

# Strong Bubbles and Common Expected Bubbles in a Finite Horizon Model\*

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## Abstract

An *expected bubble* is said to exist if it is mutual knowledge that the price of the asset is higher than the expected dividend. Similarly we call it a *strong bubble* if everyone knows that the price is higher than the maximum possible dividend. Substituting common knowledge for mutual knowledge, I develop the new concepts of a *common expected bubble* and a *common strong bubble*. In a simple finite horizon model with asymmetric information and short sales constraints following Allen, Morris and Postlewaite (1993), I show that the following results hold for any finite number of agents. First, under the implicit assumption of perfect memory, common strong bubbles never exist in any rational expectations equilibrium. Second, it is possible to have one that is both a strong bubble and a common expected bubble in a rational expectations equilibrium. Furthermore, the second result is robust to both *strongly symmetric perturbations* in beliefs and *very symmetric perturbations* in dividends. A counterexample of the first result is possible when agents are forgetful.

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# 1 Introduction

Bubbles exist in many markets, not only those where assets have fundamental values hard to determine or observe (stocks, for instance), but also some where assets have fundamental values known to be less than their prices (fiat money, for instance). How can bubbles be explained and what must be true for the existence of bubbles? Though claiming that most bubbles are irrational is much easier than interpreting bubbles in a rational way, economists have made and are still making efforts to deal with the latter.

Among the huge literature on the existence of bubbles, one strand has developed models based on the existence of some irrational agents, often called noise traders in the literature (see, for example, De Long, Shleifer, Summers and Waldmann (1990), Abreu and Brunnermeier (2003), and Zurita (2004)). Papers in this strand interpret bubbles by the interaction between the rational and the irrational.<sup>1</sup>

Another strand of the literature, has tried to model bubbles under the assumption that all agents are rational.<sup>2</sup> In such settings, an asset bubble can be explained either by the assumption of an infinite horizon or by the infinite presence of new agents (see Tirole (1982) and Tirole (1985) for example). However, in order to interpret the existence of a finite horizon bubble<sup>3</sup> in a rational expectations equilibrium with a finite number of agents, either a change of standard assumptions (for instance, symmetric information) or the introduction of specific requirements (for instance, short sales constraints) has to be made. Thus the question becomes: What is the minimum requirement for the existence of such a rational bubble?

By the well-known no-trade theorem of Milgrom and Stokey (1982), under the standard setting, if the initial allocation is efficient relative to each agent's belief, then the common knowledge of feasibility of and voluntary participation in trade will give agents no incentive to trade, no matter whether they have private information or not. If there is no trade in a finite horizon economy, there is certainly no bubble. Hence the ex ante inefficiency of the endowment allocation, or the existence of potential gains from trade, is one necessary condition for such a bubble to exist.<sup>4</sup>

Allen, Morris and Postlewaite (1993) (AMP (1993) henceforth) define two types of bubbles—expected bubbles and strong bubbles—in their finite-agent finite-horizon finite-state trade model, and show that private information about the states and short sales constraints for all agents are another two necessary conditions for the existence of strong bubbles. An *expected bubble* is said to exist if it is mutual knowledge that the price of the asset is higher than the expected dividend. They call it a *strong bubble* if everyone knows that the price is higher than the maximum possible dividend. While

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<sup>1</sup>Though the rational agents have incentive to take advantage of the irrational, it is possible that noise traders may actually earn a higher expected return than rational investors do. For details, see De Long, Shleifer, et al. (1990).

<sup>2</sup>In fact it is assumed that the rationality of the agents is common knowledge in most papers of this strand. Under the assumption of rational expectations, these two are equivalent.

<sup>3</sup>Among all the bubble phenomena, finite horizon bubbles are probably most puzzling.

<sup>4</sup>For a complete proof, see Tirole (1982).

the concept of expected bubbles provides a starting point for analysis, economists are more interested in the concept of strong bubbles.

Combining these three together with a fourth requirement that the agents' trade should not be common knowledge, AMP (1993) presented an example of strong bubbles in a rational expectations equilibrium with three agents and three periods.<sup>5</sup> This model captures the "greater fools" dynamic in the sense that because of asymmetric information, agents may hold a worthless asset at a positive price in the first period (hence a strong bubble), in hopes of selling it in the second period to someone else who thinks it may be worth something. In short, a rational bubble can exist in this setting because even though everyone knows that the asset is overpriced, they may still hold it with the belief that others might think that it is valuable.

Given the success of the Allen, Morris and Postlewaite model, economists are somewhat less than satisfied with the last assumption, the one requiring no common knowledge of trades, since many bubbles do exist in reality with the public information of agents' actions. Conlon (2004) constructed a strong bubble example in a similar setting<sup>6</sup> where there are only two agents. Since trades are automatically common knowledge for the two-agent case, this result has questioned the necessity of the assumption of no common knowledge of trades for the existence of a finite horizon bubble in a rational expectations equilibrium. Another contribution of Conlon (2004) is that the bubble in the model is not only strong but also robust to  $n$ th order knowledge, that is (all agents know that) <sup>$n$</sup>  the price is higher than any possible dividend agents will receive.

Based on the fact of the existence of  $n$ th order bubbles, one may naturally ask whether a bubble can be robust to common knowledge. In this paper, by requiring common knowledge instead of mutual knowledge, I develop two new concepts of bubbles: a common expected bubble and a common strong bubble. A *common expected bubble* is said to exist if it is common knowledge that the price of the asset is higher than the expected dividend. A *common strong bubble* is said to exist if it is common knowledge that the price of the asset is higher than the maximum possible dividend. The concept of the common strong bubble is so "strong" that it can be shown never to exist in any rational expectations equilibrium under the standard assumption of perfect memory. However, I am able to show that within the same framework as the AMP (1993) model but with common knowledge of trades, a strong bubble can exist in the case of two agents, and this bubble can still exist even when it is common knowledge that the price is higher than the expected dividend agents will receive (hence a common expected bubble). Moreover, such a bubble, both a strong bubble

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<sup>5</sup>It has been shown in that paper that there is no expected bubble in the last two periods under their framework, which will be described in Section 2; hence the minimum number of periods for the existence of a bubble is 3.

<sup>6</sup>The setting of Conlon (2004) differs from AMP (1993) in the sense that agents' information structures are determined both by the private signals they receive at the beginning of period 1 and by the public signals they receive at the beginning of every period. The information structures are chosen so that prices reveal no additional information.

and a common expected bubble, is robust to one class of symmetric perturbations in beliefs and another class of symmetric perturbations in dividends, and can exist for any finite number of agents.<sup>7</sup> This positive result itself, on the one hand, weakens the assumptions of the models of bubbles by reducing the four necessary conditions to three, and hence improves these models' applicability and powers in interpretation. On the other hand, the surprising result of the existence of common expected bubbles is somewhat counterintuitive but captures the idea that agents do not rush in face of bubbles since, given the common knowledge of the heterogeneous beliefs and the information structures, they believe that they can take advantage of it in a later period. The fact that these bubbles are robust to certain classes of symmetric perturbations provides insight into the structure of the bubbles. Another contribution of this paper lies in the understanding of the structural characteristics of models of bubbles: I show that a couple of structural conditions must be satisfied for a strong bubble to exist in a rational expectations equilibrium in a 2-agent symmetric economy. One of them is that the minimum number of states is 8. Furthermore, it is also worth noting that the implicit assumption of perfect memory plays a key role in the nonexistence result for common strong bubbles: if people might forget some information they knew before, then a common strong bubble may happen in equilibrium.

The next section of the paper introduces the basic framework following AMP (1993), gives four concepts of bubbles, and shows the nonexistence of common strong bubbles in any rational expectations equilibrium. Section 3 presents a simple example of a rational bubble with two agents; the bubble is both a strong bubble and a common expected bubble. Section 4 characterizes necessary conditions about the number of states and the structure of information partitions for the existence of strong bubbles and common expected bubbles. Section 5 shows that the bubble example we described above is robust to both strongly symmetric perturbations in beliefs and very symmetric perturbations in dividends. Section 6 gives a counterexample of the nonexistence result of common strong bubbles if we allow for imperfect memory. Section 7 shows the general results for any finite number of agents. Section 8 provides concluding remarks and directions for further study.

## 2 The Model

### 2.1 Basic Setup

The same framework is established here as in AMP (1993), except that the requirement that the trades should not be common knowledge is removed.

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<sup>7</sup>I assume that each agent is distinguished from the others in the sense that either their beliefs are heterogeneous or their information structures are different, or both. Otherwise, this result would hold trivially since each agent can be "divided" according to endowments into any finite number of subagents.

In the pure exchange economy under study, there are  $I$  ( $\geq 2$ ) risk neutral<sup>8</sup> agents ( $i = 1, 2, \dots, I$ ),  $T$  ( $\geq 3$ ) periods ( $t = 1, 2, \dots, T$ ) and  $N$  ( $\geq 2$ ) states of the world represented by  $\omega \in \Omega$ . Only 2 assets exist in the market: one riskless (money) and the other risky. There is no discount between any two periods. Each share of the risky asset will only pay a state-dependent dividend denoted by  $d(\omega)$  at the end of period  $T$ .

Agent  $i$  is endowed with  $m_i$  units of money and  $e_i$  shares of the risky asset at the beginning of period 1. In each period  $t$  and in each realized state  $\omega$ , agents can exchange claims on the risky asset at a state-and-period-dependent price  $P_t(\omega)$ . Agent  $i$ 's net trade in period  $t$  when state  $\omega$  is realized is denoted by  $x_{it}(\omega)$ , and we write  $x_i = (x_{i1}, x_{i2}, \dots, x_{iT})$ ,  $x_t = (x_{1t}, x_{2t}, \dots, x_{It})$ , and  $x = (x_1, x_2, \dots, x_I)$ . Hence agent  $i$ 's final consumption in state  $\omega$  with net trades  $x_i$  at price  $P(\omega) = (P_1(\omega), P_2(\omega), \dots, P_T(\omega))$ , denoted by  $y_i(\omega, P(\omega), x_i)$ , is equal to  $m_i + e_i P_T(\omega) + \sum_{t=1}^T x_{it}(\omega) [P_{t+1}(\omega) - P_t(\omega)]$ , where  $P_{T+1}(\omega) = d(\omega)$ . Let  $u_i(\cdot)$  be agent  $i$ 's utility function. Then agent  $i$ 's utility in state  $\omega$  with net trades  $x_i$  at price  $P(\omega)$ , is  $u_i(y_i(\omega, P(\omega), x_i))$ . For simplicity, assume that  $u_i(\cdot)$  is the identity function for all  $i$ .

Each agent  $i$  has a subjective belief about the probability distribution of the state, denoted by  $\pi_i(\omega)$ .<sup>9</sup>  $\forall i = 1, 2, \dots, I, \forall \omega \in \Omega, \pi_i(\omega) > 0$ .

## 2.2 Information Structure

At the beginning of each period  $t$ , before observing the current price and making the trade, agent  $i$ 's information about the state is represented by  $S_{it}$ , a partition of the space  $\Omega$ , and his price-and-trade-refined information is represented by  $S_{it}^{PX}$ .<sup>10</sup> We denote by  $s_{it}(\omega)$  ( $s_{it}^{PX}(\omega)$ ) the partition member in  $S_{it}$  ( $S_{it}^{PX}$ ) containing the state  $\omega$ . In other words,  $s_{it}(\omega)$  consists of all the possible states agent  $i$  believes he might be in when the state  $\omega$  is realized in period  $t$ . For example,  $s_{i1}(\omega_1) = \{\omega_1, \omega_2\}$  means that in period 1 agent  $i$  believes he might be either in  $\omega_1$  or  $\omega_2$  when  $\omega_1$  is realized.

$S_{it}^{PX}$  is determined by  $(S_{it}, P_t, x_t)$  such that

$$\forall \omega \in \Omega, s_{it}^{PX}(\omega) = s_{it}(\omega) \cap \{\omega' | P_{t'}(\omega') = P_{t'}(\omega) \text{ and } x_{t'}(\omega') = x_{t'}(\omega) \quad \forall t' \leq t\}.$$

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<sup>8</sup>Agents are assumed to be either risk averse or risk neutral in AMP (1993). Here for simplicity, I only consider the case of risk neutrality. All the results will remain valid for the risk averse case as long as the potential gain from trade is high enough.

<sup>9</sup>We may either assume same utility function with heterogeneous beliefs, or assume common prior with different utility functions, in order to give agents an incentive to trade. Here we adopt the former one and in the next version we may also consider the latter. For other approaches to induce trade, see AMP (1993) for details.

<sup>10</sup>In the AMP (1993) model, they only focus on the price-refined information  $S_{it}^P$ . In their model it is assumed that the trades are not common knowledge and hence agents cannot get additional information from trades.

