

# Nash Bargaining Theory with Non-Convexity and Unique Solution\*

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## Abstract

We introduce log-convexity for bargaining problems. We show that the class of all regular and log-convex two-player bargaining problems is the *largest* class of regular two-player bargaining problems allowing for non-convexity, on which the standard axiomatic characterization of the asymmetric Nash bargaining solution holds. Log-convexity of a bargaining problem is strictly weaker than convexity of the choice set. Indeed, the well-recognized non-convex bargaining problems arising from duopolies with asymmetric constant marginal costs are log-convex. We also compare the Nash bargaining solution with some of its extensions appeared in the literature.

KEYWORDS: Bargaining problem, duopoly, non-convexity, log-convexity, Nash bargaining solution, Nash product.

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# 1 Introduction

The bargaining theory introduced in the seminal papers of Nash (1950, 1953) postulates that a group of players chooses a payoff allocation from a set of feasible payoff allocations. The implementation of a payoff allocation requires unanimous agreement among the players. In the case of disagreement, the players end up getting some pre-determined payoff allocation known as the *status quo* or the *threat point*. A bargaining problem in the sense of Nash is thus represented by a pair consisting of a choice set in the payoff space and a threat point. A bargaining solution is defined on a class of bargaining problems, assigning a feasible payoff allocation to each bargaining problem in the class.

Nash adopted an axiomatic approach to resolve the bargaining problem:

One states as axioms several properties that it would seem natural for the solution to have and then one discovers that the axioms actually determine the solution uniquely (Nash 1953, p. 129).

Nash postulated a set of axioms that are deemed to be natural for a bargaining solution. It is remarkable that for the class of *compact* and *convex* bargaining problems, Nash axioms characterize a unique solution, which has come to be called the *Nash bargaining solution*. Furthermore, the Nash bargaining solution assigns to each bargaining problem the unique maximizer of the (symmetric) *Nash product* over the choice set. Nash bargaining model has become one of the most fruitful paradigms in game theory.<sup>1</sup>

One of the Nash axioms is that of symmetry. When it is removed, Kalai (1977) showed that for the class of compact convex bargaining problems, a bargaining solution *unique up to* specifications of payoff allocations for a “normalized problem” is characterized by the remaining axioms. This solution is known as *the asymmetric Nash bargaining solution*. A payoff allocation for the normalized problem has been customarily interpreted as representing players’ bargaining powers.<sup>2</sup> Given players’ bargaining powers, the asymmetric Nash bargaining solution assigns to each bargaining problem the unique maximizer of the Nash product weighted by the given bargaining powers (also known as the generalized Nash product). These properties are known as the standard axiomatic characterization of the asymmetric Nash bargaining solution for

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<sup>1</sup>An alternative approach to bargaining is to apply sequential, non-cooperative models of negotiation. Nash (1953, p. 129) argued that the two approaches “are complementary; each helps to justify and clarify the other.” See Binmore, Rubinstein and Wolinsky (1986) for a formal establishment of the relationship between the Nash axiomatic and the sequential non-cooperative approaches to bargaining.

<sup>2</sup>This interpretation can be traced to Shubik (1959, p. 50).

the class of compact convex bargaining problems. For applications, the uniqueness of the bargaining solution up to specifications of players' bargaining powers is useful, when there is no a priori reason why players' bargaining powers are not case-specific.

In most applications, both the choice set and threat point are not directly given. They are derived from more primitive data. As a result, the choice set needs not be convex. Indeed, the non-convexity of the choice set of feasible profit shares arising from duopolies with asymmetric constant marginal costs and concave profit functions has been well recognized in the industrial organization literature.<sup>3</sup>

The non-convexity is usually removed by allowing for randomized payoff allocations and by assuming that players are expected utility maximizers. The expected utility maximization is, however, not an automatically valid behavioral hypothesis, while randomized allocations can cause conflicts of interest at the post-realization stage and may not be realistic in some applications. It is therefore desirable to analyze the extent to which the Nash bargaining theory can be extended to allow for non-convexity.

In this paper, we contribute to the Nash bargaining theory by finding the largest class of regular two-person bargaining problems on which the standard axiomatic characterization of the asymmetric Nash bargaining solution holds. Our first main result shows that with generic bargaining powers, the maximizer of the Nash product over the choice set of a “regular” bargaining problem is *unique* if and only if the bargaining problem is “log-convex” (Theorem 1). Regularity of a bargaining problem in this paper means that the threat point is strictly dominated and the choice set is closed, comprehensive relative to the threat point, and bounded above. Log-convexity, on the other hand, requires that the Pareto frontier of the set of the log-transformed positive payoff gain allocations be *strictly* concave. The regularity requirement is mild while the log-convexity of a bargaining problem is strictly *weaker* than the convexity of the choice set. Indeed, we show that duopoly bargaining problems with asymmetric constant marginal costs and concave monopoly profit functions are log-convex but not convex.

The establishment of the standard axiomatic characterization of the asymmetric Nash solution relies on the following separation property: Given bargaining powers, all non-maximizers over a convex choice set can be separated from the unique maximizer by the tangent line to the indifference curve of the Nash product weighted by the given bargaining powers at the maximizer. With log-convexity, however, this separation property is not always valid. Nevertheless, our second main result shows that each

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<sup>3</sup>See, for example, Bishop (1960), Schmalensee (1987), and Tirole (1988, p. 242, 271).

non-maximizer is separable from the unique maximizer by the tangent line to the indifference curve of the corresponding Nash product at some feasible point different from the non-maximizer (Lemma 1). This separation result, interesting in its own right, turns out to be strong enough for the standard axiomatic characterization to be valid on our class of all regular and log-convex bargaining problems as shown by our third main result (Theorem 2). Moreover, the standard axiomatic characterization is not valid on any larger class of regular bargaining problems (Theorem 2).

Our paper is most closely related to Zhou (1997) among several other extensions of the Nash bargaining theory allowing for non-convexity. Both papers consider single-valued bargaining solutions and extend the asymmetric Nash bargaining solution without modifying the Nash axioms. Zhou showed that on the class of all regular bargaining problems, any single-valued bargaining solution satisfying all of Nash axioms except for that of symmetry is determined by the maximization of the Nash product weighted by some pre-determined bargaining powers. However, single-valued bargaining solutions are not uniquely determined by the Nash axioms on this broad class.

Intuitively, bargaining solutions are subject to stronger requirements on a larger class than on a smaller one. For example, in characterizing bargaining solutions under the Nash axioms, Zhou applied the IIA axiom to bargaining problems with choice sets equal to the comprehensive hulls of *finitely* many payoff allocations. Those bargaining problems are not log-convex. Accordingly, his characterization result or proof cannot be applied to establish the unique axiomatization of the asymmetric Nash bargaining solution on our class. Our notion of log-convexity provides a tight extension in the sense that it specifies the largest class of regular bargaining problems for the standard axiomatic characterization of the asymmetric Nash bargaining solution to hold. In comparison, Zhou's class is too broad to guarantee that Nash axioms characterize a unique bargaining solution.

Other extensions appeared in the literature are made possible through extensions of Nash axioms. Among them, Conley and Wilkie's (1996) extension is closely related to ours, in the sense that both extensions are unique and single-valued. Their axioms imply a two-step procedure for finding the solution. First, convexify the choice set via randomization and apply the Nash bargaining solution to the convexified problem. Second, use the intersection point of the segment between the threat point and the Nash bargaining solution for the convexified problem with the original Pareto frontier as the bargaining solution. The single-valuedness of their unique solution enables an interesting comparison with the asymmetric Nash bargaining solution for non-convex

bargaining problems. Herrero (1989) considered axioms that in general result in a multi-valued bargaining solution for non-convex bargaining problems. She showed that the bargaining solution satisfying her axioms can be essentially characterized by an “equal distance property”. When adjusted by players’ bargaining powers, this property turns out to be also closely related to the Nash product maximization for non-convex bargaining problems in our class.<sup>4</sup>

The rest of the paper is organized as follows. The next section briefly reviews Nash axioms and Nash bargaining solution with convexity. Section 3 introduces log-convexity of bargaining problems and provides our main results. Section 4 applies our results to non-convex bargaining problems arising from duopoly. Section 5 compares the asymmetric Nash bargaining solution for non-convex bargaining problems with the Conley-Wilkie extension, and establishes an extended equal distance property of Herrero (1989) for the asymmetric Nash bargaining solution on the class of bargaining problems in this paper. Section 6 concludes.

## 2 Nash Bargaining Solution with Convexity

The Nash bargaining theory takes a bargaining problem as primitive. A two-person bargaining problem is composed of a choice set  $S \subset \mathfrak{R}^2$  of feasible payoff allocations the players can jointly achieve with agreement, and a threat point  $d \in S$  the players end up getting in case of disagreement. A bargaining solution on a class  $\mathcal{B}$  of bargaining problems is a rule  $f$  assigning a feasible allocation  $f(S, d) = (f_1(S, d), f_2(S, d)) \in S$  to each bargaining problem  $(S, d) \in \mathcal{B}$ .

A positive affine transformation for player  $i$ ’s payoff is a mapping  $\tau_i : \mathfrak{R} \rightarrow \mathfrak{R}$  such that for some two real numbers  $a_i > 0$  and  $b_i$ ,  $\tau_i(u_i) = a_i u_i + b_i$  for all  $u_i \in \mathfrak{R}$ . Given  $\tau_1$  and  $\tau_2$ ,  $\tau(u) = (\tau_1(u_1), \tau_2(u_2))$  for all  $u \in \mathfrak{R}^2$ . Nash considered the following well-known axioms on bargaining solutions.<sup>5</sup>

*Strict Individual Rationality* (SIR): For any  $(S, d) \in \mathcal{B}$ ,  $f_i(S, d) > d_i$ ,  $i = 1, 2$ .

