Binary Data, Hierarchy of Attributes, and Multidimensional Deprivation

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Abstract

Empirical estimation of multidimensional deprivation measures has gained momentum in the last few years. Several existing measures assume that deprivation dimensions are cardinally measurable, when, in many instances, such data is not always available. In this paper, we propose a class of deprivation measures when the only information available is whether an individual is deprived in an attribute or not. The framework is then extended to a setting in which the multiple dimensions are grouped as basic attributes that are of fundamental importance for an individual's quality of life and non-basic attributes which are at a much lower level of importance. Alternative intuitive notions of priority of basic attributes are proposed. Empirical illustrations of the proposed measures are provided based on estimation of multidimensional deprivation among children in Ethiopia, India, Peru and Vietnam.

Keywords: binary data, children, deprivation, hierarchy, multiple dimensions, poverty

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1. Introduction

In the last few years, assessment of multidimensional deprivation has emerged at the forefront of poverty research. Measurement of multidimensional poverty levels are a high-priority topic

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of research with enormous policy implications (Permanyer, 2014). Theoretically, the importance of attributes of well-being, besides income, has long been recognized (e.g. Hicks and Streeten's (1979) basic needs approach, Sen's (1985) capabilities approach). However the empirical exercise of measuring multidimensional deprivation within countries has gained momentum in recent years. Starting in 2010, the United Nations Human Development Report publishes annually, ranking of more than hundred countries based on a multidimensional poverty index. The European Commission estimates multidimensional poverty among its member nations and so do statistical agencies in other countries such as Mexico, Columbia, Bhutan and Philippines. Despite the rapid pace at which the empirical literature is growing, the existing deprivation measures have not been quite amenable for estimation purposes.

In particular we question two key assumptions made in the literature. The first assumption concerns measuring the extent of deprivation. It is usually assumed that each of the multiple attributes or dimensions under consideration is cardinally measurable along real intervals. However, in many instances, it is simply difficult to collect cardinal data and only ordinal data are available. For instance, in order to measure material deprivation or an asset index, typically the only data available is whether or not a household has a working toilet, a television set and so on. A second, related assumption concerns aggregating the multiple deprivations. Again, it is typically assumed that an individual's deprivations along different dimensions are substitutable, so that, other things remaining the same, a small increase in an individual's deprivation along any dimension can be offset by a suitable simultaneous decrease in the individual's deprivation along any other dimension.[1] Yet, intuitively we often treat deprivation in certain attributes, such as food, as basic and more fundamental than deprivation in other attributes such as access to internet. Given a multiplicity of attributes, policy makers often like to prioritize by focusing primarily on the removal of deprivation in terms of some of these attributes, which are considered to be basic, and relegating to a second place the objective of removing deprivation in terms of the other attributes, which are considered to be non-basic. Thus, we believe that both the assumptions are too strong and often difficult to fulfill when working with empirical data. In this paper, we propose a class of measures of multidimensional

[1] Both the assumptions are implicitly made in several existing measures; see for example Bourguignon and Chakravarty (2003), Chakravarty and Silber (2007), Duclos, Sahn and Younger (2006), Dutta, Pattanaik and Xu (2003) and Tsui (2002).

deprivation, which dispenses with these assumptions.

Consider first the assumption of cardinal measurability of attributes along real intervals. It is a convenient assumption, which allows one to use, for each attribute, a cardinal measure of an individual's deprivation in terms of the attribute- the measure being simply the shortfall, if any, of the individual's achieved level for the attribute from a pre-specified benchmark, normalized through division by the benchmark itself. The difficulty, however, is that many important attributes, such as health, nourishment, education, and housing do not lend themselves very well to cardinal measurement. Often the reason is that what is regarded as a single attribute is itself multidimensional, and, though each of the dimensions of such an attribute may be cardinally measurable, there may not be general agreement about how to aggregate over these different dimensions of an attribute so as to have an overall cardinal measure of the attribute itself. Consider health. There are numerous dimensions of a person's health - the person's weight, her level of energy, frequency of her illnesses, her blood pressure, the level of her cholesterol, and so on. Many of these separate dimensions are amenable to cardinal measurement, but it is extremely difficult to provide a cardinal measurement of an individual's overall health. Nor is it practical to include in one's analysis all these different dimensions of health as separate attributes on their own. The same seems to be true of many other attributes including nourishment, housing, and even education. In fact, in the case of many attributes, such as health and overall nourishment, there may not even be an agreement about a sufficiently discriminating ordinal measurement of deprivation. Thus, people may fail to agree about whether a given person's health is “not quite good” or “poor” or “very poor”.

We do not claim that cardinal measurement is never possible for important attributes. In principle, leisure, seen as hours outside work, seems to be cardinally measurable. Similarly, in considering nourishment, if we focus on calorie consumption as an exclusive “proxy” for nourishment, then we shall have cardinal measurability for nourishment, though such measurability would be secured at the cost of the narrowness of our interpretation of nourishment. We would, however, like to emphasize that the assumption of cardinal measurement is demanding and unlikely to be satisfied for many crucial dimensions of deprivation. Ideally, a general theory of multidimensional deprivation should permit all different (ordinal and cardinal) forms of measurement for attributes. But, in the absence of such a general theory, it is still of interest to investigate how far one can go with just binary, ordinal measurement.

In this paper, we work with a coarse form of ordinal measurement of a person's deprivation.

Suppose for every attribute, there are exactly two levels of deprivation: either the individual is deprived along that dimension (in which case her deprivation along the dimension takes the value 1) or she is not (in which case her deprivation along the dimension takes the value 0).While such binary and ordinal measurement[2] of an individual's deprivation in terms of different attributes offers less rich information than ordinal measurement of deprivation involving more than two levels, there is likely to be much wider agreement about the information that it does offer.[3]

With this simple binary measure of an individual's deprivation in terms of each attribute, the informal basis of our group deprivation measures is given by an $n \times m$ matrix where n is the number of individuals in the group, m is the number of attributes, and, for every individual and every attribute, the corresponding entry in the matrix is either 0 or 1. The problem becomes one of aggregating such a deprivation matrix to reach a single index which reflects the overall deprivation of the group. We explore the structure of different classes of such aggregation procedures. This is one of the purposes of the paper. Two of our main results derive sufficient conditions for the index of overall multidimensional deprivation of a group to be the sum of overall deprivations of all individuals in the group, where the overall deprivation of each individual is an increasing function of the weighted average of that individual's deprivations in terms of the different attributes, the function being the same for all individuals.

A second purpose of this paper is to examine the usual assumption of substitutability between attributes for individuals and to explore alternative ways of relaxing this assumption in our framework. Even when all attributes are cardinally measurable, it is not easy to defend the position that a "small" increase in a child's malnutrition can be offset by a sufficiently large reduction in the child's deprivation in terms of access to information available through radio, television, or internet. This consideration is particularly important when there are exactly two levels of deprivation, 0 and 1, for each attribute, so that we do not have the notion of a "small" increase in an individual's deprivation in terms of an attribute. In this framework, it seems even less defensible to assume that a switch in the nourishment status of a child from 0 ("non-deprived") to 1 ("deprived") can

[2] We consider binary variables which are ordinal. For examples of binary variables which can be nominal, cardinal or ratio-scale, see Alkire et al (2015)

[3] There are relatively more examples in the literature where multidimensional measures use discrete data, for instance, see Alkire and Foster (2011), Bossert, Chakravarty and D'Ambrosio (2012), Laso de la Vega (2010) but fewer examples of measures which use binary data, see, Fusco and Dickes (2006) as an example.

be offset by a switch from 1 to 0 of her deprivation in terms of access to information.

