the sample mean tail index, $\bar{\alpha}$ as

$$\hat{A}_i = \frac{k}{n} X_{i,n,n-k}^{\bar{\alpha}}.$$

In taking the mean tail index we focus specifically on the scale parameter. Notice that equation (5.1) is equivalent to

$$\sqrt{k}(\log \frac{n}{k})^{-1}(\log \hat{A} - \log A) \sim \mathcal{N}(0, 1).$$

Since the logarithm of the scale is normally distributed with a known variance, it is convenient to use it directly in the regression against the firm characteristics as follows,

$$\log \hat{A}_i = \beta_0 + \sum_{j=1}^m (\beta_j \Gamma_i^j) + \epsilon_i, \quad i = 1, \dots, d \quad (5.2)$$

where Γ_i^j are the firm characteristics from the accounting data⁷.

5.3 Results

Table 4 reports the descriptive statistics of the firm characteristics for the year 2007 and Figure 2 illustrates a series of scatter plots of the scale parameter estimates from the 2008-2011 period against the firm level characteristics from 2007⁸. Table 5 reports the regression results. We find the size of the firm, measured by market capitalization, has a

⁷In the regression for each window, we further eliminate any firm with negative book-to-market ratio. The negative book-to-market ratio of a firm presents problems in terms of lack of economic interpretation. In the case of the book-to-market ratio, a negative value also presents problems by not permitting a log transformation.

⁸Descriptive statistics and scatter plots for the remaining periods are available upon request.

significantly negative relation to the scale parameter. Therefore firm downside tail risk is significantly lower for large firms. Intuitively, smaller firms are less able to diversify risks relative to their larger counterparts. This leaves them more susceptible to idiosyncratic shocks and less able to withstand large aggregate shocks.

The book-to-market ratio has a significant positive relation to the scale parameter over each of the 4 periods. It is significant at the 99% confidence level for the 2005-2008 period and then declines in significance over the remaining three periods (95% in 2006-2009 and 90% in 2007-2010 and 2008-2011). This result implies that “growth” stocks have higher downside tail risk than “value” stocks. Firms are able to deliver higher growth by undertaking riskier projects with higher expected return. These firms would thus bear greater downside tail risk potential than “value” firms that are involved in safer projects.

Higher leverage is found in the regressions to be positive and significant in relation to the scale parameter in 3 of 4 periods. For the period encompassing 2006-2009 and 2007-2010 it is significant at the 99% confidence level and it is significant at the 95% confidence level for the 2008-2011 period. Firms that fund their business activities with greater amounts of debt are likely to have more volatile returns. One simple explanation follows from the fact that leverage has an amplifying effect on the risk in equity return compared to that in the return on average assets (RoAA) by

$$R_i = RoAA \cdot (Leverage + 1).$$

Given the level of downside risk in RoAA, firms that have higher leverage will have more extreme downside losses.

Asymmetric information regarding investor perception of a firm’s stock, captured by the bid-ask spread, is positively and significantly related to the scale parameter in each of the four regression periods at the 99% confidence level. A greater bid-ask spread corresponds to greater uncertainty regarding the future value of the firm. Firms with higher uncertainty are more sensitive to new information regarding the firm’s future. Adjustments to a firm’s value based on the arrival of new information will come in the form of sharper price corrections. Hence, firms with higher bid-ask spreads have a greater

chance of experiencing large negative (or positive) idiosyncratic shocks.

Lastly, the coefficient on the estimated market beta is positive and significant (at the 99% confidence level) in only the 2008-2011 period. This can be explained by the single-factor model. Suppose the return of a stock follows

$$R_i = \beta_i R_M + \epsilon_i,$$

where ϵ_i is an idiosyncratic factor with scale A_{ϵ_i} , A_M is the scale parameters of the market factor. Assume that the idiosyncratic and market factor share a common tail index α , we get that the scale parameter of the stock return is

$$A_i = \beta_i^\alpha A_M + A_{\epsilon_i},$$

This equation shows a positive relation between the market beta and the scale parameter.

