

Liquidity Constraints*

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Abstract

We study economies where some assets play an essential role to finance consumption opportunities but payment arrangements are subject to a moral hazard problem. Agents can produce fraudulent assets at a positive cost, which generates an endogenous upper bound on the quantity of assets that can be exchanged for goods and services. This endogenous liquidity constraint depends on the characteristics of the assets, trading frictions, and policy. Our model offers insights for asset prices and liquidity premia, the value of currency, the rate of return dominance, and the working of monetary policy.

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1 Introduction

Liquidity constraints – the restrictions on the use of assets to finance needs for consumption or investment – play an important role for macroeconomic outcomes.¹ As shown by Bansal and Coleman (1996), Kiyotaki and Moore (2005a, 2005b, 2008), and Lagos (2007), among others, liquidity constraints can help explain asset pricing anomalies and the channel through which monetary policy affects assets' yields. The standard approach, however, is to assume, rather than explain, the restrictions on the use of assets to finance consumption or investment. The objective of this paper is to get a closer look into the microeconomic frictions that affect the ability of economic agents to exchange assets for goods and services.

The friction we emphasize is a moral hazard problem related to the ease with which the authenticity or quality of an asset can be ascertained. This friction aims to capture that, throughout history, most means of payment and financial assets have been threatened by fraudulent activities.² In contrast to existing studies (e.g., Freeman, 1985; Lester, Postlewaite, and Wright, 2008) we take seriously the notion that producing deceptive financial claims is a costly activity. This assumption is key to generate non-trivial liquidity constraints. We introduce this moral hazard problem into an economy with limited commitment, lack of enforcement, and no record keeping, similar to the one in Lagos and Wright (2005). In this economy there are centralized trades, where assets can be priced competitively, and decentralized bilateral trades that feature an essential role for assets to finance random consumption opportunities. The intricate part is the determination of the payment arrangements in matches where sellers are uninformed. We adopt a simple bargaining game and we use the methodology from Inn and Wright (2008) for signaling games with unobservable choices

¹For a literature review on liquidity constraints, see Williamson (2008).

²Dated back to the medieval Europe, individuals clipped the edge of silver and gold coins in order to deceive the recipients of those coins. During the 19th Century in U.S., vast quantities of fake banknotes were produced, making the U.S. a “nation of counterfeiters” according to Mihm (2007). In modern economies, intangible payment arrangements suffer from identity thefts, and many financial claims are subject to fraudulent activity. According to Schreft (2007), in 2006 8.4 million U.S. consumers found themselves to be victims of identity theft, and the estimate of total fraud cost from identity theft is about \$49.3billion. In terms of the recent crisis, mortgage-backed securities suffer from various mortgage frauds, the annual losses of which are estimated between \$4 billion and \$6 billion. (See http://www.fbi.gov/hq/mortgage_fraud.htm) One example of such frauds is called "property flipping". An individual purchases a property at one price and sells it to a "straw buyer" at a higher price. The straw buyer does not make the subsequent mortgage payments and the loan is foreclosed. There is also pure house stealing where con artists transfer the deed of a house into their name by obtaining the forms, forging signatures, and using fake IDs.

to select an equilibrium.

The main insight of our analysis is that the moral hazard problem induced by the possibility to produce fake assets generates an endogenous liquidity constraint. While it is feasible to transfer any quantity of one's asset holdings in a match, if the quantity offered is above a certain threshold, then the trade is rejected with some positive probability. Moreover, the probability that a trade goes through falls with the size of the trade. In equilibrium, agents never find it optimal to offer more asset than what can be accepted with certainty, which prevents fraudulent payments from taking place. The endogenous upper bound on the transfer of assets depends on the physical properties and the rate of return of the asset, trading frictions, and monetary policy. For instance, we establish that liquidity constraints are more likely to bind if the cost to produce fraudulent financial claims is low, trading frictions are large, the rate of return of the asset is low, or inflation is high.

We develop three applications of our model to illustrate the role played by endogenous liquidity constraints. We first investigate the implications of our model for asset pricing and liquidity premia by considering an economy with Lucas' (1978) trees. The model provides conditions for the emergence of liquidity premia. A liquidity premium emerges if there is a shortage of asset in the sense that neither the first-best level of consumption nor what is allowed by the liquidity constraint are achievable. Assets that are easier to counterfeit are less likely to exhibit a liquidity premium and are more likely to pay a higher rate of return. Finally, if the liquidity constraint is binding, then the liquidity premium of the asset increases with the cost to produce counterfeits and trading frictions.

Our second application focuses on how the threat of counterfeiting affects fiat monetary systems. We obtain the striking result that even though sellers can never recognize counterfeits from genuine money, the possibility to counterfeit currency does not affect the existence of a monetary equilibrium, and counterfeiting does not occur in equilibrium. However, a higher cost to produce counterfeits can raise the value of money and improve the allocations whenever the liquidity constraint binds. This justifies Central Bank's policy of spending resources to improve the recognizability of currency despite of counterfeiting being insignificant. The Friedman rule remains the optimal policy but it fails to implement the first-best allocation if the cost of counterfeiting is sufficiently small.

Our third application deals with an old question in monetary theory, the coexistence of assets with different rates of return. The purest form of the rate-of-return-dominance puzzle is represented by the coexistence of fiat money and interest-bearing, default-free, nominal bonds. We assume that only nominal bonds are threatened by fraudulent imitations. If the supply of bonds relative to the supply of money is low and the liquidity constraint on the use of bonds does not bind, then, as predicted by the rate-of-return-dominance puzzle, money and bonds are perfect substitutes and bonds do not pay interest. If the relative supply of bonds is large and the liquidity constraint binds, bonds pay interest, absent extraneous restrictions on their use as means of payment. The endogenous liquidity constraint enables an open-market purchase to lower nominal interest rates. Even so, open market operations are irrelevant since they have no effects on the allocation and welfare as long as the growth rate of money is not changed. Finally, our model rejects the Fisher hypothesis: the real interest rate depends on monetary factors.

1.1 Literature review

This paper shares the main theme of moral hazard considerations that has been used in previous studies to motivate credit or liquidity constraints. For instance, agents in Kiyotaki and Moore's (1997) environment can take the borrowed funds and run so that collateral constitutes a binding constraint on loans. Bernanke and Gertler (1989) consider a model with costly state verification where higher borrower net worth reduces the agency costs of acquiring funds to finance investments. Both papers provide important contributions in addressing the financial frictions on aggregate liquidity as an mechanism to amplify and propagate shocks to the economy. Kiyotaki and Moore (2001, 2005a, 2005b) assume limited commitments of the issuers and holders of private claims to justify constraints on debt issuance and resaleability of claims, both of which allows agents to generate funds up to an exogenous fraction of the capital or assets they hold. The moral hazard problem in Holmstrom and Tirole (1998, 2001, 2008) is described as follows. A risk-neutral entrepreneur would like to issue debt backed by an investment project. The probability of success depends on the entrepreneur's choice of where to invest the funds. There is an efficient technology that gives a high probability of success and an inefficient technology which gives a lower probability of success but provides the entrepreneur with a private benefit. There is an incentive constraint that induces

the entrepreneur to be diligent, just like our no-counterfeiting constraint. The distinction from the previous literature is that we emphasize the lack of recognizability of assets and we embody the moral hazard problem into a search-theoretic model which explicitly depicts the monetary considerations that matter for asset prices; moreover, we derive endogenously the liquidity constraints. Our approach is related to the literature on counterfeiting, which includes, e.g., Green and Weber (1996), Williamson (2002), Nosal and Wallace (2007), and Li and Rocheteau (2009).

Williamson and Wright (1994) were the first to introduce a moral hazard problem in the context of search-theoretic model of monetary exchange. Closely related to what we do, Rocheteau (2007) introduces an adverse selection problem in a monetary model with risk-free and risky assets. Our analysis illustrates that moral hazard and adverse selection problems involve different methodologies and have different implications for the form of the liquidity constraints. Lester, Postlewaite, and Wright (2008), on the other hand, focus on endogenizing the fraction of matches where the asset can be recognized to explain the liquidity of an asset.

Our model can also be viewed as providing microfoundations for some of the exogenously imposed liquidity constraints in the literature. For instance, as the cost of producing fraudulent claims goes to zero, agents stop trading the asset in uninformed matches, as in Lagos (2007) or Lester, Postlewaite, and Wright (2008). If the cost of producing fraudulent claims is not too large but the asset is abundant, then agents only spend a fraction of their asset holdings in all matches, as in Kiyotaki and Moore (2005b). Our asset pricing results complement those of Geromichalos, Licari and Suarez-Lledo (2007), without liquidity constraints, and Lagos (2007), with an exogenous liquidity constraint. Our application to the rate-of-return dominance puzzle is related to Bryant and Wallace (1979, 1980) and Aiyagari, Wallace, and Wright (1997) who emphasize costs of intermediation and legal restrictions.

2 Environment

Time is discrete, starts at $t = 0$, and continues forever. Each period has two subperiods, a morning where trades occur in a decentralized market (DM), followed by an afternoon where trades take place in competitive markets (CM). There is a continuum of infinitely lived agents divided into two types, called *buyers* and *sellers*, who differ in terms of when they produce and consume. The labels

buyers and *sellers* indicate agents' roles in the DM market. The measures of buyers and sellers are equal to 1. There are two consumption goods, one produced in the DM and the other in the CM. Consumption goods are perishable.

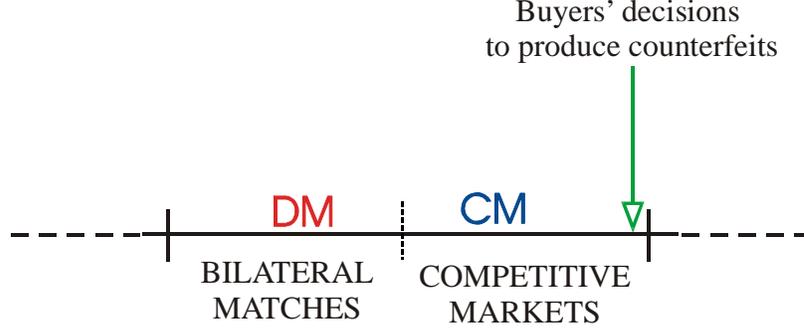


Figure 1: Timing of a representative period

Buyers and sellers are treated symmetrically in the CM: they can both produce and consume. In the DM, however, buyers only consume, while sellers only produce. The lifetime expected utility of a buyer from date 0 onward is

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t [u(q_t) + x_t - \ell_t], \quad (1)$$

where x_t is the CM consumption of period t , ℓ_t is the CM disutility of work, q_t is the DM consumption, and $\beta \in (0, 1)$ is a discount factor. The utility function $u(q)$ is twice continuously differentiable, $u(0) = 0$, $u'(q) > 0$, and $u''(q) < 0$. The production technology in the CM is linear, with labor as the only input, $y_t = \ell_t$.

The lifetime expected utility of a seller from date 0 onward is

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t [-c(q_t) + x_t - \ell_t], \quad (2)$$

where q_t is the DM production. The cost function $c(q)$ is twice continuously differentiable, $c(0) = 0$, $c'(q) > 0$, and $c''(q) \geq 0$. Let q^* denote the solution to $u'(q^*) = c'(q^*)$.

We will consider various versions of the model which differ in terms of which assets are available for trade. In the first version, agents trade Lucas' (1978) trees that yield a constant dividend flow

in terms of general goods. In the second version, the asset is fiat and has no intrinsic value. In the last version, agents can hold both fiat money and one-period nominal bonds.

In the CM, agents trade goods and assets competitively. In the DM, a fraction $\sigma \in (0, 1)$ of sellers are matched bilaterally and at random with a fraction σ of buyers. Trades are quid pro quo, and agents can transfer any asset they hold. Agents' portfolios are private information. Terms of trade are determined according to a simple bargaining game: The buyer makes an offer, which the seller accepts or rejects. If the offer is accepted, then the trade is implemented, provided that it is feasible given agents' portfolios. At the end of the DM, the matched pairs split apart. To establish an essential role for a medium of exchange, we assume no public record of individuals' trading histories.

The moral hazard problem is modeled as follows. If an asset is counterfeitable, then buyers in the CM can produce any quantities of counterfeit assets at a positive fixed cost. The utility cost of counterfeiting an asset is $k > 0$. The technology to produce counterfeits in period t becomes obsolete in period $t + 1$, so paying the cost only allows an agent to produce counterfeit assets in one period. In the DM a seller is not able to recognize the authenticity of an asset, and he does not observe the portfolio of the buyer he is matched with. Any counterfeit that would be traded in the CM is automatically detected and confiscated. Consequently, the only outlet for the counterfeit asset is the DM. Moreover, all the counterfeits produced in period t are confiscated by the government before agents enter the CM of period $t + 1$.

3 The counterfeiting game

We consider a simple counterfeiting game between a buyer and a seller chosen at random. This game starts in the CM of period $t - 1$ and ends in the CM of period t . The asset, which is perfectly divisible and durable, is traded in the CM of period t at the price ϕ_t , for all t . The asset generates $\zeta \geq 0$ units of general good at the beginning of the CM before the asset is traded. This game is general enough to accommodate different types of assets, including real or fiat assets, short-lived or long-lived assets. Agents hold no asset at the beginning of the game. Thanks to quasilinear preferences, this assumption is with no loss in generality.

The sequence of the moves are as follows: (i) In the CM of $t - 1$, the buyer chooses whether

or not to produce counterfeits; (ii) The buyer determines the quantity of CM-goods to produce in exchange for some genuine assets; (iii) During the next day, the buyer is matched with a seller with probability σ , and he makes an offer (q, d) , where q represents the output produced by the seller and d the transfer of asset (genuine or counterfeit) from the buyer to the seller; (iv) The seller decides whether to accept the offer.³

Since the production of counterfeits involves only a fixed cost, counterfeiting can be described as a binary action, $\chi \in \{0, 1\}$. If $\chi = 0$, then the buyer produces no counterfeit, while if $\chi = 1$ the buyer produces any quantity of counterfeits that is needed to fulfill his offer in the DM. The sequential structure of the game is illustrated in the game tree of Figure 2. An arc of circle indicates that the action set at a given node is infinite, while a dotted line represents an information set. (We omit the move by Nature that determines whether a buyer in the DM is matched or not, i.e., the game tree is represented for the case where $\sigma = 1$.)

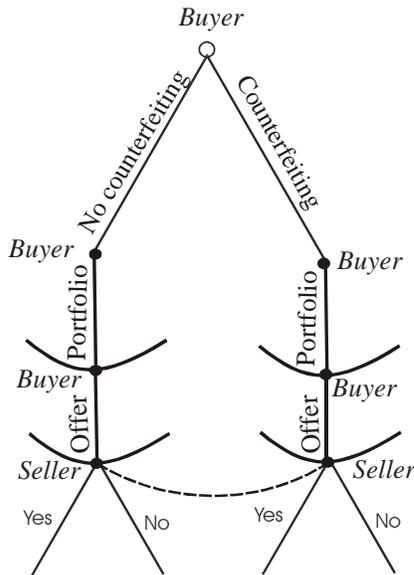


Figure 2: Original game tree

³A timing sequence that would be equally plausible is that buyers choose first their holdings of genuine assets and then whether or not to produce counterfeits; or the two decisions could be taken simultaneously. Our equilibrium notion will be immune against the timing of buyers' moves. Also, while we assume that asset holdings are private information, our analysis goes through even if we had assumed instead that asset holdings are common-knowledge in the match (see Appendix D).