It seems much more plausible to assume that some attributes are of fundamental importance for an individual's quality of life while there are other attributes which are at a much lower level of importance[4], so that an increase from 0 to 1 in an individual's deprivation in terms of an attribute of the former type cannot be offset by decreases from 1 to 0 in the individual's deprivations in terms of attributes of the latter type. In fact, it may be plausible to make even a stronger assumption, namely that, other things remaining the same, an increase (from 0 to 1) in an individual's deprivation in terms of an attribute of the former type cannot be offset by decreases in all the individuals' deprivations in terms of attributes of the latter type.

In this paper, we consider a relatively simple structure where we have exactly two categories of attributes: basic attributes and non-basic attributes.[5] In assessing overall social deprivation, each basic attribute will be assumed to have priority over the class of non-basic attributes. We formulate alternative intuitive notions of priority of basic attributes. Suppose the deprivation status, in terms of some basic attribute j, of an individual, i, changes from 1 (deprived) to 0 (non-deprived), when every other individual's deprivation status in terms of all attributes remains the same and individual i's deprivations in terms of all basic attributes other than j also remain the same, but individual i's deprivations in terms of non-basic attributes change in any way one likes. Then a weak priority of basic attributes requires that the overall group deprivation must decrease compared with the initial situation. For example, in the initial situation, suppose individual i is deprived in a basic attribute, say safe drinking water but owns a television set, a non-basic attribute (this situation is not uncommon in households in the many slums in developing countries). Now suppose there is no change in deprivation levels of other individuals (as well as no change in individual i's deprivation in terms of other basic attributes) , but individual i gains access to safe drinking water. Then regardless of whether that individual still owns a television set or not, the weak priority notion requires overall social deprivation to have reduced from the initial situation.

We also propose a second concept of priority of basic attributes which is much stronger than this. Now suppose the deprivation status of some individual i in terms of a basic attribute j changes from 1 to 0 when individual i's deprivations in terms of basic attributes other than j remain the same, the

[4] Cf. Maslow's (1943, 1954) theory of a hierarchy of human needs.

[5] The analysis can be extended to a setting with more than two tiers in our hierarchical structure for the attributes, though we shall not undertake this exercise here.

deprivations of other individuals in terms all basic attributes remain the same, but the deprivations of all individuals in terms of non-basic attributes change in any way one likes. In this case the strong concept of priority of basic attributes mandates that the overall group deprivation must be lower in the new situation. Continuing with the previous example, suppose initially, individual i lacks access to safe drinking water but individual i and all other individuals in the slum own a television set. In the latter situation, individual i gains access to safe drinking water, but no individual in the slum owns a television set any more. Then the concept of strong priority of basic attributes requires overall group deprivation in the second situation to be lower than in the initial situation. We explore the restrictions, which each of these two types of priority of basic attributes imposes on the weights to be attached to the attributes. We show that the class of such rules is in fact a subclass of the general class of measures formulated with binary data.

Finally, we provide some examples of indices belonging to the proposed class of deprivation measures.We use data from the Young Lives, an international study on childhood poverty, conducted by the University of Oxford. The study conducted household surveys among poor communities in four developing countries: Ethiopia, India, Peru and Vietnam. We compile data which is binary in nature; for example, whether a child is underweight or stunted, whether a child can read or write, and whether the household has access to electricity, sanitation and drinking water. We find that the extent of deprivation among the poor in these countries differs significantly. For instance, 65 percent of the surveyed children in India lacked access to sanitation facilities whereas in Peru only 9 percent of children were deprived of sanitation facilities. In Ethiopia, 85 percent children were illiterate, compared to only 17 percent in Vietnam. Thus it is not obvious which of the four countries had the highest incidence of deprivation in a multidimensional setting. We estimate different multidimensional deprivation measures using alternative functional forms and rank order countries according to the extent of deprivation. We use different classification of basic and non-basic attributes by varying the weighing structure of attributes. Overall, we find that Ethiopia and India had substantially higher multidimensional deprivation among children relative to Vietnam and Peru.

The plan of the paper is as follows. In Section 2, we introduce some basic notation, in Section 3, we formulate several axioms and in Section 4 we use the axioms to characterize the main class of group deprivation measures. In Section 5, we extend our analysis by introducing a hierarchy of attributes. Empirical illustrations of the proposed measures of group deprivation are given in

Section 6. We conclude in Section 7. The proofs of all the results are given in the appendix to the paper.

2. Basic notation

Let $N = \{1, \cdots, n\}$ be a given finite set of individuals with $n \geq 2$ and let $F = \{f_1, \cdots, f_m\}$ be a finite set of attributes with $m \geq 2$; we shall refer to N as the group or society under consideration. Let $M = \{1, \cdots, m\}$. Let $\mathcal{D}$ be the class of all $n \times m$ matrices, D, such that each entry in D is either 0 or 1. The elements of $\mathcal{D}$ will be denoted by $C = (c_{ij}), D = (d_{ij})$, etc., and will be called deprivation matrices. For all $D \in \mathcal{D}$, all $i \in N$ and all $j \in M$, the entry d_{ij} in the deprivation matrix D will be interpreted as i's level of deprivation in terms of attribute f_j : if $d_{ij} = 0$, then i is not deprived in terms of attribute f_j in the matrix D; on the other hand, if $d_{ij} = 1$, then i is deprived in terms of attribute f_j in deprivation matrix D. For each $i \in N$, let $d_{i\bullet} = (d_{i1}, \cdots, d_{im})$ denote the deprivation status of individual i, in D, along the m dimensions. Similarly, for each $j \in M$, let $d_{\bullet j} = (d_{1j}, \cdots, d_{nj})$ denote the vector of the n individuals' deprivation levels, in D, in terms of the given attribute f_j.

A group deprivation measure (or a deprivation measure for short) is a function h from $\mathcal{D}$ to the closed interval $[0, 1]$. For all $C, D \in \mathcal{D}, h(C) \geq h(D)$ is interpreted as the degree of deprivation that the society has under C is at least as high as the degree of deprivation under D, $h(C) > h(D)$ and $h(C) = h(D)$ being interpreted in a corresponding fashion.

Our central concern is the class of all group deprivation measures h, such that:

(1) for some increasing function $g : [0, 1] \to [0, 1]$ with $g(0) = 0$ and $g(1) = 1$, and some positive constants $\omega_1, \cdots, \omega_m$ with $\omega_1 + \cdots + \omega_m = 1$, we have $[h(C) = \frac{1}{n} \sum_{i \in N} g(\sum_{j=1}^{m} \omega_j c_{ij})$, for all $C = (c_{ij}) \in \mathcal{D}]$.

Let H denote the class of group of all group deprivation measures h which satisfy (1). For all $h \in H$, let E_h be the set of all $(g, \omega_1, ..., \omega_m)$, such that g is an increasing function from $[0, 1]$ to $[0, 1]$ with $g(0) = 0$ and $g(1) = 1$; $\omega_1, \cdots, \omega_m$ are positive constants with $\omega_1 + \cdots + \omega_m = 1$; and $[h(C) = \frac{1}{n} \sum_{i \in N} g(\sum_{j=1}^{m} \omega_j c_{ij})$ for all $C = (c_{ij}) \in \mathcal{D}]$.