Obviously  $\forall i = 1, 2, \dots, I, \forall t = 1, 2, \dots, T, \forall \omega \in \Omega, \{\omega\} \subseteq s_{it}^{PX}(\omega) \subseteq s_{it}(\omega)$ .

We assume agents have perfect memory so that

$$\forall i = 1, 2, \dots, I, \forall \omega \in \Omega, \forall t > t', s_{it}(\omega) \subseteq s_{it'}(\omega).$$

Obviously this implies that

$$\forall i = 1, 2, \dots, I, \forall \omega \in \Omega, \forall t > t', s_{it}^{PX}(\omega) \subseteq s_{it'}^{PX}(\omega).$$

It should be noted that when agents make trades to optimize their payoffs, the information they based on is  $s_{it}^{PX}(\omega)$  instead of  $S_{it}$ , since it is assumed that rational agents should make use of all the information they can obtain. As we will see, the assumption of perfect memory plays an important role in Proposition 1, which we will state at the end of this section.

## 2.3 Rational Expectations Equilibrium

Before we come to the definition of a rational expectations equilibrium, in order to be consistent with the AMP (1993) model, two concepts have to be introduced first.

**Definition 1 (Information Feasibility)** *Agent  $i$ 's net trades  $x_i$  are information feasible if in each period  $t$ ,  $x_{it}$  is measurable with respect to player  $i$ 's price-and-trade-refined information,  $S_{it}^{PX}$ . Formally,  $x_i$  are information feasible if*

$$\forall t = 1, 2, \dots, T, \forall \omega \in \Omega, s_{it}^{PX}(\omega) \subseteq \{\omega' : x_{it}(\omega') = x_{it}(\omega)\}.$$

The last part of the above expression is equivalent to  $\forall \omega', \omega'' \in s_{it}^{PX}(\omega), x_{it}(\omega') = x_{it}(\omega'')$ , which might capture more intuition than the one used in the definition. Basically, information feasibility rules out the possibility of acting differently given the same information.

**Definition 2 (No Short Sales)** *Agent  $i$ 's net trades  $x_i$  satisfy no short sales if in each period  $t$  and in each state  $\omega$  agent  $i$ 's holdings of the risky asset are non-negative. Formally,  $x_i$  satisfy no short sales if*

$$\forall t = 1, 2, \dots, T, \forall \omega \in \Omega, e_i + \sum_{s=0}^t x_{it}(\omega) \geq 0.$$

As shown in AMP (1993), this no short sales condition is necessary for the existence of a bubble in a rational expectations equilibrium. It should be noted that there is no constraint on the short sales of money.

Denote by  $j_t(\omega)$  the join of  $s_{1t}(\omega), s_{2t}(\omega), \dots, s_{It}(\omega)$ ,<sup>11</sup> and by  $m_t(\omega)$  the meet of  $s_{1t}(\omega), s_{2t}(\omega), \dots, s_{It}(\omega)$ .<sup>12</sup>

<sup>11</sup>The join  $j_t(\omega)$  of  $s_{1t}(\omega), s_{2t}(\omega), \dots, s_{It}(\omega)$  is such that (1)  $\forall i = 1, 2, \dots, I, j_t(\omega) \subseteq s_{it}(\omega)$  and (2) for all  $j'_t(\omega)$  satisfying (1),  $j'_t(\omega) \subseteq j_t(\omega)$ . It is also called the coarsest common refinement.

<sup>12</sup>The meet  $m_t(\omega)$  of  $s_{1t}(\omega), s_{2t}(\omega), \dots, s_{It}(\omega)$  is such that (1)  $\forall i = 1, 2, \dots, I, s_{it}(\omega) \subseteq m_t(\omega)$  and (2) for all  $m'_t(\omega)$  satisfying (1),  $m_t(\omega) \subseteq m'_t(\omega)$ . It is also called the finest common coarsening.

Now we are ready to give the definition of a Rational Expectations Equilibrium in this pure exchange economy.

**Definition 3 (Rational Expectations Equilibrium)**  $(P, x) \in R_+^{NT} \times R^{INT}$  is a Rational Expectations Equilibrium if

(C1)  $\forall i = 1, 2, \dots, I$ ,  $x_i$  are information feasible and satisfy no short sales. Denote the set of all such  $x_i$ 's by  $F_i(e_i, P, x_{-i}, S_i)$ , where  $S_i = (S_{i1}, S_{i2}, \dots, S_{iT})$ ,<sup>13</sup>

(C2)  $\forall i = 1, 2, \dots, I$ ,  $x_i \in \arg \max_{x'_i \in F_i(e_i, P, x_{-i}, S_i)} \sum_{\omega \in \Omega} \pi_i(\omega) u_i(y_i(\omega, P, x'_i))$ ;<sup>14</sup>

(C3)  $\forall t = 1, 2, \dots, T, \forall \omega \in \Omega, \sum_{i=1}^I x_{it}(\omega) = 0$ ;

(C4)  $\forall t = 1, 2, \dots, T, P_t(\cdot)$  is measurable with respect to  $j_t(\omega)$ . Formally,  $\forall t = 1, 2, \dots, T, \forall \omega \in \Omega, j_t(\omega) \subseteq \{\omega' : P_t(\omega') = P_t(\omega)\}$ .

Basically, (C1) describes the feasible set of trade for each agent, (C2) says that each agent maximizes his expected utility given his price-and-trade-refined information, (C3) requires that the market should clear in equilibrium, and (C4) implies that all the information contained in price is from the join of the individual information.

## 2.4 Different Concepts of Bubbles

Different definitions of bubbles will lead to different results even within the same framework. As a base line, we use the concept of an expected bubble, defined in AMP (1993). As we will see, the stronger the concept of a bubble become, the harder for it to exist in equilibrium.

**Definition 4 (Expected Bubble)** As in AMP (1993), an expected bubble is said to exist in state  $\omega$  in period  $t$  if in state  $\omega$  it is mutual knowledge that the price of the risky asset in period  $t$  is higher than the expected dividend an agent will receive, that is

$$\forall i = 1, 2, \dots, I, P_t(\omega) > \frac{1}{\sum_{\omega' \in s_{it}^{PX}(\omega)} \pi_i(\omega')} \sum_{\omega' \in s_{it}^{PX}(\omega)} \pi_i(\omega') d(\omega').$$

**Definition 5 (Strong Bubble)** As in AMP (1993), a strong bubble is said to exist in state  $\omega$  in period  $t$  if in state  $\omega$  it is mutual knowledge that the price of the risky asset in period  $t$  is higher than the maximum possible dividend an agent will receive, that is

$$\forall i = 1, 2, \dots, I, \forall \omega' \in s_{it}^{PX}(\omega), P_t(\omega) > d(\omega').$$

<sup>13</sup>Since  $\forall x_i \in F_i$ ,  $x_i$  are information feasible,  $F_i$  depends on the information structure  $S_i$ , the prices  $P$ , and other agents' trades  $x_{-i}$ . Since  $x_i$  satisfy no short sales,  $F_i$  depends on the endowment  $e_i$ . That's why it is written as  $F_i(e_i, P, x_{-i}, S_i)$ .

<sup>14</sup>Another perhaps more intuitive way to express (C2) is (C2')  $\forall i = 1, 2, \dots, I, x_i \in \arg \max_{x'_i \in F_i(e_i, P, x_{-i}, S_i)} E_i[u_i(y_i(\omega, P, x'_i)) | S_{i1}^{PX}]$ . It is easy to see that (C2') is equivalent to (C2).

As seen from above, the concept of strong bubbles strengthens the concept of expected bubbles in a way that it requires that the asset price be higher than the maximum possible dividend, not just the expected dividend. As will be seen below, another way to strengthen the concept of expected bubbles is to require common knowledge instead of mutual knowledge. This requirement is reasonable since in the real world people's behaviors do not only depend on their own beliefs, but also depend on others' beliefs, others' beliefs on their own beliefs, and so on. Therefore, we might expect to see something different when common knowledge is introduced into the concept of bubbles.

**Definition 6 (Common Expected Bubble)** *A common expected bubble is said to exist in state  $\omega$  in period  $t$  if in state  $\omega$  it is common knowledge that the price of the risky asset in period  $t$  is higher than the expected dividend an agent will receive, that is*

$$\forall i = 1, 2, \dots, I, \forall \omega' \in m_t^{PX}(\omega), P_t(\omega) > \frac{1}{\sum_{\omega'' \in s_{it}^{PX}(\omega')} \pi_i(\omega'')} \sum_{\omega'' \in s_{it}^{PX}(\omega')} \pi_i(\omega'') d(\omega'').^{15}$$

**Definition 7 (Common Strong Bubble)** *A common strong bubble is said to exist in state  $\omega$  in period  $t$  if in state  $\omega$  it is common knowledge that the price of the risky asset in period  $t$  is higher than the maximum possible dividend an agent will receive, that is*

$$\forall \omega' \in m_t^{PX}(\omega), P_t(\omega) > d(\omega').$$

## 2.5 Nonexistence of Common Strong Bubbles in Equilibrium

Among the 4 definitions above, clearly the common strong bubble is the strongest one. One may wonder if there exists such a bubble in a rational expectations equilibrium. The answer is NO, due to the following proposition. This nonexistence result is actually an immediate implication from Corollary 4.1 in Morris-Postlewaite-Shin (1995). Here we adopt a different approach to proof.

**Proposition 1** *Under the perfect memory assumption,  $\forall \omega \in \Omega, \forall t = 1, 2, \dots, T$ , it is impossible for a common strong bubble to exist in state  $\omega$  in period  $t$  in any rational expectations equilibrium.*

**Proof.** Suppose it is possible and  $\exists \omega, \exists t$  such that a common strong bubble exists in state  $\omega$  in period  $t$  in a rational expectations equilibrium. Then  $m_t^{PX}(\omega)$  is the set of states where there is common knowledge among agents when  $\omega$  is realized. Thus we have  $\forall \omega' \in m_t^{PX}(\omega), P_t(\omega) = P_t(\omega') > d(\omega')$ . By the feature of rational expectations equilibrium, there must exist some agent  $i$  for whom buying is at least as good as

<sup>15</sup> $m_t^{PX}(\omega)$  is the meet of  $s_{1t}^{PX}(\omega), s_{2t}^{PX}(\omega), \dots, s_{It}^{PX}(\omega)$ .

selling, which implies that  $P_t(\omega) \leq E_i [P_{t+1}(\omega') | \omega' \in s_{it}^{PX}(\omega)]$ . Therefore,  $P_t(\omega) \leq \max_i \max_{\omega' \in s_{it}^{PX}(\omega)} P_{t+1}(\omega') \leq \max_{\omega' \in m_t^{PX}(\omega)} P_{t+1}(\omega')$ . Since agents have perfect memory, we have  $\forall i = 1, 2, \dots, I, s_{i(t+1)}^{PX}(\omega) \subseteq s_{it}^{PX}(\omega)$ , which implies  $m_{t+1}^{PX}(\omega) \subseteq m_t^{PX}(\omega)$ . By induction we have  $P_t(\omega) \leq \max_{\omega' \in m_t^{PX}(\omega)} P_{T+1}(\omega') = \max_{\omega' \in m_t^{PX}(\omega)} d(\omega')$ . Thus  $\exists \omega^* \in m_t^{PX}(\omega)$  such that  $d(\omega^*) \geq P_t(\omega)$ , which causes a contradiction. ■

The intuition behind the nonexistence of common strong bubbles is that if it is common knowledge that the price today is higher than the highest dividend agents may receive, then agents might be better off by selling the asset instead of holding it, no matter what kind of heterogeneous beliefs they may have. Since everyone wants to sell, there cannot be a rational expectations equilibrium any more. It is worth noting that the result of Proposition 1 is independent of the assumption of common knowledge of trades. In the case of no common knowledge of trades, the result is still true. The only modification needed is replacing the price-and-trade-refined information by the price-refined information. It is also worth noting that the result of Proposition 1 crucially depends on the perfect memory assumption. If we allow for agents to forget some information they knew before, a common strong bubble may exist in a rational expectations equilibrium. Such a counterexample is presented in Section 6.

Though under the standard assumption of perfect memory there is no common strong bubble in any rational expectations equilibrium, an expected bubble, which is both strong and common expected, can exist in a rational expectations equilibrium of a three-period two-agent economy, as will be shown in the next section.

### 3 A Simple Example: Strong Bubbles and Common Expected Bubbles with Two Agents

#### 3.1 Exogenous Setting

AMP (1993) has constructed a strong bubble in a rational expectations equilibrium of a three-period three-agent economy with the assumption of no common knowledge of trades. In this section, I will provide a simple example of the existence of strong bubbles with two agents where trades become automatically common knowledge. Moreover, as will be shown, the bubble in the example will also be robust to common knowledge in the expected sense, hence a common expected bubble.