*Symmetry* (SYM): For any  $(S, d) \in \mathcal{B}$  with  $d_1 = d_2$  and  $(u_2, u_1) \in S$  whenever  $(u_1, u_2) \in S$ ,  $f_1(S, d) = f_2(S, d)$ .

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<sup>4</sup>See Kaneko (1980) for another multi-valued extension of the Nash bargaining solution, among other ones.

<sup>5</sup>Roth (1977) showed that a bargaining solution is (strictly) Pareto optimal whenever the solution satisfies SIR, INV, and IIA. Thus, with INV and IIA, Pareto optimality as Nash originally considered can be replaced by SIR.

*Invariance to Equivalent Utility Representations (INV):* For any  $(S, d) \in \mathcal{B}$  and for any positive affine transformation  $\tau : \mathfrak{R}^2 \rightarrow \mathfrak{R}^2$ ,  $f(\tau(S), \tau(d)) = \tau(f(S, d))$ .

*Independence of Irrelevant Alternatives (IIA):* For any  $(S, d), (S', d) \in \mathcal{B}$  with  $S \subseteq S'$ ,  $f(S', d) \in S$  implies  $f(S, d) = f(S', d)$ .

When  $\mathcal{B}$  is composed of compact convex bargaining problems with strictly Pareto dominated threat points, these four axioms uniquely characterize the symmetric Nash bargaining solution: To each  $(S, d) \in \mathcal{B}$  it assigns the payoff allocation determined by

$$\max_{u \in S, u \geq d} (u_1 - d_1)(u_2 - d_2). \quad (1)$$

See Nash (1953) and Roth (1979) for details.

When the symmetry axiom is removed, Kalai (1977) showed that for the class of compact convex bargaining problems, the bargaining solution that assigns the payoff allocation determined by the maximization of the Nash product weighted by bargaining powers  $\alpha$  for player 1 and  $1 - \alpha$  for player 2,

$$\max_{u \in S, u \geq d} (u_1 - d_1)^\alpha (u_2 - d_2)^{1-\alpha}, \quad (2)$$

to  $(S, d) \in \mathcal{B}$  is the unique bargaining solution satisfying SIR, INV, IIA, and the payoff allocation  $(\alpha, 1 - \alpha)$  for the normalized bargaining problem:

$$S^\circ = \{u \in \mathfrak{R}^2 \mid u_1 + u_2 \leq 1\} \text{ and } d^\circ = (0, 0). \quad (3)$$

For later reference in the paper, we have

**Definition 1** *Let  $\mathcal{B}$  be a class of bargaining problems such that  $(S^\circ, d^\circ) \in \mathcal{B}$ . A bargaining solution  $f$  on  $\mathcal{B}$  is an asymmetric Nash bargaining solution with bargaining powers  $\alpha \in (0, 1)$  for player 1 and  $(1 - \alpha)$  for player 2, if  $f$  satisfies SIR, INV, IIA, and  $f(S^\circ, d^\circ) = (\alpha, 1 - \alpha)$ . The asymmetric Nash bargaining solution is unique on  $\mathcal{B}$  if it is unique for any given bargaining powers (i.e. unique up to choices of the players' bargaining powers).*

Applying the usual proof, the maximization of the Nash product weighted by bargaining powers  $\alpha$  and  $1 - \alpha$  always results in an asymmetric Nash bargaining solution with these bargaining powers on a class, if the maximizer is unique for all bargaining

problems in the class. Consequently, with the uniqueness of the weighted Nash product maximizer for all bargaining powers, the asymmetric Nash bargaining solution with any given players' bargaining powers must be determined by (2) whenever it is unique as defined above.

### 3 Nash Bargaining Solution without Convexity

Our motivation for considering bargaining problems allowing for non-convex choice sets comes from the bargaining problem between two firms supplying a homogeneous product with constant marginal costs  $c_1 \neq c_2$ . Assume side-payments are not feasible.<sup>6</sup> Then, with market demand  $P(Q)$ , a profit distribution  $\pi = (\pi_1, \pi_2)$  is feasible if and only if there exists a pair  $q = (q_1, q_2) \geq 0$  of quantities (or output quotas) such that

$$\pi_1 = [P(Q) - c_1]q_1 \quad \text{and} \quad \pi_2 = [P(Q) - c_2]q_2.$$

Setting  $p = P(Q)$  and  $Q = D(p)$ , we obtain

$$\frac{\pi_1}{p - c_1} + \frac{\pi_2}{p - c_2} = D(p). \tag{4}$$

We defer a general discussion of the non-convexity of duopoly bargaining problems to Section 4. Below we use a numerical example to illustrate the Pareto profit frontier that implies the non-convexity of the corresponding choice set.

**Example 1:** Let  $D(p) = \max\{20 - p, 0\}$ ,  $c_1 = 0$ , and  $c_2 = 10$ . Then the monopoly profit for firm 2 is  $\pi_2 = 25$ . Given price  $10 < p < 15$  and given firm 2's profit  $\pi_2 \leq 25$ , (4) implies that firm 1's profit is given by

$$h(p, \pi_2) \equiv p(20 - p) - \frac{p}{p - 10}\pi_2.$$

Thus, fixing firm 2's profit at  $\pi_2 \leq 25$ , firm 1's maximum profit is determined by

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<sup>6</sup>Explicit transfers or bribes between firms may be too risky due to antitrust scrutiny. Binmore (2007, p. 480-481) illustrated the use of Nash bargaining solution as a tool for studying collusive behavior in a Cournot duopoly with asymmetric constant marginal costs. However, the profit choice set without side-payments is drawn as if it is convex (Figure 16.13(a)), which is not with respect to the assumed demand and cost functions. Using a linear demand function and asymmetric, quadratic average-cost functions, Mayberry, Nash and Shubik (1953) examined and compared several theories of duopoly including the Nash bargaining theory. The profit choice set in their setting is convex.

choosing  $p$  to maximize  $h(p, \pi_2)$ . The first-order condition yields

$$2(10 - p) + \frac{10\pi_2}{(p - 10)^2} = 0.$$

It follows that

$$p(\pi_2) = 10 + (5\pi_2)^{\frac{1}{3}}$$

is the price that maximizes firm 1's profit given firm 2's profit  $\pi_2$ . Hence,

$$\pi_1 = h(p(\pi_2), \pi_2) = 100 - \pi_2 - 3(5\pi_2)^{\frac{2}{3}}, \quad 0 \leq \pi_2 \leq 25,$$

characterizes the Pareto frontier of the set of feasible profit allocations. Simple calculation shows that the frontier is a strictly convex curve. Consequently, the feasible choice set of the resulting bargaining problem is not convex. Intuitively, without side-payments, production arrangements are necessarily inefficient in order for firm 2, the inefficient one, to receive positive profits, so that one unit of profit gain by firm 2 results in more than one unit of profit loss for firm 1.

In his extension of the Nash bargaining theory, Zhou (1997) showed that if a single-valued bargaining solution  $f$  satisfies SIR, INV, and IIA on the class of bargaining problems  $(S, d)$  such that  $S$  is closed, comprehensive, bounded from above, and  $d$  is strictly Pareto dominated, then  $f$  is determined by (2) for some  $\alpha \in (0, 1)$  such that  $f(S^\circ, d^\circ) = (\alpha, 1 - \alpha)$ . That is,  $f$  must be an asymmetric Nash bargaining solution with some pre-specified bargaining powers. The following example illustrates that asymmetric Nash bargaining solutions are not unique on this broad class of bargaining problems.

**Example 2:** Let  $a$  and  $b$  be positive numbers such that  $a + b < 1$ . Consider bargaining problem  $(S, d^\circ)$  where  $S$  consists of payoff allocations  $u = (u_1, u_2)$  bounded from above by the following two segments. One segment connects points  $(1, 0)$  and  $(a, b)$  and the other connects  $(a, b)$  and  $(0, 1)$  (see Figure 1). It is clear that  $S$  is non-convex, although it is closed, comprehensive, and bounded from above. Simple calculation shows that given bargaining power  $\alpha \in (0, 1)$  for player 1, there exists an indifference curve of the Nash product that is tangent to the Pareto frontier of  $S$  at payoff allocations

$$\left( \alpha, \frac{(1 - \alpha)b}{1 - a} \right)$$



and

$$\left( \frac{\alpha a}{1-b}, 1-\alpha \right)$$

if and only if  $\alpha \in (a, 1-b)$  and

$$\left( \frac{a}{1-b} \right)^\alpha = \left( \frac{b}{1-a} \right)^{1-\alpha}.$$

Thus, given  $a > 0$  and  $b > 0$  such that  $a+b < 1$ , there is a unique bargaining power satisfying the preceding equation, which is given by

$$\alpha = \frac{\ln(1-a) - \ln b}{\ln(1-a) - \ln b + \ln(1-b) - \ln a}. \quad (5)$$

Consequently, for the bargaining problem in this example, the Nash product maximization problem (2) with player 1's bargaining power  $\alpha$  as in (5) has multiple solutions if  $\alpha \in (a, 1-b)$ . Since  $a+b < 1$ ,  $\alpha > a$  if and only if

$$h(b) \equiv (1-a)\ln(1-a) - (1-a)\ln b - a\ln(1-b) + a\ln a > 0.$$

Since

$$h'(x) = -\frac{1-a}{x} + \frac{a}{1-x} < 0$$

for  $x \in (0, 1-a)$  and  $h(x)|_{x=1-a} = 0$ , it must be  $h(b) > 0$ . By similar reasoning,  $\alpha < 1-b$ . Figure 1 illustrates multiple solutions to Nash product maximization for the case of  $(a, b) = (0.3, 0.2)$ , where the unique  $\alpha$  is approximately 0.5609.<sup>7</sup>

By Zhou's (1997) result, single-valued Nash bargaining solutions are selections of Nash product maximizers. It is worth mentioning that to be a single-valued Nash bargaining solution, a selection needs to be consistent across the bargaining problems. For example, suppose that a bargaining solution  $f$  selects the lower tangency point for the bargaining problem  $(S, d^\circ)$  in this example. Now consider a new bargaining problem  $(S'^\circ)$ , where  $S'$  is obtained from  $S$  by cutting off a small portion around the lower right corner without removing the two tangent points. Then,  $f$  must also select the lower tangency point for  $(S'^\circ)$ .