Suppose the group deprivation measure h satisfies (1). Then the weighted average, $\sum_{j=1}^{m} \omega_j c_{ij}$ figuring in (1) may be thought of as individual i's overall "nominal" deprivation when i has the deprivation vector $c_{i\cdot}$, such "nominal" deprivation being the multi-dimensional counterpart of the notion of an individual's normalized "shortfall" from the poverty benchmark, which is used in the literature on the measurement of income poverty. Thus, though for every individual attribute, there are only two levels of deprivation, 0 and 1, the overall nominal deprivation of an individual can have many different levels and hence the notion of the "depth" of an individual's overall nominal deprivation is non-trivial. The expression $g(\sum_{j=1}^{m} \omega_j c_{ij})$ figuring in (1) has the obvious interpretation as individual i's overall "real" deprivation when individual i has the deprivation vector $c_{i\cdot}$.

In the following section, we shall provide axiomatic characterization of the class, H, of group deprivation measures and discuss the formal structures of several subclasses of H. One of these subclasses is the class, H', of all group deprivation measures h,such that:

(2) for some increasing function $g : [0,1] \to [0,1]$ with $g(0) = 0$ and $g(1) = 1$, and some positive constants $\omega_1, \cdots, \omega_m$ with $\omega_1 + \cdots + \omega_m = 1$, we have $[h(C) = \frac{1}{n} \sum_{i \in N} \sum_{j=1}^{m} \omega_j c_{ij}$, for all $C = (c_{ij}) \in \mathcal{D}]$.

It is clear that $H' \subseteq H$ and that, when the weights $\omega_1, \cdots, \omega_m$ for the different attributes are such that $[h(C) = \frac{1}{n} \sum_{i \in N} \sum_{j=1}^{m} \omega_j c_{ij})$, for all $C = (c_{ij}) \in \mathcal{D}]$, we have a very specific individual deprivation function g under which the overall nominal deprivation of an individual coincides with her overall "real" deprivation so that the deprivation of the group or the society is simply the average of the overall nominal deprivations of all individuals. Such a group deprivation measure can be viewed as the multidimensional counterpart of the average normalized shortfall familiar in the literature on the measurement of income poverty.

Furthermore, for any $C = (c_{ij}),\ D = (d_{ij}) \in \mathcal{D}$, any $p \in N$ and any $k \in M$, we say that, (i) C and D are (pk)-variant if $c_{pk} \neq d_{pk}$ and $c_{ij} = d_{ij}$ for all $ij \neq pk$; that is, C and D are identical except the pk-th elements, and (ii) C and D are (pk)-invariant if $c_{pk} = d_{pk}$; that is, C and D have the same pk-th elements. A deprivation matrix $D = (d_{ij}) \in \mathcal{D}$ is said to be a simple deprivation matrix if, for some $p \in N$, $d_{i\bullet}$ is the zero vector for all $i \in N \backslash \{p\}$; that is, D is such that there are at least $n-1$ individuals each of whom is not deprived in terms of any attribute.

For all $x \in \{0,1\}^m$, and all $i \in N$, let $C(i,x)$ denote the simple deprivation matrix such that $c_{i\bullet} = x$ and, for every $k \in N - \{i\}, c_{k\bullet}$ is the zero vector. For all $x, y \in \{0,1\}^m$, let $x \geq_{dom} y$ if $h(C(i,x)) \geq h(C(i,y))$ for all $i \in N$ and $x >_{dom} y$ if $h(C(i,x)) > h(C(i,y))$ for all $i \in N$.

For any two deprivation matrices C and D, we say that C dominates D, to be denoted by

$C \bowtie^{dom} D$, if, (i) for each $j \in M$, there is a permutation σ^j over N such that $c_{ij} = d_{\sigma^j(i)j}$ for all $i \in N$, and (ii) for some permutation π over N, $[c_{i\bullet} \geq_{dom} d_{\pi(i)\bullet}$ for all $i \in N]$ and $[c_{i\bullet} >_{dom} d_{\pi(i)\bullet}$ for some $i \in N]$.

3. A class of deprivation measures using binary data

In this section, we axiomatically characterize the class H of group deprivation measures.

3.1. Axioms

We introduce the following properties that are to be imposed on a measure h.

Normalization For all $D = (d_{ij})$ and for all $\delta \in \{0, 1\}$, if $[d_{ij}= \delta$ for all $i \in N$ and all $j \in M]$, then $h(D) = \delta$.

Anonymity. Let σ be a bijection from N to N. Then, for all $C, D \in \mathcal{D}$, if $c_{i\bullet} = d_{\sigma(i)\bullet}$ for all $i \in N$, then $h(C) = h(D)$.

Monotonicity. For all $C = (c_{ij}),\ D = (d_{ij})$, if $(c_{ij} \geq d_{ij}$ for all $i \in N$ and all $j \in M)$ and $C \neq D$, then $h(C) > h(D)$.

Independence. For all $C, D, C', D' \in \mathcal{D}$, and for all $k \in N$, if $[(c_{i\bullet} = d_{i\bullet}$ and $c'_{i\bullet} = d'_{i\bullet}$ for all $i \in N \backslash \{k\}$ and $(c_{k\bullet} = c'_{k\bullet}, d_{k\bullet} = d'_{k\bullet})]$, then $h(C) - h(D) = h(C') - h(D')$.

Additivity(I). For all simple deprivation matrices, $C, D, C', D' \in \mathcal{D}$, and for all $p \in N$ and all $q \in M$, if $c_{pq} = d_{pq} = 1,\ c'_{pq} = d'_{pq} = 0$, C and C' are (pq)-variant, D and D' are (pq)-variant, then $h(C) \geq h(D) \Leftrightarrow h(C') \geq h(D')$.

Additivity (II). For all $C, D, C', D' \in \mathcal{D}$, all $p \in N$, and all $q \in M$, if $c_{pq} = d_{pq} = 1,\ c'_{pq} = d'_{pq} = 0$, C and C' are (pq)-variant, and D and D' are (pq)-variant, then $C \bowtie^{dom} D \Rightarrow C' \bowtie^{dom} D'$.

Normalization is straightforward: if no one in the group N is deprived along any dimension, then the overall deprivation index for N is 0, and if everyone in N is deprived along every dimension, then the overall deprivation index for N is 1. Anonymity requires that the interchange of any two rows of a deprivation matrix does not affect the overall deprivation. It essentially says that the name of an individual has no significance in measuring overall deprivation of the society. It may be noted that in the literature on measuring multi-dimensional deprivation, Anonymity is also called Symmetry. Monotonicity requires that, if every individual under C is as at least deprived as under

D and some individual under C is deprived while the same individual is non-deprived under D, then the overall deprivation level under C is higher than that under D. These three properties are fairly standard in the literature on multi dimensional approach to deprivation, see, among others, Bourguignon and Chakravarty (2003), and Tsui (2002).

Intuitively, Independence requires that the overall deprivation measure is separable with respect to individuals' deprivations: if two deprivation matrices differ only with respect to a single individual's deprivations along the m dimensions, then the difference between the overall deprivations under the two deprivation matrices is independent of all other individuals' deprivations. This property has its root in the literature on measuring deprivation in a uni-dimensional framework, see, for example, Chakraborty, Pattanaik and Xu (2008). In a multi- dimensional framework, it has often been invoked in the discussion of the row-first, column-second procedures for measuring overall deprivation, see, for example, Dutta, Pattanaik and Xu (2003), Pattanaik, Reddy and Xu (2012), and Tsui (2002).[6]

The basic idea underlying Additivity (I) and Additivity (II) is fairly simple and intuitive, though they differ with respect to the applicability of this basic idea to various deprivation situations. The main concern is how the comparison of two deprivation matrices, C and D, changes when the same individual along a given dimension under C and D switches from the same status of being deprived (resp. being non-deprived) to the same status of being non-deprived (resp. being deprived): it requires that the comparison of C and D after such change be analogous to the comparison of C and D before such change. Additivity (I) confines the comparisons to simple deprivation matrices while Additivity (II) applies the comparisons to deprivation matrices that have the dominance relation.These additivity properties essentially require that, the measurement of an individual's overall deprivation is somewhat separable with respect to different attributes. Neither of the two properties, Additivity (I) and Additivity (II), implies the other.