There are 2 agents ( $A$  and  $B$ ), 3 periods (1, 2, and 3) and 8 states ( $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7$  and  $\omega_8$ ). Only 2 assets exist in the market: one is money and the other is called a risky asset. Each share of the risky asset will pay a dividend of amount 4 at the end of period 3 if the state is either  $\omega_1$  or  $\omega_4$ , and will pay nothing otherwise, as shown in the table below.

State	$\omega_1$	$\omega_2$	$\omega_3$	$\omega_4$	$\omega_5$	$\omega_6$	$\omega_7$	$\omega_8$
$d(\omega)$	4	0	0	4	0	0	0	0

Each agent is endowed with  $m_i$  unit of money and 1 share of the risky asset at the beginning of period 1. Agents can trade in each of period 1, 2, and 3. In period 3, after the trade is made, the dividend is realized, and then the consumption takes place.

Keeping in mind that the asymmetric information is the key to generating strong bubbles, we achieve this goal by giving agents different information structures. Remind that agent  $i$ 's ( $i = A, B$ ) information about the state in period  $t$  ( $t = 1, 2, 3$ ) is represented by  $S_{it}$ , a partition of the space  $\Omega$ . The specific structures of  $S_{it}$ 's are given by

$$\begin{aligned}
S_{A1} &= \{\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_8\}, \{\omega_6, \omega_7\}\} \\
S_{B1} &= \{\{\omega_1, \omega_2, \omega_4, \omega_5, \omega_6, \omega_8\}, \{\omega_3, \omega_7\}\} \\
S_{A2} &= \{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5\}, \{\omega_6, \omega_7\}, \{\omega_8\}\} \\
S_{B2} &= \{\{\omega_4, \omega_5, \omega_6\}, \{\omega_1, \omega_2\}, \{\omega_3, \omega_7\}, \{\omega_8\}\} \\
S_{A3} &= S_{B3} = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}, \{\omega_5\}, \{\omega_6\}, \{\omega_7\}, \{\omega_8\}\}.
\end{aligned}$$

At first glance, this particular structure of information may seem complicated, but as our analysis goes on, the reason why it is set in this form will become clear. So far, there are at least three observations. First, in period 3, each agent is perfectly informed of what the realized state is and hence there is no asymmetric information then. Second, in period 2, agent  $A$  receives more information only when he observed  $\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_8\}$  in period 1, and agent  $B$  receives more information only when he observed  $\{\omega_1, \omega_2, \omega_4, \omega_5, \omega_6, \omega_8\}$  in period 1. Third, in period 1, if the state  $\omega_7$  is realized, each agent knows that he will receive no dividend for sure.<sup>16</sup> Hence if the price is positive in period  $t = 1$  in state  $\omega = \omega_7$ , there will be a strong bubble, and that is part of what we are going for. The state where there is a strong bubble is called a *bubble state*.

There are different approaches to generate potential gains from trade. Instead of assuming different marginal utility levels across the states, here we let agents have heterogeneous beliefs, as shown in the table below with weight  $W = \frac{1}{16}$ .

State	$\omega_1$	$\omega_2$	$\omega_3$	$\omega_4$	$\omega_5$	$\omega_6$	$\omega_7$	$\omega_8$
$\pi_A$	2	1	1	1	2	1	1	7
$\pi_B$	1	2	1	2	1	1	1	7

Also, the structure of the beliefs may seem complicated for now, but it will become clear why it serves for the existence of a bubble in a rational expectation equilibrium. So far, it is easy to observe that within the two states where there will be a dividend of 4, agent  $A$  puts a higher weight on state  $\omega_1$ , and agent  $B$  puts a higher weight on state

<sup>16</sup>Take agent  $A$  into consideration for example. When  $\omega_7$  is realized, agent  $A$  will have observed the event  $\{\omega_6, \omega_7\}$ . Since in either state  $\omega_6$  or  $\omega_7$ , there is no dividend payment, agent  $A$  knows that he will receive no dividend with probability 1.

$\omega_4$ . They put the same weight on state  $\omega_7$ , and state  $\omega_8$ , respectively. The weights they put on events  $\{\omega_1, \omega_2, \omega_3\}$  and  $\{\omega_4, \omega_5, \omega_6\}$  are also symmetric.

### 3.2 A Rational Expectations Equilibrium with a Bubble

Recall the standard definition given in the last section, and in our example a rational expectations equilibrium will be a vector  $(P, x) \in R_+^{3 \times 8} \times R^{2 \times 3 \times 8}$  such that

- (C1)  $\forall i = A, B$ , net trades  $x_i$  are information feasible and satisfy no short sales;
- (C2)  $\forall i = A, B$ ,  $x_i$  maximize player  $i$ 's expected payoff with respect to his own price-and-trade-refined information;
- (C3)  $\forall t = 1, 2, 3, \forall n = 1, \dots, 8, x_{At}(\omega_n) + x_{Bt}(\omega_n) = 0$ ;
- (C4)  $\forall t = 1, 2, 3, \forall n, m = 1, \dots, 8, j_t(\omega_n) \subseteq \{\omega_m : P_t(\omega_m) = P_t(\omega_n)\}$ .

Although there are multiple rational expectations equilibria for this example, the one with the equilibrium prices and trades given in the following two tables is what we are interested in - the one in which there is a strong bubble and a common expected bubble.

State	$\omega_1$	$\omega_2$	$\omega_3$	$\omega_4$	$\omega_5$	$\omega_6$	$\omega_7$	$\omega_8$
$P_1(\omega)$	1	1	1	1	1	1	1	1
$P_2(\omega)$	2	2	2	2	2	2	0	0
$P_3(\omega)$	4	0	0	4	0	0	0	0

$$\forall \omega \in \Omega, x_{A1}(\omega) = x_{B1}(\omega) = x_{A3}(\omega) = x_{B3}(\omega) = 0$$

State	$\omega_1$	$\omega_2$	$\omega_3$	$\omega_4$	$\omega_5$	$\omega_6$	$\omega_7$	$\omega_8$
$x_{A2}(\omega)$	1	1	1	-1	-1	-1	0	0
$x_{B2}(\omega)$	-1	-1	-1	1	1	1	0	0
$x_{A2}(\omega) + x_{B2}(\omega)$	0	0	0	0	0	0	0	0

#### 3.2.1 Price-and-Trade-Refined Information

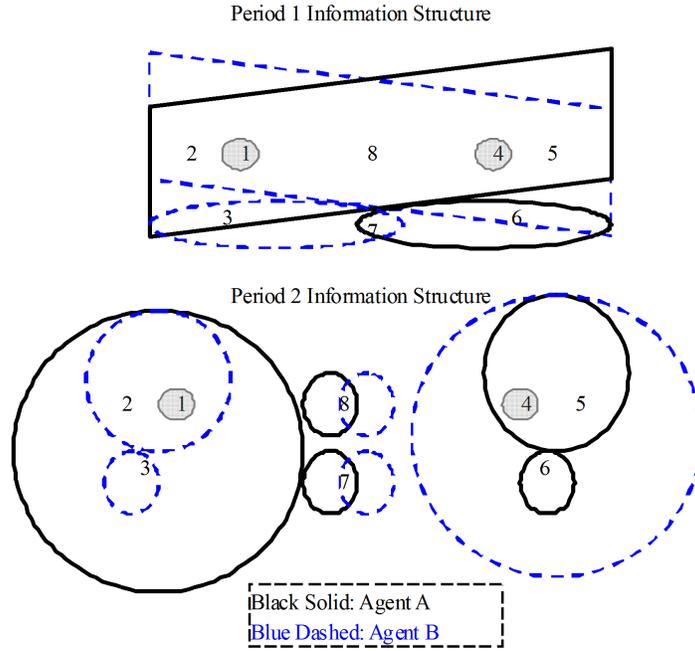
First, derive the price-and-trade-refined information for each agent in each period. It is easy to observe from the price table that  $P_1(\omega) = 1 \forall \omega \in \Omega$  and from the trade table that  $x_{A1}(\omega) = x_{B1}(\omega) = 0 \forall \omega \in \Omega$ . This implies that the prices and trades in period 1 reveal no information. Hence  $S_{A1}^{PX} = S_{A1}$ ,  $S_{B1}^{PX} = S_{B1}$ . Since in period 3, all agents already have full information about the state before observing the prices and making the trades,<sup>17</sup> the prices and trades in period 3 again, reveal no information. Hence  $S_{A3}^{PX} = S_{A3}$ ,  $S_{B3}^{PX} = S_{B3}$ . The only new information revealed by prices and trades in period 2 is that agents know where they are for sure when the state  $\omega_7$  is realized.

<sup>17</sup>Actually there is no trade in period 3 in the equilibrium under study.

Hence agents' price-and-trade-refined information in period 2 is the following, with the original information structure attached below for comparison.

$$\begin{aligned}
 S_{A2}^{PX} &= \{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5\}, \{\omega_6\}, \{\omega_7\}, \{\omega_8\}\} \\
 S_{B2}^{PX} &= \{\{\omega_4, \omega_5, \omega_6\}, \{\omega_1, \omega_2\}, \{\omega_3\}, \{\omega_7\}, \{\omega_8\}\} \\
 S_{A1} &= \{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5\}, \{\omega_6, \omega_7\}, \{\omega_8\}\} \\
 S_{B1} &= \{\{\omega_4, \omega_5, \omega_6\}, \{\omega_1, \omega_2\}, \{\omega_3, \omega_7\}, \{\omega_8\}\}.
 \end{aligned}$$

The following graph may give more intuition about the information structure than the mathematical expression does. In the graph, agent  $A$ 's information sets are described by the black solid curves; agent  $B$ 's information sets are described by the blue dotted curves; dividend paying states are emphasized in gray color.



It is worth noting that in period 2, with the price-and-trade-refined information, agent  $A$  is better informed than agent  $B$  when event  $\{\omega_4, \omega_5, \omega_6\}$  happens, and agent  $B$  is better informed than agent  $A$  when event  $\{\omega_1, \omega_2, \omega_3\}$  happens. We will see soon that the subgroup of states  $\{\omega_4, \omega_5, \omega_6\}$  is where agent  $A$  takes advantage of agent  $B$  by selling the asset he believes is overpriced to agent  $B$ , and similarly, the subgroup of states  $\{\omega_1, \omega_2, \omega_3\}$  is where agent  $B$  takes advantage of agent  $A$ .

### 3.2.2 The Existence of Strong Bubbles and Common Expected Bubbles

Second, note that there is a strong bubble in period 1 in state  $\omega_7$  since for agent  $A$ ,  $s_{A1}^{PX}(\omega_7) = \{\omega_6, \omega_7\}$ ,  $P_1(\omega_7) = 1 > 0 = d(\omega_6) = d(\omega_7)$ , and for agent  $B$ ,  $s_{B1}^{PX}(\omega_7) =$

$\{\omega_3, \omega_7\}$ ,  $P_1(\omega_7) = 1 > 0 = d(\omega_3) = d(\omega_7)$ . In short, a strong bubble exists in period 1 in  $\omega_7$  because in that state every agent knows the asset is worthless but with a positive current price.

In this example,  $m_1^{PX}(\omega_7) = \Omega$ . To see that this bubble is robust to common knowledge in the expected sense, we need to check that  $\forall i = A, B, \forall \omega \in \Omega, 1 > \frac{1}{\sum_{\omega' \in s_{i1}^{PX}(\omega)} \pi_i(\omega')} \sum_{\omega' \in s_{i1}^{PX}(\omega)} \pi_i(\omega') d(\omega')$ . There are four cases:

- 1  $\omega = \omega_7$ : Agent  $A$  observes the event  $\{\omega_6, \omega_7\}$ , and agent  $B$  observes the event  $\{\omega_3, \omega_7\}$ . Each of them will deduce that the expected dividend in period 3 will be  $\frac{1}{2}0 + \frac{1}{2}0 = 0$ , which is less than the current price.
- 2  $\omega = \omega_6$ : Agent  $A$  observes the event  $\{\omega_6, \omega_7\}$ , and his expected dividend in period 3 is 0, less than the current price. Agent  $B$  observes  $\Omega \setminus \{\omega_3, \omega_7\}$ , and his expected dividend in period 3 is  $\frac{3}{14}4 + \frac{11}{14}0 = \frac{6}{7}$ , less than the current price.
- 3  $\omega = \omega_3$ : Agent  $B$  observes the event  $\{\omega_3, \omega_7\}$ , and his expected dividend in period 3 is 0, less than the current price. Agent  $A$  observes  $\Omega \setminus \{\omega_6, \omega_7\}$ , and his expected dividend in period 3 is  $\frac{3}{14}4 + \frac{11}{14}0 = \frac{6}{7}$ , less than the current price.
- 4  $\omega_n \in \Omega \setminus \{\omega_3, \omega_6, \omega_7\}$ , Agent  $A$  observes the event  $\Omega \setminus \{\omega_6, \omega_7\}$ , and agent  $B$  observes the event  $\Omega \setminus \{\omega_3, \omega_7\}$ . Each of them will deduce that the expected dividend in period 3 will be  $\frac{3}{14}4 + \frac{11}{14}0 = \frac{6}{7}$ , which is less than the current price.