A pure strategy of the buyer in the counterfeiting game is a list $\langle \chi, a(\chi), o \rangle$ that specifies the decision to produce counterfeits, χ , the holdings of genuine asset as a function of χ , $a : \{0, 1\} \rightarrow \mathbb{R}_+$, and a mapping $o : \mathbb{R}_+ \times \{0, 1\} \rightarrow \mathbb{R}_{2+}$ that generates an offer for all histories (χ, a) . A pure strategy for the seller is an acceptance rule $\mu : \mathbb{R}_{2+} \rightarrow \{0, 1\}$ that specifies whether a given offer is accepted ($\mu = 1$) or rejected ($\mu = 0$). In the following we will allow agents to play behavioral strategies. The Bernoulli payoff of the buyer in the counterfeiting game is

$$U_t^b(a, \chi, q, d, \mu) = -k\mathbb{I}_{\{\chi=1\}} - \phi_{t-1}a + \beta \{u(q) - (\phi_t + \zeta)d\mathbb{I}_{\{\chi=0\}}\} \mathbb{I}_{\{\mu=1\}} + \beta(\phi_t + \zeta)a, \quad (3)$$

where \mathbb{I}_Z is an indicator function equal to one if property Z holds, and feasibility requires $d \leq a$ if $\chi = 0$. (We restrict buyers to make feasible offers given their own asset holdings.) If the buyer chooses to produce counterfeits ($\chi = 1$), then he incurs the fixed cost k . In order to hold a units of genuine asset, he must produce $\phi_{t-1}a$ units of the CM good in period $t - 1$. In the subsequent period the buyer enjoys the utility of consumption, $u(q)$, and he gives up d units of asset provided that the offer is accepted ($\mu = 1$). A unit of asset in the DM is worth $\phi_t + \zeta$ units of general good since the asset generates a dividend ζ and can be resold in the CM at the ex-dividend price ϕ_t . The transfer of asset reduces the buyer's payoff only if the units of asset transferred are genuine, i.e., the buyer did not produce counterfeits in $t - 1$. If the buyer is unmatched, $q = d = 0$.

Similarly, the Bernoulli payoff of the seller is

$$U_t^s(\chi, q, d, \mu) = \beta \{-c(q) + (\phi_t + \zeta)d\mathbb{I}_{\{\chi=0\}}\} \mathbb{I}_{\{\mu=1\}}. \quad (4)$$

We assume that sellers do not accumulate assets in the CM of $t - 1$. It is easy to show that they have no incentives to do so if $\phi_{t-1} > \beta(\phi_t + \zeta)$ (i.e., when the rate of return of the asset is less than the discount rate), since the seller's asset holdings are not observable and hence do not affect the terms of trade offered by the buyer. If the seller in the DM accepts the buyer's offer, $\mu = 1$, he suffers the disutility of producing, $c(q)$, and receives d units of asset. Each unit of asset is worth $\phi_t + \zeta$ unit of the CM good provided that the buyer did not produce counterfeits, $\chi = 0$.

A sequential equilibrium of the counterfeiting game is a pair of (behavioral) strategies that satisfy sequential rationality and consistency of beliefs with strategies.⁴ In order to check the

⁴A sequential equilibrium of an extensive game with imperfect information is composed of a profile of behavioral

sequential rationality of the seller's acceptance rule, one needs to specify the seller's belief regarding the buyer's action, χ , conditional on the offer (q, d) being made. Sequential equilibrium imposes little discipline on those beliefs, which can lead to a plethora of equilibria. For instance, any offer that satisfies $-\phi_{t-1}d + \beta\sigma\{u(q) - (\phi_t + \zeta)d\} + \beta(\phi_t + \zeta)d \geq 0$ is part of an equilibrium in which all other offers are attributed to counterfeiters and hence are rejected (since counterfeited claims are confiscated, and hence are valueless). However, belief systems where all offers except one are accepted are clearly unappealing. For instance, consider the offers (q, d) such that

$$-\phi_{t-1}d + \beta\{\sigma u(q) + (1 - \sigma)(\phi_t + \zeta)d\} > -k + \beta\sigma u(q). \quad (5)$$

The left side of (5) is the expected payoff of a genuine buyer who accumulates d units of asset in order to consume q units of output in the DM. The right side of (5) is the expected payoff from the same offer (q, d) if the buyer is a counterfeiter. If (5) holds, then it is easy to show that buyers prefer to accumulate genuine assets instead of producing counterfeits irrespective of sellers' acceptance rule. Hence, accumulating genuine assets and offering (q, d) dominates producing counterfeits and offering (q, d) . Provided that the seller does not believe that the buyer would play dominated strategies, such offers should be attributed to genuine buyers. More generally, when observing an offer (q, d) , a seller might want to infer who (a genuine buyer or a counterfeiter) had incentives to make such an offer, given sellers' acceptance rule. And the sellers' acceptance rule needs to be optimal given buyers' incentives. We will capture this forward induction logic in the following.

We adopt a notion of strategic stability according to which any equilibrium of the game represented in Figure 2 should also be an equilibrium of strategically equivalent games. Given that there is no strict guideline regarding the order of the buyers' moves, the advantage of using the notion of strategic stability lies in the fact that the outcome would not depend on some strategically irrelevant details of the model such as the timing of the buyer's moves. We will consider in the following the reverse-ordered game where the buyer makes first an offer (q, d) ; e.g., the buyer writes his offer in a sealed envelope before making any choice in the CM, and then he chooses whether to produce a counterfeit (See Figure 3).⁵ Note that the order of the buyers' moves does not affect the payoffs,

strategies and a belief system such that strategies are sequentially rational given the belief system, and the belief system is consistent with the strategies. For a definition of the consistency requirement see Osborne and Rubinstein (1994, Definition 224.2).

⁵This methodology, called the *reordering invariance refinement*, was developed by In and Wright (2008) for

that are given by (3) and (4) in both games, and it does not convey any information to the seller, since in both games the seller only observes the offer, irrespective of the order of the buyer's moves. The benefit from considering this reverse-ordered game is that it captures the forward-induction logic previously described and subgame perfection is sufficient to solve the game. Moreover, as we will see, it predicts a unique outcome, which is also an outcome of the original game.

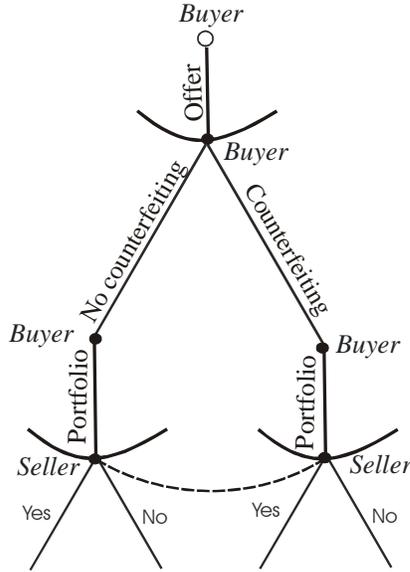


Figure 3: Reverse ordered game

The timing in the reverse-ordered game is as follows. First, the buyer determines his DM offer (e.g., he posts an offer in the CM of $t - 1$ for the DM of t); Second, he decides whether or not produce counterfeit assets; Third, he chooses how many genuine assets to accumulate; Fourth, the seller accepts or rejects the offer. We restrict the strategy space of the buyer so that following an offer (q, d) and a decision to counterfeit χ , the buyer's choice of asset holdings must be such that $a \geq d$ if $\chi = 0$. This condition says that the buyer must always be able to execute the offer chosen at the beginning of the game.⁶

signaling games with unobservable choices. It is based on the invariance condition of strategic stability from Kohlberg and Mertens (1986). It states that the solution of a game should also be the solution of any game with the same reduced normal form.

⁶This assumption eliminates equilibria where sellers would reject an offer simply because they believe that buyers do not have enough assets to execute this offer.

A behavioral strategy of the buyer in the reverse ordered game is a triple $\langle \mathbb{F}, \eta(q, d), \mathbb{G}(q, d, \chi) \rangle$ where \mathbb{F} is the distribution from which the buyer draws his offer, $1 - \eta$ is the probability that the buyer produces counterfeits conditional on the offer (q, d) being made, \mathbb{G} is the distribution for the choice of asset holdings conditional on the history $((q, d), \chi)$. The next lemma shows that buyers accumulate genuine assets or produce counterfeits but not both. Let $\delta_{\{x\}}$ denote the Dirac measure that assigns unit measure to the singleton $\{x\}$.

Lemma 1 *Assume that $\phi_{t-1} > \beta(\phi_t + \zeta)$. Any optimal strategy of the buyer is such that the distribution of probabilities for the choice of asset holdings obeys $\mathbb{G}((q, d), 1) = \delta_{\{0\}}$ and $\mathbb{G}((q, d), 0) = \delta_{\{d\}}$ for all offers (q, d) .*

Under the assumption $\phi_{t-1} > \beta(\phi_t + \zeta)$ it is costly to hold the asset: the rate of return of the asset is less than the discount rate. As a consequence, buyers will never find it optimal to bring more asset than what they intend to spend. Moreover, if buyers incur the fixed cost to produce counterfeits, they have no incentives to accumulate genuine asset holdings, $\mathbb{G}((q, d), 1) = \delta_{\{0\}}$. In the case where $\phi_{t-1} = \beta(\phi_t + \zeta)$ the choice of asset holdings of the buyer is payoff irrelevant provided that the buyer holds at least what he intends to spend in the DM (d units if he is a genuine buyer and 0 unit if he is a counterfeiter). From Lemma 1, we can reduce the buyer's strategy to a pair of distribution of offers and probability to produce counterfeits, $\langle \mathbb{F}, \eta(q, d) \rangle$, since the choice of asset holdings can be inferred from the offer and the decision to counterfeit.

The game is solved by backward induction. We first take as given the terms of trade in a match, (q, d) . Given these terms of trade, we look for a Nash equilibrium of the game where the buyer chooses to accumulate genuine assets or produce counterfeits to execute the transfer specified by the offer, and the seller decides whether to accept or reject the offer. Let $\pi \in [0, 1]$ denote the probability that a seller accepts the offer (q, d) and $\eta \in [0, 1]$ the probability that a buyer chooses to accumulate genuine asset instead of producing counterfeits. Given η , the decision of sellers to accept or reject an offer satisfies

$$\begin{array}{rcl}
 & > 0 & = 1 \\
 -c(q) + \eta(\phi_t + \zeta)d < 0 & \implies \pi & = 0 \\
 & = 0 & \in [0, 1]
 \end{array} \quad . \quad (6)$$

The seller must be compensated for his disutility of producing q units of output. When evaluating the expected value of the transfer of asset, the seller takes into account the probability that he faces an honest buyer, which is given by η . With probability η the assets are genuine and worth $\phi_t + \zeta$ units of output each; with complementary probability, $1 - \eta$, they are counterfeits and worth nothing (since they are confiscated before the CM of period t).

Given π , a buyer is willing to accumulate genuine assets under the offer (q, d) if

$$-\phi_{t-1}d + \beta \{ \sigma \pi [u(q) - (\phi_t + \zeta) d] + (\phi_t + \zeta) d \} \geq -k + \beta \sigma \pi u(q). \quad (7)$$

Equation (7) has a similar interpretation as (5). Simplifying (7), the decision rule to produce counterfeits is given by

$$\left\{ \begin{array}{l} < & = 1 \\ \{ [\phi_{t-1} - \beta (\phi_t + \zeta)] + \beta \sigma \pi (\phi_t + \zeta) \} d > k \implies \eta & = 0 \\ = & \in [0, 1] \end{array} \right. \quad (8)$$

The left side of (8) reveals two gains from producing counterfeits: The first term represents the cost due to the difference between the purchase price of the asset in $t - 1$ and the discounted resale price in t that a counterfeiter avoids by not holding genuine assets, and the second term is the saving of the cost of transferring genuine units of asset if he finds a seller who accepts the offer.

A Nash equilibrium of the subgame following the offer (q, d) is a pair $(\pi, \eta) \in [0, 1]^2$ that satisfies (6) and (8). In the following we review the Nash equilibria such that $\pi > 0$. (Any offer such that $\pi = 0$ is equivalent to the no-trade offer.) From (6) and (8) an equilibrium where the offer is accepted and buyers do not produce counterfeits, $(\pi, \eta) = (1, 1)$, requires

$$c(q) \leq (\phi_t + \zeta)d \leq \frac{k (\phi_t + \zeta)}{\phi_{t-1} - \beta(1 - \sigma) (\phi_t + \zeta)}. \quad (9)$$

According to (9), the transfer of asset must be sufficiently large to compensate the seller for his disutility of work, but it must not be too high to give buyers' incentives to produce counterfeits. There are Nash equilibria where the offer is partially accepted and some counterfeiting takes place, i.e., $(\pi, \eta) \in (0, 1)^2$. From (6) and (8),

$$\eta = \frac{c(q)}{(\phi_t + \zeta)d}, \quad (10)$$

$$\pi = \frac{k - [\phi_{t-1} - \beta (\phi_t + \zeta)] d}{\beta \sigma (\phi_t + \zeta) d}. \quad (11)$$

The condition $\eta \in (0, 1)$ implies $c(q) < (\phi_t + \zeta) d$. The condition $\pi \in (0, 1)$ implies $(\phi_t + \zeta) d \in \left(\frac{k(\phi_t + \zeta)}{\phi_{t-1} - (1-\sigma)\beta(\phi_t + \zeta)}, \frac{k(\phi_t + \zeta)}{\phi_{t-1} - \beta(\phi_t + \zeta)} \right)$. If the real value of the asset transfer is neither too small nor too large, and if the output is sufficiently low relative to the asset transfer, then buyers choose to produce counterfeits with positive probability and the offer is rejected by sellers with positive probability. According to (10) a buyer is more likely to produce counterfeits if he offers a large transfer of asset and if he asks for little output. According to (11) an offer is more likely to be rejected if it involves a large transfer of asset. There are Nash equilibria where the offer is always accepted, but some buyers produce counterfeits, $\pi = 1$ and $\eta < 1$. This is the case if $[\phi_{t-1} - \beta(1 - \sigma)(\phi_t + \zeta)] d = k$ and $-c(q) + \eta(\phi_t + \zeta)d \geq 0$. Finally, sellers can reject some offers even if there is no counterfeiting, i.e., $\pi \in (0, 1)$ and $\eta = 1$. This is the case if $c(q) = (\phi_t + \zeta) d$ and $(\phi_t + \zeta) d \leq \frac{k(\phi_t + \zeta)}{\phi_{t-1} - \beta(\phi_t + \zeta)}$. The different Nash equilibria of the subgame following an offer (q, d) are represented in Figure 4.