[6] A row-first, column-second procedure corresponds to the procedure that first aggregates each individual's deprivations along the identified dimensions into an deprivation level, and then the overall deprivation for the society is obtained by aggregating those individual deprivation levels.

4. Characterizing group deprivation measures

To begin with, we present the following result which shows that the overall deprivation of the society is the sum of individuals' overall deprivations along the m dimensions if certain axioms are imposed on the overall deprivation of the society. The proofs of the results are given in the Appendix.

Proposition 1. *A deprivation measure h satisfies Normalization, Monotonicity, Anonymity and Independence if and only if there exists an increasing function $\varphi : \{0,1\}^m \to [0,1]$ with $\varphi(0,\cdots,0) = 0$, $\varphi(1,\cdots,1) = 1$, such that, for all $C = (c_{ij}) \in \mathcal{D}$, $h(C) = \frac{1}{n}\sum_{i\in N}\varphi(c_{i\bullet})$.*

Therefore, the combination of Normalization, Anonymity, Monotonicity and Independence to be imposed on a deprivation measure h implies that h is additive across individuals. Our next result characterizes H when the number of attributes is not greater than 4. See the appendix for the proof of this result.

Proposition 2. *Suppose $m \leq 4$. A group deprivation measure h belongs to H if and only if it satisfies Normalization, Monotonicity, Anonymity, Independence and Additivity (I).*

It may be noted the result in Proposition 2 holds for $m \leq 4$. When $m \geq 5$, Normalization, Anonymity, Monotonicity, Independence, and Additivity (I), together, do not guarantee that $h \in H$. An example in the Appendix illustrates the point. The example suggests that, in search for an additive measure, Additivity (I) needs to be replaced. It turns out that if Additivity (I) is replaced by Additivity (II), then we can obtain the result for $m \geq 5$, as reported in Proposition 3. The proof can be found in the appendix.

Proposition 3. *Suppose that n is large relative to m. A deprivation measure h belongs to H if and only if it satisfies Normalization, Monotonicity, Anonymity, Independence and Additivity (II).*

The requirement that n is large relative to m is not stringent in practice. The reason for such requirement is implicit in the proof of the statement (A5) in the appendix where we need a sufficient number of individuals to construct the corresponding deprivation matrices C and D.

5. Introducing a hierarchy of attributes

The basic class of deprivation measures that we have focused on is given by $h(C) = \frac{1}{n}\sum_{i\in N} g(\sum_{j\in M}\omega_j c_{ij})$, where $h(C)$ is the index of group deprivation when the deprivation matrix is C, $g(\sum_{j\in M}\omega_j c_{ij})$

is the overall deprivation of individual i with deprivation vector $c_{i\bullet}$, and $\omega_1, ..., \omega_m$ are the weights for the different attributes. $\sum_{j \in M} \omega_j c_{ij}$ can be interpreted as the overall "nominal deprivation" of individual i; $\sum_{j \in M} \omega_j c_{ij}$ is thus the counterpart of the notion of an individual's normalized shortfall from the poverty benchmark, which figures in the literature on the measurement of income poverty. When we start with binary (0-1) deprivation data for each individual and each attribute, $\sum_{j \in M} \omega_j c_{ij}$ provides a plausible cardinal measure of the depth of overall nominal deprivation of individual i. In this section, we explore the structure of several subclasses of H by introducing a distinction between basic and non-basic attributes.

5.1. A characterization of H'

We first provide a characterization of the class H', i.e., the class of all deprivation measures h which satisfy (2). To do this, we introduce a new property.

Strong Additivity (I). For all $C, D, C', D' \in \mathcal{D}$, all $p \in N$, and all $q \in M$, if $c_{pq} = d_{pq} = 1$, $c'_{pq} = d'_{pq} = 0$, C and C' are (pq)-variant, and D and D' are (pq)-variant, then $h(C) \geq h(D) \Leftrightarrow h(C') \geq h(D')$.

Like Additivity (I), Strong Additivity (I), which was introduced in Pattanaik, Reddy and Xu (2012), imposes restrictions on how the comparison of two deprivation matrices, C and D, changes when the same individual along a given dimension under C and D switches from the same status of being deprived (resp. being non-deprived) to the same status of being non-deprived (resp. being deprived). But the difference between the two arises from the fact that Additivity (I) applies when the initial deprivation matrices are simple deprivation matrices but Strong Additivity (I) stipulates no such restriction for the initial matrices. Strong Additivity (I) implies Additivity (I) but there is no such implication relation in either direction between Strong additivity (I) and Additivity (II).

Proposition 4. *Let h be a group deprivation measure. $h \in H'$ if and only if h satisfies Normalization, Monotonicity, Anonymity, Independence and Strong Additivity (I).*

The class of measures characterized in Proposition 4 has also been obtained in Bossert, Chakravarty, and D'Ambrosio (2013) in a different setting where they deal with a richer domain by including variable societies.

5.2. Basic dimensions and priority of basic dimensions

In addition to H', we now consider subclasses of H which are based on a simple two-fold distinction between what may be called "basic" attributes and "non-basic" attributes.[7] In the introductory section, we sketched the intution of a simple framework where the policy maker distinguishes between basic and non-basic attributes and gives priority to each basic attribute over the entire group of non-basic attributes; we also described two distinct concepts of such priority. We now formally introduce these two notions of priority and study the implications of each of them for group deprivation measures in H. Let F_B denote the set of dimensions that are regarded as basic, and let F_{NB} denote the set of non-basic dimensions. We assume $F_B \neq \emptyset$. Let M_B denote the set of all $j \in M$, such that $f_j \in F_B$, and let M_{NB} denote $M \backslash M_B$. m_B and m_{NB} denote the cardinalities of M_B and M_{NB}, respectively.

We first introduce a property of group deprivation measures, which embodies a weaker notion of priority of basic attributes. What this property requires is that, if the deprivation status of an individual, i, along a basic dimension changes from non-deprived to deprived while her deprivation status remains unchanged along every other basic dimensions and every other individual's deprivation vector remains the same, then, irrespective of any changes in i's deprivation status along non-basic dimensions, the overall group deprivation must increase. Formally,

Weak Priority of Basic Attributes (WPBA). For all $C, D \in \mathcal{D}$, all $i \in N$, all $j \in M_B$, if $[c_{k\bullet} = d_{k\bullet}$ for all $k \in N \setminus \{i\}]$, $[c_{ij} = 1, (c_{ij'} = 0$ for all $j' \in M_{NB})]$ and $[d_{ij} = 0, (d_{ij'} = 1$ for all $j' \in M_{NB}), (d_{ip} = c_{ip}$ for all $p \in M_B \setminus \{j\})]$, then $h(C) > h(D)$.