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<sup>7</sup>It can be also shown that for any given  $\alpha \in (0, 1)$ , there is a continuum of pairs of  $(a, b)$  such that the equation holds and hence the maximization problem (2) has two solutions.

### 3.1 Log-Convexity of Bargaining Problems

Fix a bargaining problem  $(S, d)$  and take the logarithmic transformation of the positive payoff gain allocations to get

$$V(S, d) = \left\{ v \in \mathbb{R}^2 \mid \exists u \in S : \begin{array}{l} u_1 > d_1, u_2 > d_2, \\ v_1 \leq \ln(u_1 - d_1), \\ v_2 \leq \ln(u_2 - d_2). \end{array} \right\}.$$

Then, the maximization problem (2) is equivalent to

$$\max_{v \in V(S, d)} \alpha v_1 + (1 - \alpha)v_2. \quad (6)$$

Since  $\ln(u_i - d_i)$  is increasing and concave over  $u_i \in (d_i, \infty)$ , for  $i = 1, 2$ , the convexity of  $S$  implies the convexity of  $V(S, d)$ . But, the converse is not necessarily true. Indeed, we show in Section 4 that for any strictly Pareto dominated threat point  $d$  in the choice set  $S$  of the duopoly bargaining problem with constant asymmetric marginal costs and concave profit functions as illustrated in Example 1,  $V(S, d)$  is convex with the Pareto frontier being strictly concave even though  $S$  itself is not.

For the rest of the paper we consider bargaining problems  $(S, d)$  that satisfy the following basic conditions.

*Regularity:*  $S$  is closed, bounded above,  $d$ -comprehensive (i.e.  $u \in S$  whenever  $d \leq u \leq u'$  for some  $u' \in S$ ), and  $d$  is strictly Pareto dominated by an allocation in  $S$ .

We show in the sequel that the following condition is critical for the uniqueness of the asymmetric Nash bargaining solution:

*Log-Convexity:*  $V(S, d)$  is convex and its Pareto frontier does not contain any segment with normal  $(\theta, 1 - \theta)$  for any  $\theta \in (0, 1)$ .

The log-convexity imposes strict concavity on the Pareto frontier of  $V(S, d)$  but allows for possible vertical and horizontal lines at the two polar points of the frontier. Thus, it is weaker than  $V(S, d)$  itself being strictly convex. It can be verified that the bargaining problem in Example 2 is regular but fails to be log-convex.

Before presenting our main results, we make the following observation about the log-convexity that will be useful in the proof of Theorem 2 below. Let  $(S, d)$  and  $(S', d)$

be two regular bargaining problems with the same threat point  $d$ . Then  $V(S \cap S', d) = V(S, d) \cap V(S', d)$ . This implies that  $(S \cap S', d)$  is regular and log-convex whenever both  $(S, d)$  and  $(S', d)$  are.<sup>8</sup>

The uniqueness of the asymmetric Nash bargaining solution, as specified in Definition 1, requires that the maximization problem (2) or equivalently the maximization problem (6) have a unique solution for any bargaining power  $\alpha \in (0, 1)$  for player 1 (see the proof of Theorem 2). It turns out that this imposes exact restrictions on  $V(S, d)$  to make it satisfy the log-convexity.

**Theorem 1** *Let  $(S, d)$  be a regular bargaining problem. Then, (6) has a unique solution for all  $\alpha \in (0, 1)$  if and only if  $(S, d)$  is log-convex.*

The sufficiency of Theorem 1 is straightforward. The proof for the necessity is given in the Appendix. Our idea is to show that the closure of the convex hull of  $V(S, d)$  equals  $V(S, d)$ . The difficulty of the proof lies in the fact that  $V(S, d)$  is not necessarily compactly generated (i.e.,  $V(S, d) = C - \mathfrak{R}_+^2$  for some compact set  $C \subseteq \mathfrak{R}^2$ ), although  $V(S, d)$  is closed, comprehensive, and bounded from above. For example,

$$V = \{v \in \mathfrak{R}^2 \mid v_1 < 1, v_2 \leq 1/(v_1 - 1)\}$$

is clearly closed, comprehensive, and bounded above, but is not compactly generated. Consequently, the convex hull of  $V(S, d)$  is not necessarily closed, in which case it is contained in but not equal to its closure. As a result, we cannot directly apply the Caratheodry theorem to first represent a point on the boundary of the closure of the convex hull of  $V(S, d)$  by at most three points in  $V(S, d)$ , and then apply the separating hyperplane theorem and the uniqueness of the solution for (6) to show either the representation is degenerate, or the boundary point is Pareto dominated by a point in  $V(S, d)$ .

Instead, our proof proceeds in the following way. First, we separate a point on the boundary from the convex hull of  $V(S, d)$  by a hyperplane with a non-negative normal. Second, we approximate the point by a sequence of points in the convex hull. By applying the Caratheodry theorem to represent the points in the sequence by those

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<sup>8</sup>Indeed, it is straightforward to verify that  $(S \cap S', d)$  is regular and  $V(S \cap S', d) \subseteq V(S, d) \cap V(S', d)$ . On the other hand, take any  $v \in V(S, d) \cap V(S', d)$ ,  $u \in S$ , and  $u' \in S'$  such that  $u_i > d_i$ ,  $u'_i > d_i$ ,  $v_i \leq \ln(u_i - d_i)$ , and  $v_i \leq \ln(u'_i - d_i)$  for  $i = 1, 2$ . Set  $\underline{u}_i = \min\{u_i, u'_i\}$ . Then,  $v_i \leq \ln(\underline{u}_i - d_i)$  and  $d \ll \underline{u} \leq u, u'$ . By the regularity,  $\underline{u} \in S \cap S'$ . This shows  $v \in V(S \cap S', d)$ . Hence,  $V(S \cap S', d) \supseteq V(S, d) \cap V(S', d)$ .

in  $V(S, d)$  and then considering a limit point of the sequence of the representations, the uniqueness of the solution for (6) enables us to show that the boundary point of the convex hull is either on the boundary of  $V(S, d)$  or is Pareto dominated in  $V(S, d)$ .

### 3.2 Existence and Uniqueness

We are now ready to present our extension of the standard axiomatic characterization of the asymmetric Nash bargaining solution. Recall that the uniqueness of the asymmetric Nash bargaining solution is specified in Definition 1 in Section 2.

**Theorem 2** *The asymmetric Nash bargaining solution on the class of all regular and log-convex bargaining problems is unique and determined by the maximization of the generalized Nash product. The uniqueness is lost on any larger class of regular bargaining problems.*

The first part of Theorem 2 extends the standard axiomatic characterization of the asymmetric Nash bargaining solution to a wider class of bargaining problems that are regularity and log-convex. The second part shows that this class is the largest one. The second part of Theorem 2 follows from Theorem 1. To prove the first part, we need a separation result. Specifically, let  $(T, d)$  be a regular and log-convex bargaining problem. Without loss of generality, we assume  $d = d^\circ = (0, 0)$ . Let  $\alpha \in (0, 1)$  be given and let  $x \in T$  be the solution for (6) over  $T$ . Given any point  $y \in T$  with  $y \neq x$  and  $y \gg d^\circ$ , we show in Lemma 1 below that  $y$  can be separated from  $x$  by the tangent line to the indifference curve of the Nash product weighted by bargaining powers  $\alpha$  and  $1 - \alpha$  at a point  $z \neq y$ , such that  $y$  is below the tangent line while  $x$  is not.

**Lemma 1** *Suppose  $(T, d^\circ)$  is a regular and log-convex bargaining problem. Let  $\alpha \in (0, 1)$  and  $x = \arg \max_{u \in T: u \geq d^\circ} u_1^\alpha u_2^{1-\alpha}$ . Then, for any payoff allocation  $y \in T$  with  $y \gg d^\circ$  and  $y \neq x$ , there exists  $z \in T$  such that*

$$\alpha z_2 x_1 + (1 - \alpha) z_1 x_2 \geq z_1 z_2 \tag{7}$$

and

$$\alpha z_2 y_1 + (1 - \alpha) z_1 y_2 < z_1 z_2. \tag{8}$$

Before proving the lemma, we wish to remark the following. First, (7) and (8) together imply that the maximizer  $x \in T$  and the non-maximizer  $y \in T$  are separated

by the tangent line at point  $z \in T$  with  $z \neq y$ . Second, when  $T$  is convex, Lemma 1 holds trivially by simply letting  $z = x$ .