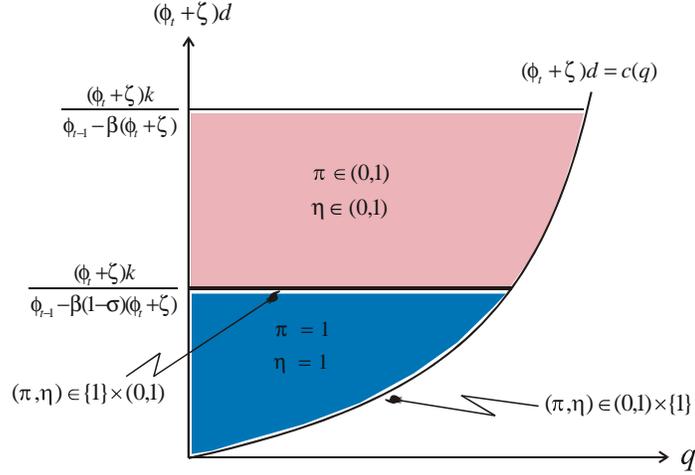


Figure 4: Equilibria of the subgame following (q, d)

Next, we move upward in the game tree to determine the offer(s) (q, d) proposed by the buyer at the beginning of the game. The offer made by the buyer solves

$$(q, d) \in \text{supp}(\mathbb{F}) \subseteq \arg \max \left\{ -k [1 - \eta(q, d)] - [\phi_{t-1} - \beta(\phi_t + \zeta)] \eta(q, d) d + \sigma \beta [u(q) - \eta(q, d) (\phi_t + \zeta) d] \pi(q, d) \right\}, \quad (12)$$

where $[\eta(q, d), \pi(q, d)]$ is an equilibrium of the subgame following the offer (q, d) .

An equilibrium of the counterfeiting game is a list of strategies, $[\eta(q, d), \pi(q, d)]$, for all the subgames following an offer, and a distribution of offers, \mathbb{F} , that satisfy (6), (8), and (12).

Proposition 1 (*Endogenous liquidity constraints*)

The equilibrium offer solution to (12) is such that $\pi = 1$ and $\eta = 1$, and it satisfies

$$(q, d) \in \arg \max \left\{ - [\phi_{t-1} - \beta(\phi_t + \zeta)] d + \sigma\beta [u(q) - (\phi_t + \zeta) d] \right\} \quad (13)$$

$$s.t. \quad -c(q) + (\phi_t + \zeta) d \geq 0, \quad (14)$$

$$d \leq \frac{k}{\phi_{t-1} - \beta(1 - \sigma)(\phi_t + \zeta)}. \quad (15)$$

Following an offer (q, d) , the seller's belief that he is facing a genuine buyer is given by $\eta(q, d)$ and his decision to accept the offer is $\pi(q, d)$ where $[\eta(q, d), \pi(q, d)]$ is an equilibrium of the subgame following the offer (q, d) . Provided that (14) holds, the decision of a seller to accept an offer is

$$\pi(q, d) = \min \left\{ \frac{\{k - [\phi_{t-1} - \beta(\phi_t + \zeta)] d\}^+}{\beta\sigma(\phi_t + \zeta) d}, 1 \right\},$$

where $\max(x, 0) = \{x\}^+$. The seller's probability to accept an offer decreases in the transfer d , a measure of the size of trade. This is related to the notion that larger trades are more costly and take more time to implement than smaller ones. A standard explanation in the market micro-structure literature (e.g., Easley and O'Hara, 1987) is that informed traders want to maximize the value of their inside information by trading large quantities. Similarly, in our model the chance for a larger trade to be implemented is smaller because they are more likely to come from opportunistic buyers. In equilibrium, buyers never find it optimal to make an offer that has a positive chance to get rejected.

The program (13)-(15) that determines the equilibrium outcome is similar to the one in the monetary models of Lagos and Rocheteau (2008), Geromichalos, Licari, and Suarez-Lledo (2007), Lester, Postlewaite and Wright (2007) and Lagos (2007), except that it incorporates an endogenous liquidity constraint, (15), that specifies an upper bound on the transfer of assets.⁷ The endogenous liquidity constraint (15) depends on the cost of producing counterfeits, the rate of return of the

⁷One can see from equation (5) that the result of an upper bound on the transfer of assets is robust to alternative productive technologies for producing counterfeits, provided that the average cost decreases with the amounts of counterfeits.

asset, agents' patience, and the extent of the search frictions. This shows that liquidity constraint are not invariant to the characteristics and physical properties of the asset and to the frictions in the environment. The higher the cost of producing counterfeits, the less likely the liquidity constraint (15) will be binding. For instance, if there are no search frictions, $\sigma = 1$, and the asset price is constant, $\phi_t = \phi_{t-1}$, the upper bound on the transfer of asset is exactly equal to k , i.e., $\phi d \leq k$.⁸ Starting from $\sigma < 1$, if the search frictions are reduced, the upper bound on the transfer of asset is lowered. Reducing trading frictions exacerbates the moral hazard problem because the trade surplus of a counterfeiter, $u(q)$, is greater than the match surplus of a genuine buyer, $u(q) - c(q)$; i.e., the payoff of the counterfeiter increases by more than the payoff of the genuine buyer. Finally, the upper bound on the transfer of asset is an increasing function of the rate of return of the asset, $\frac{\phi_t + \zeta}{\phi_{t-1}}$. As the rate of return of the asset decreases, the cost of holding the asset is higher, which raises buyer's incentives to produce counterfeits for a given size of the trade. The rate of return of the asset will depend on the extent to which the asset can be used in the DM, and will be endogenized in the next section.

4 Liquidity premia⁹

To illustrate the implications of the model for how liquidity considerations matter for asset prices, we consider an asset similar to a Lucas' (1978) tree. Each tree generates a constant flow of general goods, $\zeta > 0$, and there is a fixed supply, A , of trees. One can think of agents as trading claims on trees. Counterfeiting in this context means that agents have the possibility to produce fake claims or produce claims on unproductive trees.¹⁰

The counterfeiting game described in Section 3 is repeated in every period. Because of quasi-linear preferences, the choice of asset holdings of a buyer in the CM and his subsequent actions in the game are independent of the buyer's wealth and his history, which is private information, when

⁸In the absence of search frictions, our upper bound on the transfer of assets is reminiscent to the pledgeable income in Holmstrom and Tirole (1998, 2008).

⁹This section extends the discussion in Rocheteau (2008) to allow for long-lived assets and search frictions.

¹⁰This description is consistent with the circulation of banknotes in the 19th century, where counterfeits of genuine notes coexisted with genuine notes issued by bad banks, banks with few specie on hand to redeem their outstanding notes (see Mihm 2007). Also, while we consider the case where the productive asset is in fixed supply, we could alternatively consider a situation where capital is produced, as in Lagos and Rocheteau (2008).

he enters the CM. The lifetime utility of a buyer upon entering the CM of period $t - 1$ with a units of asset is

$$W_{t-1}^b(a) = (\phi_{t-1} + \zeta)a + T + U_t^b + \beta W_t^b(0). \quad (16)$$

According to (16) upon entering the CM the buyer enjoys a dividend flow ζ per unit of asset he owns, he can sell his a units of asset at the competitive price ϕ_{t-1} , he receives a lump-sum transfer T (in term of the general good), and enjoys the payoff U_t^b from the counterfeiting game and the continuation value in period- t CM, $W_t^b(0)$. We consider $T = 0$ in this section, and $T > 0$ in the application of fiat money of next section. Similarly, since sellers get no surplus in the DM and have no strict incentives to hold the asset across periods, the expected lifetime utility of a seller is $W_{t-1}^s(a) = (\phi_{t-1} + \zeta)a$.

We will focus on stationary equilibria where the price of the asset is constant over time, $\phi_{t-1} = \phi_t = \phi$. We denote $\phi^* = \zeta/r$ the fundamental price of the asset as defined by the discounted sum of its dividends, and $R = \frac{\phi + \zeta}{\phi}$ the (gross) rate of return of the asset. From (13)-(15) the buyer's problem can be rewritten as

$$(q, d) \in \arg \max \{-r(\phi - \phi^*)d + \sigma[u(q) - (\phi + \zeta)d]\} \quad (17)$$

$$\text{s.t.} \quad -c(q) + (\phi + \zeta)d \geq 0, \quad (18)$$

$$d \leq \frac{k}{\phi - \beta(1 - \sigma)(\phi + \zeta)}. \quad (19)$$

According to (17) the cost of holding the asset in a stationary equilibrium is the difference between the price of the asset and its fundamental value in flow terms. It is clear that $\phi \geq \phi^*$ for the buyer's problem to have a (bounded) solution. In order to determine the market-clearing price we characterize the correspondence for the aggregate demand for the asset, $A^d(\phi)$.

Lemma 2 *For all $\phi \geq \phi^*$, the correspondence $A^d(\phi)$ is non-empty and upper-hemi continuous.*

1. *If $\phi = \phi^*$, then*

$$A^d(\phi^*) = \left[\min \left(\frac{c(q^*)}{\phi^* + \zeta}, \frac{k}{\sigma\phi^*} \right), \infty \right].$$

2. *If $\phi > \phi^*$, then $A^d(\phi)$ is single-valued and it is decreasing in ϕ .*

The asset demand correspondence is illustrated in Figure 5 for the case $\frac{k}{\sigma\phi^*} < \frac{c(q^*)}{\phi^*+\zeta}$. The market clearing condition requires $A \in A^d(\phi)$. A stationary equilibrium is defined as a triple (q, d, ϕ) where (q, d) solves (17)-(19) and ϕ is such that $A \in A^d(\phi)$.

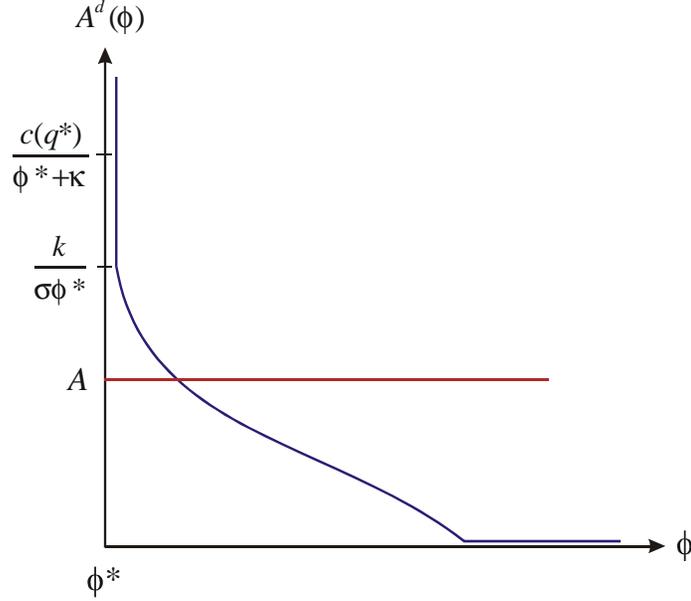


Figure 5: Asset demand correspondence

Proposition 2 (Liquidity and allocations)

There exists a unique equilibrium.

1. If $k \geq \frac{\sigma c(q^*)}{1+r}$ and $A \geq \frac{c(q^*)}{\phi^*+\zeta}$, then $q = q^*$, $d = \frac{c(q^*)}{\phi^*+\zeta}$.
2. If $k < \frac{\sigma c(q^*)}{1+r}$ and $A \geq \frac{k}{\sigma\phi^*}$, then $q = c^{-1}\left(\frac{(1+r)k}{\sigma}\right) < q^*$ and $d = \frac{rk}{\sigma\zeta}$.
3. If $A < \min\left(\frac{c(q^*)}{\phi^*+\zeta}, \frac{k}{\sigma\phi^*}\right)$, then $q < q^*$ and $d = A$.

If there is no shortage of the asset, in the sense that the supply of the asset is sufficiently large to allow agents to trade the socially-efficient quantity of output in the DM, $A \geq \frac{c(q^*)}{\phi^*+\zeta}$, and if the cost to produce counterfeits is sufficiently high so that the liquidity constraint does not bind, $k \geq \frac{\sigma c(q^*)}{1+r}$, then the economy achieves the first best allocation. If the asset is abundant but the moral hazard

problem is severe, $k < \frac{\sigma c(q^*)}{1+r}$, the liquidity constraint is binding, which reduces the output with respect to the first-best benchmark. In this case buyers spend an endogenous fraction, $\theta = \frac{rk}{\sigma \zeta A}$, of their asset holdings. The turnover of the asset increases with the cost to produce counterfeits but decreases with the ease with which the asset can be traded. In the limiting case where fraudulent claims on the asset can be produced at no cost, the asset ceases to be traded in the DM and the market shuts down.¹¹ Finally, if the shortage of asset is to such an extent that neither the first-best quantities nor what is allowed by the liquidity constraint at the fundamental price of the asset are achievable, then buyers spend all their asset holdings in the DM, $\theta = 1$, and output is inefficiently low, $q < q^*$.

Proposition 3 (Liquidity premia)

1. If $A \geq \min\left(\frac{c(q^*)}{\phi^* + \zeta}, \frac{k}{\sigma \phi^*}\right)$, then $\phi = \phi^*$, and $R = \beta^{-1}$.
2. If $A < \min\left(\frac{c(q^*)}{\phi^* + \zeta}, \frac{k}{\sigma \phi^*}\right)$, then $\phi = \min(\phi^u, \phi^c) > \phi^*$, and $R < \beta^{-1}$, where

$$\frac{r(\phi^u - \phi^*)}{\phi^u + \zeta} = \sigma \left[\frac{u' \circ c^{-1}((\phi^u + \zeta)A)}{c' \circ c^{-1}((\phi^u + \zeta)A)} - 1 \right], \quad (20)$$

and

$$\phi^c = \frac{\frac{k}{A} + \beta(1 - \sigma)\zeta}{1 - \beta(1 - \sigma)}. \quad (21)$$

A liquidity premium emerges if there is not enough asset to buy either the first-best level of consumption or the maximum allowed by the liquidity constraint at the fundamental price of the asset, $A < \min\left(\frac{c(q^*)}{\phi^* + \zeta}, \frac{k}{\sigma \phi^*}\right)$. In this case, the measured rate of return of the asset is below the rate of time preference. The asset generates a “convenience yield” because the marginal unit of the asset held by buyers serves as means of payment in the DM. As revealed by (20) and (21), the expression for the asset price can take two forms depending on whether or not the liquidity constraint binds. In both cases, and in contrast with frictionless asset pricing models, the liquidity premium depends negatively on the supply of the asset. If the liquidity constraint binds and buyers spend all their asset holdings in the DM, then the asset price increases with the cost to produce counterfeits. Moreover, when the liquidity constraint binds, the asset is priced at $\phi^c < \phi^u$; this

¹¹This limiting case was studied by Freeman (1983) and Lester, Postlewaite, and Wright (2007, 2008).

indicates that the moral hazard problem tends to lower the asset price and increase the measured return. Suppose that initially $A < \min\left(\frac{c(q^*)}{\phi^* + \zeta}, \frac{k}{\sigma\phi^*}\right)$ so that the asset price is above its fundamental value. If an advance in the technology makes it less costly to produce imitations of the asset, i.e., k falls below $\sigma\phi^*A$, then the liquidity premium of the asset price bursts and its rate of return increases.¹²

Our model has also implications for the relationship between trading impediments and asset prices.¹³

Proposition 4 (*Asset prices and trading frictions*)

If $A < \min\left(\frac{c(q^*)}{\phi^* + \zeta}, \frac{k}{\phi^*}\right)$ and

$$\frac{r\left(\frac{k}{A} - \phi^*\right)}{\frac{k}{A} + \zeta} < \frac{u' \circ c^{-1}(k + \zeta A)}{c' \circ c^{-1}(k + \zeta A)} - 1, \quad (22)$$

then there is $\bar{\sigma} < 1$ such that:

1. For all $\sigma < \bar{\sigma}$, $\phi = \phi^u$ and $\frac{\partial\phi}{\partial\sigma} > 0$.
2. For all $\sigma > \bar{\sigma}$, $\phi = \phi^c$ and $\frac{\partial\phi}{\partial\sigma} < 0$.