The next property embodies of a much stronger notion of priority of basic attributes. It requires that, if the deprivation status of an individual, i,along a basic dimension changes from non-deprived to deprived while her deprivation status remains unchanged along every other basic dimensions and every other individual's deprivation status for every basic dimension remains unchanged, then, irrespective of any changes in the deprivation status of i and the other individuals along non-basic dimensions, the overall group deprivation must increase.

[7] An interesting paper by Esposito and Chiappero-Martinetti (2010) also explores hierarchy among the different attributes. Esposito and Chiappero-Martinetti's notion of hierarchy, however, has a somewhat different structure from ours.

Strong Priority of Basic Attributes (SPBA). For all $C, D \in \mathcal{D}$, all $i \in N$, all $j \in M_B$, if $[c_{ij} = 1,\ c_{pk} = 0$ for all $pk \neq ij]$, $[d_{pk} = 0$ for all $p \in N$ and all $k \in M_B]$, $[d_{pk} = 1$ for all $p \in N$ and all $k \in M_{NB}]$, then $h(C) > h(D)$.

Our next proposition clarifies the implications of WPBA for group deprivation measures in H. The proof of the proposition is given in the appendix.

Proposition 5. *Let $h \in H$. Then h satisfies WPBA if and only if, for all $(g, \omega_1, ..., \omega_m) \in E_h$, $\min\{\omega_j : j \in M_B\} > \sum_{j' \in M_{NB}} \omega_{j'}$.*

Since g has the obvious interpretation as a function which specifies the overall deprivation of an individual given her vector of deprivations in terms of the different attributes, the restriction on the weights, $\omega_1, \cdots, \omega_m$,which go with g, in Proposition 5 amounts to evaluating an individual's overall deprivation in a lexicographic fashion: for all $c_{i\bullet}, d_{i\bullet}$, $g(c_{i\bullet}) \geq g(d_{i\bullet}) \Leftrightarrow (\sum_{j \in M_B} \omega_j c_{ij}, \sum_{j \in M_{NB}} \omega_j c_{ij}) \geq_{lex} (\sum_{j \in M_B} \omega_j d_{ij}, \sum_{j \in M_{NB}} \omega_j d_{ij})$, where $\geq_{lex}$ is the standard lexicographic relation defined over $[0, \infty)^2$. To see this, let $c_{i\bullet}, d_{i\bullet}$ be such that, for some $j' \in M_B$, $c_{ij'} = 1$ and $c_{ij} = 0$ for all $j \in M \setminus \{j'\}$, and ($d_{ij} = 0$ for all $j \in M_B$ and $d_{ij} = 1$ for all $j \in M_{NB}$). Then, $g(c_{i\bullet}) = g(\omega_{j'})$ and $g(d_{i\bullet}) = g(\sum_{j \in M_{NB}} \omega_j)$. Note that, from the above results, we have $\omega_{j'} > \sum_{j \in M_{NB}} \omega_j$ implying that $g(c_{i\bullet}) > g(d_{i\bullet})$. On the other hand, $(\sum_{j \in M_B} \omega_j c_{ij}, \sum_{j \in M_{NB}} \omega_j c_{ij}) = (\omega_{j'}, 0)$, and $(\sum_{j \in M_B} \omega_j d_{ij}, \sum_{j \in M_{NB}} \omega_j d_{ij}) = (0, \sum_{j \in M_{NB}} \omega_j)$. Then, $(\omega_{j'}, 0) >_{lex} (0, \sum_{j \in M_{NB}} \omega_j)$.

With SPBA, we can obtain a result similar to Proposition 5, though the weights attached to various dimensions get further restricted.

Finally, we establish a result, Proposition 6, which suggests that a class of lexicographic rankings of deprivation matrices is subsumed in our results; the proof can be found in the appendix. First, we define the lexicographic relation $\geq_{lex}$ over $[0,1]^2$ as usual: for all $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in [0,1]^2$, $(\alpha_1, \beta_1) \geq_{lex} (\alpha_2, \beta_2) \Leftrightarrow [\alpha_1 > \alpha_2$ or $(\alpha_1 = \alpha_2$ and $\beta_1 \geq \beta_2)]$. For a given function g, and a given set, $\{\omega_1, \cdots, \omega_m\}$, of weights, we define

$$\sigma \equiv \min\{|\sum_{i \in N} g(\sum_{j \in M_B} \omega_j c_{ij}) - \sum_{i \in N} g(\sum_{j \in M_B} \omega_j d_{ij})| \neq 0 : \text{ for all } C, D \in \mathcal{D}\}.$$

Proposition 6. *Let $h \in H$. Suppose there exists $(g, \omega_1, \cdots, \omega_m) \in E_h$,such that g is a convex and differentiable function and $\sum_{j \in M_{NB}} \omega_j < \frac{\sigma}{2ng'(1)}$. Then, for all $C, D \in \mathcal{D}$, $h(C) \geq h(D)$ if and only if*

$$(\sum_{i \in N} g(\sum_{j \in M_B} \omega_j c_{ij}), \sum_{i \in N} g(\sum_{j \in M_{NB}} \omega_j c_{ij})) \geq_{lex} (\sum_{i \in N} g(\sum_{j \in M_B} \omega_j d_{ij}), \sum_{i \in N} g(\sum_{j \in M_{NB}} \omega_j d_{ij}))$$

Thus, every possible ranking of deprivation matrices in the lexicographic framework where the overall deprivation function for each individual is suitably restricted can be subsumed in the class of rankings of deprivation matrices in our results for the non-lexicographic framework with a suitable choice of weights for the attributes in the non-lexicographic framework.

5.3. Deprivation-Decreasing Switch and the function g

Propositions 2 and 3 provide characterizations of H, and, by definition, for all $h \in H$ and all $(g, \omega_1, \cdots, \omega_m) \in E_h$, g is an increasing function of the weighted sum of an individual's deprivations along the different dimensions. From Propositions 2 and 3, we do not know much about the function g beyond the fact that it is increasing. If, however, one has further specific intuition about how the overall deprivation of the society should respond to certain changes in individual deprivations, then the function g can be further restricted.

Assume that the society consists of two individuals and that there are three attributes. Suppose $h \in H$ and $(g, \omega_1, \cdots, \omega_m) \in E_h$. Consider the following deprivation matrices:

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then, $h(C) = \frac{1}{2}[g(\omega_1 + \omega_2 + \omega_3) + g(0)] = \frac{1}{2}g(\omega_1 + \omega_2 + \omega_3)$ and $h(D) = \frac{1}{2}[g(\omega_1 + \omega_3) + g(\omega_2)]$. Note that, in C, individual 1 is deprived along each dimension, while individual 2 is non-deprived in each dimension. Intuitively, individual 1 is (unambiguously) more deprived overall than individual 2 in C. Suppose attribute f_2 is the least important among the three attributes, that is, $w_2 < w_1$ and $w_2 < w_3$. In that case, individual 1 is more deprived overall than individual 2 in D also. Suppose, starting with C, the deprivation matrix of the society changes to D. Then, this transition from C to D can be viewed as a "transfer" of 1's deprivation in terms of f_2 to individual 2, the deprivation of both individuals in terms of every other attribute, as well as the ranking of the two individuals in terms of overall deprivation, remaining the same. One may feel that, in this case, the change reduces the overall deprivation of the society, i.e., $h(C) = \frac{1}{2}g(\omega_1+\omega_2+\omega_3) > h(D) = \frac{1}{2}[g(\omega_1+\omega_3)+g(\omega_2)]$. In general, consider the following axiom for group deprivation measures in H.