**Proof.** Since  $(T, d^\circ)$  satisfies log-convexity condition, it follows from Theorem 1 that  $x = \arg \max_{u \in T: u \geq d^\circ} u_1^\alpha u_2^{1-\alpha}$  is uniquely determined. Let  $y \in T$  be such that  $y \gg d^\circ$  and  $y \neq x$ . Then, from the uniqueness of the maximizer,

$$x_1^\alpha x_2^{1-\alpha} > y_1^\alpha y_2^{1-\alpha},$$

or equivalently,

$$\alpha(\ln y_1 - \ln x_1) < (1 - \alpha)(\ln x_2 - \ln y_2).$$

Assume without loss of generality  $x_1 < y_1$  (the other case can be analogously proved). The locations of  $x$  and  $y$  are illustrated in Figure 2(a). Notice with  $x_1 < y_1$ , the previous inequality implies

$$\frac{\alpha}{1 - \alpha} < \frac{\ln x_2 - \ln y_2}{\ln y_1 - \ln x_1}. \quad (9)$$

Since  $x, y \in T$  and  $d^\circ = (0, 0)$ , it follows from the log-convexity

$$t(\ln x_1, \ln x_2) + (1 - t)(\ln y_1, \ln y_2) = (\ln x_1^t y_1^{1-t}, \ln x_2^t y_2^{1-t}) \in V(T, d^\circ)$$

for  $t \in [0, 1]$ . This is the segment between  $(\ln x_1, \ln x_2)$  and  $(\ln y_1, \ln y_2)$  in Figure 2(b). Thus, by the definition of  $V(T, d^\circ)$ , there exists  $u(t) \in T$  such that  $(\ln x_1^t y_1^{1-t}, \ln x_2^t y_2^{1-t}) \leq (\ln u_1(t), \ln u_2(t))$ , or equivalently,  $(x_1^t y_1^{1-t}, x_2^t y_2^{1-t}) \leq (u_1(t), u_2(t))$ . Since  $x, y \gg d^\circ$  implies  $(x_1^t y_1^{1-t}, x_2^t y_2^{1-t}) \gg d^\circ$ , from the regularity condition it follows

$$z(t) \equiv (x_1^t y_1^{1-t}, x_2^t y_2^{1-t}) \in T$$

for all  $t \in [0, 1]$  as illustrated in Figure 2(c).

For  $t \in (0, 1)$ , the absolute value of the slope of the tangent line of Nash product  $u_1^\alpha u_2^{1-\alpha}$  at  $z(t)$  is

$$\frac{\alpha x_2^t y_2^{1-t}}{(1 - \alpha) x_1^t y_1^{1-t}}$$

and that of the segment between  $z(t)$  and  $y$  is

$$\frac{x_2^t y_2^{1-t} - y_2}{y_1 - x_1^t y_1^{1-t}}.$$

Thus, the tangent line of Nash product  $u_1^\alpha u_2^{1-\alpha}$  at  $z(t)$  is flatter than the segment between  $z(t)$  and  $y$  if and only if <sup>9</sup>

$$\frac{\alpha x_2^t y_2^{1-t}}{(1-\alpha)x_1^t y_1^{1-t}} < \frac{x_2^t y_2^{1-t} - y_2}{y_1 - x_1^t y_1^{1-t}} \Leftrightarrow \frac{\alpha x_2^t}{(1-\alpha)x_1^t} < \frac{x_2^t - y_2^t}{y_1^t - x_1^t}.$$

By L'Hôpital's rule and (9), the above inequalities hold as  $t \rightarrow 0$ . Now choose  $t \in [0, 1]$  such that the tangent line of Nash product  $u_1^\alpha u_2^{1-\alpha}$  at point  $z(t)$ ,

$$\alpha z_2(t)u_1 + (1-\alpha)z_1(t)u_2 = z_1(t)z_2(t), \quad (10)$$

is flatter than the segment between  $z(t)$  and  $y$ . Since  $x$  is Pareto optimal,  $x_1 < y_1$  necessarily implies  $x_2 > y_2$ . Thus,  $z_1(t) < y_1$  and  $z_2(t) > y_2$ . Consequently,  $y$  is below the tangent line in (10). On the other hand, as the unique solution for (6) over choice set  $T$ ,  $x$  cannot lie below the indifference curve of  $u_1^\alpha u_2^{1-\alpha}$  at  $z(t)$ . (See Figure 2(c) for an illustration.) Therefore, (7) and (8) are established by setting  $z = z(t)$ . ■

As remarked above, Lemma 1 extends the separation between the maximizer of the indifference curve of Nash product over a convex choice set and the rest of the choice set by the tangent line at the maximizer. This separation is applied in the usual proof of the uniqueness of Nash bargaining solution. In the case with a regular and log-convex choice set, the separation in Lemma 1 is weaker in the sense that it may be point-dependent (i.e.,  $z$  changes as point  $y$  changes in the choice set). But, as is clear from the proof of Theorem 2 below, the separation in Lemma 1 is strong enough for the standard axiomatic characterization of the asymmetric Nash bargaining solution to hold on our class.

**Proof.** Let  $\mathcal{B}$  be the class of all regular and log-convex bargaining problems. By Theorem 1, the Nash product maximizer in (2) is unique on  $\mathcal{B}$  for all bargaining powers of the players. Thus, as noticed before, the usual proof can be applied to show that the Nash product maximization results in an asymmetric Nash bargaining solution for any given bargaining powers of the players. This establishes the existence.

We now prove the uniqueness by way of contradiction. Suppose that  $g$  is an asymmetric bargaining solution on  $\mathcal{B}$  with bargaining power  $\alpha \in (0, 1)$  for player 1. By Definition 1,  $g$  satisfies SIR, INV, IIA, and  $g(S^\circ, d^\circ) = (\alpha, 1 - \alpha)$ . If  $g$  is not determined by Nash product maximization (2), then there exists a bargaining problem

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<sup>9</sup>Since  $x_1 < y_1$ , the condition is equivalent to  $y$  lying below the tangent line.

$(S, d) \in \mathcal{B}$  such that  $g(S, d) \neq f(S, d)$ , where  $f$  denotes the asymmetric bargaining solution resulting from Nash product maximization (2) such that  $f(S^\circ, d^\circ) = (\alpha, 1 - \alpha)$  (same bargaining powers as with solution  $g$ ). Consider affine transformation:

$$\tau(u) = (\tau_1(u_1), \tau_2(u_2)) = \left( \frac{\alpha(u_1 - d_1)}{f_1(S, d) - d_1}, \frac{(1 - \alpha)(u_2 - d_2)}{f_2(S, d) - d_2} \right), \quad u \in \mathfrak{R}^2.$$

We have  $\tau(f(S, d)) = (\alpha, 1 - \alpha)$ ,  $\tau(d) = d^\circ$ , and  $(\tau(S), \tau(d)) \in \mathcal{B}$ .

Let  $T = \tau(S)$ ,  $x = \tau(f(S, d))$ , and  $y = \tau(g(S, d))$ . It follows that  $x = \arg \max_{u \in T: u \geq d^\circ} u_1^\alpha u_2^{1-\alpha}$ ,  $y \in T$ ,  $y \neq x$ , and  $y \gg d$ . By Lemma 1, there exists  $z \in T$  such that  $y$  is below the tangent line

$$\alpha z_2 u_1 + (1 - \alpha) z_1 u_2 = z_1 z_2$$

of Nash product  $u_1^\alpha u_2^{1-\alpha}$  at  $z$ . Let  $S'$  denote the choice set formed by this tangent line together with the two axes. Clearly,  $(S', d^\circ)$  is regular and log-convex and  $y \in S'$ . By INV and  $g(S^\circ, d^\circ) = (\alpha, 1 - \alpha)$ , we have

$$g(S', d^\circ) = z.$$

Since  $(S' \cap \tau(S), d^\circ)$  is regular and log-convex and since  $z \in S' \cap \tau(S)$ , it follows from IIA that

$$g(S' \cap \tau(S), d^\circ) = z.$$

On the other hand, since  $y = \tau(g(S, d)) = g(\tau(S), \tau(d)) \in \tau(S)$  and  $y \in S'$ , we also have

$$g(S' \cap \tau(S), d^\circ) = y.$$

It must be  $y = z$ , which contradicts  $y \neq z$ . This establishes the uniqueness.

Finally, let  $\mathcal{B}'$  be a larger class of regular bargaining problems than  $\mathcal{B}$ . Then, there is a bargaining problem  $(S', d') \in \mathcal{B}'$  which is not log-convex. By the regularity, Nash product maximization is well-defined for all bargaining problems in  $\mathcal{B}'$  and all bargaining powers of the players. However, by Theorem 1,  $(S', d')$  has multiple generalized Nash product maximizers for some bargaining power  $\alpha' \in (0, 1)$  for player 1. Consider bargaining solutions  $f^1$  and  $f^2$  on  $\mathcal{B}'$  with player 1's bargaining power equal to  $\alpha'$ , where for  $(S, d) \in \mathcal{B}'$ ,

$$f^1(S, d) = \arg \max_{u \in \text{NPM}(S, d)} u_1$$

and

$$f^2(S, d) = \arg \max_{u \in \text{NPM}(S, d)} u_2.$$

Here,  $\text{NPM}(S, d)$  denotes the set of the corresponding generalized Nash product maximizers for bargaining problem  $(S, d)$ . By the Pareto optimality, we have  $f^1(S', d^2(S', d'))$ . Moreover, as single-valued selections of Nash product maximizers,  $f^1$  and  $f^2$  satisfy SIR, INV, IIA, and  $f^1(S^\circ, d^\circ) = f^2(S^\circ, d^\circ) = (\alpha', 1 - \alpha')$ . This concludes that both  $f^1$  and  $f^2$  are asymmetric Nash bargaining solutions with bargaining powers  $\alpha'$  and  $1 - \alpha'$ . ■

### 3.3 A Subclass of Log-Convex Bargaining Problems

For Nash product maximization with generic bargaining powers to have a unique solution, the log-convexity quantifies how non-convex the IR portion of a bargaining problem can be allowed. When the frontier is smooth, the log-convexity is implied by the elasticity of the IR frontier being monotonically decreasing as shown in the following proposition.