To conclude, the heterogeneity in the scale parameter is explained by the firm-level characteristics. From the regression results, we find that book-to-market ratio, bid-ask spread, and the market beta are positive and significantly related to the scale parameter, while size and leverage are negative and significantly related to the scale. These results provide alternative evidence for tail scale heterogeneity.

6 Conclusion

This paper analyzes cross-sectional differences in tail risk across equity returns and further identifies drivers of these tail risks. If stock returns are assumed to follow a heavy-tailed distribution, then the downside tail risk is determined by two parameters: the tail index and scale. In the case of Gaussian distribution, this would only be one parameter (variance). Moore et al. (2012) show that if large downside losses are of sufficient concern to investors then the downside tail distribution of asset returns will share a homogeneous tail index. This is referred to as the “tail index equivalence hypothesis”. We empirically test this hypothesis by implementing two novel testing procedures, the *Minmax Tail*

Shape and *Benchmark Tail Shape* tests, over a sample of 230 US non-financial firms. We do not reject tail index equivalence between the years 2008 and 2011. During this period, the heterogeneity in downside tail risk is attributed to the scale of the tail distribution. The cross-sectional drivers of tail risk are the following firm-level characteristics: the size, book-to-market ratio, leverage, bid-ask spread and market beta. Since expected returns have to compensate for the variation in tail scale, the same factors are found to explain the scale as in the case of expected returns.

Identification of the firm level drivers of downside tail risk help investors evaluate potential tail risks when holding equities from different firms. Based on the evaluation of firm-level characteristics, investors can construct a portfolio of equities which accommodates both their desired return and downside risk appetite. From a corporate finance viewpoint, our result allows firm managers to better control their downside tail risk in order to attract long-term risk averse investors.

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7 Appendix

7.1 Proof of Proposition 2.1

We begin by reviewing Proposition 7.2.3 in De Haan and Ferreira (2006). All the other referenced theorems or examples used in this proof are also from this book. Under the bivariate EVT setup, denote F_1 and F_2 as the marginal distribution of (X_1, X_2) and $U_i = \left(\frac{1}{1-F_i}\right)^\leftarrow$ as the marginal quantile function for $i = 1, 2$. Then, with proper conditions on the k series, under a proper Skorokhod construction, there exists a Gaussian process $W(x, y)$ on $[0, T] \times [0, T]$ for any finite T , such that

$$\sup_{0 \leq x, y \leq T} \left| \sqrt{k} \left(\frac{1}{k} \sum_{t=1}^n 1_{1-F_1(X_{1,t}) \leq kx/n \text{ or } 1-F_2(X_{2,t}) \leq ky/n} - \frac{n}{k} \left\{ 1 - F \left(U_1 \left(\frac{n}{kx} \right), U_2 \left(\frac{n}{ky} \right) \right) \right\} - W(x, y) \right) \right| \rightarrow 0. \quad (7.1)$$

holds almost surely. Here, the process $W(x, y)$ is a mean zero process with covariance structure

$$\text{Cov}(W(x_1, y_1), W(x_2, y_2)) = x_1 \wedge x_2 + y_1 \wedge y_2 - R(x_1, y_1) - R(x_2, y_2) + R(x_1 \vee x_2, y_1 \vee y_2).$$

By taking $y = 0$ in (7.1), we get that

$$\sup_{0 \leq x, y \leq T} \left| \sqrt{k} \left(\frac{1}{k} \sum_{t=1}^n 1_{1-F_1(X_{1,t}) \leq kx/n} - x \right) - W(x, 0) \right| \rightarrow 0.$$

almost surely. Comparing this expression with the proof of Theorem 5.1.2 and 5.1.4, we observe that the process $W(x, 0)$ is a standard Brownian motion that is used to construct the limit process of the tail empirical process. Thus from Example 5.1.5, we get that

$$\sqrt{k} \left(\frac{1}{\hat{\alpha}_1} - \frac{1}{\alpha_1} \right) \xrightarrow{P} \frac{1}{\alpha_1} \left(\int_0^1 W(u, 0) \frac{du}{u} - W(1, 0) \right).$$