Therefore, the bubble in period 1 in state  $\omega_7$  is a common expected bubble. Actually, the reader can check that in our example the common expected bubble exists in period 1, not only in state  $\omega_7$ , but also in any other state.

### 3.2.3 Check of Equilibrium Conditions

Last, check that the prices and trades described above constitute a rational expectations equilibrium. We check all four conditions step by step.

**Check (C1):** We observe from the trade table that the minimum amount of trade in period 2 is  $-1$ . By the fact that there is no trade in either period 1 or 3 and that each agent is endowed with 1 share of the risky asset, the no short sales condition is satisfied for  $x_A$  and  $x_B$ . To see if the  $x_i$ 's are information feasible, it suffices to only look at period 2 since no trade occurs either in period 1 or 3. In period 2, actually each agent's action remains the same given the same price-and-trade-refined information.<sup>18</sup> This implies that  $x_A$  and  $x_B$  also satisfy the information feasibility condition.

<sup>18</sup>Take agent  $A$  for example.

$$\begin{aligned} \forall \omega = \omega_6, s_{A2}^{PX}(\omega) &= \{\omega_6\} \subset \{\omega_4, \omega_5, \omega_6\} = \{\omega' : x_{A2}(\omega') = x_{A2}(\omega)\}, \\ \forall \omega \in \{\omega_4, \omega_5\}, s_{A2}^{PX}(\omega) &= \{\omega_4, \omega_5\} \subset \{\omega_4, \omega_5, \omega_6\} = \{\omega' : x_{A2}(\omega') = x_{A2}(\omega)\}, \\ \forall \omega \in \{\omega_1, \omega_2, \omega_3\}, s_{A2}^{PX}(\omega) &= \{\omega_1, \omega_2, \omega_3\} = \{\omega' : x_{A2}(\omega') = x_{A2}(\omega)\}, \\ \forall \omega \in \{\omega_7, \omega_8\}, s_{A2}^{PX}(\omega) &= \{\omega\} \subset \{\omega_7, \omega_8\} = \{\omega' : x_{A2}(\omega') = x_{A2}(\omega)\}. \end{aligned}$$

**Check (C2):** Maximization of the expected payoff at the beginning of period 1 under the constraints of information feasibility and no short sales, is equivalent to maximization of the expected payoff in each period given the current price–and-trade–refined information under the same constraints.

In period 3, each agent has no incentive to trade since the price is exactly equal to the dividend for every state.

In period 2, there are in total 4 cases:

- (p2-i)  $\forall i \in \{A, B\}$ , if agent  $i$  observes the event  $\{\omega_7\}$  or  $\{\omega_8\}$ , he knows that with probability 1 the price in period 3 will be 0, which is equal to the current price, thus he is indifferent between trading or not in period 2, so the equilibrium trade of 0 maximizes his expected payoff in this case.
- (p2-ii) If agent  $A$  observes the event  $\{\omega_1, \omega_2, \omega_3\}$  (or if agent  $B$  observes the event  $\{\omega_4, \omega_5, \omega_6\}$ ), he will deduce that the expected price in period 3 will be  $\frac{1}{2}4 + \frac{1}{4}0 + \frac{1}{4}0 = 2$ , which is equal to the current price, thus he is indifferent between trading or not in period 2, so the equilibrium trade of 1 maximizes his expected payoff in this case.
- (p2-iii) If agent  $A$  observes the event  $\{\omega_4, \omega_5\}$  (or if agent  $B$  observes the event  $\{\omega_1, \omega_2\}$ ), he will deduce that the expected price in period 3 will be  $\frac{1}{3}4 + \frac{2}{3}0 = \frac{4}{3}$ , which is less the current price 2, thus he has an incentive to sell any of the asset he owns in period 2, so under the short sales constraint and given there is no trade in period 1, the equilibrium trade of  $-1$  maximizes his expected payoff in this case.
- (p2-iv) If agent  $A$  observes the event  $\{\omega_6\}$  (or if agent  $B$  observes the event  $\{\omega_3\}$ ), he knows that with probability 1 the price in period 3 will be 0, which is less the current price 2, thus he has an incentive to sell any of the asset he owns in period 2, so under the short sales constraint and given there is no trade in period 1, the equilibrium trade of  $-1$  maximizes his expected payoff in this case.

In period 1, there are 2 cases:

- (p1-i) If agent  $A$  observes the event  $\{\omega_6, \omega_7\}$  (or if agent  $B$  observes the event  $\{\omega_3, \omega_7\}$ ), he will deduce that the expected price in period 2 will be  $\frac{1}{2}2 + \frac{1}{2}0 = 1$ , which is equal to the current price, thus he is indifferent between trading or not in period 1, so the equilibrium trade of 0 maximizes his expected payoff in this case.
- (p1-ii) If agent  $i$  observes the event other than the one described in (p1-i), he will deduce that the expected price in period 2 will be  $\frac{2 \times 2 + 1 \times 3}{14}2 + \frac{7}{14}0 = 1$ , which is equal to the current price, thus he is indifferent between trading or not in period 1, so the equilibrium trade of 0 maximizes his expected payoff in this case.

The above analysis guarantees that condition (C2) is satisfied.

**Check (C3) and (C4):** It is seen that the market clears in each period in each state from the table of trades, hence (C3) is satisfied. Note that  $P_1(\omega) = 1 \forall \omega \in \Omega$ , hence  $P_1(\cdot)$  is measurable with respect to  $j_1(\cdot)$ . Also note that  $j_3(\omega) = \{\omega\} \forall \omega \in \Omega$ , hence  $P_3(\cdot)$  is measurable with respect to  $j_3(\omega)$ . To see  $P_2(\cdot)$  is measurable with respect to  $j_2(\omega)$ , note that  $\forall n = 1, \dots, 6$ ,  $j_2(\omega_n) \subseteq \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\} = \{\omega : P_2(\omega) = P_2(\omega_n) = 2\}$ , and  $\forall n = 7, 8$ ,  $j_2(\omega_n) \subseteq \{\omega_7, \omega_8\} = \{\omega : P_2(\omega) = P_2(\omega_n) = 0\}$ . This completes the check that the prices and trades given in the example constitute a rational expectations equilibrium.

### 3.3 Discussion

We have shown that, in a simple finite horizon model with asymmetric information and short sales constraints, a strong bubble and a common expected bubble can exist in the same period in the same state in a rational expectations equilibrium with common knowledge of trades, under the same basic setting as in AMP (1993).

It is worthwhile to make some remarks about this simple example.

- (1) The initial distribution of the asset is not efficient. To see this, with zero-trade, each agent's expected payoff

$$m_i + \sum_{\omega \in \Omega} \pi_i(\omega) \left[ e_i P_T(\omega) + \sum_{t=1}^T x_{it}(\omega) [P_{t+1}(\omega) - P_t(\omega)] \right]$$

would have been  $m_i + \frac{3}{4}$ , while in the equilibrium, each agent's expected payoff is  $m_i + 1$ . Thus our example does not violate the no-trade theorem and the necessary condition of ex ante inefficiency is satisfied here. In fact, as the analysis has shown, in our example those who gain from the trade are the sellers whenever the trade takes place.

- (2) The social welfare is maximized in the rational expectation equilibrium with bubbles if there is no initial endowment of money. Note that in our example the social welfare is maximized when in every state the social planner gives all the assets to the agent who puts the highest weight on that state. Hence the maximum social welfare should be  $\frac{9}{8}(m_1 + m_2) + 2$ . When either agent has positive endowment of money, the social welfare of the equilibrium outcome is not maximized. However, if each agent is endowed with no money, then the social welfare is maximized in equilibrium. To put it in another way, if the social planner is only allowed to reallocate on the risky asset, then the equilibrium maximizes the sum of the utilities of the agents. This implies a surprising observation that the rational bubbles do not necessarily lead to inefficiency.
- (3) The short sale constraints are binding in period 2 for the sellers whenever the trade takes place. In the cases of (p2-iii) and (p2-iv), where agents play the seller's

role, since the expected price for the asset is higher than the current price, agents would like to take advantage of this and sell as much as they can. If there were no short sales constraints, an equilibrium would not have been reached under the current price. This is where the no short sales assumption plays its role.

- (4) The asymmetric information functions in such a way that even though all agents know that the asset is overpriced, they are still willing to hold the asset as long as the information on overpricing is not common knowledge in the strong sense. It is this feature that makes a bubble possible in a rational expectations equilibrium.
- (5) For simplicity, the example is constructed in such a way that even though trade is common knowledge, it reveals no additional information to either agent.

## 4 Structural Characteristics for the Existence of Bubbles

Assume there are only two agents. There is no trade in the first period and information becomes perfect in the last period. The dividend can only take two values,  $\forall \omega, d(\omega) \in \{0, D\}$  where  $D > 0$ .

**Claim 1** *Under the perfect memory assumption, suppose there is a bubble in period  $t$  in state  $\omega$  in a rational expectations equilibrium in economy with state set  $\Omega$ . Then there is also a bubble in equilibrium in the subeconomy with state set  $m_t^{PX}(\omega)$ .*

**Claim 2** *Under the perfect memory assumption, for a strong bubble to exist in a rational expectations equilibrium in a 2-agent 3-period economy, there must be at least 2 states with positive dividends, that is*

$$|\{\omega \in \Omega | d(\omega) > 0\}| \geq 2.$$

**Proof.** Suppose a strong bubble exists in period 1 in state  $\omega^*$ .

Consider agent  $A$  first. Since  $P_1(\omega^*) > \max_{\omega \in s_{A1}^{PX}(\omega^*)} d(\omega) = 0$  and  $P_1(\omega^*) = E_A [P_2(\omega') | \omega' \in s_{A1}^{PX}(\omega^*)]$ , the fact that agent  $A$  is willing to hold the asset implies that  $\exists \omega^A \in s_{A1}^{PX}(\omega^*)$  such that  $P_2(\omega^A) \geq P_1(\omega^*) > 0$ . Since  $s_{A2}^{PX}(\omega^A) \subseteq s_{A1}^{PX}(\omega^A) = s_{A1}^{PX}(\omega^*)$ , when  $\omega^A$  is realized, in period 2 agent  $A$  knows for sure that he will receive nothing. Give  $P_2(\omega^A) > 0$ , it must be the case that when  $\omega^A$  is realized, in period 2 agent  $B$ 's expected return is nonzero. This implies that  $\exists \omega^{AB} \in s_{B2}^{PX}(\omega^A)$  such that  $d(\omega^{AB}) > 0$ . Since in equilibrium in period 2 agent  $A$  will always sell in state  $\omega^A$  and agent  $B$  cannot tell the difference between  $\omega^A$  and  $\omega^{AB}$ , it must be the case that in equilibrium in period 2 agent  $A$  will always sell in state  $\omega^{AB}$  as well.

Then consider agent  $B$ , and we have similar results.  $\exists \omega^B \in s_{B1}^{PX}(\omega^*)$  such that  $P_2(\omega^B) > 0$  and when  $\omega^B$  is realized, in period 2 agent  $B$  knows for sure that he will receive nothing. This implies that  $\exists \omega^{BA} \in s_{A2}^{PX}(\omega^B)$  such that  $d(\omega^{BA}) > 0$  and in equilibrium in period 2 agent  $B$  will always sell in state  $\omega^{BA}$ .