**Proposition 1** *Let  $(S, d^\circ)$  be a regular bargaining problem such that the IR frontier of  $S$  is given by a  $C^2$  function,  $u_j = \psi(u_i)$ , over  $u_i \in [0, \bar{u}_i]$  for some number  $\bar{u}_i < \infty$  with  $\psi(\bar{u}_i) = 0$  and for  $i \neq j$ . If*

$$(R) \quad \psi'(u_i) < 0 \text{ and } e(u_i) \equiv \frac{u_i \psi'(u_i)}{\psi(u_i)} \text{ is strictly decreasing over } (0, \bar{u}_i),$$

*then  $(S, d^\circ)$  is log-convex.*

**Proof.** Notice first for any pair  $v = (v_i, v_j)$  in the frontier of  $V(S, d^\circ)$  there exists  $u_i \in (0, \bar{u}_i)$  such that  $v_i = \ln(u_i)$  and  $v_j = \ln[\psi(u_i)]$ . That is, the frontier of  $V(S, d^\circ)$  is parameterized by  $u_i \in (0, \bar{u}_i)$ . Thus, the slope of the frontier at  $v$  is given by

$$e(u_i) = \frac{u_i \psi'(u_i)}{\psi(u_i)}.$$

Notice  $\psi'(u_i) < 0$  implies that the frontier of  $V(S, d^\circ)$  is downward sloping. Hence, it is strictly concave if  $e'(u_i) < 0$ . ■

Note that  $e(u_i)$  in Proposition 1 is the elasticity of the IR frontier of a bargaining problem with respect to  $u_i$ . Thus, it is relatively easier to remember and check condition



( $R$ ) than the log-convexity for some applications. A stronger sufficient condition is either  $\psi$  is log-concave, or  $u_i\psi'(u_i)$  is decreasing, in addition to  $\psi$  being decreasing.

With a generic threat point  $d$ , condition ( $R$ ) takes the following form,

$$\psi'(u_i) < 0 \text{ and } e(u_i) \equiv \frac{(u_i - d_i)\psi'(u_i)}{\psi(u_i) - d_j} \text{ is strictly decreasing over } (d_i, \bar{u}_i).$$

Given  $\psi' < 0$  over  $(d_i, \bar{u}_i)$ ,  $(S, d)$  satisfying the above inequality implies that  $(S, d')$  also satisfies it for all  $d' \in S$  with  $d' \geq d$  such that  $d'_j < \psi(\bar{u}_i)$ .

Moreover, as  $d$  varies in  $S$ , the log-convexity of  $(S, d)$  puts further restrictions on  $S$ . A natural question is whether  $S$  is necessarily convex when  $(S, d)$  is log-convex for all strictly Pareto dominated threat points  $d \in S$ . Example 1 provides a counter example and Proposition 2 below illustrates a class of bargaining problems that is log-convex but not convex for all threat points that are strictly Pareto dominated.

## 4 An Application: Bargaining Problems Arising in Duopolies

As Example 1 illustrates, the profit choice set of the duopoly bargaining problem is not convex. In this section, we show that for a general demand function and asymmetric constant marginal costs, the duopoly bargaining problem without side-payments belongs to the class of regular and log-convex bargaining problems. We assume without loss of generality that firms' constant marginal costs  $c_1$  and  $c_2$  satisfy  $c_1 < c_2$  and  $D(c_2) > 0$ . For  $i = 1, 2$ , let  $\bar{\pi}_i(p) = (p - c_i)D(p)$  denote firm  $i$ 's *monopoly profit* function and  $\bar{p}_i$  its *monopoly price*. Notice  $c_1 < c_2$  implies  $\bar{p}_1 < \bar{p}_2$ . To make bargaining non-trivial, we further assume  $c_2 < \bar{p}_1$ .

Let  $\pi = (\pi_1, \pi_2)$  be a profit distribution between the firms. By (4), the feasible set of profit allocations for the two firms is given by <sup>10</sup>

$$\Pi = \left\{ \pi \left| \exists p : \begin{array}{l} c_2 < p < D^{-1}(0) \\ \frac{\pi_1}{p-c_1} + \frac{\pi_2}{p-c_2} = D(p) \end{array} \right. \right\}.$$

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<sup>10</sup>Notice that prices  $p \leq c_2$  or  $p \geq D^{-1}(0)$  are Pareto dominated by prices in  $(c_2, D^{-1}(0))$ .

Given price  $p > c_2$  and firm 2's profit  $\pi_2$ , it follows from (4) that

$$h(p, \pi_2) \equiv \bar{\pi}_1(p) - \frac{\pi_2(p - c_1)}{p - c_2} \quad (11)$$

is the profit for firm 1 should they agree to let firm 2 receive profit  $\pi_2$ . Thus, a pair  $\pi \in \Pi$  is on the Pareto frontier if and only if

$$\pi_1 = \max_{c_2 < p < D^{-1}(0)} h(p, \pi_2).$$

The supporting price,  $p(\pi_2)$ , is determined by the first-order condition

$$h_1(p, \pi_2) = \bar{\pi}'_1(p) + \frac{(c_2 - c_1)\pi_2}{(p - c_2)^2} = 0$$

and the Pareto frontier is given by  $\pi_1 = h(p(\pi_2), \pi_2)$  for  $\pi_2 \in [0, \bar{\pi}_2(\bar{p}_2)]$ . Note that the second-order condition is satisfied if  $\bar{\pi}_1(p)$  is concave.

Our next result shows that under the assumed conditions on the cost and profit functions,  $(\Pi, d)$  is a non-convex bargaining problem, but is regular and log-convex whenever  $d \geq 0$  is strictly Pareto dominated.

**Proposition 2** *Assume  $c_1 < c_2$  and  $\bar{\pi}_1(p)$  is twice continuously differentiable and concave over the finite range of prices at which demand is positive. Then,  $\Pi$  is non-convex, but  $(\Pi, d)$  is regular and log-convex for all threat points  $d \geq 0$  that are strictly Pareto dominated in  $\Pi$ .*

The proof of Proposition 2 is given in the Appendix. Proposition 2 and Theorem 2 together imply that the duopoly bargaining problem  $(\Pi, d)$  has a unique Nash bargaining solution under the assumed conditions. One can thus perform comparative static analysis of the Nash bargaining solution with respect to bargaining powers, marginal costs, and threat points.

## 5 Comparison

To simplify, we consider bargaining problems  $(S, d)$  with  $d = d^\circ$  and  $(S, d^\circ)$  satisfying condition  $(R)$ . If  $\psi$  is concave, then  $S$  is convex in which case extended Nash bargaining solutions reduce to the Nash bargaining solution. Correspondingly, the focus of our comparison is on non-convex bargaining problems.

Under Conley and Wilkie's (1996) extension, the bargaining solution is determined by the intersection of the Pareto frontier of the choice set with the segment connecting the threat point and the Nash bargaining solution for the convexified problem. Although they only consider extension of the symmetric Nash bargaining solution, their method can be applied to extend the asymmetric Nash bargaining solution. Given  $(S, d)$  and  $\alpha \in (0, 1)$ , we denote by  $f^N(S, d|\alpha)$  the asymmetric Nash bargaining solution with bargaining powers  $\alpha$  for player 1 and  $1 - \alpha$  for player 2 and by  $f^{CW}(S, d|\alpha)$  the Conley-Wilkie's extension.

**Proposition 3** *Suppose  $(S, d^\circ)$  satisfies the regularity condition and  $(R)$  with  $\psi'' > 0$ . Then there exists  $\hat{\alpha} \in (0, 1)$  such that (a)  $f^N(S, d^\circ|\hat{\alpha}) = f^{CW}(S, d^\circ|\hat{\alpha})$ , (b)  $f_1^N(S, d^\circ|\alpha) < f_1^{CW}(S, d^\circ|\alpha)$  and  $f_2^N(S, d^\circ|\alpha) > f_2^{CW}(S, d^\circ|\alpha)$  for  $\alpha < \hat{\alpha}$ , and (c)  $f_1^N(S, d^\circ|\alpha) > f_1^{CW}(S, d^\circ|\alpha)$  and  $f_2^N(S, d^\circ|\alpha) < f_2^{CW}(S, d^\circ|\alpha)$  for  $\alpha > \hat{\alpha}$ .*

The proof of Proposition 3 is in the Appendix. Proposition 3 implies that the Nash bargaining solution without convexification is more responsive to the relative bargaining powers than the Conley-Wilkie bargaining solution. The intuition appears to be that the Nash bargaining solution without convexification tends to utilize more local curvature of the Pareto frontier of the choice set than the Conley-Wilkie solution does. For the class of duopoly bargaining problems with  $c_1 < c_2$  discussed in Section 4, we can further show that  $\hat{\alpha} > 1/2$ , implying that in the presence of asymmetric marginal costs and equal bargaining powers, the Nash bargaining solution favors the less efficient firm as compared to the Conley-Wilkie bargaining solution.