This is equivalent to

$$\sqrt{k} \left(\frac{\hat{\alpha}_1}{\alpha_1} - 1 \right) \xrightarrow{P} - \left(\int_0^1 W(u, 0) \frac{du}{u} - W(1, 0) \right) =: N_1.$$

Similarly, we get that

$$\sqrt{k} \left(\frac{\hat{\alpha}_2}{\alpha_2} - 1 \right) \xrightarrow{P} - \left(\int_0^1 W(0, v) \frac{dv}{v} - W(0, 1) \right) =: N_2.$$

Since W is a bivariate Gaussian process, we get that (N_1, N_2) follows a bivariate normal distribution, with mean zero, and marginal variance 1. The last step is to calculate their covariance. The calculation is as follows

$$\begin{aligned} \text{Cov}(N_1, N_2) &= \text{Cov} \left(\int_0^1 W(u, 0) \frac{du}{u} - W(1, 0), \int_0^1 W(0, v) \frac{dv}{v} - W(0, 1) \right) \\ &= \int_0^1 \int_0^1 R(u, v) \frac{du}{u} \frac{dv}{v} - \int_0^1 R(1, v) \frac{dv}{v} - \int_0^1 R(u, 1) \frac{du}{u} + R(1, 1) \\ &= \int_{0 \leq u \leq v \leq 1} R(u, v) \frac{du}{u} \frac{dv}{v} - \int_0^1 R(u, 1) \frac{du}{u} \\ &\quad + \int_{0 \leq v \leq u \leq 1} R(u, v) \frac{du}{u} \frac{dv}{v} - \int_0^1 R(1, v) \frac{dv}{v} \\ &\quad + R(1, 1) \\ &=: I_1 + I_2 + R(1, 1). \end{aligned}$$

Due to the homogeneity of the R function, we have that

$$\begin{aligned} \int_{0 \leq u \leq v \leq 1} R(u, v) \frac{du}{u} \frac{dv}{v} &= \int_0^1 dv \int_0^v R(u/v, 1) \frac{du}{u} \\ &= \int_0^1 dv \int_0^1 R(u, 1) \frac{du}{u} \\ &= \int_0^1 R(u, 1) \frac{du}{u}. \end{aligned}$$

Hence $I_1 = 0$. Symmetrically, we get that $I_2 = 0$. Thus $\text{Cov}(N_1, N_2) = R(1, 1)$. \square

7.2 Simulation of the limit distribution

The test procedures, described in Subsection 3.1, require simulated observations from the limit distribution of the test statistics. That is based on simulating a multivariate normal distributed random vector $(N_1, \dots, N_d, N_M)^T$ with mean zero and covariance Σ which consists of all bivariate τ measure between each pair of stocks. A conventional procedure would be a two-step approach: first estimate the covariance matrix; second employ a proper simulation technique to simulate a multivariate normal distribution with the estimated covariance matrix. Instead of using such a two-step approach, we provide a fast simulation procedure in one step without pre-estimating the covariances. This procedure is feasible thanks to the structure of the covariance matrix to be estimated.

Recall that the estimator for $\tau_{i,j}$ for $i, j = 1, 2, \dots, d, M$ is given as

$$\begin{aligned}\hat{\tau}_{i,j} &= \frac{1}{k} \sum_{t=1}^n 1_{X_{i,t} > X_{i,(n-k)}} \text{ and } 1_{X_{j,t} > X_{j,(n-k)}} \\ &= \frac{1}{k} \left(1_{X_{i,1} > X_{i,(n-k)}}, \dots, 1_{X_{i,n} > X_{i,(n-k)}} \right) \left(1_{X_{j,1} > X_{j,(n-k)}}, \dots, 1_{X_{j,n} > X_{j,(n-k)}} \right)^T.\end{aligned}$$

Notice that this expression is also valid in the case that $i = j$, as in this case $\hat{\tau}_{i,i} = 1 = \tau_{i,i}$. By writing each element in the estimator of Σ as the expression above, we get that

$$\hat{\Sigma} = \frac{1}{k} I_{n \times (d+1)}^T I_{n \times (d+1)},$$

where $I_{n \times (d+1)} = (I_{t,i})_{t=1, \dots, n; i=1, \dots, d, M}$ with $I_{t,i} = 1_{X_{i,t} > X_{i,(n-k)}}$ is an indicator matrix indicating whether the i -th stock on day t is having a loss exceeding a threshold corresponding to the i -th shock, indicated by $X_{i,(n-k)}$.