Proposition 4 shows that there is a non-monotonic relationship between asset prices and trading frictions. For low values of σ , the liquidity constraint does not bind because the difficulty to meet a trading partner in the DM reduces buyers' incentives to produce counterfeits. In that case, if buyers meet sellers more frequently in the DM, then the value of holding a marginal unit of asset goes up. However, above a threshold for σ , the liquidity constraint binds and the asset price falls. This also suggests that moral hazard considerations may matter more when the market is more liquid, in the sense that the turnover of the asset is high.

¹²This result is in accordance with the analysis from the Federal Bureau of Investigation (http://www.fbi.gov/publications/financial/fcs_report2007/financial_crime_2007.htm#Mortgage) for mortgage backed securities:

If fraudulent practices become systemic within the mortgage industry and mortgage fraud is allowed to become unrestrained, it will ultimately place financial institutions at risk and have adverse effects on the stock market. Investors may lose faith and require higher returns from mortgage backed securities.

¹³There is an extensive literature on transaction costs and asset prices. In the context of markets with search frictions, the relationship between asset prices and trading frictions is explored in Duffie, Garleanu, and Pedersen (2005) and Weill (2008).

4.1 Extension with multiple assets

The model can be extended to the case of multiple assets (as shown in Appendix C). Suppose, for instance, that there are two assets, labelled 1 and 2, both of which are subject to the moral hazard problem. Then, the determination of the terms of trade in the DM is given by

$$(q, d_1, d_2) \in \arg \max \left\{ -r \sum_{i=1}^2 (\phi_i - \phi_i^*) d_i + \sigma \left[u(q) - \sum_{i=1}^2 (\phi_i + \zeta_i) d_i \right] \right\} \quad (23)$$

$$\text{s.t.} \quad -c(q) + \sum_{i=1}^2 (\phi_i + \zeta_i) d_i \geq 0, \quad (24)$$

$$d_i \leq \frac{k_i}{\phi_i - \beta(1 - \sigma)(\phi_i + \zeta_i)}, \quad i = 1, 2. \quad (25)$$

>From (25) each asset is subject to a liquidity constraint which depends on the properties of the asset. If the liquidity constraints are not binding, then the two assets will have the same rate of return. If at least one of the liquidity constraint binds, then both assets can have different rates of return, and this rate of return differential depends on the physical properties of the asset (k_1 and k_2) as well as the supplies of the assets and trading frictions. We will elaborate more on this point in Section 6.

5 Counterfeiting and the value of money

In this section we apply the counterfeiting game discussed in Section 3 to an intrinsically useless object ($\zeta = 0$), fiat money. We study how the moral hazard problem affects the value of money and the conduct of monetary policy.

The supply of money, M_t , is growing at a constant growth rate $\gamma \equiv \frac{M_{t+1}}{M_t} > \beta$. We focus on stationary equilibria where the real value of money is constant over time, $\phi_{t+1}M_{t+1} = \phi_tM_t$. Consequently, the rate of return of money is $\frac{\phi_{t+1}}{\phi_t} = \gamma^{-1} < \beta$. Since it is costly to hold the asset, all buyers will hold the quantity that they expect to spend in the DM, and this quantity is the unique

solution to the buyer's problem in Proposition 1, i.e.,

$$(q, d) \in \arg \max \{-\nu\phi_t d + \sigma [u(q) - \phi_t d]\} \quad (26)$$

$$\text{s.t.} \quad -c(q) + \phi_t d \geq 0, \quad (27)$$

$$\phi_t d \leq \frac{k}{\beta(\nu + \sigma)}, \quad (28)$$

where $\nu = \frac{\gamma - \beta}{\beta}$ is the cost of holding real balances. One novelty with respect to the previous section is that policy, through the growth rate of money supply, has a direct effect on the liquidity constraint, (28). If the money growth rate increases, then it is more costly to hold genuine money and, as a consequence, the liquidity constraint on the transfer of real balances becomes more restrictive.

The value of money, ϕ_t , is determined by the market-clearing condition in the CM according to which

$$a_t = d_t = M_t. \quad (29)$$

A stationary equilibrium is then a list $\langle q, \{d_t\}_{t=1}^{\infty}, \{\phi_t\}_{t=1}^{\infty} \rangle$ that solves (26)-(28) and (29). The equilibrium is monetary if $\phi_t > 0$ at all dates.

Proposition 5 *(The threat of counterfeiting and the value of money)*

There exists a monetary equilibrium if and only if

$$\frac{u'(0)}{c'(0)} > 1 + \frac{\nu}{\sigma}. \quad (30)$$

Let $\bar{k} = \beta(\nu + \sigma)c(\hat{q})$ where $\frac{u'(\hat{q})}{c'(\hat{q})} = 1 + \frac{\nu}{\sigma}$.

1. If $k \geq \bar{k}$ then

$$\frac{u'(q)}{c'(q)} = 1 + \frac{\nu}{\sigma}, \quad (31)$$

$$\phi_t = \frac{c(q)}{M_t}. \quad (32)$$

2. If $k < \bar{k}$ then

$$q = c^{-1}\left(\frac{k}{\beta(\nu + \sigma)}\right), \quad (33)$$

$$\phi_t = \frac{k}{M_t\beta(\nu + \sigma)}. \quad (34)$$

Proposition 5 shows the following remarkable result: fiat money can be valued even though sellers do not have the technology to distinguish genuine units of money from counterfeits. The possibility of counterfeiting does not threaten the existence of a monetary equilibrium: the cost of producing counterfeits, $k > 0$, is absent from (30). In particular, if the Inada conditions hold, $\frac{u'(0)}{c'(0)} = +\infty$, then a monetary equilibrium always exists.¹⁴

Proposition 5 does not imply that the lack of recognizability of the currency is innocuous. As shown in the next Corollary, the mere possibility of counterfeiting affects the equilibrium allocations even if there is no counterfeiting in equilibrium, provided the fixed cost of producing counterfeits is not too large.

Corollary 1 (*Trading frictions, moral hazard, and the value of money*)

1. For all $k < \bar{k}$,

$$\frac{\partial q}{\partial k} > 0, \quad \frac{\partial \phi_t}{\partial k} > 0, \quad \frac{\partial q}{\partial \sigma} < 0, \quad \frac{\partial \phi_t}{\partial \sigma} < 0.$$

2. For all $k > \bar{k}$, then q and ϕ are independent of k , and $\frac{\partial q}{\partial \sigma} > 0$, $\frac{\partial \phi_t}{\partial \sigma} > 0$.

If the threat of counterfeiting is binding ($k < \bar{k}$), then an increase in the cost of producing counterfeits raises output and the value of money. Policies that make it harder to counterfeit fiat money, such as the use of special paper and ink, and the frequent redesign of the currency, have real effects even though no counterfeiting takes place. As in Proposition 4, a reduction in the trading frictions lowers output and the value of money. This result is in contrast with the positive relationship between ϕ_t and σ when the liquidity constraint is not binding. Intuitively, as the trading frictions are reduced, a potential counterfeiter has a higher chance to pass a counterfeit, which raises the incentives of an opportunistic behavior.

We now ask whether the optimal monetary policy is affected by the threat of counterfeiting. We measure social welfare as the discounted sum of the match surpluses in the DM,

$$\mathcal{W} = \sigma \frac{u(q) - c(q)}{1 - \beta}. \tag{35}$$

¹⁴Using models of commodity money, Li (1995) found that a commodity that is subject to the recognizability problem can still serve as a media of exchange due to the low storability cost. However, this result is in contrast with the findings in Nosal and Wallace (2007, Proposition 2) that the set of parameter values under which fiat money is valued shrinks as the cost of producing counterfeits increases.

Proposition 6 (*Optimal monetary policy and the threat of counterfeiting*)

Suppose (30) holds. Then, $\frac{\partial q}{\partial \nu} < 0$ and $\frac{\partial \mathcal{W}}{\partial \nu} < 0$. The Friedman rule achieves the first best if and only if

$$k \geq \beta \sigma c(q^*). \tag{36}$$

The Friedman rule is optimal even in the presence of a recognizability problem. This is so because the quantities traded in the DM decrease with the inflation rate irrespective of whether or not the liquidity constraint is binding. According to (36), it achieves the first-best allocation only if the cost of producing counterfeits, k , is large enough. If (36) does not hold, the quantity traded in the DM is too low even at the Friedman rule. This suggests that the welfare cost from deviating from the Friedman rule will be higher when the threat of counterfeiting is binding.

To conclude this section, we illustrate how the game-theoretic foundations for the endogenous liquidity constraint matter for policy analysis. Suppose that we impose an exogenous constraint on the transfer of money in the DM, $d \leq \theta m$. This constraint is analogous to the one used in Kiyotaki and Moore (2005b). Then, the buyer's choice of terms of trade would solve

$$(q, m) \in \arg \max \{-\nu \phi_t m + \sigma [u(q) - \phi_t \theta m]\} \quad \text{s.t.} \quad -c(q) + \phi_t \theta m \geq 0.$$

The first-order condition to this problem is

$$\frac{u'(q)}{c'(q)} = 1 + \frac{\nu}{\theta \sigma}.$$

In this case the Friedman rule achieves the first-best allocation for any θ . Moreover, a monetary equilibrium exists if $\frac{u'(0)}{c'(0)} > 1 + \frac{\nu}{\theta \sigma}$. So the liquidity constraint makes it less likely that a monetary equilibrium will exist. Both results fail to hold in our model.

6 Rate of return dominance

In this section we introduce a second asset, one-period nominal bonds, that can compete with money as a medium of exchange. Bonds can be counterfeited at a positive fixed cost $k > 0$ while fiat money is perfectly recognizable.¹⁵ We will demonstrate that our model can offer an explanation

¹⁵In Appendix C we briefly discuss a general case where both fiat money and bonds are subject to counterfeiting.

for the rate of return dominance puzzle—the central issue in monetary theory—according to which fiat money and bonds coexist even though bonds pay interest. We will study the implications of the model for the determination of the nominal interest rate and the effects of open-market operations.¹⁶

Bonds are perfectly divisible and payable to the bearer. They are issued by the government in the CM at the price ρ in terms of money, and are redeemed for one unit of money in the next CM.¹⁷ The implicit nominal interest rate is $i_t = \frac{1}{\rho_t} - 1$. If $\rho_t < 1$, then bonds are sold at a discount, i.e., they pay interest. If $\rho_t = 1$, then bonds and money are perfect substitutes. Both the supply of money, M_t , and the supply of bonds, B_t , are growing at the constant rate γ , so that the ratio $\frac{M_t}{B_t}$ is constant. Counterfeited bonds are not redeemed and they are destroyed with probability one when they mature. The interest payments on bonds is financed by lump-sum taxes paid by buyers in the CM. The budget constraint of the government is

$$T_t + \phi_t B_{t-1} + \phi_t M_{t-1} = \phi_t \rho B_t + \phi_t M_t,$$

where T_t is the lump-sum transfer to buyers in the CM (expressed in terms of the general good). We will focus on stationary equilibria where the real value of asset holdings is constant; i.e., $\phi_t M_t = \phi_{t+1} M_{t+1}$ (so that the rate of return of money, $\frac{\phi_{t+1}}{\phi_t}$, is constant and equal to $\frac{1}{\gamma}$), and the price of newly-issued bonds is constant.

The (reverse-ordered) game of the previous section is extended as follows. First, the buyer chooses an offer (q, d_m, d_b) to make in the DM, where d_m is the transfer of money and d_b is the transfer of bonds. Second, buyers decide whether to produce counterfeit bonds ($\chi = 1$) or not ($\chi = 0$). Third, buyers choose a portfolio of genuine bonds and money, b and m . The portfolio is private information. Fourth, in the subsequent period each buyer is matched with a seller with probability σ . The seller decides whether or not to accept the offer set by the buyer at the beginning of the game. Following the same logic as the one in the previous sections (see Appendix B), the

¹⁶Historically bonds were produced on paper, just like banknotes. There are also instances where interest-bearing bonds would circulate as money, like the war bonds issued in Arkansas at the beginning of the 1860's (see, e.g., Burdekin and Weidenmier 2008). Nowadays most bonds are no longer produced as physical pieces of paper, but their transfer necessitates information about the identity and account of the bond holder and this information is more difficult to authenticate than fiat money.

¹⁷We focus on equilibria where buyers redeem their bonds when they mature: matured bonds do not keep circulating across periods. Shi (2005) provides a method to refine away equilibria with circulating matured bonds.

following proposition characterizes the offer made by a buyer.

Proposition 7 (*Endogenous liquidity constraints in a dual asset economy*)

The equilibrium of the counterfeiting game is such that $\pi = 1$ and $\eta = 1$ and the buyer's offer satisfies

$$(q, d_m, d_b) \in \arg \max \left\{ - \left(\frac{\gamma - \beta}{\beta} \right) \phi_t d_m - \left(\frac{\rho\gamma - \beta}{\beta} \right) \phi_t d_b + \sigma [u(q) - \phi_t (d_m + d_b)] \right\} \quad (37)$$

$$s.t. \quad -c(q) + \phi_t (d_m + d_b) \geq 0, \quad (38)$$

$$\phi_t d_b \leq \frac{k}{\rho\gamma - \beta(1 - \sigma)}. \quad (39)$$

Moreover, if $\rho\gamma > \beta$, then $b = d_b$; if $\rho\gamma = \beta$, then $b \geq d_b$.

According to (37) the buyer chooses an offer in order to maximize his expected payoff in the DM net of the cost of holding money and bonds. The cost of holding money is $\frac{\gamma - \beta}{\beta}$, while the cost of holding bonds is $\frac{\rho\gamma - \beta}{\beta}$. Notice that the cost of holding bonds is positive when the price of bonds, ρ , is greater than the fundamental value of bonds, β/γ . According to (38) the offer must be acceptable by sellers given their beliefs that bonds are genuine. According to (39) the upper bound on the quantity of bonds that buyers can transfer in the DM is proportional to the cost of producing counterfeits, k , and it is decreasing with the price of bounds. If bonds are sold at a higher price, the opportunity cost of holding genuine bonds is higher, and hence, buyers have stronger incentives to produce counterfeits.

Lemma 3 For all $\gamma > \beta$ and $\rho \leq 1$ the problem (37)-(39) has a solution. If $\rho < 1$, then (d_m, d_b) is unique. If $\rho = 1$, then $d_m + d_b$ is unique.