Since in equilibrium in period 2 agent  $A$  always sells in state  $\omega^{AB}$  and agent  $B$  always sells in state  $\omega^{BA}$ ,  $\omega^{AB} \neq \omega^{BA}$ . ■

**Claim 3** *For a strong bubble to exist in a symmetric rational expectations equilibrium in a 2-agent symmetric economy, there must be at least 2 states with positive dividends, that is*

$$|\{\omega \in \Omega | d(\omega) > 0\}| \geq 2.$$

**Proof.** By AMP(1993), for a strong bubble to exist in a rational expectations equilibrium, there must be potential gains from trade. And these gains will be distributed to the agents in each trade. But since there is no constraint on the short sales of money, in each trade the agent who is buying the asset won't receive any gains, otherwise he would be buying as much as he can, in which situation there would be no equilibrium. Therefore, the agents receive the gains only if they play the role of sellers. Since it is a symmetric economy, each agent has a positive probability to sell the asset. Consider Agent  $A$  first. Suppose he is better off by selling the asset in period  $t$  in state  $\omega_A$ . Then in period  $t$  there must be a state with positive dividend, denoted by  $\omega_A^B$ , from which agent  $B$  cannot tell the difference to  $\omega_A$ . Since agent  $B$  is buying in period  $t$  in state  $\omega_A^B$ , this implies that agent  $A$  is selling in period  $t$  in  $\omega_A^B$ . By symmetry, in period  $t$ , there exists another state  $\omega_B^A$  with positive dividend, where agent  $A$  is buying and agent  $B$  is selling. Obviously  $\omega_A^B \neq \omega_B^A$ . ■

**Claim 4** *Under the perfect memory assumption, for a strong bubble to exist in a rational expectations equilibrium in a 2-agent symmetric economy, for each agent, at least one price-and-trade-refined information set contains at least 3 states, including one with positive dividend, that is*

$$\forall i, \exists t, \exists \omega \text{ such that } |s_{it}^{PX}(\omega)| \geq 3 \text{ and } \max_{\omega' \in s_{it}^{PX}(\omega)} d(\omega') > 0.$$

**Proof.** Let  $\omega^*$  be the bubble state. Suppose in period 1 agent  $A$  cannot tell difference between  $\omega^*$  and  $\omega_A$ , both of which are zero-dividend states. And without loss of generality, suppose in period  $t$  in state  $\omega_A$  agent  $A$  can sell the asset at a positive price. This implies that in period  $t$  agent  $B$  cannot tell difference between  $\omega_A$  and some positive-dividend state  $\omega_A^B$ , or  $\omega_A \in s_{Bt}^{PX}(\omega_A^B)$ . Since agent  $B$  will be buying in period  $t$  in state  $\omega_A^B$ , agent  $A$  must be selling, hence in period  $t$  there must exist some zero-dividend state  $\omega'$  such that  $\omega' \in s_{At}^{PX}(\omega_A^B)$ . If  $\omega' \in s_{Bt}^{PX}(\omega_A^B)$ , we are done. Suppose not, then there must exist some positive-dividend state  $\omega''$  such that  $\omega'' \in s_{Bt}^{PX}(\omega')$ . And this would again imply that there exists some zero-dividend state  $\omega'''$  such that  $\omega''' \in s_{At}^{PX}(\omega'')$ . If  $\omega''' \in s_{At}^{PX}(\omega')$  or  $\omega''' \in s_{Bt}^{PX}(\omega')$ , we are done. If not, we can follow

the same logic. Since the number of states is finite, and  $s_{At}^{PX}(\omega_A)$  does not contain any positive-dividend states, at the end we will find a price-and-trade-refined information set which contains at least 3 states including one with positive dividend. By symmetry this is also true for agent  $B$ . ■

**Claim 5** *Under the perfect memory assumption, for a strong bubble to exist in a rational expectations equilibrium in a 2-agent symmetric economy, there must be at least 8 states, that is*

$$|\Omega| \geq 8.$$

**Proof.** Suppose not and there are only 7 states instead. Assume in period  $t$  agent  $i$  has a price-and-trade-refined information set  $\{\omega_{i1}, \omega_{i2}, \omega_{i3}\}$  and the bubble state is  $\omega^*$ . This implies  $P_t(\omega^*) = 0$  and  $P_t(\omega_{ik}) > 0$  for  $i = A, B$  and  $k = 1, 2, 3$ . It is easy to know that in period 1 for agent  $A$ ,  $s_{A1}^{PX}(\omega^*) \subset \{\omega^*, \omega_{B1}, \omega_{B2}, \omega_{B3}\}$ . Without loss of generality, assume  $\omega_{B1} \in s_{A1}^{PX}(\omega^*)$ . Since there is no trade in period 1, the equilibrium price should be equal to agent  $A$ 's expected price. This implies  $P_1(\omega_{B1}) < P_t(\omega_{ik})$  from agent  $A$ 's perspective.

Now consider agent  $B$ . It is easy to know that in period 1 for agent  $B$ ,  $\{\omega_{B1}, \omega_{B2}, \omega_{B3}\} \subset s_{B1}^{PX}(\omega_{B1}) \subset \{\omega_{A1}, \omega_{A2}, \omega_{A3}, \omega_{B1}, \omega_{B2}, \omega_{B3}\}$ . But this would imply  $P_1(\omega_{B1}) = P_t(\omega_{ik})$  from agent  $B$ 's perspective.

Therefore, there must be at least 8 states. ■

**Claim 6** *For a common expected bubble to exist in period  $t$  in state  $\omega$ , it must be the case that the current price is higher than every agent's expected dividend across the meet of the information partition containing  $\omega$ , that is*

$$\forall i = 1, 2, \dots, I, P_t(\omega) > E_i[d(\omega') | \omega' \in m_t^{PX}(\omega)].$$

**Proof.** By the definition of common expected bubbles,  $\forall i = 1, 2, \dots, I, \forall \omega' \in m_t^{PX}(\omega), P_t(\omega) > E_i[d(\omega'') | \omega'' \in s_{it}^{PX}(\omega')]$ .

Since  $E_i[d(\omega') | \omega' \in m_t^{PX}(\omega)]$  is weighted average of  $E_i[d(\omega'') | \omega'' \in s_{it}^{PX}(\omega')]$ , immediately we have  $P_t(\omega) > E_i[d(\omega') | \omega' \in m_t^{PX}(\omega)]$ . ■

It turns out that the example of strong bubbles and common expected bubbles we have presented in the previous section is actually the simplest one with minimum number of states.

## 5 Robust Analysis for the Symmetric Case

According to AMP (1993), by the nature of the model, such a bubble is not robust, neither to perturbations in beliefs nor to perturbations in dividends. However, for an economy with symmetric structure, we find that the equilibria with these bubbles, though are not robust to perturbations in a general sense, but might be robust to perturbations in a symmetric sense.

In this section, we focus on three-period models with a symmetric setting.

**Definition 8 (Symmetry)** *The model has a symmetric setting if for any  $i, j = 1, 2, \dots, I$ , there exists a bijective mapping  $L$  from  $\{1, 2, \dots, N = |\Omega|\}$  to  $\{1, 2, \dots, N\}$  such that for any  $t = 1, 2, 3$ ,*

- (1)  $S_{it} = S_{jt}|L$ , where  $S_{jt}|L$  is  $j$ 's relabelled information partition at  $t$  under  $L$ ;
- (2)  $\pi_i(\omega_n) = \pi_j(\omega_{L(n)})$ ;
- (3)  $d(\omega_n) = d(\omega_{L(n)})$ ;
- (4)  $(m_i, e_i) = (m_j, e_j)$ .

Basically equation (1) means that it is information-symmetric. Similarly it is belief-symmetric by (2), dividend-symmetric by (3), and endowment-symmetric by (4).

It should be noted that the symmetry assumption is more than assuming symmetry w.r.t information, symmetry w.r.t. dividend, symmetry w.r.t. belief, and symmetry w.r.t. endowment, respectively. That is because we require the same mapping  $L$  for conditions (1)-(3) to be satisfied.

We call  $(\omega_n, \omega_{L(n)})$  a symmetric pair of states for agent  $i$  and  $j$  if  $L(L(n)) = n$ .

Recall that a state where there is a strong bubble is called a *bubble state*, denoted by  $\omega^*$ .

A zero-dividend state is called a *bubble-related state* for agent  $i$ , denoted by  $\omega^{*,i}$ , if (1) it is not a bubble state and (2) agent  $i$  cannot tell the difference between this state and the bubble state in the first period. Note there may be more than one bubble-related state for agent  $i$ .

A zero-dividend state is called a *dummy state*,  $\omega^D$ , if when this state is realized (1) no agents are sure about their future payoff in the first period and (2) all of them know that the asset is worthless in the second period. A dummy state is necessary for a strong bubble to exist in equilibrium in our model because of the equilibrium conditions.

In a three-period model, bubble bursts in the second period, which implies that in a bubble state all agents know that the asset is worthless in the second period. This is the same feature between bubble state and dummy state. The difference is that in the first period in a bubble state all agents know that the asset is worthless while in a dummy state they are not sure about the value of the asset.

For instance, in the example of bubbles in Section 3, the setting is symmetric, and for  $i = A, j = B$ , we have the relabelling function

$$L(n) = \begin{cases} n + 3 & \text{if } n = 1, 2, 3 \\ n - 3 & \text{if } n = 4, 5, 6 \\ n & \text{if } n = 7, 8 \end{cases} .$$

It is easy to see that  $(\omega_1, \omega_4)$ ,  $(\omega_2, \omega_5)$ ,  $(\omega_3, \omega_6)$ ,  $(\omega_7, \omega_7)$ ,  $(\omega_8, \omega_8)$  are symmetric pairs of states. Here  $\omega_7$  is the bubble state,  $\omega_6$  is the bubble-related state for agent  $A$ ,  $\omega_3$  is the bubble-related state for agent  $B$ , and  $\omega_8$  is the dummy state.

Now we are ready to give a definition to the symmetric perturbation.

**Definition 9 (Symmetric Perturbation)** For a model with symmetric setting, a perturbation  $\eta : \Omega \rightarrow R$  is Symmetric if for any symmetric pair of states  $(\omega_m, \omega_n)$ ,  $m, n \in \{1, 2, \dots, N\}$ ,

$$\eta(\omega_m) = \eta(\omega_n).$$

Even though mathematically symmetric perturbations are of measure zero when we consider the whole family of perturbations, it does make economic sense to look at this particular type of perturbations. First, economic systems function in a way that same or similar shocks are received in symmetric states. Second, symmetric states may be generated by the same fundamental factor, and hence should be perturbed by the same amount.

In addition to symmetric perturbations, we can have even stronger concepts for perturbations.

**Definition 10 (Very Symmetric Perturbation)** For a model with symmetric setting, a perturbation  $\eta : \Omega \rightarrow R$  is Very Symmetric if (1) it is Symmetric; and (2) for the bubble state  $\omega^*$  and the dummy state  $\omega^D$ ,

$$\eta(\omega^*) = \eta(\omega^D).$$

It is straightforward to see from the definition that a very symmetric perturbation requires same perturbations not only for a symmetric pair of states, but also for a pair of bubble state and dummy state. Since in both bubble state and dummy state the asset is worthless and this becomes agents' mutual knowledge in the second period, it is reasonable to think about the situation where the dividend perturbations for a pair of bubble state and dummy state are the same.

**Definition 11 (Strongly Symmetric Perturbation)** For a model with symmetric setting, a perturbation  $\eta : \Omega \rightarrow R$  is Strongly Symmetric if (1) it is Symmetric; (2) for the bubble state  $\omega^*$  and the dummy state  $\omega^D$

$$\eta(\omega^*) = -\eta(\omega^D);$$

and (3) for any  $i = 1, 2, \dots, I$ , for the bubble state  $\omega^*$  and all the bubble-related state(s)  $\omega^{*,i}$

$$\frac{\eta(\omega^*)}{\pi_i(\omega^*)} = \frac{\sum \eta(\omega^{*,i})}{\sum \pi_i(\omega^{*,i})}.$$

A strongly symmetric perturbation is different from a very symmetric perturbation in two ways. First, for the pair of bubble state and dummy state, the former requires the same amount toward opposite directions while the latter requires the same amount toward the same direction. Second, for the bubble state and all the bubble-related state(s), the former requires the amount proportional on the prior while the latter has no restriction on it. A strongly symmetric perturbation makes sense when we consider

a perturbation in beliefs. Condition (2) can be interpreted as the following: if you increase the probability for the bubble state, you have to decrease the probability for the dummy state by the same amount. Condition (3) is reasonable because it requires the perturbation in beliefs does not affect agents' beliefs of having a strong bubble when the bubble state is realized in the first period.

As will be shown next, these two particular types of symmetric perturbations are of our interest because they play an important role in the robust analysis.

To illustrate the results, we use the example of bubbles in Section 3 for perturbation analysis. Recall that the equilibrium is characterized by the price table and the trade table below.

State	$\omega_1$	$\omega_2$	$\omega_3$	$\omega_4$	$\omega_5$	$\omega_6$	$\omega_7$	$\omega_8$
$P_1(\omega)$	1	1	1	1	1	1	1	1
$P_2(\omega)$	2	2	2	2	2	2	0	0
$P_3(\omega)$	4	0	0	4	0	0	0	0

$$\forall \omega \in \Omega, x_{A1}(\omega) = x_{B1}(\omega) = x_{A3}(\omega) = x_{B3}(\omega) = 0$$

State	$\omega_1$	$\omega_2$	$\omega_3$	$\omega_4$	$\omega_5$	$\omega_6$	$\omega_7$	$\omega_8$
$x_{A2}(\omega)$	1	1	1	-1	-1	-1	0	0
$x_{B2}(\omega)$	-1	-1	-1	1	1	1	0	0
$x_{A2}(\omega) + x_{B2}(\omega)$	0	0	0	0	0	0	0	0

There are potentially two ways to make perturbations: one is through belief distribution and the other is through dividend distribution.