With expressing the estimator of the covariance matrix $\hat{\Sigma}$ as a matrix product, we have the following simulation procedure for the random vector $(N_1, \dots, N_d, N_M)^T$. First, simulate n independently and identically standard normally distributed random variables $(G_1, \dots, G_n)^T$. Subsequently, calculate

$$(N_1, \dots, N_d, N_M)^T = \frac{1}{\sqrt{k}} (G_1, \dots, G_n) \cdot I_{n \times (d+1)}.$$

It is obvious that the simulated $(d+1)$ -dimensional random vector follows a multivariate normal distribution with mean zero and covariance matrix $\hat{\Sigma}$. By repeating the above procedure, we obtain a simulated sample of the random vector $(N_1, \dots, N_d, N_M)^T$ with the desired distribution. After this, it is straightforward to obtain simulated random variables following the limit distributions of the test statistics.

8 Tables and Figures

Table 1: Test Results on Tail Index Equivalence Hypothesis

λ	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
Sample 1											
$\alpha = 3$											
Minmax	3.6	2.8	2.6	3.1	2.9	4.5	3.6	3.8	4.5	3.0	4.7
Benchmark	5.3	6.1	6.0	5.2	6.0	5.4	4.2	3.8	2.5	2.0	1.0
$\alpha = 4$											
Minmax	2.7	2.7	1.0	1.9	2.6	2.1	2.7	2.9	3.3	2.5	2.5
Benchmark	5.0	6.2	5.0	6.0	5.6	4.2	3.6	3.7	2.6	2.1	0.8
Sample 2											
$\alpha_1 = 3, \alpha_2 = 4$											
Minmax	14.4	14.6	14.3	13.3	14.5	15.5	13.7	12.6	10.1	7.1	5.0
Benchmark	16.0	18.4	18.2	15.9	12.4	10.1	9.3	6.8	3.1	1.1	1.3
$\alpha_1 = 4, \alpha_2 = 3$											
Minmax	13.7	17.0	14.6	14.8	15.1	13.9	12.6	9.1	6.8	3.3	1.5
Benchmark	4.3	4.0	3.3	2.8	3.8	1.7	2.4	1.7	1.2	0.7	0.5
$\alpha_1 = 3, \alpha_2 = 5$											
Minmax	42.0	40.2	43.1	41.5	44.4	42.7	46.2	38.1	35.1	26.0	16.2
Benchmark	33.7	35.6	31.4	32.5	27.1	24.3	19.9	14.3	10.8	6.9	3.6
$\alpha_1 = 5, \alpha_2 = 3$											
Minmax	40.2	39.9	41.5	39.0	34.5	35.1	26.6	17.3	9.9	6.2	3.3
Benchmark	5.0	5.6	5.7	5.9	6.0	4.3	4.2	2.2	2.1	1.7	0.6

Note: This table reports the percentage of simulations in which the p -value falls below the 5% level, under the null hypothesis of tail index equivalence. The data are generated as in Sample 1 and Sample 2 in Subsection 2.2. For each given λ and sample setup, we simulate 1000 samples with each sample consisting of 1000 observations. The parameter λ indicates the tail dependence between the simulated observations. $\lambda = 0$ corresponds to tail independence, while the increase of λ corresponds to an increase in tail dependence.