If $\rho = 1$, then bonds and money are perfect substitutes and the composition of the buyer's portfolio is indeterminate. To overcome this indeterminacy, we will focus on symmetric equilibria where buyers hold the same portfolio. The market-clearing condition for the money market is

$$d_m = M_t. \quad (40)$$

The demand for genuine bonds is d_b if $\rho\gamma > \beta$, and $d_b \leq b$ if $\rho\gamma = \beta$. Consequently, the clearing of the bond market at a symmetric equilibrium requires

$$d_b = B_t \text{ if } \rho\gamma > \beta \quad (41)$$

$$d_b \leq B_t \text{ if } \rho\gamma = \beta. \quad (42)$$

An equilibrium is a list $\langle q, \rho, \{d_{m,t}\}_{t=1}^{\infty}, \{d_{b,t}\}_{t=1}^{\infty}, \{\phi_t\}_{t=1}^{\infty} \rangle$ that solves (37)-(42).

Proposition 8 (Allocations and prices in a dual asset economy)

Suppose $\frac{u'(0)}{c'(0)} > 1 + \frac{\gamma - \beta}{\beta\sigma}$. There exists a monetary equilibrium where the output traded in the DM solves

$$\frac{u'(q)}{c'(q)} = 1 + \frac{\gamma - \beta}{\beta\sigma}. \quad (43)$$

(i) If $\frac{B_t}{M_t + B_t} \leq \frac{k}{c(q)[\gamma - \beta(1 - \sigma)]}$, then

$$\rho = 1, \quad (44)$$

$$\phi_t = \frac{c(q)}{M_t + B_t}. \quad (45)$$

(ii) If $\frac{B_t}{M_t + B_t} \geq \frac{k}{c(q)\beta\sigma}$, then

$$\rho = \frac{\beta}{\gamma}, \quad (46)$$

$$\phi_t = \frac{c(q) - \frac{k}{\beta\sigma}}{M_t}. \quad (47)$$

(iii) If $\frac{k}{[\gamma - \beta(1 - \sigma)]c(q)} < \frac{B_t}{M_t + B_t} < \frac{k}{\beta\sigma c(q)}$, then

$$\rho = \frac{1}{\gamma} \left[\frac{k}{c(q)} \frac{M_t + B_t}{B_t} + \beta(1 - \sigma) \right], \quad (48)$$

$$\phi_t = \frac{c(q)}{M_t + B_t}. \quad (49)$$

First, consider an economy where government bonds cannot be counterfeited, $k \rightarrow \infty$. From Proposition 8 (i), the price of newly-issued bonds is $\rho = 1$. Bonds are perfect substitutes for fiat money and they do not pay interest. In this case, from (45), the value of money decreases if the total stock of liquid assets, $M_t + B_t$, increases. Second, suppose that bonds can be counterfeited

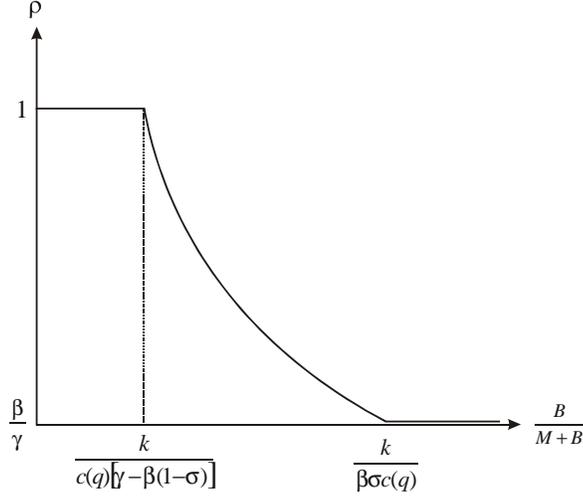


Figure 6: Price of bonds

at no cost, $k \rightarrow 0$. From Proposition 8 (ii), $\rho = \beta/\gamma$ so that the nominal interest rate, $i_t = \nu$, is approximately equal to the sum of the rate of time preference and the inflation rate. Bonds pay interest in order to compensate agents for their rate of time preference and for the depreciation of the value of money over time.

More generally, bonds pay interest provided that the supply of bonds is relatively large, $\frac{B_t}{M_t+B_t} > \frac{k}{[\gamma-\beta(1-\sigma)]c(q)}$. Bonds are more likely to be sold at a discount if the cost to produce counterfeits is low and if trading frictions are not too severe. As the moral hazard problem becomes more severe (k is lower), or as the trading frictions are reduced (σ is higher), less transfer of real quantity of bonds is allowed by the liquidity constraint, which leads to a higher yield of bonds. If the supply of bonds is sufficiently large relative to the cost of producing counterfeits, $\frac{B_t}{M_t+B_t} \geq \frac{k}{c(q)\beta\sigma}$, then the value of fiat money is affected by the recognizability of bonds as captured by k .

The model can be used to study the effects of open market operations on the nominal interest rate and the allocations. We interpret an open-market operation as a change in the ratio $\frac{B_t}{M_t}$.

Corollary 2 (Irrelevance of open-market operations)

If $\frac{k}{[\gamma-\beta(1-\sigma)]c(q)} < \frac{B_t}{M_t+B_t} < \frac{k}{\beta\sigma c(q)}$, then $\frac{\partial i}{\partial(\frac{B_t}{M_t})} > 0$. Moreover, $\frac{\partial q}{\partial(\frac{B_t}{M_t})} = \frac{\partial \mathcal{W}}{\partial(\frac{B_t}{M_t})} = 0$.

Provided that the ratio $\frac{B_t}{M_t+B_t}$ is neither too large nor too small, an open market sale ($\frac{B_t}{M_t}$

is increased) raises the nominal interest rate. The reason is as follows. If the buyer receives an additional bond, under the previously prevailing market price of bonds, he cannot spend it in the DM. The price of bonds thus must decrease to reflect this illiquidity up to the point where the liquidity constraint binds again.

Although the conduct of monetary policy affects the interest rate, it has no effect on the real allocation and welfare.¹⁸ When bonds are relatively scarce (case (i) in Proposition 8), money and bonds are perfect substitutes and $\rho = 1$. Obviously, in that case a change in the composition of money and bonds is irrelevant. When bonds are more abundant, the constraint on the transfer of bonds is binding (cases (i) and (ii) in Proposition 8). An open market operation affects the price of bonds, but the output is still determined by (43) so that the marginal benefit of an additional unit of real balances is equal to its cost.

Finally, our model challenges the Fisher hypothesis according to which nominal interest rates rise one-for-one with anticipated inflation. The Fisher hypothesis relies on the assumption that real interest rates are independent of monetary factors. Let R_b denote the real gross interest rate on bonds. By definition, $R_b = \frac{\phi_t}{\rho\phi_{t-1}} = \frac{1}{\gamma\rho}$. From Proposition 8,

$$\begin{aligned} R_b &= \frac{1}{\gamma} \text{ if } \frac{B_t}{M_t + B_t} \leq \frac{k}{c(q)[\gamma - \beta(1 - \sigma)]} \\ &= \frac{1}{\beta} \text{ if } \frac{B_t}{M_t + B_t} \geq \frac{k}{c(q)\beta\sigma} \\ &= \frac{1}{\frac{k}{c(q)} \frac{M_t + B_t}{B_t} + \beta(1 - \sigma)} \text{ otherwise.} \end{aligned}$$

The Fisher hypothesis is only valid when bonds are abundant, $\frac{B_t}{M_t + B_t} \geq \frac{k}{c(q)\beta\sigma}$ (case (ii) in Proposition 8). In this case, bonds are illiquid at the margin, so the real interest rate is equal to the rate of time preference, as implied by a standard cash-in-advance model. In contrast, if bonds are scarce, $\frac{B_t}{M_t + B_t} \leq \frac{k}{c(q)[\gamma - \beta(1 - \sigma)]}$, then money and bonds are perfect substitute, so the real interest rate is equal to the rate of return of currency, the inverse of the money growth rate. Hence, the nominal interest rate is zero and is independent of the money growth rate. For the intermediate level of the supply of bonds, the real interest rate depends on both the money growth rate and the relative

¹⁸This result is related to the assertion of the irrelevance of government's portfolio given a path of fiscal policy in Wallace (1981), where the equilibrium allocation is not influenced by the open market operations.

supply of bonds. As the inflation rate increases, the real interest rate decreases. Consequently, the nominal interest rate will not increase by as much as the inflation rate.

7 Conclusion

We have provided game-theoretic foundations for the constraints that affect the ability of economic agents to exchange assets for goods and services, and we have studied the macroeconomic implications of such constraints for asset prices, the value of money, and the conduct of monetary policy. Liquidity constraints emerge due to the moral hazard problem associated with agents' ability to produce fake assets at a positive cost. These constraints depend on the physical properties of the assets, their rates of return, trading frictions, and policy.

We showed that liquidity premia depend on the recognizability of assets, the supply of assets, and trading frictions. The finding of a negative relationship between the liquidity premium and the supply of the asset is consistent with the study of the convenience yield of Treasury securities by Krishnamurthy and Vissing-Jorgensen (2008). One implications for the current financial crisis we derived is this: Assets that are more vulnerable to moral hazard considerations have a lower liquidity premium and generate a higher rate of return. As is well-known now liquidity in the market for mortgage-backed securities was adversely effected when fraudulent activities in the origination of mortgage loans became widespread.

If the asset subject to fraudulent imitation is currency, it is shown that the existence of a fiat money system is robust even when agents cannot distinguish genuine money from counterfeits. This is consistent with the view that "counterfeiting of U.S. currency was economically insignificant and thus did not pose a threat to the U.S. monetary system." (G.A.O., 1996). The possibility of counterfeiting, however, can prevent the Friedman rule from achieving the first-best allocation. In a version of the model with risk-free nominal bonds and fiat money, the moral hazard problem on payments can explain the rate of return dominance puzzle, the liquidity effect of open-market purchases, and the lack of validity of the Fisher hypothesis.

As outlined throughout the paper, our methodology is flexible enough to accommodate various extensions: the asset subject to a moral hazard problem could be produced capital, multiple assets may suffer different degrees of counterfeiting threat, and the moral hazard problem can take different

forms that are often considered in the corporate finance literature.

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Appendix A.

Proof of Lemma 1 Consider the subgame following the history $((q, d), 1)$, i.e., the buyer chose the offer (q, d) and decided to produce counterfeits, $\chi = 1$. The buyer's expected payoff if he accumulates a units of genuine assets is

$$-k - [\phi_{t-1} - \beta(\phi_t + \zeta)] a + \beta\sigma u(q) \mathbb{E}\mathbb{I}_{\{\mu=1\}},$$

where the expectation is with respect to the seller's strategy. Recall that any offer is feasible when $\chi = 1$ since the buyer can produce any quantity of counterfeits. The seller's decision to accept or reject an offer is independent of a since asset holdings are not observable. Since $\phi_{t-1} > \beta(\phi_t + \zeta)$, the optimal choice of asset holdings is $a = 0$, i.e., $\mathbb{G}((q, d), 1) = \delta_{\{0\}}$. Similarly, the buyer's expected payoff following the history $((q, d), 0)$ (i.e., the buyer chooses not to produce counterfeits) is

$$- [\phi_{t-1} - \beta(\phi_t + \zeta)] a + \beta\sigma \{u(q) - (\phi_t + \zeta)d\} \mathbb{E}\mathbb{I}_{\{\mu=1\}},$$

where $a \geq d$ for the offer to be feasible. Given $\phi_{t-1} > \beta(\phi_t + \zeta)$, the optimal choice of asset holdings is such that $a = d$, i.e., $\mathbb{G}((q, d), 0) = \delta_{\{d\}}$.

Proof of Proposition 1 The proof proceeds in two steps. First, we consider the set of offers such that $\pi(q, d) = \eta(q, d) = 1$ in order to show that the buyers' payoff is at least equal to what is implied by the problem (13)-(15). Second, we will show that any other offer that does not satisfy $\pi(q, d) = \eta(q, d) = 1$ generates a payoff less than U^* .

1. **Lower bound for the buyer's payoff.** From (9) the supremum of the buyer's payoff among all offers (q, d) such that $\pi(q, d) = \eta(q, d) = 1$ is

$$U^* = \max_{q,d} \left\{ - [\phi_{t-1} - \beta(\phi_t + \zeta)] d + \sigma\beta [u(q) - (\phi_t + \zeta) d] \right\}$$

$$\text{s.t. } c(q) \leq (\phi_t + \zeta)d \leq \frac{k(\phi_t + \zeta)}{\phi_{t-1} - \beta(1 - \sigma)(\phi_t + \zeta)}.$$

The buyer's payoff is continuous in q and d , and it is maximized over a compact set. So a solution exists. If the offer (q, d) is such that $c(q) < (\phi_t + \zeta)d < \frac{k(\phi_t + \zeta)}{\phi_{t-1} - \beta(1-\sigma)(\phi_t + \zeta)}$, then the equilibrium of the subgame following (q, d) is unique and such that $\pi(q, d) = \eta(q, d) = 1$. So for any collection of strategies $\langle \pi(q, d), \eta(q, d) \rangle$ the supremum of the buyer's payoff given by (12) is not less than U^* .

2. Ruling out offers that do not satisfy $\pi(q, d) = \eta(q, d) = 1$.

(a) Offers such that $(\pi, \eta) \in (0, 1)^2$.

>From (10)-(11), $\eta = \frac{c(q)}{(\phi_t + \zeta)d}$ and $\pi = \frac{k - [\phi_{t-1} - \beta(\phi_t + \zeta)]d}{\beta\sigma(\phi_t + \zeta)d}$. The condition $\eta \in (0, 1)$ implies $c(q) < (\phi_t + \zeta)d$. The condition $\pi \in (0, 1)$ implies

$$(\phi_t + \zeta)d \in \left(\frac{k(\phi_t + \zeta)}{\phi_{t-1} - (1-\sigma)\beta(\phi_t + \zeta)}, \frac{k(\phi_t + \zeta)}{\phi_{t-1} - \beta(\phi_t + \zeta)} \right). \quad (50)$$

Since the buyer is indifferent between producing a counterfeit or not his payoff is equal to

$$-k + \beta\sigma\pi u(q) = -k + \frac{k - [\phi_{t-1} - \beta(\phi_t + \zeta)]d}{(\phi_t + \zeta)d} u(q),$$

where we have used $\pi = \frac{k - [\phi_{t-1} - \beta(\phi_t + \zeta)]d}{\beta\sigma(\phi_t + \zeta)d}$ to obtain the right side of the equality. For a given d this expression is monotonically increasing in q . So the supremum of the buyer's payoff among all offers such that $(\pi, \eta) \in (0, 1)^2$ is such that $c(q) = (\phi_t + \zeta)d$ and $\eta = 1$. Substituting $d = \frac{c(q)}{(\phi_t + \zeta)}$, this supremum can be written as

$$-k + \left[k - \left(\frac{\phi_{t-1}}{\phi_t + \zeta} - \beta \right) c(q) \right] \frac{u(q)}{c(q)}.$$

Since $u'' < 0$, $c'' \geq 0$ and $u(0) = c(0) = 0$, $\frac{u'(q)q}{u(q)} < 1 \leq \frac{c'(q)q}{c(q)}$ and $u(q)/c(q)$ is decreasing in q . Moreover, $\frac{\phi_{t-1}}{\phi_t + \zeta} \geq \beta$ implies that the term in square brackets is non-increasing in q . Hence, the buyer's payoff is decreasing in q , which from (50) implies that the supremum is such that $(\phi_t + \zeta)d = c(q) = \frac{k(\phi_t + \zeta)}{\phi_{t-1} - (1-\sigma)\beta(\phi_t + \zeta)}$ and $\pi = 1$. The supremum is no greater than U^* since it corresponds to an offer such that $\pi = \eta = 1$, and it is not achieved.