## 5.1 Belief Perturbation

State	$\omega_1$	$\omega_2$	$\omega_3$	$\omega_4$	$\omega_5$	$\omega_6$	$\omega_7$	$\omega_8$
$\pi_A$	$2 + \varepsilon_{A,1}$	$1 + \varepsilon_{A,2}$	$1 + \varepsilon_{A,3}$	$1 + \varepsilon_{A,4}$	$2 + \varepsilon_{A,5}$	$1 + \varepsilon_{A,6}$	$1 + \varepsilon_{A,7}$	$7 + \varepsilon_{A,8}$
$\pi_B$	$1 + \varepsilon_{B,1}$	$2 + \varepsilon_{B,2}$	$1 + \varepsilon_{B,3}$	$2 + \varepsilon_{B,4}$	$1 + \varepsilon_{B,5}$	$1 + \varepsilon_{B,6}$	$1 + \varepsilon_{B,7}$	$7 + \varepsilon_{B,8}$

Suppose the original equilibrium is the one with a bubble in period 1 in state  $\omega_7$ , which was shown previously. Now for each state  $\omega_n$ , the associated belief  $\pi_i(\omega_n)$  (or denoted by  $\pi_{i,n}$  for simplicity) for agent  $i$  ( $i = A, B$ ) is perturbed by a very small amount  $\varepsilon_{i,n}$ , where  $\sum_{1 \leq n \leq 8} \varepsilon_{i,n} = 0$ ,  $i = A, B$ . Suppose the information structure remains the same and the agents trade the same way in the new equilibrium as before, then it suffices to have the new equilibrium prices satisfy the following equations, denoted by  $BP$ .

$$\begin{aligned}
P_3(\omega_n) &= d_n, n = 1, 2, \dots 8. \\
P_2(\omega_{n=1,2,3}) &= \frac{d_1(\pi_{A,1} + \varepsilon_{A,1}) + d_2(\pi_{A,2} + \varepsilon_{A,2}) + d_3(\pi_{A,3} + \varepsilon_{A,3})}{\pi_{A,1} + \pi_{A,2} + \pi_{A,3} + \varepsilon_{A,1} + \varepsilon_{A,2} + \varepsilon_{A,3}}. \\
P_2(\omega_{n=4,5,6}) &= \frac{d_4(\pi_{B,4} + \varepsilon_{B,4}) + d_5(\pi_{B,5} + \varepsilon_{B,5}) + d_6(\pi_{B,6} + \varepsilon_{B,6})}{\pi_{B,4} + \pi_{B,5} + \pi_{B,6} + \varepsilon_{B,4} + \varepsilon_{B,5} + \varepsilon_{B,6}}. \\
P_2(\omega_n) &= d_n, n = 7, 8. \\
P_1(\omega_{n,1 \leq n \leq 8}) &= \frac{P_2(\omega_6)(\pi_{A,6} + \varepsilon_{A,6}) + P_2(\omega_7)(\pi_{A,7} + \varepsilon_{A,7})}{\pi_{A,6} + \pi_{A,7} + \varepsilon_{A,6} + \varepsilon_{A,7}} \\
&= \frac{P_2(\omega_3)(\pi_{B,3} + \varepsilon_{B,3}) + P_2(\omega_7)(\pi_{B,7} + \varepsilon_{B,7})}{\pi_{B,3} + \pi_{B,7} + \varepsilon_{B,3} + \varepsilon_{B,7}} \\
&= \frac{\sum_{1 \leq n \leq 8, n \neq 6, 7} P_2(\omega_n)(\pi_{A,n} + \varepsilon_{A,n})}{\sum_{1 \leq n \leq 8, n \neq 6, 7} (\pi_{A,n} + \varepsilon_{A,n})} \\
&= \frac{\sum_{1 \leq n \leq 8, n \neq 3, 7} P_2(\omega_n)(\pi_{B,n} + \varepsilon_{B,n})}{\sum_{1 \leq n \leq 8, n \neq 3, 7} (\pi_{B,n} + \varepsilon_{B,n})}.
\end{aligned}$$

### 5.1.1 Strongly Symmetric Perturbations

If the perturbation is strongly symmetric, then by definition we have the following conditions.

$$\begin{aligned}
\varepsilon_{A,1} &= \varepsilon_{B,4}, \varepsilon_{B,1} = \varepsilon_{A,4}, \\
\varepsilon_{A,2} &= \varepsilon_{B,5}, \varepsilon_{B,2} = \varepsilon_{A,5}, \\
\varepsilon_{A,3} &= \varepsilon_{B,6}, \varepsilon_{B,3} = \varepsilon_{A,6}, \\
\varepsilon_{A,7} &= \varepsilon_{B,7}, \varepsilon_{A,8} = \varepsilon_{B,8}, \\
\varepsilon_{A,6} &= \varepsilon_{A,7} = -\varepsilon_{A,8}.
\end{aligned}$$

Keep in mind that  $P_2(\omega_7) = P_2(\omega_8) = 0$  in our example, which means that the price of the asset is zero in both bubble state and dummy state in period 2. Note that  $P_2(\omega^*) = P_2(\omega^D) = 0$  is not necessarily true in general, but it always holds for a three-period model with a strong bubble in equilibrium.

Consider the prices specified below. It is easy to check that these prices automatically satisfy the set of equations  $BP$ .

$$\begin{aligned}
P_3(\omega_n) &= d_n, n = 1, 2, \dots 8. \\
P_2(\omega_{n,1 \leq n \leq 6}) &= \begin{cases} \frac{4^{(2+\varepsilon_{A,1})}}{4+\varepsilon_{A,1}+\varepsilon_{A,2}+\varepsilon_{A,3}} & \text{if } 1 \leq n \leq 6 \\ 0 & \text{if } n = 7, 8 \end{cases}. \\
P_1(\omega_{n,1 \leq n \leq 8}) &= \frac{2(2 + \varepsilon_{A,1})}{4 + \varepsilon_{A,1} + \varepsilon_{A,2} + \varepsilon_{A,3}}.
\end{aligned}$$

This implies that we have found a new equilibrium with the above equilibrium prices. The last step is to check whether the coexistence of a strong bubble and a common expected bubble is still true in period 1 in state  $\omega_7$ . The answer is yes as long as the perturbation is sufficiently small such that  $P_1 > \frac{6}{7}$ ,<sup>19</sup> or  $-4\varepsilon_{A,1} + 3\varepsilon_{A,2} + 3\varepsilon_{A,3} < 2$ . This can be guaranteed by assuming  $\max_{i=A,B,1 \leq n \leq 8} |\varepsilon_{i,n}| < \frac{1}{5}$ . Therefore, the bubble in our example is robust to any strongly symmetric perturbations in beliefs if  $\max_{i=A,B,1 \leq n \leq 8} |\varepsilon_{i,n}| < \frac{1}{5}$ .

It is also worth noting that this result can also be applied to a more general case where the overpriced asset is not necessarily worthless. As long as the dividend in the bubble state and dummy state are the same ( $d_7 = d_8$ ), hence  $P_2(\omega_7) = P_2(\omega_8)$ , then we still have the same result.

## 5.2 Dividend Perturbation

State	$\omega_1$	$\omega_2$	$\omega_3$	$\omega_4$	$\omega_5$	$\omega_6$	$\omega_7$	$\omega_8$
$d(\omega)$	$4 + \delta_1$	$\delta_2$	$\delta_3$	$4 + \delta_4$	$\delta_5$	$\delta_6$	$\delta_7$	$\delta_8$

Suppose the original equilibrium is the one we studied before. Now for each state  $\omega_n$ , the associated dividend  $d_n$  is perturbed by a very small amount  $\delta_n$ . Suppose the information structure remains the same and the agents trade the same way in the new equilibrium as before, then it suffices to have the new equilibrium prices satisfy the following equations, denoted by  $DP$ .

$$\begin{aligned}
P_3(\omega_n) &= d_n + \delta_n, n = 1, 2, \dots, 8. \\
P_2(\omega_{n=1,2,3}) &= \frac{\pi_{A,1}(d_1 + \delta_1) + \pi_{A,2}(d_2 + \delta_2) + \pi_{A,3}(d_3 + \delta_3)}{\pi_{A,1} + \pi_{A,2} + \pi_{A,3}}. \\
P_2(\omega_{n=4,5,6}) &= \frac{\pi_{B,4}(d_4 + \delta_4) + \pi_{B,5}(d_5 + \delta_5) + \pi_{B,6}(d_6 + \delta_6)}{\pi_{B,4} + \pi_{B,5} + \pi_{B,6}}. \\
P_2(\omega_n) &= d_n + \delta_n, n = 7, 8. \\
P_1(\omega_{n,1 \leq n \leq 8}) &= \frac{\pi_{A,6}P_2(\omega_6) + \pi_{A,7}P_2(\omega_7)}{\pi_{A,6} + \pi_{A,7}} \\
&= \frac{\pi_{B,3}P_2(\omega_3) + \pi_{B,7}P_2(\omega_7)}{\pi_{B,3} + \pi_{B,7}} \\
&= \frac{\sum_{1 \leq n \leq 8, n \neq 6, 7} \pi_{A,n}P_2(\omega_n)}{\sum_{1 \leq n \leq 8, n \neq 6, 7} \pi_{A,n}} \\
&= \frac{\sum_{1 \leq n \leq 8, n \neq 3, 7} \pi_{B,n}P_2(\omega_n)}{\sum_{1 \leq n \leq 8, n \neq 3, 7} \pi_{B,n}}.
\end{aligned}$$

<sup>19</sup>The number  $\frac{6}{7}$  was obtained when we check the existence of a common expected bubble in Section 3.

### 5.2.1 Very Symmetric Perturbations

If the perturbation is very symmetric, then by definition we have the following equations.

$$\begin{aligned}\delta_1 &= \delta_4, \\ \delta_2 &= \delta_5, \\ \delta_3 &= \delta_6, \\ \delta_7 &= \delta_8.\end{aligned}$$

Consider the prices specified below. It is easy to check that these prices automatically satisfy the set of equations  $DP$ .

$$\begin{aligned}P_3(\omega_n) &= d_n + \delta_n, n = 1, 2, \dots, 8. \\ P_2(\omega_{n,1 \leq n \leq 6}) &= \begin{cases} 2 + \frac{2\delta_1 + \delta_2 + \delta_3}{4} & \text{if } 1 \leq n \leq 6 \\ \delta_n & \text{if } n = 7, 8 \end{cases} . \\ P_1(\omega_{n,1 \leq n \leq 8}) &= 1 + \frac{2\delta_1 + \delta_2 + \delta_3 + 4\delta_7}{8}.\end{aligned}$$

This implies that we have found a new equilibrium with the above equilibrium prices. The last step is to check whether the coexistence of a strong bubble and a common expected bubble is still true in period 1 in state  $\omega_7$ . Similarly, the answer is yes as long as the perturbation is sufficiently small such that  $P_1 > \frac{6}{7}$ , or  $2\delta_1 + \delta_2 + \delta_3 + 4\delta_7 > -\frac{8}{7}$ . This can be guaranteed by assuming  $\max_{1 \leq n \leq 8} |\delta_n| < \frac{1}{7}$ . Therefore, the bubble in our example is robust to any very symmetric perturbations in dividends if  $\max_{1 \leq n \leq 8} |\delta_n| < \frac{1}{7}$ .

Similarly here we don't necessarily require that  $d_7 = d_8 = 0$ . The result holds as long as  $d_7 = d_8$ , which implies  $P_2(\omega_7) = P_2(\omega_8)$ .

## 6 An Example of Common Strong Bubbles with Agents of Imperfect Memory

In Section 2, we have pointed out that the nonexistence result for common strong bubbles relies heavily upon the assumption of perfect memory. Once this standard assumption is relaxed, that is to say agents might forget some information they originally knew, then it is possible to have a common strong bubble in a rational expectations equilibrium. A simple counterexample is constructed below.

### 6.1 Exogenous Setting

The same as before, there are 2 agents ( $A$  and  $B$ ), 3 periods (1, 2, and 3) and 8 states ( $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7$  and  $\omega_8$ ). There are only 2 assets: money and the risky asset.

The dividend distribution over states for the risky asset remains exactly the same as in Section 3, and is shown in the table below.