Table 2: Summary Statistics of the Tail Index and Test on Equivalence

	MIN	MEAN	MAX	STD	$\hat{\alpha}_m$	$\hat{\tau}$	$\hat{\tau}_m$	$\hat{\phi}_m$	$\hat{\phi}_b$
00-03	1.39	2.75	4.56	0.45	3.82	0.09	0.15	0.00	0.00
01-04	1.55	2.74	4.24	0.44	3.60	0.10	0.17	0.00	0.00
02-05	1.63	2.75	4.14	0.44	3.68	0.10	0.19	0.00	0.00
03-06	1.38	2.84	4.92	0.51	3.36	0.08	0.15	0.00	0.00
04-07	1.28	2.83	4.01	0.47	3.11	0.11	0.20	0.01	0.15
05-08	1.36	2.27	3.60	0.41	1.90	0.26	0.40	0.04	0.41
06-09	1.38	2.49	4.29	0.41	2.29	0.26	0.42	0.37	0.65
07-10	1.56	2.57	3.84	0.41	2.51	0.26	0.41	0.10	0.54
08-11	1.79	2.66	4.26	0.40	2.52	0.27	0.43	0.42	0.57

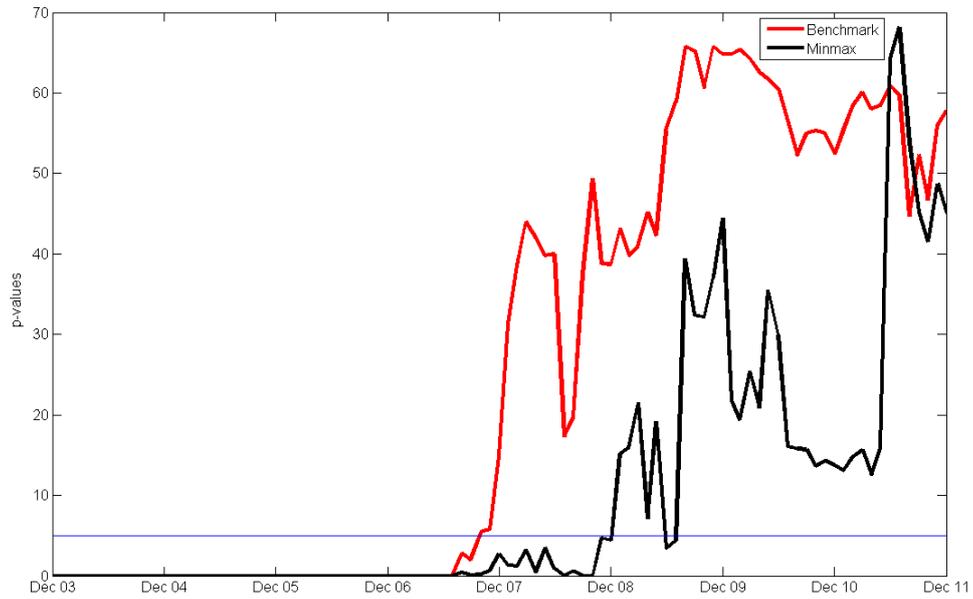
Note: This table reports a summary of statistics of the tail indexes of stock returns for each rolling window. The dataset consists of 230 US equities and the benchmark S&P 500 Composite Index, with daily returns from 01.01 2000 to 31.12.2011. $\hat{\alpha}_M$ is the tail index estimate for S&P 500 Index. $\hat{\tau}$ is the average value of tail dependence estimates across all pairs of equities, whereas $\hat{\tau}_M$ is the average value of tail dependence estimates between equities and S&P 500 Index. The $\hat{\phi}_m$ and $\hat{\phi}_b$ columns report the p -values based on the test statistics in Subsection 2.1 for testing the tail index equivalence. The m and b indicate the *Minmax Tail Shape* and *Benchmark Tail Shape* tests respectively.

Table 3: Summary Statistics of the Scale Parameter and Test on Equivalence

	MIN	MEAN	MAX	STD	\hat{A}_m	$\hat{\phi}_m$	$\hat{\phi}_b$
00-03	0.20	8.37	322.59	23.93	1.17	0.00	0.00
01-04	0.17	5.79	119.70	13.42	0.71	0.00	0.00
02-05	0.19	3.64	62.38	7.15	0.49	0.00	0.00
03-06	0.13	2.70	55.65	5.42	0.10	0.00	0.00
04-07	0.09	2.45	47.74	4.44	0.10	0.00	0.00
05-08	0.15	2.09	34.21	3.23	0.18	0.00	0.00
06-09	0.19	4.45	47.87	6.58	0.42	0.00	0.00
07-10	0.23	5.62	77.18	8.98	0.61	0.00	0.00
08-11	0.23	8.25	256.02	21.18	0.70	0.00	0.00