Conditions  $(R)$  together with the regularity condition also guarantees a geometric property of the asymmetric Nash bargaining solution given in Proposition 4 below. Given  $\alpha \in (0, 1)$ , let  $I_1(\alpha)$  and  $I_2(\alpha)$  denote the horizontal and vertical intercepts of the tangent line to the Pareto frontier of  $S$  at  $f^N(S, d^\circ|\alpha)$ , respectively. We use  $\|\cdot\|$  to denote the Euclidean norm.

**Proposition 4** *For any  $\alpha \in (0, 1)$ ,*

$$\alpha\|f^N(S, d^\circ|\alpha) - I_2(\alpha)\| = (1 - \alpha)\|f^N(S, d^\circ|\alpha) - I_1(\alpha)\|$$

*for any regular bargaining problem satisfying  $(R)$ .*

Proposition 4 states that the distances between the asymmetric Nash bargaining solution and the two intercepts are proportional to the associated bargaining powers. In the case of equal bargaining powers, this geometric (equal distance) property

is the main feature in Herrero's (1989) characterization of the multi-valued bargaining solution under a different set of axioms and has also been noted in Mas-Colell, Whinston and Green (1995, Example 22.E.3, p. 842) for compact convex bargaining problems. The *proportional distance property* of the asymmetric Nash bargaining solution in Proposition 4 represents an extension of the equal distance property to the asymmetric cases and a large class of regular bargaining problems satisfying  $(R)$ .

## 6 Conclusion

It is well-known that on the class of compact convex bargaining problems Nash axioms without that of symmetry determine a unique bargaining solution that is parameterized by players' bargaining powers. The parametrization by players' bargaining powers is useful in applications, for it offers flexibility in fitting data by choosing suitable bargaining powers. In this paper, we have extended the standard axiomatic characterization of the asymmetric Nash bargaining solution to the class of all regular and log-convex bargaining problems. We have shown that this is largest class for such extension to hold.

Bargaining between duopolists with asymmetric constant marginal costs and concave monopoly profit functions for the firms results in non-convex bargaining problems, which are regular and convex. The uniqueness and single-valuedness of the Nash bargaining solution make it possible to conduct comparative static analysis with respect to bargaining powers, threat points and determinants of the choice sets such as the marginal costs for duopoly bargaining problems. They also enable interesting comparisons with other single-valued bargaining solutions that appeared in the literature.

## Appendix: Proofs

Before proceeding with the proof of Theorem 1, we make the following observations. First, the logarithmic transformation  $(v_1, v_2) = (\ln(u_1 - d_1), \ln(u_2 - d_2))$  is a homeomorphism from  $S \cap \{(u_1, u_2) \mid u_1 > d_1, u_2 > d_2\}$  onto  $V(S, d)$ . It follows that the boundary  $\partial V(S, d)$  of  $V(S, d)$  must be the homeomorphic image of that of  $S \cap \{(u_1, u_2) \mid u_1 > d_1, u_2 > d_2\}$ ; that is,

$$\partial V(S, d) = \left\{ (\ln(u_1 - d_1), \ln(u_2 - d_2)) \mid u \in \partial S : \begin{array}{l} u_1 > d_1, \\ u_2 > d_2. \end{array} \right\}.$$

Thus,  $V(S, d)$  is closed. Second, because  $S$  is bounded above, so is  $V(S, d)$ , implying  $\partial V(S, d) \neq \emptyset$ . Third, for any  $(u_1, u_2) \in S \cap \{(u_1, u_2) \mid u_1 > d_1, u_2 > d_2\}$ , the intervals  $((d_1, u_2), (u_1, u_2)]$  and  $((u_1, d_2), (u_1, u_2)]$  are transformed into  $((-\infty, \ln(u_2 - d_2)), (\ln(u_1 - d_1), \ln(u_2 - d_2))]$  and  $((\ln(u_1 - d_1), -\infty), (\ln(u_1 - d_1), \ln(u_2 - d_2))]$  in  $V(S, d)$ , respectively. As a result,  $V(S, d) = V(S, d) - \mathfrak{R}_+^2$ . Thus, by moving the origin properly, we can assume without loss of generality that for some constant  $h > 0$ ,

$$V(S, d) \subset -(h, h) - \mathfrak{R}_+^2, \tag{A1}$$

so that the closed cone generated by  $V(S, d)$  and any line passing through the origin with a normal vector in  $\mathfrak{R}_+^2$  have no point in common other than the origin.

**Lemma 2** *Let  $U \subseteq \mathfrak{R}^2$  be closed, convex, and comprehensive (i.e.  $U = U - \mathfrak{R}_+^2$ ). If the boundary  $\partial U$  of  $U$  is non-empty, then  $U = \partial U - \mathfrak{R}_+^2$ .*

**Proof.** Notice  $\partial U - \mathfrak{R}_+^2 \subseteq U - \mathfrak{R}_+^2 = U$  is automatic. Conversely, let  $u$  be any interior point of  $U$ . Then, there exists a number  $\epsilon > 0$  such that  $(u_i + \delta, u_j) \in U$  for  $i \neq j$  and for all  $\delta \in [0, \epsilon)$ . Set  $\delta_i = \sup\{\delta \mid (u_i + \delta, u_j) \in U\}$ . Since  $U$  is convex and comprehensive,  $\delta_i < \infty$  for at least one  $i$ , for otherwise  $U = \mathfrak{R}^2$  which contradicts  $\partial U \neq \emptyset$ . Assume without loss of generality  $\delta_1 < \infty$ . Then,  $u^1 = (u_1 + \delta_1, u_2) \in \partial U$  because  $U$  is closed. It follows that  $u = u^1 - (\delta_1, 0) \in \partial U - \mathfrak{R}_+^2$ . This shows  $U \subseteq \partial U - \mathfrak{R}_+^2$ . ■

**Proof of Theorem 1:** The sufficiency of Theorem 1 is trivial. To prove the necessity, we only need to show that  $V(S, d)$ , which we simply denote by  $V$ , is convex because the necessity of the rest of the conditions for log-convexity is obvious.

Let  $\bar{V}$  denote the closure of the convex hull of  $V(S, d)$ . Notice  $\bar{V}$  also satisfies  $\bar{V} = \bar{V} - \mathfrak{R}_+^2$ ; hence, from Lemma 2,  $\bar{V} = \partial \bar{V} - \mathfrak{R}_+^2$ . For any boundary point  $\bar{v} \in \partial \bar{V}$ , it

follows from the separation theorem and the comprehensiveness of  $\bar{V}$  that there exists  $a \in \mathfrak{R}_+^2$  such that

$$a \cdot \bar{v} = \max_{v \in \bar{V}} a \cdot v = \max_{v \in V} a \cdot v. \quad (A2)$$

Without loss of generality, we may take  $a = (\theta, 1 - \theta)$  for some  $\theta \in [0, 1]$ .

There exist sequences  $\{v^k(n)\}_n$  in  $V$  for  $k = 1, 2, 3$ ,  $\{\lambda(n)\}_n$  in  $\mathfrak{R}_+^3$ , and  $\{\epsilon(n)\}_n$  in  $\mathfrak{R}$  such that <sup>11</sup>

$$\bar{v} = \lambda_1(n)v^1(n) + \lambda_2(n)v^2(n) + \lambda_3(n)v^3(n) + \epsilon(n), \quad (A3)$$

$$\lambda_1(n) + \lambda_2(n) + \lambda_3(n) = 1, \quad (A4)$$

$$\epsilon(n) \rightarrow 0. \quad (A5)$$

By the non-nativity of  $\lambda(n)$  and (A4), we may assume

$$\lambda(n) \rightarrow \bar{\lambda} \in \mathfrak{R}_+^3 \text{ with } \bar{\lambda}_1 + \bar{\lambda}_2 + \bar{\lambda}_3 = 1. \quad (A6)$$

By (A1), the sequences  $\{\lambda_k(n)v^k(n)\}_n$ ,  $k = 1, 2, 3$ , are bounded above and by (A3), (A5), and (A6), they are also bounded below. Hence, we may assume

$$\lambda_k(n)v^k(n) \rightarrow \bar{w}^k \in \mathfrak{R}^2, \quad k = 1, 2, 3 \Rightarrow \bar{v} = \bar{w}^1 + \bar{w}^2 + \bar{w}^3. \quad (A7)$$

Suppose first  $\bar{\lambda}_k > 0$  for  $k = 1, 2, 3$ . Then, since  $V$  is closed, (A3) and (A5) – (A7) imply

$$v^k(n) \rightarrow \bar{v}^k \in V(S, d), \quad k = 1, 2, 3, \quad \bar{v} = \bar{\lambda}_1 \bar{v}^1 + \bar{\lambda}_2 \bar{v}^2 + \bar{\lambda}_3 \bar{v}^3. \quad (A8)$$

Suppose now  $\bar{\lambda}_k = 0$  for some  $k$  but  $\bar{\lambda}_{k'} > 0$  for  $k' \neq k$ . Without loss of generality, assume  $\bar{\lambda}_1 = 0$ ,  $\bar{\lambda}_2 > 0$ , and  $\bar{\lambda}_3 > 0$ . In this case,

$$\bar{v} = \bar{w}^1 + \bar{\lambda}_2 \bar{v}^2 + \bar{\lambda}_3 \bar{v}^3, \quad \bar{\lambda}_2, \bar{\lambda}_3 > 0, \quad \bar{\lambda}_2 + \bar{\lambda}_3 = 1. \quad (A9)$$

By (A2) and (A7),

$$a \cdot \bar{w}^1 = \lim_{n \rightarrow \infty} a \cdot \lambda_1(n)v^1(n) = \lim_{n \rightarrow \infty} \lambda_1(n)a \cdot v^1(n) \leq \lim_{n \rightarrow \infty} \bar{\lambda}_1 a \cdot \bar{v} = 0.$$

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<sup>11</sup>The reason is as follows. Since  $\bar{v}$  is in  $\bar{V}$ , there exists a sequence  $\{\bar{v}(n)\}_n$  in the convex hull of  $V$  such that  $\bar{v}(n) \rightarrow \bar{v}$ . By Carathéodory Theorem, for each  $n$ , there exist  $v^1(n), v^2(n), v^3(n)$  in  $V$  and  $\lambda(n) \in \mathfrak{R}_+^3$  satisfying (A4) such that  $\bar{v}(n) = \lambda_1(n)v^1(n) + \lambda_2(n)v^2(n) + \lambda_3(n)v^3(n)$ . Finally, (A3) and (A5) are established by setting  $\epsilon(n) = \bar{v} - \bar{v}(n)$  for all  $n$ .