(b) Offers such that $(\pi, \eta) \in \{1\} \times (0, 1)$.

>From (6) and (8) $d = \frac{k}{\phi_{t-1} - \beta(1-\sigma)(\phi_t + \zeta)}$ and $-c(q) + \eta(\phi_t + \zeta)d \geq 0$. Since the buyer is indifferent between being a counterfeiter or a genuine buyer, the supremum of the buyer's payoff among such offers is

$$\sup \left\{ -k + \beta\sigma u(q) : c(q) \leq \frac{\eta(\phi_t + \zeta)k}{\phi_{t-1} - \beta(1-\sigma)(\phi_t + \zeta)} \text{ and } \eta \in (0, 1) \right\} \leq U^*.$$

Since the buyer's payoff is increasing in q and the upper bound on q increases with η it is immediate that the supremum corresponds to $\eta = 1$ and $c(q) = \frac{(\phi_t + \zeta)k}{\phi_{t-1} - \beta(1-\sigma)(\phi_t + \zeta)}$. It corresponds to $\pi = \eta = 1$ and hence it is no greater than U^* . Moreover, it is not achieved.

(c) Offers such that $(\pi, \eta) \in (0, 1) \times \{1\}$.

>From (6) and (8) $c(q) = (\phi_t + \zeta)d$ and $(\phi_t + \zeta)d \leq \frac{k(\phi_t + \zeta)}{\phi_{t-1} - \beta(1-\sigma\pi)(\phi_t + \zeta)}$. Using $d = \frac{c(q)}{(\phi_t + \zeta)}$, the expected payoff of a genuine buyer is

$$- \left(\frac{\phi_{t-1}}{\phi_t + \zeta} - \beta \right) c(q) + \sigma\beta\pi [u(q) - c(q)].$$

Let $\hat{q} = \arg \max \left\{ - \left(\frac{\phi_{t-1}}{\phi_t + \zeta} - \beta \right) c(q) + \sigma\beta [u(q) - c(q)] \right\}$ where we set $\pi = 1$ in the expression of the buyer's payoff. If $c(\hat{q}) \leq \frac{k(\phi_t + \zeta)}{\phi_{t-1} - \beta(1-\sigma)(\phi_t + \zeta)}$, then the supremum of the buyer's payoff among offers such that $(\pi, \eta) \in (0, 1) \times \{1\}$ corresponds to $q = \hat{q}$ and $\pi = 1$ and it is equal to U^* . Notice that this is the solution of the problem when the incentive constraint does not bind. If $c(\hat{q}) > \frac{k(\phi_t + \zeta)}{\phi_{t-1} - \beta(1-\sigma)(\phi_t + \zeta)}$, then the constraint on the transfer of assets binds and the supremum is such that $c(q) = (\phi_t + \zeta)d = \frac{k(\phi_t + \zeta)}{\phi_{t-1} - \beta(1-\sigma\pi)(\phi_t + \zeta)}$

The buyer's payoff is

$$-k + \left\{ k - \left(\frac{\phi_{t-1}}{\phi_t + \zeta} - \beta \right) c(q) \right\} \frac{u(q)}{c(q)}$$

where we have used $\sigma\pi = \frac{k(\phi_t + \zeta) - [\phi_{t-1} - \beta(\phi_t + \zeta)]c(q)}{\beta(\phi_t + \zeta)c(q)}$, which is derived from the expression $c(q) = (\phi_t + \zeta)d = \frac{k(\phi_t + \zeta)}{\phi_{t-1} - \beta(1-\sigma\pi)(\phi_t + \zeta)}$. Since $\frac{u(q)}{c(q)}$ is decreasing in q , the supremum is decreasing in q and it corresponds to $c(q) = \frac{k(\phi_t + \zeta)}{\phi_{t-1} - \beta(1-\sigma)(\phi_t + \zeta)}$ and $\pi = 1$. To summarize, the supremum of the buyer's payoff among offers such that $(\pi, \eta) \in (0, 1) \times \{1\}$ is equal to U^* , but it is not achieved since it requires $\pi = 1$.

>From all the cases discussed above, the supremum of the problem in (12) is U^* , and it is achieved by the offer that solves (13)-(15).

Proof of Lemma 2 The objective function in the problem (17)-(19) is continuous and it is maximized over a compact set. >From the Theorem of the Maximum (see, e.g., Theorem 3.6 in Stokey, Lucas and Prescott 1989), $A^d(\phi)$ is non-empty and it is upper-hemi continuous. Recall that if $\phi > \phi^*$, $A^d(\phi)$ corresponds to the values of d that solve (17)-(19); otherwise, if $\phi = \phi^*$, it corresponds to all the values such that $a \geq d$.

(i) **The case $\phi = \phi^*$.** If (19) does not bind, then $q = q^*$ and $d = \frac{c(q^*)}{\phi^* + \zeta}$. If (19) binds, then $d = \frac{k}{\sigma\phi^*}$. Moreover, if $\phi = \phi^*$, then buyers are indifferent to hold assets that they do not spend in the DM. Consequently,

$$A^d(\phi^*) = \left[\min \left(\frac{c(q^*)}{\phi^* + \zeta}, \frac{k}{\sigma\phi^*} \right), \infty \right).$$

(ii) **The case $\phi > \phi^*$.** If (19) does not bind, then there is a unique solution to (17)-(19) and it is such that

$$-\frac{r(\phi - \phi^*)}{\phi + \zeta} + \sigma \left[\frac{u' \circ c^{-1}((\phi + \zeta)d)}{c' \circ c^{-1}((\phi + \zeta)d)} - 1 \right] \leq 0,$$

with an equality if $d > 0$. Since $\frac{\phi - \phi^*}{\phi + \zeta}$ is increasing in ϕ and $\frac{u'}{c'}$ is decreasing, the transfer of assets, d , is a decreasing function of ϕ , and it is strictly decreasing if $d > 0$. If (19) binds, then $d = \frac{k}{\phi - \beta(1 - \sigma)(\phi + \zeta)}$, which is also decreasing in ϕ . Moreover, if $\phi > \phi^*$ buyers do not want to hold more assets than what they intend to spend in the DM, $a = d$. (See Lemma 1.) Consequently, $A^d(\phi)$ is single-valued and it is decreasing in ϕ .

Proof of Propositions 2 and 3 >From Lemma 2, the asset demand correspondence is upper-hemi continuous, convex-valued, and any selection is decreasing in ϕ . Moreover, $\infty \in A^d(\phi^*)$ and $A^d(\phi)$ tends to $\{0\}$ as ϕ tends to infinity. Hence, there is a unique ϕ such that $A \in A^d(\phi^*)$. See Figure 5 for an illustration of the argument.

(i) The condition $k \geq \frac{\sigma c(q^*)}{1+r}$ implies $\frac{k}{\sigma\phi^*} \geq \frac{c(q^*)}{\phi^* + \zeta}$. If $A \geq \frac{c(q^*)}{\phi^* + \zeta}$, then $A \in A^d(\phi^*) = \left[\min \left(\frac{c(q^*)}{\phi^* + \zeta}, \frac{k}{\sigma\phi^*} \right), \infty \right]$ and hence $\phi = \phi^*$, and $R = \beta^{-1}$. Moreover, the solution to (17)-(19) is such that $q = q^*$ since

$\frac{c(q^*)}{\phi^* + \zeta} \leq \frac{k}{\sigma \phi^*}$ implies that (19) does not bind. From the seller's participation constraint at equality, $d = \frac{c(q^*)}{\phi^* + \zeta}$, and since $a \geq d$ if $\phi = \phi^*$, we have $A(\phi^* + \zeta) \geq c(q^*)$, which implies that q^* is incentive-feasible.

(ii) The reasoning is similar as in (i) except that (19) binds, which implies $q < q^*$. From (19) at equality, $d = \frac{rk}{\sigma \zeta}$.

(iii) If $A < \min\left(\frac{c(q^*)}{\phi^* + \zeta}, \frac{k}{\sigma \phi^*}\right)$, then $A \notin A^d(\phi^*)$ and, hence, $\phi > \phi^*$ and $R < \beta^{-1}$. If (19) does not bind, then $d = A$ where

$$d \in \arg \max_d \left\{ -r(\phi - \phi^*)d + \sigma [u \circ c^{-1}((\phi + \zeta)d) - (\phi + \zeta)d] \right\}.$$

The first-order condition corresponds to (20) and the solution is interior because of market clearing. The liquidity constraint (19) does not bind if

$$\frac{k}{\phi^u - \beta(1 - \sigma)(\phi^u + \zeta)} \geq A = \frac{k}{\phi^c - \beta(1 - \sigma)(\phi^c + \zeta)},$$

where the equality of right side comes from the definition of ϕ^c in (21). Therefore, $\phi^u \leq \phi^c$. If (19) binds, then $\phi^u \geq \phi^c$, and $d = A = \frac{k}{\phi - \beta(1 - \sigma)(\phi + \zeta)}$ which gives (21). Consequently, $\phi = \min(\phi^u, \phi^c)$. In order to show that $q < q^*$, suppose that $q = q^*$. Then, from (20), $\phi^u = \phi^*$, and since $\phi = \min(\phi^u, \phi^c) \geq \phi^*$, this implies $\phi = \phi^*$. But since $(\phi^* + \zeta)A < c(q^*)$, $q < q^*$. A contradiction.

Proof of Proposition 4 > From (20) and (21) if $\sigma = 0$, then $\phi^u = \phi^*$ and $\phi^c = \frac{\frac{k}{A} + \beta\zeta}{1 - \beta} > \phi^*$. Moreover, ϕ^u is increasing in σ and ϕ^c is decreasing in σ . So if $\phi^c < \phi^u$ at $\sigma = 1$, then, by the continuity of ϕ^u and ϕ^c , there is a threshold $\bar{\sigma} < 1$ such that for all $\sigma < \bar{\sigma}$, $\phi = \phi^u$ and for all $\sigma > \bar{\sigma}$, $\phi = \phi^c$, since from Proposition 3 $\phi = \min(\phi^u, \phi^c)$. The condition $\phi^c < \phi^u$ at $\sigma = 1$ holds if the left side of (20) evaluated at $\phi = \phi^c = \frac{k}{A}$ is less than the right side of (20) evaluated at $\phi = \phi^c = \frac{k}{A}$, which gives (22).

Proof of Proposition 5 Let $\hat{q} = \arg \max \{-\nu c(q) + \sigma [u(q) - c(q)]\}$, i.e.,

$$-\nu c'(\hat{q}) + \sigma [u'(\hat{q}) - c'(\hat{q})] \leq 0, \quad "=" \text{ if } \hat{q} > 0. \quad (51)$$

If the constraint (28) is not binding, then $\phi_t d = c(\hat{q})$, and otherwise $\phi_t d = \frac{k}{\beta(\nu+\sigma)}$. Consequently,

$$\phi_t d = \min \left[c(\hat{q}), \frac{k}{\beta(\nu+\sigma)} \right].$$

>From the market-clearing condition (29),

$$\phi_t = \frac{\min \left[c(\hat{q}), \frac{k}{\beta(\nu+\sigma)} \right]}{M_t}. \quad (52)$$

>From (52) and the assumption $k > 0$, money is valued in every period if and only if $\hat{q} > 0$. From (51) $\hat{q} > 0$ if and only if

$$-\nu c'(0) + \sigma [u'(0) - c'(0)] > 0,$$

which gives (30).

The threshold \bar{k} is defined as the value of k such that $\phi_t M_t = c(\hat{q}) = \frac{k}{\beta(\nu+\sigma)}$. If (30) holds then $\hat{q} = \arg \max \{-\nu c(q) + \sigma [u(q) - c(q)]\} > 0$ and hence, $\bar{k} > 0$. If $k > \bar{k}$, then $\phi_t M_t = c(\hat{q})$ and $q = \hat{q}$, which gives (31)-(32). If $k < \bar{k}$, then $\phi_t M_t = \frac{k}{\beta(\nu+\sigma)}$ and $q = c^{-1}(\phi_t M_t)$, which gives (33)-(34).

Proof of Proposition 6 >From (31) and (33) it is immediate that $\frac{\partial q}{\partial \nu} < 0$. From (35),

$$\frac{\partial \mathcal{W}}{\partial \nu} = \sigma \frac{u'(q) - c'(q)}{1 - \beta} \frac{\partial q}{\partial \nu}.$$

Since $q < q^*$ for all $\nu > 0$, $u'(q) - c'(q) > 0$, which implies $\frac{\partial \mathcal{W}}{\partial \nu} < 0$. As a consequence, the Friedman rule is optimal. The Friedman rule achieves the first best if and only if $k \geq \bar{k}$ at $\nu = 0$. According to the definition of \bar{k} in Proposition 5 this implies $k \geq \beta \sigma c(\hat{q})$ where $\frac{u'(\hat{q})}{c'(\hat{q})} = 1$, i.e., $\hat{q} = q^*$.

Proof of Lemma 3 The solution for q to (37)-(39) is such that $q \leq q^*$. Otherwise the buyer could reduce q to q^* and $d_m + d_b$ so that the seller's payoff is unchanged, which would raise his own payoff while. Moreover, (38) holds at equality. So (q, d_m, d_b) lies in the compact set $[0, q^*] \times \left[0, \frac{c(q^*)}{\phi_t}\right] \times \left[0, \frac{k}{\phi_t[\rho\gamma - \beta(1-\sigma)]}\right]$. Since the objective function is continuous, from the theorem of maximization, there is a solution to the buyer's problem.

Using that (38) holds at equality, the objective function can be rewritten as

$$-\left(\frac{\gamma - \beta}{\beta}\right) \phi_t d_m - \left(\frac{\rho\gamma - \beta}{\beta}\right) \phi_t d_b + \sigma \{u \circ c^{-1}[\phi_t(d_m + d_b)] - \phi_t(d_m + d_b)\} \quad (53)$$

If $\rho < 1$, the cost of holding bonds is less than the cost of holding money. If (39) does not bind, then $d_m = 0$ and d_b solves

$$\max_{d_b} \left\{ -\left(\frac{\rho\gamma - \beta}{\beta}\right) \phi_t d_b + \sigma \{u \circ c^{-1}(\phi_t d_b) - \phi_t d_b\} \right\};$$

if (39) binds, then d_b solves (39) at equality and d_m is the unique value that maximizes (53). If $\rho = 1$, then money and bonds are perfect substitutes and $d_m + d_b$ is uniquely determined by the value that maximizes (53).