Note: This table presents a summary statistics of the scale parameter for each rolling window. The dataset consists of 230 US equities and the benchmark S&P 500 Composite Index, with daily returns from 01.01 2000 to 31.12.2011. \hat{A}_M is the scale parameter estimate for S&P 500 Index. The $\hat{\phi}_m$ and $\hat{\phi}_b$ columns report the p -values based on the test statistics in Subsection 4.1 for testing the tail risk equivalence. The m and b indicate the *Minmax Tail Risk* and *Benchmark Tail Risk* tests respectively.

Figure 1: p - values: Four-year rolling window analysis



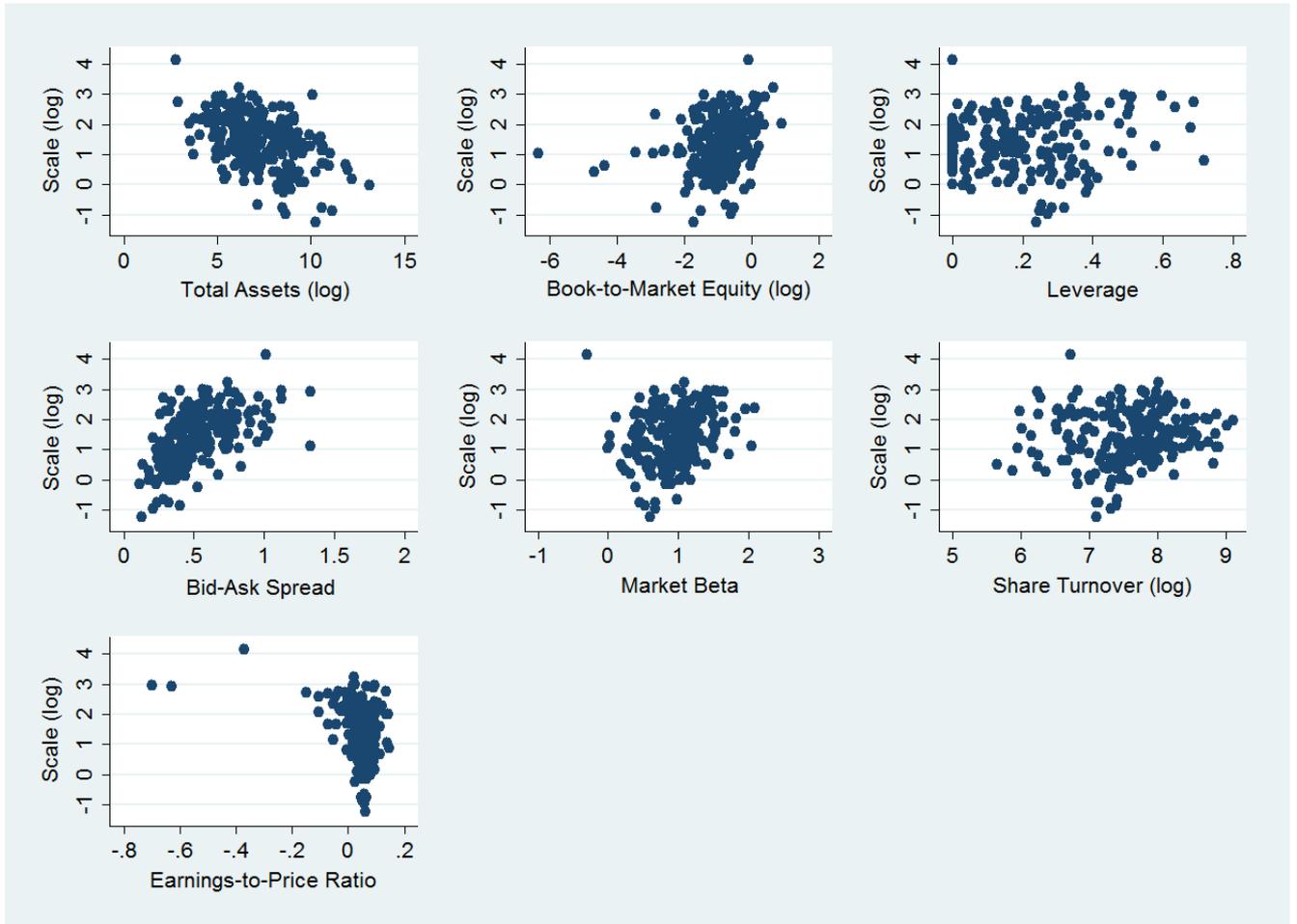
Note: The figure plots the p -values for both *Minmax* and *Benchmark* tests for each rolling window. The dataset consists of 230 US equities and the benchmark S&P 500 Composite Index with daily returns from 01.01 2000 to 31.12.2011. We move the four-year rolling window monthly.