Deprivation-Decreasing Switch. Let $h \in H$. Then, for all $(g, \omega_1, \cdots, \omega_m) \in E_h$, for all $C, D \in \mathcal{D}$, and for all $i, i' \in N$ and all $j \in M$, if $[(c_{k\bullet} = d_{k\bullet}$ for all $k \in N\backslash\{i, i'\}]$, $c_{i\bullet}$ and $d_{i\bullet}$ are identical except that $c_{ij} = 1$ and $d_{ij} = 0$, $c_{i'\bullet}$ and $d_{i'\bullet}$ are identical except that $c_{i'j} = 0$ and $d_{i'j} = 1$, and $(g(c_{i\bullet}) > g(c_{i'\bullet})$ and $g(d_{i\bullet}) \geq (g(d_{i'\bullet})]$, then $h(C) > h(D)$.

The intuition underlying the axiom, deprivation-decreasing switch, is as follows. Suppose we start with a situation where, individual i is deprived in terms of f_j, but individual i' is not, and, further, i's overall deprivation is higher than that of i'. Now suppose the two individuals switch their deprivation statuses in terms of f_j so that, after the switch, i becomes non-deprived in terms of f_j and i' becomes deprived in terms of f_j, but there is no change in the deprivation status of either i or i' in terms of any attribute other than f_j and there is no change in the deprivation status of any individual other than i and i' in terms of any attribute. Further, suppose that, even after the change, i's overall deprivation is at least as great as that of i'. Then what the axiom of deprivation-decreasing switch stipulates is that the overall deprivation of the society decreases as a result of the change. This intuition is similar to that of "prioritarianism"[8] where the most deprived individuals are given some priority[9].

The following result shows the implication of Deprivation-Decreasing Switch for group deprivation measures in H; we omit the proof of the result which is fairly straightforward. Before stating the result formally, we introduce a notation. For a given vector $\omega = (\omega_1, \cdots, \omega_m) \in (0, 1)^m$ with $\sum_{j\in M} \omega_j = 1$, let $R_\omega = \{t \in [0, 1] : t = \sum_{j\in M} \omega_j x_{\omega j}$ for some $x_{\omega j} \in \{0, 1\}^m\}$.

Proposition 7. *Let $h \in H$. h satisfies Deprivation-Decreasing Switch if and only if, for all $(g, \omega_1, \cdots, \omega_m) \in E_h$, g has the following property:*

(d-convexity): for all $\alpha, \beta \in R_\omega$ and all $\gamma > 0$ with $\alpha - \gamma \in R_\omega$ and $\beta + \gamma \in R_\omega$,[$\alpha - \gamma \geq \beta + \gamma$] $\Rightarrow$ [$g(\alpha) - g(\alpha - \gamma) > g(\beta + \gamma) - g(\beta)$].

[8] See, for example, Bosmans, Ooghe and Lauwers (2013), and Parfit (1997).

[9] The intuition underlying deprivation-decreasing switch may depend on whether attributes are *substitutes* or *complements*, and this intuition may not be entirely compelling when some attributes are "complements" of each other. For a discussion of this point, see Bourguignon and Chakravarty (2003, 2009), and Pattanaik, Reddy, and Xu (2012).

It may be noted that, if g is increasing and convex, then g satisfies d-convexity. For example, a power function $g(t) = t^{\alpha}$, with $\alpha > 1$, satisfies d-convexity.

6. An empirical illustration

In this section, we provide empirical illustrations of deprivation indices that belong to the class of deprivation measures characterized in the previous sections. We measure deprivation among children for the following reasons. First, a family's income often fails to reflect a child's deprivation in terms of nutritional intake, health, access to education, and so on. Hence international organizations such as the UNICEF have adopted a multidimensional approach to measure child deprivation (UNICEF Policy Brief, 2011). Second, data on attributes reflecting a child's well-being is often difficult to collect and is not always available in a cardinal form. It is compiled from household surveys, with questions to the caregivers as to whether a child has any long term health problems, does the child receive adequate meals, whether a child works in the informal sector, and so on. Thus a class of deprivation measures using only binary data is of particular relevance.

We use data from the Young Lives study conducted by the Department of International Development at the University of Oxford. The study conducts large-scale household survey of children and their primary caregivers in poorer communities in four countries, Ethiopia, India, Peru and Vietnam. It is a panel study which follows two cohorts of children-the older cohort of children was born in 1994-1995, and the younger cohort of children was born in 2001-2002. We compile data on the younger cohort using the 2009 round, so that the children in our sample are between age 7 and 9 years old.[10]

6.1. Deprivation Attributes

Deprivation among children is measured in terms of seven attributes: (1) weight-for- age; (2) height-for-age; (3) access to clean drinking water; (4) sanitation (access to flush or septic toilets); (5) literacy; (6) access to electricity; and (7) access to consumer durables. For each of these attributes, a child's deprivation takes exactly one of two values: 1 (deprived) and 0 (not deprived).

[10] The Young Lives datasets from the household and child surveys are available from the UK Data Service for 2002 (Round 1), 2006 (Round 2), and 2009 (Round 3). The datasite notes that 2013 (Round 4) data will be made available in the archive in late-2015.

The standards prescribed by the World Health Organization are used to determine whether a child is underweight (weight-for-age more than two standard deviations below the benchmark), or stunted (height-for-age more than two standard deviations below the benchmark). Literacy, i.e., the ability to read and write without difficulty is determined by tests conducted by the person in charge of the survey. So far as access to consumer durables is concerned, the survey list for all countries includes radio, television, bicycle, motorbike, automobile, landline phone, and mobile phone.[11] We consider a child deprived in terms of access to consumer durables if he/she has access to less than three consumer durables in the relevant list. Table 1 provides a summary of the indicators and the benchmarks used.

Table 1: Attributes indicating Deprivation among Children

Attributes	A child is deprived if
Weight	Underweight, i.e., the weight-for-age z score $<$ -2 std. dev.
Height	Stunted, i.e., the height-for-age z score $<$ -2 std. dev.
Drinking water	No access to tap drinking water
Sanitation	No access to flush/septic toilet
Electricity	No electricity in the house
Literacy	Unable read or write without problems
Consumer Durables	Household has less than 3 consumer durables

Table 2 lists the percent of children deprived in each attribute in the four countries. There seems to be substantial variability across countries in terms of the incidence of deprivation. About 45 percent children in India were underweight, 35 percent in Ethiopia and only 5 percent in Peru. Roughly 20 percent children in each country were stunted. Few (3 percent) children in India had no access to drinking water or electricity. But a majority of children (81 percent) in Vietnam and (50 percent) in Ethiopia lived in households with no drinking water or electricity. Only 9 percent children in Peru lived in households with no sanitation compared to 65 percent children in India, where according to the 2011 census nearly half of the population had no access to a latrine.

[11] In addition to the common items listed, the surveys include refrigerator and fan for India and Vietnam, sofa and bedstead for Ethiopia, and refrigerator, iron, blender, and stove for Peru.

Table 2: Percent of Deprived Children across Countries

Attributes	Ethiopia	India	Peru	Vietnam
Weight	35	45	05	25
Height	21	29	20	20
Drinking water	51	03	20	81
Sanitation	43	65	09	38
Electricity	52	03	14	03
Literacy	85	63	40	17
Consumer Durables	54	41	23	09

Source: The Young Lives Dataset, 2009

Percent of children who had difficulty in reading and writing was high in most countries, with Ethiopia having highest (85 percent) rate, and Vietnam with the least (17 percent) illiteracy rate. Relatively few children in Vietnam (9 percent) but more than 50 percent children in Ethiopia lived in households with two or less consumer durables. Given the significant variability in the incidence of deprivation in each attribute, we are interested in estimating measures of multidimensional deprivation in each country.