Proof of Proposition 8 >From (37)-(39), and taking into account that buyers can choose to hold more bonds than what they spend in bilateral matches when $\rho\gamma = \beta$, since the cost of holding bonds is zero, the buyer's problem becomes

$$\begin{aligned} (q, d_m, d_b, b) \in \arg \max & \left\{ -\left(\frac{\gamma - \beta}{\beta}\right) \phi_t d_m - \left(\frac{\rho\gamma - \beta}{\beta}\right) \phi_t b + \sigma [u(q) - \phi_t(d_m + d_b)] \right\} \\ \text{s.t.} & \quad -c(q) + \phi_t(d_m + d_b) = 0, \\ & \quad \phi_t d_b \leq \frac{k}{\rho\gamma - \beta(1 - \sigma)}, \quad d_b \leq b. \end{aligned}$$

Replacing q by its expression given by the seller's participation constraint. Notice that the objective function is concave. The Lagrangian associated with this problem is

$$\begin{aligned} \mathcal{L} = & -\left(\frac{\gamma - \beta}{\beta}\right) \phi_t d_m - \left(\frac{\rho\gamma - \beta}{\beta}\right) \phi_t b + \sigma \{u \circ c^{-1}[\phi_t(d_m + d_b)] - \phi_t(d_m + d_b)\} \\ & + \lambda \left(\frac{k}{\rho\gamma - \beta(1 - \sigma)} - \phi_t d_b \right) + \mu \phi_t (b - d_b) \end{aligned}$$

where the Lagrange multiplier λ is associated with the liquidity constraint and the Lagrange multiplier μ with the feasibility constraint on the transfer of bonds. The first-order (necessary and sufficient) condition with respect to d_m is

$$-\frac{\gamma - \beta}{\beta} + \sigma \left(\frac{u'(q)}{c'(q)} - 1 \right) \leq 0, \quad (54)$$

with an equality if the equilibrium is monetary. The assumption $\frac{u'(0)}{c'(0)} > 1 + \frac{\gamma-\beta}{\beta\sigma}$ implies $q > 0$, and hence (43). The first-order condition with respect to d_b is

$$\sigma \left(\frac{u'(q)}{c'(q)} - 1 \right) - \lambda - \mu \leq 0.$$

If the solution is interior, and using (54) at equality,

$$\frac{\gamma - \beta}{\beta} = \lambda + \mu. \quad (55)$$

It can easily be checked that $d_b = 0$ only if $\rho = 1$ in which case buyers are indifferent between holding money and bonds so that (55) still holds at equality. Finally, the first-order condition with respect to b , assuming an interior solution (since in equilibrium the bonds market must clear) is

$$\mu = \frac{\rho\gamma - \beta}{\beta}. \quad (56)$$

Together with (55) this gives

$$\lambda = \frac{\gamma(1 - \rho)}{\beta}. \quad (57)$$

We consider the following two cases:

1. The constraint (39) is not binding ($\lambda = 0$).

>From (57) $\rho = 1$. From (56), $\mu = \frac{\gamma-\beta}{\beta} > 0$ and hence $d_b = b = B_t$. From (38) at equality, $\phi_t(M_t + B_t) = c(q)$ and hence (45). The constraint (39) is not binding if

$$\frac{B_t}{M_t + B_t} c(q) \leq \frac{k}{\gamma - \beta(1 - \sigma)}.$$

2. The constraint (39) binds, $\lambda > 0$ and $\phi_t d_b = \frac{k}{\rho\gamma - \beta(1 - \sigma)}$.

There are two subcases to consider.

- (a) $d_b \leq b = B_t$ is not binding ($\mu = 0$).

>From (56), $\rho\gamma = \beta$ that leads to (46). From (38) at equality, $\phi_t(M_t + d_b) = c(q)$ which gives (47). The condition $d_b \leq B_t$ requires

$$\frac{k}{\beta\sigma} \leq \frac{B_t}{M_t + B_t} c(q).$$

(b) $d_b \leq b = B_t$ is binding ($\mu > 0$).

Then, (38) and (39) at equality give (48) and (49). From (56) and (57), since $\lambda > 0$ and $\mu > 0$, $\rho \in (\frac{\beta}{\gamma}, 1)$ which implies

$$\frac{k}{[\gamma - \beta(1 - \sigma)]c(q)} < \frac{B_t}{M_t + B_t} < \frac{k}{\beta\sigma c(q)}.$$

Appendix B. Derivation of the liquidity constraint (39)

The Bernoulli payoffs to the buyer and seller in the counterfeiting game are

$$\begin{aligned} U^b(m, b, \chi, q, d_m, d_b, \mu) &= -k\mathbb{I}_{\{\chi=1\}} - \phi_{t-1}(m + \rho b) \\ &\quad + \beta \left\{ u(q) - \phi_t(d_m + d_b\mathbb{I}_{\{\chi=0\}}) \right\} \mathbb{I}_{\{\mu=1\}} + \beta\phi_t(m + b), \end{aligned} \quad (58)$$

where feasibility requires $d_m \leq m$ and $d_b \leq b$ if $\chi = 0$, and

$$U^s(\chi, q, d_m, d_b, \mu) = \beta \left\{ -c(q) + \phi_t(d_m + d_b\mathbb{I}_{\{\chi=0\}}) \right\} \mathbb{I}_{\{\mu=1\}}, \quad (59)$$

respectively. Notice from (58) that the price of newly-issued bonds in terms of general goods is $\rho\phi_{t-1}$ while the price of matured bonds is ϕ_t (since a matured bond is redeemed for one unit of money). The transfer of bonds in the DM lowers the utility of the buyer only if they are genuine bonds ($\chi = 0$). A behavioral strategy of the buyer is a list $\langle \mathbb{F}, \eta(q, d_m, d_b), \mathbb{G}(q, d_m, d_b, \chi) \rangle$ where \mathbb{F} is the distribution from which the buyer draws his offer, $1 - \eta$ is the probability that the buyer produces counterfeits conditional on the offer (q, d_m, d_b) being made, \mathbb{G} is the distribution for the choice of money and genuine bond holdings conditional on the history (q, d_m, d_b, χ) .

Lemma 4 *Assume $\rho\gamma > \beta$. Any optimal strategy of the buyer is such that the distribution from which the buyer chooses a portfolio satisfies $\mathbb{G}(q, d_m, d_b, 0) = \delta_{\{(d_m, d_b)\}}$ and $\mathbb{G}(q, d_m, d_b, 1) = \delta_{\{(d_m, 0)\}}$.*

Proof. Rewrite the buyer's payoff, (58), as

$$\begin{aligned} U_t^b(m, b, \chi, q, d_m, d_b, \mu) &= -k\mathbb{I}_{\{\chi=1\}} - (\gamma - \beta)\phi_t m - (\rho\gamma - \beta)\phi_t b \\ &\quad + \beta\sigma \left\{ u(q) - \phi_t(d_m + d_b\mathbb{I}_{\{\chi=0\}}) \right\} \mathbb{I}_{\{\mu=1\}}. \end{aligned}$$

Since $\gamma - \beta > 0$ holding money is costly and hence, conditional on the offer (q, d_m, d_b) being made, the optimal strategy of the buyer is $m = d_m$. (We assume that the buyer is committed to execute the offer he has made at the beginning of the game.) If $\rho\gamma > \beta$, then holding (genuine) bonds is costly. So if the buyer has the technology to produce counterfeits, $\chi = 1$, then $b = 0$; if the buyer does not have the technology to produce counterfeits, $\chi = 0$, then $d_b = b$. ■

Since $\gamma \geq \rho\gamma > \beta$ it is costly to hold money and, as a consequence, buyers will not hold more money than they intend to spend in the DM. Similarly, since $\rho\phi_{t-1} > \beta\phi_t$ if $\rho\gamma > \beta$, then it is costly to hold bonds and buyers will not hold more genuine bonds than the amount that they will spend in the DM. In contrast, if $\rho\gamma = \beta$, then buyers are willing to hold more bonds than what they spend in the DM, but the excess of bonds holdings does not affect their payoffs. As a consequence of Lemma 4, we can ignore the buyer's choice of bonds holdings and focus only on the transfer of bonds in the DM.

The (reverse-ordered) counterfeiting game is solved using the same logic as in Section 3. Consider a subgame following an offer (q, d_m, d_b) . The seller's decision to accept the offer is given by

$$\begin{aligned} & > 0 & & = 1 \\ -c(q) + \phi_t(\eta d_b + d_m) < 0 & \implies \pi & = 0 & . \\ & = 0 & & \in [0, 1] \end{aligned} \quad (60)$$

According to (60), the seller receives both money and bonds, but only money can be recognized with certainty. Hence, sellers will rationally anticipate that bonds are genuine with probability η . Given π , a buyer is willing to accumulate genuine money if

$$\begin{aligned} & -(\rho\gamma - \beta)\phi_t d_b - (\gamma - \beta)\phi_t d_m + \beta\sigma\{u(q) - \phi_t(d_m + d_b)\}\pi \geq \\ & -k - (\gamma - \beta)\phi_t d_m + \beta\sigma\{u(q) - \phi_t d_m\}\pi. \end{aligned}$$

where we have used Lemma 4 to simplify the buyers' payoffs, i.e., if $\rho\gamma > \beta$, then an honest buyer ($\chi = 0$) accumulates just enough bonds to make his payment in the DM, whereas if $\rho\gamma = \beta$, then the quantity of bonds that a buyer holds beyond what is necessary to make the payment in the DM does not affect his payoff. By producing counterfeits the buyer saves both the cost of holding bonds, $(\rho\gamma - \beta)\phi_t d_b$, and the value of transferring those bonds in the DM if conducting a trade, $\phi_t d_b$, but he incurs the fixed cost of producing counterfeits, k . Hence, the decision to produce counterfeit bonds obeys the following condition:

$$\begin{aligned} & < & & = 1 \\ (\rho\gamma - \beta + \beta\sigma\pi)\phi_t d_b > k & \implies \eta & = 0 & \\ & = & & \in [0, 1] \end{aligned} \quad (61)$$

Given a collection of Nash equilibria $\langle \pi(q, d_m, d_b), \eta(q, d_m, d_b) \rangle$ for each subgame following an offer (q, d_m, d_b) , the optimal offer of a buyer solves

$$(q, d_m, d_b) \in \text{supp}\mathbb{F} \subset \arg \max \{ -k [1 - \eta(q, d_m, d_b)] - (\rho\gamma - \beta) \phi_t d_b \eta(q, d_m, d_b) \\ - (\gamma - \beta) \phi d_m + \sigma\beta [u(q) - \phi_t d_m - \eta(q, d_m, d_b) \phi_t d_b] \pi(q, d_m, d_b) \}. \quad (62)$$

An equilibrium of the counterfeiting game is a collection of Nash equilibria for all the subgames following an offer, $\langle \pi(q, d_m, d_b), \eta(q, d_m, d_b) \rangle$, and a distribution of offers, \mathbb{F} , that satisfy (60), (61), and (62).

Proof of Proposition 7 The proof follows the same logic as in the proof of Proposition 1. Consider equilibria of the subgame following an offer (q, d_m, d_b) such that $\pi = 1$ and $\eta = 1$. From (60) and (61), $-c(q) + \phi_t(d_b + d_m) \geq 0$ and $\phi_t d_b \leq \frac{k}{\rho\gamma - \beta(1 - \sigma)}$, which corresponds to (38) and (39). If these inequalities are strict, the Nash equilibrium is unique. Among all these equilibria the supremum of the buyer's payoff is reached for the solution to (37)-(39). Denote this supremum U^* . Hence, any solution to (62) must give a payoff at least equal to U^* .

In the rest of the proof we will show that any other offer such that $(\pi, \eta) \neq (1, 1)$ generates a payoff that is less than U^* . Suppose that the buyer makes an offer such that $\pi \in (0, 1)$ and $\eta \in (0, 1)$. From (60) and (61),

$$\eta = \frac{c(q) - \phi_t d_m}{\phi_t d_b}, \quad (63)$$

$$\pi = \frac{k - (\rho\gamma - \beta)\phi_t d_b}{\beta\sigma\phi_t d_b}. \quad (64)$$

According to (64) by reducing his transfer of bonds a buyer can raise the probability that his offer is accepted, and hence he can raise his expected payoff, $-k - (\gamma - \beta)\phi_t d_m + \beta\sigma\pi [u(q) - \phi_t d_m]$. Consequently, the supremum of the buyer's payoff among offers that satisfy (63) and (64) is such that $\pi = 1$ and/or $\eta = 1$. We will consider these two types of equilibria in the following.

Consider an offer such that $(\pi, \eta) \in \{1\} \times (0, 1)$. From (60) and (61), $\phi_t d_b = \frac{k}{\rho\gamma - \beta(1 - \sigma)}$ and $-c(q) + \phi_t(d_m + \eta d_b) \geq 0$. The supremum of the buyer's payoff among such offers is

$$\sup \left\{ -k - (\gamma - \beta)\phi_t d_m + \beta\sigma [u(q) - \phi_t d_m] : c(q) \leq \phi_t d_m + \frac{\eta k}{\rho\gamma - \beta(1 - \sigma)} \text{ and } \eta \in (0, 1) \right\}.$$

The solution corresponds to $\eta = 1$ and $c(q) = \phi_t d_m + \frac{k}{\rho\gamma - \beta(1 - \sigma)}$, and it is no greater than U^* . Moreover, it is not achieved.

Finally, consider offers such that $(\pi, \eta) \in (0, 1) \times \{1\}$. From (60) and (61), $\phi_t(d_m + d_b) = c(q)$ and $\phi_t d_b \leq \frac{k}{\rho\gamma - \beta(1 - \sigma\pi)}$. The buyer's payoff is

$$-(\gamma - \beta)\phi_t d_m - (\rho\gamma - \beta)\phi_t d_b + \beta\sigma\pi [u(q) - c(q)].$$

The buyer's expected payoff is increasing with π . Suppose $\pi = 1$. If the supremum of the buyer's payoff is such that $\phi_t d_b \leq \frac{k}{\rho\gamma - \beta(1 - \sigma)}$ does not bind, then it corresponds to $\pi = 1$ and it is equal to U^* . If the constraint binds, then $\phi_t d_b = \frac{k}{\rho\gamma - \beta(1 - \sigma\pi)}$ and the buyer's payoff is

$$U(\pi) = \sup_q \left\{ -(\gamma - \beta)c(q) + \frac{\gamma(1 - \rho)k}{\rho\gamma - \beta(1 - \sigma\pi)} + \beta\sigma\pi [u(q) - c(q)] \right\}.$$

The derivative with respect to π is

$$\begin{aligned} U'(\pi) &= \frac{1}{\pi} \left\{ -\frac{\beta\sigma\pi(1 - \rho)\gamma}{\rho\gamma - \beta(1 - \sigma\pi)} \phi_t d_b + \beta\sigma\pi [u(q) - c(q)] \right\} \\ &\geq \frac{1}{\pi} \{ -(\gamma - \beta)c(q) + \beta\sigma\pi [u(q) - c(q)] \} \\ &\geq 0, \end{aligned}$$

where we have used that $\phi_t d_b \leq c(q)$, $(1 - \rho)\gamma \leq \gamma - \beta$, and $\frac{\beta\sigma\pi}{\rho\gamma - \beta(1 - \sigma\pi)} \leq 1$ in order to obtain the first inequality. The second inequality results from the fact that $\max_q \{ -(\gamma - \beta)c(q) + \beta\sigma\pi [u(q) - c(q)] \} \geq 0$. So the supremum corresponds to $\pi = 1$; it is no greater than U^* and it is not achieved.