State	$\omega_1$	$\omega_2$	$\omega_3$	$\omega_4$	$\omega_5$	$\omega_6$	$\omega_7$	$\omega_8$
$d(\omega)$	4	0	0	4	0	0	0	0

Agents' endowment are the same as before, that is  $m_i$  unit of money and 1 unit of the risky asset. What differs from the previous example is the information structure. In period 1, both agents receive the same information, represented by  $S_{A1}$  and  $S_{B1}$  respectively, where  $S_{A1} = S_{B1}$ . When it comes to period 2, both agents forget everything they knew in period 1, and then they get to receive some new information, represented by  $S_{A2}$  and  $S_{B2}$  respectively. In this case,  $S_{i2}$  is no longer necessarily a finer partition than  $S_{i1}$  is, for  $i = A, B$ . In period 3, again as before, each agent is perfectly informed of what the realized state is. The structures for the information partitions are given by

$$\begin{aligned}
S_{A1} &= S_{B1} = \{\{\omega_2, \omega_3, \omega_5, \omega_6, \omega_8\}, \{\omega_1, \omega_4, \omega_7\}\} \\
S_{A2} &= \{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5\}, \{\omega_6, \omega_7\}, \{\omega_8\}\} \\
S_{B2} &= \{\{\omega_4, \omega_5, \omega_6\}, \{\omega_1, \omega_2\}, \{\omega_3, \omega_7\}, \{\omega_8\}\} \\
S_{A3} &= S_{B3} = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}, \{\omega_5\}, \{\omega_6\}, \{\omega_7\}, \{\omega_8\}\}.
\end{aligned}$$

The heterogeneous belief about the probability distribution of the state, for each agent, is shown in the table below with weight  $W = \frac{1}{16}$ .

State	$\omega_1$	$\omega_2$	$\omega_3$	$\omega_4$	$\omega_5$	$\omega_6$	$\omega_7$	$\omega_8$
$\pi_A$	2	1	1	1	2	1	3	5
$\pi_B$	1	2	1	2	1	1	3	5

## 6.2 A Rational Expectations Equilibrium with Common Strong Bubbles

A similar calculation and check procedure will show that the above economy has a rational expectations equilibrium that is the same as the one we studied in Section 3. It is characterized by the price table and the trade table below.

State	$\omega_1$	$\omega_2$	$\omega_3$	$\omega_4$	$\omega_5$	$\omega_6$	$\omega_7$	$\omega_8$
$P_1(\omega)$	1	1	1	1	1	1	1	1
$P_2(\omega)$	2	2	2	2	2	2	0	0
$P_3(\omega)$	4	0	0	4	0	0	0	0

$$\forall \omega \in \Omega, x_{A1}(\omega) = x_{B1}(\omega) = x_{A3}(\omega) = x_{B3}(\omega) = 0$$

State	$\omega_1$	$\omega_2$	$\omega_3$	$\omega_4$	$\omega_5$	$\omega_6$	$\omega_7$	$\omega_8$
$x_{A2}(\omega)$	1	1	1	-1	-1	-1	0	0
$x_{B2}(\omega)$	-1	-1	-1	1	1	1	0	0
$x_{A2}(\omega) + x_{B2}(\omega)$	0	0	0	0	0	0	0	0

Now it is time to look for the common strong bubbles in such an equilibrium. Observe that in period 1 in any state from the set  $\{\omega_2, \omega_3, \omega_5, \omega_6, \omega_8\}$ , it is common knowledge that the dividend in period 3 will be 0. Given a positive price 1, it is exactly the case that it is common knowledge that the price of the risky asset is higher than the maximum possible dividend an agent will receive, and hence there is a common strong bubble in period 1 in any state from the set  $\{\omega_2, \omega_3, \omega_5, \omega_6, \omega_8\}$ .

This counterexample shows that perfect memory is an important necessary assumption for the nonexistence of common strong bubbles. In the real world, it is arguable that not all people have perfect memory. Therefore, a common strong bubble may exist in an economy of the real life. This seems to be a surprising result, and it provides an alternative explanation for the existence of bubbles by the assumption of imperfect memory, instead of the assumption of noise traders.

## 7 General Results

In Section 3, an example of a rational bubble that is both a strong bubble and a common expected bubble is presented in a rational expectations equilibrium with 2 agents. Furthermore, as will be shown next, this result holds for any finite number of agents.

Let  $S^F \equiv \{\{\omega\} \mid \omega \in \Omega\}$ , and  $S^F$  is called the perfect information structure for  $\Omega$ . Before constructing bubble examples, we shall make some restrictions on the agents' information structure so as to avoid trial bubbles from duplications.

**Assumption 1 (Different Information Structure)**  $\forall i, j = 1, \dots, I, \forall t = 1, \dots, T, S_{it}, S_{jt} \neq S^F \Rightarrow S_{it} \neq S_{jt}$ .

The assumption of Different Information Structure says that as long as agents don't have perfect information, there must be somewhere their information differs from each other. This assumption rules out the possibility of duplicating identical agents.

**Assumption 2 (Distinct Information Everywhere)**  $\forall i, j = 1, \dots, I, \forall t = 1, \dots, T, \forall \omega \in \Omega, s_{it}(\omega), s_{jt}(\omega) \neq \{\omega\} \Rightarrow s_{it}(\omega) \neq s_{jt}(\omega)$ .

The assumption of Distinct Information Everywhere says that as long as agents don't have perfect information, their information differs from each other everywhere. It is easy to know that Assumption 2 is much stronger than Assumption 1. Assumption 2 implies Assumption 1, but not vice versa.

**Assumption 3 (Common Knowledge of Trades)**  $\forall i = 1, \dots, I, \forall t = 1, \dots, T$ ,  $x_{it}$  is common knowledge.

Based on the assumptions above, two propositions can be made on the existence of strong bubbles in a rational expectations equilibrium.

**Proposition 2** *Under Assumption 1 and 3, for any  $I \geq 2$ , there exists an economy under the framework described in Section 2, with  $I$  agents, 3 periods and  $3I + 2$  states, presenting a bubble, both strong and common expected, in a rational expectations equilibrium.*

**Proof.** See Appendix 1. ■

**Proposition 3** *Under Assumption 2 and 3, for any  $I \geq 2$ , there exists an economy under the framework described in Section 2, with  $I$  agents, 3 periods and  $I \cdot \max\{3, I\} + 2$  states, presenting a bubble, both strong and common expected, in a rational expectations equilibrium.*

**Proof.** See Appendix 2. ■

The strong bubble part of the result is not new, and has been analyzed by AMP (1993) and Conlon (2004). However, by presenting a bubble, not only strong but also common expected, the above propositions provide a new answer to what properties of bubbles we can expect to have in a rational world. The common expected bubble part of the result is surprising since it is somewhat counterintuitive that an expected bubble can be robust to common knowledge in a rational expectations equilibrium. But actually it is the common knowledge of the heterogeneous beliefs and the information structures that guarantees that agents have no incentive to rush in face of bubbles, because by rational expectations they know that they can take advantage of it in a later period.

It should also be noted that the conclusions above are independent of the assumption of no common knowledge of trade. In Proposition 3 of AMP (1993), the assumption of no common knowledge of trades was argued as a necessary condition for the existence of bubbles in a rational expectations equilibrium. The idea of the argument is the following: Geanakoplos (1992) has argued that with common knowledge of trades, agents would have behaved in the same way without the private part of their information (originally stated as "common knowledge of actions negates asymmetric information about events"), and then there would be no strong bubbles since there is no asymmetric information about the states. However, as pointed out by Conlon (2004), the

conclusion that there are no strong bubbles is only true for the new economy where every agent has the same information, which is the common part of their original information. The bubble may still exist in the original economy since in period 1 there is no trade and hence agents still have their private information.

## 8 Conclusion

Based on the work of Allen, Morris and Postlewaite (1993), Conlon (2004), and many others, this paper develops two new concepts of rational bubbles: a common expected bubble and a common strong bubble, and shows that in a finite-state finite-horizon model the following results hold for any finite number of agents. First, there is no common strong bubble in any rational expectations equilibrium under the perfect memory assumption. Second, there exists a three-period economy with asymmetric information and short sales constraints, where an expected bubble can exist in a rational expectations equilibrium, and moreover this bubble, not only a strong bubble, but also a common expected bubble, is robust to both strongly symmetric perturbations in beliefs and very symmetric perturbations in dividends. The first result partially answers what properties a bubble cannot have in a rational world, and the second result tells more about what a bubble might look like, given the results in AMP (1993) and Conlon (2004). A counterexample of the first result is provided when agents are forgetful, and this implies that the implicit perfect memory assumption is key to the result of nonexistence of common strong bubbles. The necessary structural conditions in Section 4 provide insight into the structural characteristics of models of bubbles. One important condition is that for a strong bubble to exist in equilibrium the minimum number of states is 8.

One direction for future work will be to show the coexistence of common expected bubbles and higher order strong bubbles for any finite number of agents, following Conlon (2004) in which an example of higher order bubbles is constructed for the two-agent case. Another direction will be to introduce some irrational agents into the model and to see whether a common strong bubble can exist in such a setting. Since bubbles modeled in this paper are not robust to perturbations of agents' beliefs in a general sense, introducing noise into the model might be another good direction. It might also be important and potentially interesting to test the theory on the existence of rational bubbles by conducting experimental work.

## Appendix

### Appendix 1:

Proof to Proposition 2:

Write  $\Omega = \{\omega_n | n = 1, 2, \dots, 3I + 2\}$ . Let  $\Omega_D \equiv \{\omega_n \in \Omega | n = 3i - 2, i = 1, 2, \dots, I\}$ ,  $\Omega_{2W} \equiv \{\omega_n \in \Omega | n = 3i - 1, i = 1, 2, \dots, I\}$ ,  $\Omega_i \equiv \{\omega_{3i-2}, \omega_{3i-1}, \omega_{3i}\}$ ,  $\Omega_i^- \equiv \Omega_i \setminus \{\omega_{3i}\} = \{\omega_{3i-2}, \omega_{3i-1}\}$ ,  $i = 1, 2, \dots, I$ .

Each share of the risky asset will pay a dividend of amount 4 at the end of period 3 if the state  $\omega \in \Omega_D$  and will pay nothing otherwise. Each agent is endowed with  $I$  units of money and 1 share of the risky asset at the beginning of period 1.

The specific structures of  $S_{it}$ 's are given by

$$\begin{aligned}
 S_{11} &= \{\Omega \setminus \{\omega_{3I}, \omega_{3I+1}\}, \{\omega_{3I}, \omega_{3I+1}\}\} \\
 S_{i1} &= \{\Omega \setminus \{\omega_{3i-3}, \omega_{3I+1}\}, \{\omega_{3i-3}, \omega_{3I+1}\}\} \quad \forall i = 2, \dots, I \\
 S_{12} &= \{\Omega_1, \Omega_2, \dots, \Omega_{I-1}, \Omega_I^-, \{\omega_{3I}\}, \{\omega_{3I+1}\}, \{\omega_{3I+2}\}\} \\
 S_{i2} &= \{\Omega_1, \dots, \Omega_{i-2}, \Omega_i, \dots, \Omega_I, \Omega_{i-1}^-, \{\omega_{3i-3}\}, \{\omega_{3I+1}\}, \{\omega_{3I+2}\}\} \quad \forall i = 2, \dots, I \\
 S_{i3} &= S^F \quad \forall i = 1, 2, \dots, I.
 \end{aligned}$$

The agents' beliefs about the states are given by the following functions.

$$\pi_i(\omega_n) = \begin{cases} 2W & \text{if } n = 3i - 2 \text{ or } \omega_n \in \Omega_{2W} \setminus \{\omega_{3i-1}\} \\ (4I - 1)W & \text{if } n = 3I + 2 \\ W & \text{otherwise} \end{cases} \quad \forall i = 1, 2, \dots, I, W = \frac{1}{8I}.$$

To see that the belief of agent  $i$  is well defined, note that the number of elements in  $\Omega_{2W}$  is  $I$ , hence there are  $I$  states which are put with probability  $2W$ . Since there is only one state with probability  $(3I + 2)W$ , the number of the states with probability  $W$  is  $3I + 2 - I - 1 = 2I + 1$ . Thus,  $\sum_{\omega \in \Omega} \pi_i(\omega) = I \times 2W + 1 \times (4I - 1)W + (2I + 1) \times W = 8IW = 1$ .