Table 4: Descriptive Statistics: Firm Characteristics, 2007

Variable	MEAN	STD	MIN	MAX
Market Cap (log)	7.192	1.863	2.766	13.131
Book-to-Market Equity (log)	-0.932	0.825	-6.354	0.888
Leverage	0.177	0.163	0	0.718
Earnings-to-Price	0.038	0.082	-0.698	0.147
Share Turnover (log)	7.54	0.702	5.213	9.111
Bid-Ask Spread	0.533	0.24	0.116	1.925
Market Beta	0.958	0.395	-0.296	2.283

Note: The table reports the summary statistics of the firm characteristics: Market Capitalization (log), Book-to-Market Equity (log), Leverage, Bid-Ask Spread, Market Beta, Share Turnover (log), Earnings-to-Price Ratio in the end of 2007.

Figure 2: Scatter Plots: Scale (2009-2011) vs. Firm Characteristics (2007)



Note: The figure presents the scatter plots of the scale (log), estimated from the daily returns in the period 2008-2011, against the firm characteristics calculated from 2007 year end data: Market Capitalization (log), Book-to-Market Equity (log), Leverage, Bid-Ask Spread, Market Beta, Share Turnover (log), Earnings-to-Price Ratio.

Table 5: The Determinants of Downside Tail Risk

	(1)	(2)	(3)	(4)
	2004	2005	2006	2007
Size	-0.131*** (-5.68)	-0.158*** (-5.27)	-0.196*** (-6.18)	-0.174*** (-6.13)
BE/ME	0.136*** (3.03)	0.133** (2.45)	0.134* (1.88)	0.102* (1.76)
Leverage	0.314 (1.21)	1.057*** (3.22)	1.386*** (4.58)	0.643** (2.38)
E/P	0.498 (0.94)	0.435 (0.75)	0.230 (0.23)	-1.328* (-1.71)
Turnover	0.0105 (0.22)	0.105 (1.56)	0.178** (2.48)	0.0648 (1.01)
BidAsk	1.108*** (6.80)	0.994*** (4.87)	0.863*** (3.38)	1.103*** (4.26)
Beta	0.0245 (0.34)	0.0566 (0.55)	0.0730 (0.85)	0.510*** (3.52)
Constant	0.563 (1.56)	0.712 (1.57)	0.650 (1.31)	1.134** (2.40)
Observations	217	219	222	225
R^2	0.444	0.348	0.372	0.455

t statistics in parentheses

* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$

Note: The above table lists the results of four regressions over four sample periods. The dependent variable is the logarithm of the scale parameter is estimated over a four year window, e.g. 2005-2008, 2006-2009, 2007-2010, and 2008-2011. The dependent variables are taken as year-end data for the years preceding the estimation of the dependent variable, e.g. 2004, 2005, 2006, and 2007. These are measured as follows: Size is the logarithm value of market capitalization (ME), BE/ME is the logarithm value of the book-to-market equity, Leverage is the debt-to-equity ratio, and E/P is the earnings-to-price ratio. Turnover is the share turnover and BidAsk is the bid-ask difference. Beta is the market beta estimated from CAPM model.