6.2. Illustrations of Deprivation Measures

Using our analytical framework, we now measure and compare the overall deprivation of children in each country (rank 1 denotes the highest level of deprivation, rank 2 denotes the second highest level of deprivation, and so on). We have $n = 2000$, number of children in each country, and $m = 7$ attributes.[12] Recall that our basic result for the calculation of overall deprivation of a group is given by $h(C) = \frac{1}{n}\sum_{i\in N} g(\sum_{j=1}^{m} \omega_j c_{ij})$ for all $C \in \mathcal{D}$ where $c_{ij} = 1$, if the i-th individual is deprived in attribute $f_j$and $c_{ij} = 0$ otherwise, w_j is the weight attached to attribute f_j and g specifies individual's overall deprivation function. We shall consider two different forms for the function g, namely: $g(t) = t$ and $g(t) = t^2$.

[12] The exact sample size for each country after removing missing values is: 1875 Ethiopia, 1911 India, 1916 Peru and 1904 Vietnam.

It may be recalled that earlier we suggested the interpretation of $\sum_{j=1}^{m} w_{ð}`---c_{ij}$ as individual i's overall nominal deprivation and the interpretation of $g(\sum_{j=1}^{m} w_j c_{ij})$ as individual i's overall real deprivation. Given these interpretations, $g(t) = t$ will amount to eliminating the distinction between an individual's nominal deprivation and her real deprivation. Then in this case, the overall deprivation, $\sum_{i \in N} g(\sum_{j=1}^{m} w_j c_{ij})$, of the society will be the counterpart of the "average of normalized shortfalls" in the literature on the measurement of a group's income poverty. Similarly, given our interpretation of $\sum_{j=1}^{m} w_j c_{ij}$ as individual i's overall nominal deprivation, overall social deprivation for the case where $g(t) = t^2$ will be analogous to the "mean of squared normalized shortfalls" in the literature on the measurement of a group?s income poverty.

6.2.1. Substitutability between Attributes

We start by making no distinction between basic attributes and non-basic attributes. Then the deprivation measure $h(C)$ with $g(t) = t$ illustrates the class of measures which satisfy Normalization, Monotonicity, Anonymity, Independence and Strong Additivity (I) (Proposition 4) and the deprivation measure $h(C)$ with $g(t) = t^2$ illustrates the class of measures which satisfy Normalization, Monotonicity, Anonymity, Independence, Additivity (II) and Deprivation-decreasing switch (Propositions 3 and 7).

Table 3: Substitutability between Attributes

Case	Attributes (weights)		Countries	$g(t)=t$	Rank	$g(t)=t^2$	Rank
0	Weight	Electricity	Ethiopia	0.485	1	0.294	1
	Height	Literacy	India	0.359	2	0.179	2
	Drinking water	Durables	Vietnam	0.275	3	0.121	3
	Sanitation	(1/7)	Peru	0.191	4	0.080	4
	(1/7)						
I	Weight	Drinking water	Ethiopia	0.440	1	0.256	1
	Height	Sanitation	India	0.362	2	0.194	2
	(2/9)	Electricity	Vietnam	0.263	3	0.122	3
		Literacy	Peru	0.178	4	0.072	4
		Durables					
		(1/9)					
II	Weight	Electricity	Ethiopia	0.478	1	0.281	1
	Height	Durables	India	0.382	2	0.201	2
	Drinking water	(1/12)	Vietnam	0.311	3	0.149	3
	Sanitation		Peru	0.190	4	0.077	4
	Literacy						
	(2/12)						

Source: Authors' calculation

Table 3 contains estimates of the deprivation measures and the resulting country ranking. Suppose all attributes are weighed equally, i.e., $w_1 = ... = w_7 = \frac{1}{7}$. In this case (Case 0), when we use the function $g(t) = t$, the overall deprivation index $h(C)$ gives the sum of deprivations $\left(\sum_{i\in N}\sum_{j=1}^{m} c_{ij}\right)$ as a proportion of the maximum deprivations possible present in a country $(n \times m)$. Ethiopia has the highest deprivation level, followed by India, Vietnam and Peru has the lowest deprivation level. Now suppose we assign different weights to attributes. Note that the weights are assigned such that they do not impose any lexicographic ordering among the attributes, i.e. the weights do not satisfy the conditions specified in Section 5. In case I being underweight or stunted is given greater weight whereas in case II all attributes except access to electricity and durables are given greater weights. The values of the deprivation index in each country varies as we alternate the weights on the attributes but there is no change in country ranking in terms of the extent of multidimensional deprivation.

6.2.2. Weak Priority of Basic Attributes

We now make a distinction between basic attributes and non-basic attributes. The deprivation measure $h(C)$ with $g(t) = t$ illustrates the class of measures in Proposition 4 and $h(C)$ with $g(t) = t^2$ in Proposition 7, which in addition to the properties listed in Propositions 4 and 7 also satisfy the property of weak priority of basic attributes (WPBA).

Table 4: Non-substitutability between Basic and Non-Basic Attributes

Case	Basic (weights)	Non-Basic (weight)	Countries	$g(t) = t$	Rank	$g(t) = t^2$	Rank
I	Weight	Drinking water	Ethiopia	0.365	2	0.223	2
	Height	Sanitation	India	0.367	1	0.235	1
	(6/17)	Electricity	Vietnam	0.244	3	0.140	3
		Literacy	Peru	0.155	4	0.071	4
		(1/17)					
II	Weight	Electricity	Ethiopia	0.475	1	0.278	1
	Height	Durables	India	0.391	2	0.211	2
	Drinking water	(1/17)	Vietnam	0.325	3	0.162	3
	Sanitation		Peru	0.190	4	0.077	4
	Literacy						
	(3/17)						
III	Weight	Electricity	Ethiopia	0.355	2	0.220	2
	Height	Durables	India	0.376	1	0.251	1
	(3/8)	(1/32)	Vietnam	0.255	3	0.153	3
	Drinking water		Peru	0.152	4	0.073	4
	Sanitation						
	Literacy						
	(1/16)						
IV	Weight	Electricity	Ethiopia	0.299	2	0.222	2
	Height	Durables	India	0.374	1	0.293	1
	(15/32)	(1/128)	Vietnam	0.230	3	0.172	3
	Drinking water		Peru	0.135	4	0.082	4
	Sanitation						
	Literacy						
	(1/64)						

Source: Authors' calculation.

In Table 4 we revisit case I and case II as listed in Table 3. Unlike in Table 3, however the weights chosen in Table 4 are such that they ensure a weak, lexicographic priority of basic attributes over non-basic attributes. For instance in case I, we treat the anthropometric attributes as basic and all the other attributes as non-basic.The weights are chosen such that, suppose w_1, w_2 are weights attached to weight-for-age and height-for-age respectively, then $\min\{w_1, w_2\} > (w_3 + w_4 + w_5 + w_6 + w_7), w_j > 0$ for all j and $\sum_{j \in M} \omega_j = 1$. A lexicographic priority implies the following. Suppose a child's deprivation status for a basic attribute, say, weight-for-age, changes from normal to underweight, while her deprivation status remains unchanged for the other basic attribute, namely, height-for-age. Then, no favorable changes of her deprivation status in non-basic attributes can possibly offset the unfavorable change in her status with respect to weight. On comparing the rankings in Table 3 and 4, we find that in case I, there is a switch in ranking between Ethiopia and India. Recall that in case I, the weight of each of the basic attribute in Table 3 is about 0.22 percent, whereas in Table 4 the weight of each basic attribute is increased to 0.35 percent. Hence the extent of multidimensional deprivation India which had the highest proportion of children who are underweight and stunted, exceeds that in Ethiopia. In case II, the ranking is preserved in both the tables.