On the other hand, by (A2) and (A9),

$$a \cdot \bar{w}^1 = a \cdot \bar{v} - [\bar{\lambda}^2(a \cdot \bar{v}^2) + \bar{\lambda}^3(a \cdot \bar{v}^3)] \geq 0.$$

It follows that  $a \cdot \bar{w}^1 = 0$ . This shows that  $\bar{w}^1$  lies on the line having normal vector  $a = (\theta, 1 - \theta) \in \mathfrak{R}_+^2$  and passing through the origin. Furthermore, as the limit of  $\{\lambda_1(n)v^1(n)\}$ ,  $\bar{w}^1$  is in the closed cone generated by  $V$ . Thus, by (A1), we must have  $\bar{w}^1 = 0$ . Suppose finally  $\bar{\lambda}_k = 1$  for some  $k$ . Assume without loss of generality  $\bar{\lambda}_3 \neq 0$ . In this case, a similar proof as before shows  $\bar{w}^1 = \bar{w}^2 = 0$ .

In summary, by letting  $\bar{v}^k$  be arbitrary element in  $V$  when  $\bar{\lambda}_k = 0$ , the preceding analysis establishes

$$\bar{v} = \bar{\lambda}_1 \bar{v}^1 + \bar{\lambda}_2 \bar{v}^2 + \bar{\lambda}_3 \bar{v}^3. \quad (A10)$$

When  $\bar{\lambda}_k > 0$ , (A2) and (A10) together imply

$$a \cdot \bar{v}^k = \max_{v \in V} a \cdot v,$$

which implies  $\bar{v}^k \in \partial V$ . If  $a \in \mathfrak{R}_{++}^2$ , then the above equality and the assumption that (6) has a unique solution imply  $\bar{v}^k = \bar{v}$  whenever  $\bar{\lambda}_k > 0$ . Hence,  $\bar{v} \in \partial V \subset V$ . If  $a = (1, 0)$ , then

$$\bar{v}_1 = a \cdot \bar{v} = a \cdot \bar{v}^k = \bar{v}_1^k.$$

Assume without loss of generality  $\bar{\lambda}_1 > 0$  and  $\bar{v}_2^1 = \max\{\bar{v}_2^k \mid \bar{\lambda}_k > 0\}$ . Then, by (A10),

$$\bar{v}_2 = \bar{\lambda}_1 \bar{v}_2^1 + \bar{\lambda}_2 \bar{v}_2^2 + \bar{\lambda}_3 \bar{v}_2^3 \leq \bar{v}_2^1.$$

Consequently,  $\bar{v} \leq \bar{v}^1$ . This shows  $\bar{v} \in V$ . If  $a = (0, 1)$ , then a similar proof establishes  $\bar{v} \in V$ .

We have shown that  $\bar{v} \in V$  for any  $\bar{v} \in \partial \bar{V}$ . Thus, by Lemma 2,

$$V \subseteq \bar{V} = \partial \bar{V} - \mathfrak{R}_+^2 \subseteq V - \mathfrak{R}_+^2 = V.$$

This concludes that  $V = \bar{V}$  implying that  $V$  contains its convex hull. Hence,  $V$  is convex. ■

**Proof of Proposition 2:** The non-convexity of  $\Pi$  was shown in Tirole (1988, p. 242, 271) under one extra assumption that  $\bar{\pi}_2(p)$  is concave. The non-convexity of  $\Pi$  can also be shown without the concavity of  $\bar{\pi}_2(p)$ . To see this, notice that the Pareto

frontier of  $\Pi$  is given by the function

$$\pi_1 = \psi(\pi_2) = h(p(\pi_2), \pi_2), \quad \pi_2 \in [0, \bar{\pi}_2(\bar{p}_2)].$$

By (11),

$$h_2(p, \pi_2) = -\frac{p - c_1}{p - c_2}, \quad h_{22}(p, \pi_2) = 0, \quad h_{12}(p, \pi_2) = \frac{c_2 - c_1}{(p - c_2)^2}, \quad (\text{A11})$$

and

$$h_{11}(p, \pi_2) = \bar{\pi}_1''(p) - \frac{2(c_2 - c_1)\pi_2}{(p - c_2)^3}, \quad p'(\pi_2) = -\frac{h_{12}(p(\pi_2), \pi_2)}{h_{11}(p(\pi_2), \pi_2)}. \quad (\text{A12})$$

It follows from the envelope theorem, (A11), and (A12) that

$$\psi'(\pi_2) = h_2(p(\pi_2), \pi_2), \quad (\text{A13})$$

and

$$\psi''(\pi_2) = h_{12}(p(\pi_2), \pi_2)p'(\pi_2) = -\frac{[h_{12}(p(\pi_2), \pi_2)]^2}{h_{11}(p(\pi_2), \pi_2)} > 0, \quad (\text{A14})$$

where the strict inequality holds since  $h_{11}(p, \pi_2) < 0$  due to the concavity of  $\bar{\pi}_1(p)$ .

Next, we show that  $(\Pi, d)$  is log-convex for all threat point  $d \geq 0$  that are strictly Pareto dominated in  $\Pi$ . Without loss of generality, assume  $d = d^\circ$ . By Proposition 1, it suffices to show that  $\psi'(\pi_2) < 0$  and  $\pi_2\psi'(\pi_2)$  is decreasing in  $\pi_2$ .

By (A11) and (A13),  $\psi'(\pi_2) < 0$  if  $p(\pi_2) > c_2$ . Suppose  $p(\pi_2) < \bar{p}_1$ . The strict concavity of  $\bar{\pi}_1(p)$  implies that  $\bar{\pi}_1'(p) > 0$  for  $p \in [p(\pi_2), \bar{p}_1]$ , which in turn implies  $h_1(p, \pi_2) > 0$  for  $p \in [p(\pi_2), \bar{p}_1]$ . Consequently,  $p(\pi_2)$  cannot be Pareto optimal. Thus,  $p(\pi_2) \geq \bar{p}_1 > c_2$ , where the last inequality holds by assumption. By the strict concavity of  $\bar{\pi}_1(p)$  and (A12),

$$-h_{11}(p, \pi_2) > \frac{2(c_2 - c_1)\pi_2}{(p - c_2)^3} > 0 \quad (\text{A15})$$

for any  $\pi_2 > 0$  along the frontier. It then follows from (A11)-(A15) that

$$\begin{aligned} (\pi_2\psi'(\pi_2))' &= \psi'(\pi_2) + \pi_2\psi''(\pi_2) \\ &= h_2(p(\pi_2), \pi_2) + \pi_2 \frac{[h_{12}(p(\pi_2), \pi_2)]^2}{-h_{11}(p(\pi_2), \pi_2)} \\ &< -\frac{p(\pi_2) - c_1}{p(\pi_2) - c_2} + \frac{c_2 - c_1}{2[p(\pi_2) - c_2]} \\ &= -\frac{2p(\pi_2) - c_1 - c_2}{2[p(\pi_2) - c_2]} \\ &< 0 \end{aligned}$$



for any  $\pi_2 > 0$  along the frontier, where the last inequality holds since  $p(\pi_2) \geq \bar{p}_1 > c_2 > c_1$ . ■

**Proof of Proposition 3:** By Proposition 1,  $(S, d^\circ)$  is regular and log-convex. Thus, by Theorem 2, there is a unique Nash bargaining solution determined by (2), where the first-order condition is given by

$$-\frac{u_1 \psi'(u_1)}{\psi(u_1)} = \frac{\alpha}{1 - \alpha}. \quad (\text{A16})$$

The Nash bargaining solution  $(f_1^N(S, d^\circ|\alpha), f_2^N(S, d^\circ|\alpha))$  is determined by (A16) and  $u_2 = \psi(u_1)$ . It follows that the ratio of the two payoff gains is

$$R^N(\alpha) \equiv \frac{f_2^N(S, d^\circ|\alpha)}{f_1^N(S, d^\circ|\alpha)} = -\frac{1 - \alpha}{\alpha} \psi'(f_1^N(S, d^\circ|\alpha)).$$

Moreover, the left-hand side of (A16) is strictly increasing in  $u_1$  by (R) and the right-hand side is strictly increasing in  $\alpha$ . It follows that  $f_1^N(S, d^\circ|\alpha)$  is strictly increasing in  $\alpha$  for any interior solution.