The equilibrium with the prices and trades given below is what we look for - the one in which there is a strong and common expected bubble in period 1 in state  $\omega_{3I+1}$ .

$$\begin{aligned}
 P_1(\omega) &= 1 \quad \forall \omega \in \Omega. \\
 P_2(\omega_n) &= \begin{cases} 0 & \text{if } n = 3I + 1 \text{ or } n = 3I + 2 \\ 2 & \text{otherwise} \end{cases} \\
 P_3(\omega_n) &= \begin{cases} 4 & \text{if } n \in \Omega_D \\ 0 & \text{otherwise} \end{cases}
 \end{aligned}$$

$$\forall i = 1, 2, \dots, I, \forall \omega \in \Omega, x_{i1}(\omega) = x_{i3}(\omega) = 0.$$

$$x_{i2}(\omega_n) = \begin{cases} I - 1 & \text{if } \omega_n \in \Omega_i \\ 0 & \text{if } n = 3I + 1 \text{ or } n = 3I + 2 \quad \forall i = 1, 2, \dots, I. \\ -1 & \text{otherwise} \end{cases}$$

Observe that neither the prices nor the trades reveal any additional information with the settings above.

It can be similarly checked following the procedures described in the two-agent example that the above prices and trades constitute a rational expectations equilibrium. And since in period 1 in state  $\omega_{3I+1}$ , each agent knows that he will receive nothing at the end of period 3, given the positive price of 1 in period 1, there exists a strong bubble in this equilibrium.

Note that  $m_1^{PX}(\omega_{3I+1}) = \Omega$ . To see that this bubble is robust to common knowledge in the expected sense, we need to check that  $\forall i = 1, 2, \dots, I, \forall \omega, 1 > \frac{1}{\sum_{\omega' \in s_{i1}^{PX}(\omega)} \pi_i(\omega')} \sum_{\omega' \in s_{i1}^{PX}(\omega)} \pi_i(\omega') d(\omega')$ . Note that for agent 1 (or agent  $i, i \geq 2$ ), either

he will observe  $\{\omega_{3I}, \omega_{3I+1}\}$  (or  $\{\omega_{3i-3}, \omega_{3I+1}\}$ ), or he will observe  $\Omega \setminus \{\omega_{3I}, \omega_{3I+1}\}$  (or  $\Omega \setminus \{\omega_{3i-3}, \omega_{3I+1}\}$ ). If it is the first case, his expected dividend will be  $\frac{1}{2}0 + \frac{1}{2}0 = 0$ ; If it is the second case, his expected dividend will be  $\frac{I+1}{8I-2}4 + \frac{7I-3}{8I-2}0 = \frac{2I+2}{4I-1}$ . In either case, the expected dividend is less than the price. Therefore, the bubble in period 1 in state  $\omega_{3I+1}$  is a common expected bubble.

However it should be noted under the structure above,  $\forall \omega_n \in \Omega \setminus \{\omega_{3I+1}, \omega_{3I+2}\}$ , in period 2 in state  $\omega_n$  there are always  $(I-1)$  agents who observe the same event  $\Omega_i = \{\omega_{3i-2}, \omega_{3i-1}, \omega_{3i}\}$ <sup>19</sup> where  $i$  is determined such that  $\omega_n \in \Omega_i$ . Obviously this violates Assumption 2. In order to ensure that agents' information differs from each other everywhere when there is no perfect information, the number of the states has to be large enough to guarantee the existence of bubbles.

## Appendix 2:

Proof to Proposition 3:

The case of 2 agents has already been shown in section 3. Here it suffices to consider the case when  $I \geq 3$ .

Write  $\Omega = \{\omega_n | n = 1, 2, \dots, I^2 + 2\}$ . Let  $\Omega_D \equiv \{\omega_n \in \Omega | n = I(i-1) + 1, i = 1, 2, \dots, I\}$ ,  $\Omega_{(I-1)W} \equiv \{\omega_n \in \Omega | n = I(i-1) + 2, i = 1, 2, \dots, I\}$ ,  $\Omega_j \equiv \{\omega_n \in \Omega | (j-1) + 1 \leq n \leq Ij\}$ ,  $\Omega_j^{-k} \equiv \Omega_j \setminus \{\omega_{I(j-1)+k}\}$ ,  $j, k = 1, 2, \dots, I$ .

Again, each share of the risky asset will pay a dividend of amount 4 at the end of period 3 if the state  $\omega \in \Omega_D$  and will pay nothing otherwise. Each agent is endowed with  $I$  units of money and 1 share of the risky asset at the beginning of period 1.

<sup>19</sup>Though there is one agent observing  $\{\omega_n\}$  or  $\Omega_i \setminus \{\omega_n\}$ ,  $\Omega_i$  is common knowledge in this case. And this feature holds also for the constructed example under proposition.

Let  $a_{ij}$  be the  $i$ th row and  $j$ th column element of the following  $I \times I$  matrix. Hence  $\omega_{I(j-1)+a_{ij}}$  is the  $a_{ij}$ th element in  $\Omega_j$ .

$$\begin{bmatrix} & 2 & 3 & \cdots & I-1 & I \\ I & & 2 & \cdots & I-2 & I-1 \\ I-1 & I & & 2 & \cdots & I-2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 3 & 4 & \cdots & I & & 2 \\ 2 & 3 & \cdots & I-1 & I & \end{bmatrix}$$

The specific structures of  $S_{it}$ 's are given by

$$\begin{aligned} S_{i1} &= \{\Omega \setminus \{\omega_{Ik_i}, \omega_{I^2+1}\}, \{\omega_{Ik_i}, \omega_{I^2+1}\}\} \text{ where } k_i \text{ is determined by } a_{ik_i} = I \\ S_{i2} &= \{\{\omega_{I(j-1)+a_{ij}} : 1 \leq j \leq I, j \neq i\} \cup \{\Omega_j^{-a_{ij}} : 1 \leq j \leq I, j \neq i\}\} \cup \{\Omega_i, \{\omega_{I^2+1}\}, \{\omega_{I^2+2}\}\} \\ S_{i3} &= S^F \quad \forall i = 1, 2, \dots, I. \end{aligned}$$

The agents' beliefs about the states are given by the following functions.  $\forall i = 1, 2, \dots, I$ ,

$$\pi_i(\omega_n) = \begin{cases} (I-1)W & \text{if } n = I(i-1) + 1 \text{ or } \omega_n \in \Omega_{(I-1)W} \setminus \{\omega_{I(i-1)+2}\} \\ (2I(I-1) - 1)W & \text{if } n = I^2 + 2 \\ W & \text{otherwise} \end{cases}, W = \frac{1}{4I(I-1)}.$$

To see that the belief of agent  $i$  is well defined, note that the number of elements in  $\Omega_{(I-1)W}$  is  $I$ , hence there are  $I$  states which are put with probability  $(I-1)W$ . Since there is only one state with probability  $(2I(I-1) - 1)W$ , the number of the states with probability  $W$  is  $I^2 + 2 - I - 1 = I(I-1) + 1$ . Thus,  $\sum_{\omega \in \Omega} \pi_i(\omega) = I \times (I-1)W + 1 \times (2I(I-1) - 1)W + (I(I-1) + 1) \times W = 4I(I-1)W = 1$ .

The equilibrium with the prices and trades given below is what we look for - the one in which there is a strong and common expected bubble in period 1 in state  $\omega_{I^2+1}$ .

$$\begin{aligned} P_1(\omega) &= 1 \quad \forall \omega \in \Omega. \\ P_2(\omega_n) &= \begin{cases} 0 & \text{if } n = I^2 + 1 \text{ or } n = I^2 + 2 \\ 2 & \text{otherwise} \end{cases} \\ P_3(\omega_n) &= \begin{cases} 4 & \text{if } n \in \Omega_D \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$$\forall i = 1, 2, \dots, I, \forall \omega \in \Omega, x_{i1}(\omega) = x_{i3}(\omega) = 0.$$

$$x_{i2}(\omega_n) = \begin{cases} I-1 & \text{if } \omega_n \in \Omega_i \\ 0 & \text{if } n = I^2 + 1 \text{ or } n = I^2 + 2 \\ -1 & \text{otherwise} \end{cases} \quad \forall i = 1, 2, \dots, I.$$

Observe that neither the prices nor the trades reveal any additional information with the settings above.

It can be similarly checked following the procedures described in the two-agent example that the above prices and trades constitute a rational expectations equilibrium. And since in period 1 in state  $\omega_{I^2+1}$ , each agent knows that he will receive nothing at the end of period 3, given the positive price of 1 in period 1, there exists a strong bubble in this equilibrium.

Note that  $m_1^{PX}(\omega_{I^2+1}) = \Omega$ . To see that this bubble is robust to common knowledge in the expected sense, we need to check that  $\forall i = 1, 2, \dots, I, \forall \omega \in \Omega, 1 > \frac{1}{\sum_{\omega' \in s_{i1}^{PX}(\omega)} \pi_i(\omega')} \sum_{\omega' \in s_{i1}^{PX}(\omega)} \pi_i(\omega') d(\omega')$ . Note that for agent 1, either he will observe  $\{\omega_{Ik_i}, \omega_{I^2+1}\}$ , or he will observe  $\Omega \setminus \{\omega_{Ik_i}, \omega_{I^2+1}\}$ . If it is the first case, his expected dividend will be  $\frac{1}{2}0 + \frac{1}{2}0 = 0$ ; If it is the second case, his expected dividend will be  $\frac{2(I-1)}{4I(I-1)-2}4 + \frac{4I(I-1)-2-2(I-1)}{4I(I-1)-2}0 = \frac{4}{2I-\frac{1}{I-1}}$ . In either case, the expected dividend is less than the price. Therefore, the bubble in period 1 in state  $\omega_{I^2+1}$  is a common expected bubble.

## References

- [1] Abreu, D. and M. K. Brunnermeier (2003). "Bubbles and Crashes." *Econometrica* 71(1): 173-204.
- [2] Allen, F., S. Morris, and A. Postlewaite. (1993). "Finite Bubbles with Short Sale Constraints and Asymmetric Information." *Journal of Economic Theory* 61(2): 206-229.
- [3] Aumann, R. J. (1976). "Agreeing to Disagree." *The Annals of Statistics* 4(6): 1236-1239.
- [4] Blanchard, O. J. and M. W. Watson (1982). "Bubbles, Rational Expectations, and Financial Markets." NBER Working Paper #945.
- [5] Conlon, J. R. (2004). "Simple Finite Horizon Bubbles Robust to Higher Order Knowledge." *Econometrica* 72(3): 927-936.
- [6] De Long, J. B., A. Shleifer, L. H. Summers, and R. J. Waldmann (1990). "Noise Trader Risk in Financial Markets." *The Journal of Political Economy* 98(4): 703-738.
- [7] Diba, B. T. and H. I. Grossman (1987). "On the Inception of Rational Bubbles." *The Quarterly Journal of Economics* 102(3): 697-700.
- [8] Diba, B. T. and H. I. Grossman (1988). "Explosive Rational Bubbles in Stock Prices?" *The American Economic Review* 78(3): 520-530.
- [9] Diba, B. T. and H. I. Grossman (1988). "The Theory of Rational Bubbles in Stock Prices." *The Economic Journal* 98(392): 746-754.
- [10] Doblus-Madrid, A. (2009). "A Robust Model of Bubbles with Multidimensional Uncertainty", Working Paper.
- [11] Geanakoplos, J. (1992). "Common Knowledge." *The Journal of Economic Perspectives* 6(4): 53-82.
- [12] Geanakoplos, J. and H. M. Polemarchakis (1981). "We Can't Disagree Forever." *Journal of Economic Theory* 28(1): 192-200.
- [13] Grossman, S. J. and J. E. Stiglitz (1980). "On the Impossibility of Informationally Efficient Markets." *The American Economic Review* 70(3): 393-408.
- [14] Jovanovic, B. (2007). "Bubbles in Prices of Exhaustible Resources." NBER Working Paper #13320.
- [15] LeRoy, S. F. (2004). "Rational Exuberance." *Journal of Economic Literature* 42(3): 783-804.

- [16] Lux, T. (1995). "Herd Behaviour, Bubbles and Crashes." *The Economic Journal* 105(431): 881-896.
- [17] Milgrom, P. and N. Stokey (1982). "Information, Trade and Common Knowledge." *Journal of Economic Theory* 26(1): 17-27.
- [18] Morris, S., A. Postlewaite, and H. S. Shin (1995). "Depth of Knowledge and the Effect of Higher Order Uncertainty." *Economic Theory* 6: 453-467.
- [19] Tirole, J. (1982). "On the Possibility of Speculation under Rational Expectations." *Econometrica* 50(5): 1163-1181.
- [20] Tirole, J. (1985). "Asset Bubbles and Overlapping Generations." *Econometrica* 53(6): 1499-1528.
- [21] Zurita, F. (2004). "On the Limits to Speculation in Centralized versus Decentralized Market Regimes." *Journal of Financial Intermediation* 13(3): 378-408.