Appendix C: Two counterfeitable assets

In this Appendix we consider the case where bonds and money can be counterfeited at positive fixed costs $k_b \in (0, +\infty)$ and $k_m \in (0, +\infty)$, respectively. We will prove the following proposition.

Proposition 9 *There is no counterfeiting in equilibrium and the buyer's offer satisfies*

$$(q, d_m, d_b) \in \arg \max \left\{ - \left(\frac{\gamma - \beta}{\beta} \right) \phi_t d_m - \left(\frac{\rho\gamma - \beta}{\beta} \right) \phi_t d_b + \sigma [u(q) - \phi_t (d_m + d_b)] \right\} \quad (65)$$

$$s.t. \quad -c(q) + \phi_t (d_m + d_b) \geq 0, \quad (66)$$

$$\phi_t d_m \leq \frac{k_m}{\gamma - \beta(1 - \sigma)}, \quad (67)$$

$$\phi_t d_b \leq \frac{k_b}{\rho\gamma - \beta(1 - \sigma)}. \quad (68)$$

The buyer's decisions to produce counterfeit money and bonds are represented by the variables $\chi_m \in \{0, 1\}$ and $\chi_b \in \{0, 1\}$, respectively. The buyer's Bernoulli payoff is

$$\begin{aligned} U^b(m, b, \chi_m, \chi_b, q, d_m, d_b, \mu) &= -k_m \mathbb{I}_{\{\chi_m=1\}} - k_b \mathbb{I}_{\{\chi_b=1\}} - \phi_{t-1} (m + \rho b) \\ &\quad + \beta \sigma \{ u(q) - \phi_t (d_m \mathbb{I}_{\{\chi_m=0\}} + d_b \mathbb{I}_{\{\chi_b=0\}}) \} \mathbb{I}_{\{\mu=1\}} + \beta \phi_t (m + b), \end{aligned}$$

and the seller's Bernoulli payoff is

$$U^s(\chi_m, \chi_b, q, d_m, d_b, \mu) = \beta \{ -c(q) + \phi_t (d_m \mathbb{I}_{\{\chi_m=0\}} + d_b \mathbb{I}_{\{\chi_b=0\}}) \} \mathbb{I}_{\{\mu=1\}}.$$

To solve for an equilibrium, we use the reverse ordered game. First, the buyer determines his DM offer; Second, he decides whether or not to counterfeit money and/or bonds; Third, he chooses his portfolio of genuine assets; Fourth, the seller accepts or rejects the offer. Following an offer (q, d) and a decision to counterfeit (χ_m, χ_b) , the buyer's choice of genuine assets satisfy $m \geq d_m$ if $\chi_m = 0$, and $b \geq d_b$ if $\chi_b = 0$.

Consider a subgame following an offer (q, d_m, d_b) . Let ω_0 (resp., $\omega_b, \omega_m, \omega_2$) denote the probability for the buyer to choose $(\chi_b, \chi_m) = (0, 0)$ (resp., $(1, 0), (0, 1), (1, 1)$). By definition, $\omega_0 + \omega_m + \omega_b + \omega_2 = 1$. A necessary condition for an offer to be accepted is $\omega_2 < 1$; that is, in equilibrium a seller will never accept an offer in which both assets are counterfeited with probability one. It is clear that producing a counterfeit of an asset with probability one cannot be part

show that any other offer (q, d_m, d_b) that does not satisfy $\pi = \omega_0 = 1$ generates a payoff less than what is implied by (65)-(68).

Equilibria such that $\Omega_0 > \max(\Omega_m, \Omega_b, \Omega_2)$. In such equilibria, $\omega_0 = 1$ and $\omega_m = \omega_b = \omega_2 = 0$. The conditions $\Omega_0 > \Omega_m$ and $\Omega_0 > \Omega_b$ imply

$$\begin{aligned}\phi_t d_m &< \frac{k_m}{\gamma - \beta + \beta\sigma\pi}, \\ \phi_t d_b &< \frac{k_b}{\rho\gamma - \beta + \beta\sigma\pi}.\end{aligned}$$

It also implies $2\Omega_0 > \Omega_m + \Omega_b = \Omega_0 + \Omega_2$, and hence $\Omega_0 > \Omega_2$. The buyer's expected payoff is $-(\rho\gamma - \beta)\phi_t d_b - (\gamma - \beta)\phi_t d_m + \beta\sigma\pi[u(q) - \phi_t(d_m + d_b)]$. If $\pi < 1$, then $c(q) = \phi_t(d_b + d_m)$. An infinitesimal increase in d_m or d_b makes the offer acceptable, $\pi = 1$, and raises the buyer's payoff. Among all offers such that $\pi = 1$, the supremum of the buyer's payoff is reached for the solution to (65)-(68). Denote this supremum U^* . Finally, notice that when the following three conditions are satisfied,

$$\begin{aligned}\phi_t d_m &< \frac{k_m}{\gamma - \beta + \beta\sigma}, \\ \phi_t d_b &< \frac{k_b}{\rho\gamma - \beta + \beta\sigma}, \\ c(q) &< \phi_t(d_b + d_m),\end{aligned}$$

then the equilibrium of the subgame following the offer (q, d_m, d_b) is unique. Consequently, the supremum of the buyer's payoff in any equilibrium of the counterfeiting game is equal or greater than U^* .

Equilibria such that $\Omega_m = \Omega_0 > \max(\Omega_b, \Omega_2)$. In such equilibria $\omega_0 + \omega_m = 1$ and $\omega_b = \omega_2 = 0$. By (74), $\Omega_b + \Omega_m = \Omega_0 + \Omega_2$, and thus, $\Omega_b = \Omega_2$. The conditions $\Omega_m = \Omega_0$ and $\Omega_0 > \Omega_b$ imply

$$\begin{aligned}\pi &= \frac{k_m - (\gamma - \beta)\phi_t d_m}{\beta\sigma\phi_t d_m}, \\ \phi_t d_b &< \frac{k_b}{\rho\gamma - \beta + \beta\sigma\pi}.\end{aligned}$$

The buyer's expected payoff is $-k_m - (\rho\gamma - \beta)\phi_t d_b + \beta\sigma\pi[u(q) - \phi_t d_b]$. If $\pi < 1$, then reducing d_m will increase π and the buyer's payoff is increased as well. If $\pi = 1$, then (69) implies

$$c(q) \leq \phi_t[d_b + (1 - \omega_m)d_m].$$

For given d_b , the supremum of the buyer's payoff among offers such that $\Omega_m = \Omega_0 > \max(\Omega_b, \Omega_2)$ corresponds to the highest value of q consistent with the above inequality, which is obtained when $\omega_m = 0$ and hence $\omega_0 = 1$. Consequently, the optimal offer among equilibria such that $\Omega_m = \Omega_0 > \max(\Omega_b, \Omega_2)$ satisfies $\pi = \omega_0 = 1$ and hence the buyer's payoff is no greater than U^* .

Equilibria such that $\Omega_b = \Omega_0 > \max(\Omega_m, \Omega_2)$. In such equilibria $\omega_0 + \omega_b = 1$ and $\omega_m = \omega_2 = 0$. Since $\Omega_b + \Omega_m = \Omega_0 + \Omega_2$, then $\Omega_m = \Omega_2$. The conditions $\Omega_m = \Omega_2$ and $\Omega_0 > \Omega_m$ imply

$$\begin{aligned} \pi &= \frac{k_b - (\rho\gamma - \beta)\phi_t d_b}{\beta\sigma\phi_t d_b}, \\ \phi_t d_m &< \frac{k_m}{\gamma - \beta + \beta\sigma\pi}. \end{aligned}$$

The buyer's expected payoff is $-k_b - (\gamma - \beta)\phi_t d_m + \beta\sigma\pi[u(q) - \phi_t d_m]$. If $\pi < 1$, then if d_b is reduced, π increases and the buyer's payoff also increases. If $\pi = 1$, then (69) implies

$$c(q) \leq \phi_t[d_m + (1 - \omega_b)d_b].$$

For given d_m , the supremum of the buyer's payoff among offers such that $\Omega_b = \Omega_0 > \max(\Omega_m, \Omega_2)$ corresponds to the highest value of q consistent with the above inequality, which is obtained when $\omega_b = 0$. Consequently, the optimal offer among equilibria such that $\Omega_b = \Omega_0 > \max(\Omega_m, \Omega_2)$ satisfies $\pi = \omega_0 = 1$ and hence the buyer's payoff is no greater than U^* .

Equilibria such that $\Omega_m = \Omega_b = \Omega_0 = \Omega_2$. This requires

$$\pi = \frac{k_b - (\rho\gamma - \beta)\phi_t d_b}{\beta\sigma\phi_t d_b} = \frac{k_m - (\gamma - \beta)\phi_t d_m}{\beta\sigma\phi_t d_m}.$$

If d_b and d_m are such that $\pi < 1$, then the buyer's expected payoff, $-k_b - k_m + \beta\sigma\pi u(q)$, can be raised by reducing both d_b and d_m . Suppose next that d_b and d_m are such that $\pi = 1$. Then,

$$c(q) \leq \phi_t[(1 - \omega_b - \omega_2)d_b + (1 - \omega_m - \omega_2)d_m].$$

The supremum of the buyer's payoff, $-k_b - k_m + \beta \sigma u(q)$, among offers such that $\Omega_m = \Omega_b = \Omega_0 = \Omega_2$ corresponds to $\omega_b = \omega_m = \omega_2 = 0$. Consequently, the optimal offer among equilibria such that $\Omega_m = \Omega_b = \Omega_0 = \Omega_2$ satisfies $\pi = \omega_0 = 1$ and hence the buyer's payoff is no greater than U^* .

Infeasible cases We rule out the following cases:

1. $\Omega_0 < \max(\Omega_m, \Omega_b, \Omega_2)$

As indicated above, there cannot be an equilibrium where counterfeiting any or both assets occurs with probability one. So we can rule out the cases $\omega_2 = 1$, $\omega_2 + \omega_b = 1$, and $\omega_2 + \omega_m = 1$. Suppose $\omega_2 + \omega_m + \omega_b = 1$ with $\Omega_m = \Omega_b = \Omega_2$. From (74) $\Omega_0 = \Omega_m = \Omega_b = \Omega_2$. A contradiction.

2. $\Omega_2 = \Omega_m = \Omega_0 > \Omega_b$; $\Omega_2 = \Omega_b = \Omega_0 > \Omega_m$; $\Omega_m = \Omega_b = \Omega_0 > \Omega_2$.

Consider an equilibrium such that $\Omega_2 = \Omega_m = \Omega_0 > \Omega_b$. Suppose $\Omega_m = \Omega_0$. Then, $\Omega_b + \Omega_m = \Omega_0 + \Omega_2$ implies $\Omega_b = \Omega_2$. A contradiction. By the same reasoning we can dismiss equilibria such that $\Omega_2 = \Omega_b = \Omega_0 > \Omega_m$ or $\Omega_m = \Omega_b = \Omega_0 > \Omega_2$.

Appendix D. Observable portfolios

In the body of the paper we assume that asset holdings are unobservable in a match and hence cannot convey information about the buyer's decision of whether or not to produce counterfeits. In this appendix we will show that the outcome of the counterfeiting game is unaffected if the buyer's choice of asset holdings is common-knowledge in the match. For simplicity, we assume that sellers do not accumulate asset holdings. One can interpret this assumption as a restriction to the set of equilibria where the sellers' acceptance rule and buyers' offer are independent of the seller's asset holdings (in which case the seller has no strict incentives to accumulate asset). Following the methodology of Inn and Wright (2008), we consider the reverse ordered game where observable action are chosen first. The timing of this game is as follows:

1. The buyer chooses the offer (q, d) and his asset holdings (genuine or fake), a . If $a < d$, then the offer, which is not feasible, is interpreted as no trade.
2. The buyer chooses whether to accumulate genuine assets or produce counterfeits.
3. If a match occurs, the seller observes (q, d, a) and decides whether to accept or reject the offer.

If $\phi_{t-1} > \beta(\phi_t + \zeta)$, then it is costly to accumulate genuine asset holdings. Therefore, if a buyer chooses to produce counterfeits, he will never find it optimal to accumulate genuine assets as well. So in the following we can reduce the buyer's strategy in a subgame following (q, d, a) to the probability η of accumulating genuine assets.

Consider the subgame following the choice of (q, d, a) . Given π , a buyer is willing to accumulate genuine assets if

$$-\{\phi_{t-1} - \beta(\phi_t + \zeta)\}a + \beta\pi\{u(q) - (\phi_t + \zeta)d\} \geq -k + \beta\sigma\pi u(q).$$

The decision rule to produce counterfeits is given by

$$\begin{array}{rcl} & < & = 1 \\ \{\phi_{t-1} - \beta(\phi_t + \zeta)\}a + \beta\pi(\phi_t + \zeta)d > k & \implies & \eta = 0 \\ & = & \in [0, 1] \end{array} .$$

Consider triplets (q, d, a) such that $\pi = \eta = 1$. The conditions for such equilibria are

$$\begin{aligned} -c(q) + (\phi_t + \zeta)d &\geq 0, \\ \{\phi_{t-1} - \beta(\phi_t + \zeta)\}a + \beta(\phi_t + \zeta)d &\leq k. \end{aligned}$$

If (q, d, a) is such that $\pi = \eta = 1$, then (q, d, d) is also such that $\pi = \eta = 1$ since $\phi_{t-1} \geq \beta(\phi_t + \zeta)$. If an offer and a choice of portfolio are attributed to a genuine buyer with certainty, then the same offer and a portfolio with no excess asset holdings, $a = d$, are also attributed to a genuine buyer. Moreover, if $\phi_{t-1} > \beta(\phi_t + \zeta)$, then the offer (q, d, d) is strictly preferred by the buyer.

Consider triplets (q, d, a) such that $(\pi, \eta) \in (0, 1)^2$. Then,

$$\begin{aligned} \eta &= \frac{c(q)}{(\phi_t + \zeta)d} \\ \pi &= \frac{k - \{\phi_{t-1} - \beta(\phi_t + \zeta)\}a}{\beta(\phi_t + \zeta)d}. \end{aligned}$$

The probability of an offer being accepted is decreasing in a . Since it is costly for a buyer to hold excess asset holdings when $\phi_{t-1} > \beta(\phi_t + \zeta)$, the seller will believe that a buyer who holds more asset than what he intends to spend is more likely to be a counterfeiter. So if $\phi_{t-1} > \beta(\phi_t + \zeta)$, then it will never be optimal for a buyer to choose $a > d$. We can follow the same reasoning for Nash equilibria such that $\pi = 1$ and $\eta < 1$ or $\pi \in (0, 1)$ and $\eta = 1$. Provided that $\phi_{t-1} > \beta(\phi_t + \zeta)$ it is never optimal for a buyer to choose $a > d$. Given this result, one can follow the proof of Proposition 1 to show that the offer is the same as in the case where money holdings are not observable.