The frontier of the convex hull of  $S$  is a straight line given by

$$u_2 = \frac{\bar{u}_2}{\bar{u}_1}(\bar{u}_1 - u_1).$$

Applying Nash bargaining on the convex hull yields a solution  $u = (\alpha \bar{u}_1, (1 - \alpha) \bar{u}_2)$ . This solution and the Conley-Wilkie's extended Nash bargaining solution have the same ratio of the payoff gains for the two players, which is given by

$$R^{CW}(\alpha) \equiv \frac{f_2^{CW}(S, d^\circ|\alpha)}{f_1^{CW}(S, d^\circ|\alpha)} = \frac{1 - \alpha}{\alpha} \frac{\bar{u}_2}{\bar{u}_1}.$$

Comparing the two solutions yields

$$\frac{R^N(\alpha)}{R^{CW}(\alpha)} = \frac{-\psi'(f_1^N(S, d^\circ|\alpha))}{\bar{u}_2/\bar{u}_1}.$$

The strict convexity of  $\psi$  and the monotonicity of  $f_1^N(S, d^\circ|\alpha)$  at the interior solution imply that  $-\psi'(f_1^N(S, d^\circ|\alpha))$  is strictly decreasing in  $\alpha$ . Thus, there exists  $\hat{\alpha} \in (0, 1)$  such that (a)  $R^N(\hat{\alpha}) = R^{CW}(\hat{\alpha})$ , (b)  $R^N(\alpha) > R^{CW}(\alpha)$  if  $\alpha < \hat{\alpha}$ , and (c)  $R^N(\alpha) < R^{CW}(\alpha)$  if  $\alpha > \hat{\alpha}$ . The claims then follow since both solutions are on the frontier.

**Proof of Proposition 4:** Let  $z \in S$  be an allocation on the Pareto frontier. The line that is tangent to the frontier at  $z$  can be represented by

$$u_2 = \psi(z_1) + \psi'(z_1)(u_1 - z_1).$$

The tangent line intersects the horizontal axis at  $I_1 = (z_1 + \psi(z_1)/\psi'(z_1), 0)$  and the vertical axis at  $I_2 = (0, \psi(z_1) - z_1\psi'(z_1))$ . The distance between  $z$  and  $I_2$  is given by

$$\begin{aligned} \|z - I_2\|^2 &= (z_1)^2 + (z_1\psi'(z_1))^2 \\ &= (z_1)^2[1 + (\psi'(z_1))^2]. \end{aligned}$$

The distance between  $z$  and  $I_1$  is given by

$$\begin{aligned} \|z - I_1\|^2 &= (\psi(z_1)/\psi'(z_1))^2 + (\psi(z_1))^2 \\ &= \frac{(\psi(z_1))^2[1 + (\psi'(z_1))^2]}{(\psi'(z_1))^2}. \end{aligned}$$

It follows that

$$\frac{\|z - I_2\|^2}{\|z - I_1\|^2} = \left( \frac{z_1\psi'(z_1)}{\psi(z_1)} \right)^2.$$

The proportional distance property states that

$$\frac{\|z - I_2\|}{\|z - I_1\|} = \frac{1 - \alpha}{\alpha},$$

which is equivalent to

$$\left( \frac{1 - \alpha}{\alpha} - \frac{z_1\psi'(z_1)}{\psi(z_1)} \right) \left( \frac{1 - \alpha}{\alpha} + \frac{z_1\psi'(z_1)}{\psi(z_1)} \right) = 0.$$

Since  $\psi'(z_1) < 0$ , the above equality is then equivalent to

$$-\frac{z_1\psi'(z_1)}{\psi(z_1)} = \frac{\alpha}{1 - \alpha}$$

which is the first-order condition for an interior solution to the Nash product maximization problem (2). The maximization has a unique solution since  $V(S, d^0)$  is regular and log-convex under the assumptions. ■

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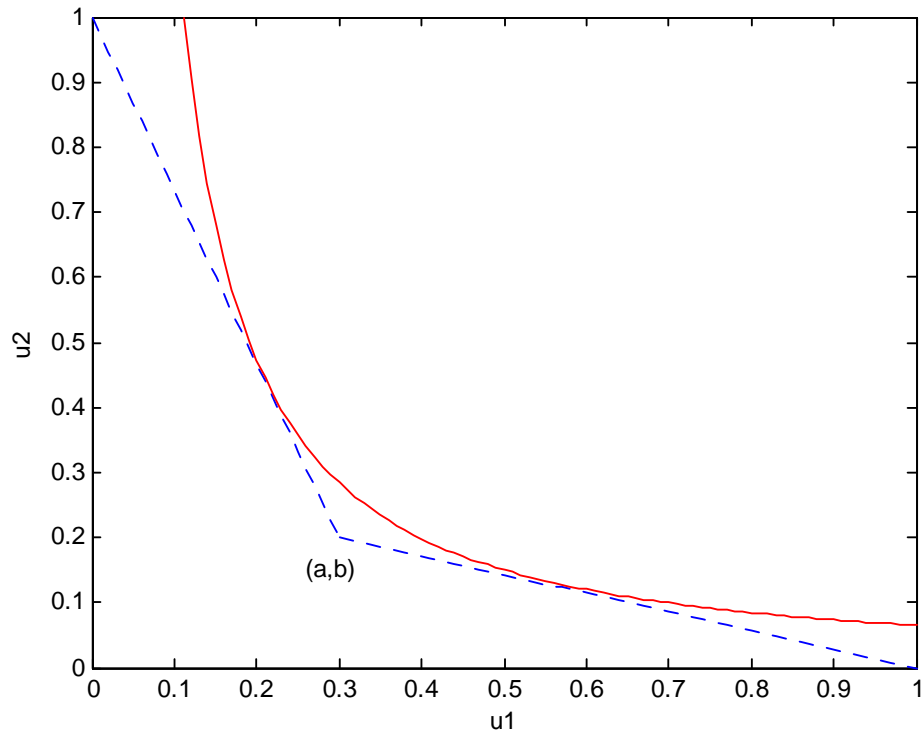


Figure 1: Multiple Solutions to Nash Product Maximization

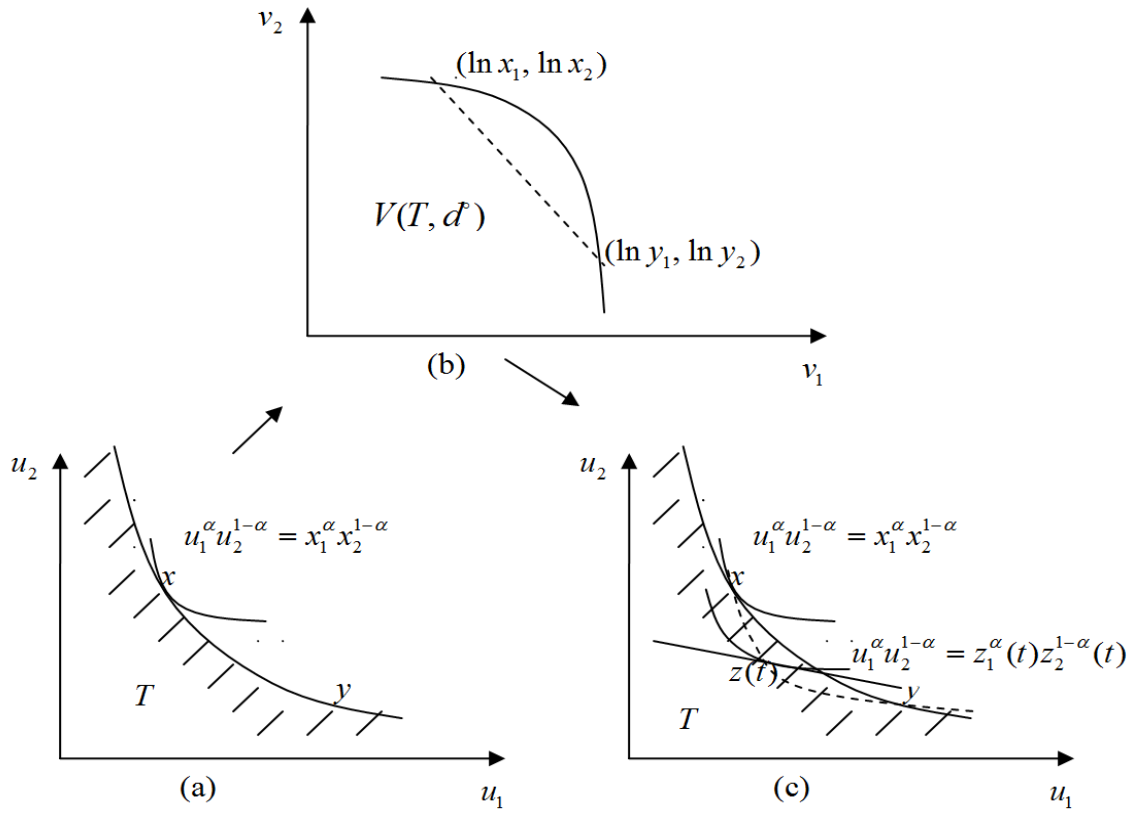


Figure 2: Separation between the Maximizer  $x$  of Nash Product  $u_1^\alpha u_2^{1-\alpha}$  over Choice Set  $T$  and Point  $y \in T$  by the Tangent Line of the Nash Product at Point  $z(t) \in T$  with  $z(t) \neq y$ .