In cases I and II in Table 4, each of the basic attributes have equal weights. However lexicographic orderings can take many forms depending on the weights assigned. In cases III and IV, we assign higher weights to the anthropometric attributes within the basic attributes. Compared to case II, we find that there is again a switch in the ranking between Ethiopia and India as we place increasingly higher weight on the anthropometric attributes in cases III and IV respectively. Of course, we could repeat this for other identifications of basic and non-basic attributes.

6.2.3. Strong Priority of Basic Attributes

Finally we measure deprivation when the basic attributes, as a group, have lexicographic priority over the group of non-basic attributes in the assessment of overall group deprivation (see Proposition 6). In addition to the previously mentioned properties, the deprivation measures now satisfy the property of SPBA. SPBA requires that, the change of one individual's deprivation along any given basic dimension cannot be compensated by the changes of all the individuals' deprivations along all non-basic dimensions. Thus the weights attached to the attributes need to be further restricted. In particular, the sum of weights of the non-basic attributes should satisfy the inequality: $\sum_{j \in M_{NB}} \omega_j < \frac{\sigma}{2ng'(1)}$.

Consider deprivation measures based on the g function where $g(t) = t$. Suppose each attribute within a class is given equal weight and suppose that there are k basic attributes. Then the above inequality can be simplified as: $\sum_{j\in M_{NB}} \omega_j < \frac{1}{2nk+1}$ and the sum of weights of the basic attributes $\sum_{j\in M_B} \omega_j > \frac{2nk}{2nk+1}$.[13]

Table 5: Strong non-substitutability between Basic and Non-Basic Attributes

Case	Basic (weights)	Non-Basic (weight)	Countries	$g(t) = t$	Rank
I	Weight	Drinking water	Ethiopia	0.281	2
	Height	Sanitation	India	0.373	1
	(4000/8001)	Electricity	Vietnam	0.222	3
		Literacy	Peru	0.129	4
		Durables			
		(1/40005)			
II	Weight	Electricity	Ethiopia	0.468	1
	Height	Durables	India	0.413	2
	Drinking water	(1/40002)	Vietnam	0.361	3
	Sanitation		Peru	0.188	4
	Literacy				
	(4000/20001)				

Source: Authors' calculation.

In Table 5, we use case I and II from the previous tables; in case I the two anthropometric attributes are treated as basic attributes ($k = 2$) whereas in case II, we extend the list to five basic attributes ($k = 5$). Given a large sample size ($n = 2000$), the weight attached to each non-basic attribute is significantly small (< 0.000025) in both the cases. In case I, the weights on the basic anthropometric attributes sum to almost one (0.99987).The weights attached to basic attributes now dramatically differ from those attached to the non-basic attributes. This is because SPBA implies that if a healthy child's status changes to underweight, then this additional deprivation

[13] It may be noted that, in our Proposition 6 , the group size n is fixed, while in our empirical illustrations in Table 5, we are comparing societies with different sizes. It can be checked that, for different group sizes n_1, n_2, n_3, n_4, if $\sum_{j\in M_B} \omega_j > \max\{\frac{2n_1k}{2n_1k+1}, \frac{2n_2k}{2n_2k+1}, \frac{2n_3k}{2n_3k+1}, \frac{2n_4k}{2n_4k+1}\}$, then the result of Proposition 9 continues to hold, and therefore, we can apply the result of Proposition 9 to different group sizes.

in a basic dimension cannot be compensated by additional achievement in all of the non-basic attributes of all the children, including the child herself. In Table 5, case I, we find that India's deprivation level is the highest since India had the largest percent of children who were underweight and stunted among all countries; in Case II we find that the ranking changes back to Ethiopia as the most deprived, followed by India, Vietnam and finally Peru.

7. Conclusion

In this paper, we explored the structure of several classes of group deprivation measures, using an analytical framework which required, in a sense minimal information on deprivation dimensions. The proposed measures were based on data where each individual's deprivation in terms of any given attribute is assumed to take one of two values, 0 and 1. Furthermore, intrinsically and for policy purposes there is a need to rank order the multiple attributes. Our framework is flexible to allow for such hierarchy of attributes. We introduced a simple distinction between basic and non-basic attributes, and showed that when the weights attached to non-basic attributes are sufficiently small, the individual deprivation function takes a lexicographic form with priority being given to the basic attributes. Finally, we provided an empirical illustration by measuring our overall group deprivation measures for children in four developing countries. Overall we found that the extent of multidimensional deprivation among children was higher in India and Ethiopia. lower in Vietnam, and least in Peru. The ranking of India and of Ethiopia was subject to change depending upon the classification of attributes, especially when weight-for-age and height-for-age were treated as priority basic attributes. Further investigation is required to extend our analysis to cover the more general case where we have binarily ordinal measurement of an individual's deprivation for only some of the attributes and more discriminating measurement of deprivation for the other attributes.

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Appendices

Appendix A. Proofs of Propositions

A. Proof of Proposition 1

Proof. It may be checked that, each measure of the class, $h(C) = \frac{1}{n}\sum_{i \in N} \varphi(c_{i\bullet})$, where $\varphi : \{0,1\}^m \to [0,1]$ is an increasing function, satisfies Normalization, Monotonicity, Anonymity and Independence. Therefore, we need only to show that, if a measure h satisfies Normalization, Monotonicity, Anonymity and Independence, then there exists an increasing function $\varphi : \{0,1\}^m \to [0,1]$ such that, for all $C = (c_{ij})_{n \times m} \in \mathcal{D}$, $h(C) = \frac{1}{n}\sum_{i \in N} \varphi(c_{i\bullet})$.

Let h satisfy Normalization, Monotonicity, Anonymity and Independence, and let $C = (c_{ij})_{n \times m} \in \mathcal{D}$. Consider

$$h\begin{pmatrix} c_{1\bullet} \\ c_{2\bullet} \\ \vdots \\ c_{n\bullet} \end{pmatrix} - h\begin{pmatrix} 0_{1\bullet} \\ c_{2\bullet} \\ \vdots \\ c_{n\bullet} \end{pmatrix} \text{ and } h\begin{pmatrix} c_{1\bullet} \\ 0_{2\bullet} \\ \vdots \\ 0_{n\bullet} \end{pmatrix} - h\begin{pmatrix} 0_{1\bullet} \\ 0_{2\bullet} \\ \vdots \\ 0_{n\bullet} \end{pmatrix}$$

By Normalization,

$$h\begin{pmatrix} 0_{1\bullet} \\ 0_{2\bullet} \\ \vdots \\ 0_{n\bullet} \end{pmatrix} = 0$$

and by Independence

$$h\begin{pmatrix} c_{1\bullet} \\ c_{2\bullet} \\ \vdots \\ c_{n\bullet} \end{pmatrix} - h\begin{pmatrix} 0_{1\bullet} \\ c_{2\bullet} \\ \vdots \\ c_{n\bullet} \end{pmatrix} = h\begin{pmatrix} c_{1\bullet} \\ 0_{2\bullet} \\ \vdots \\ 0_{n\bullet} \end{pmatrix} - h\begin{pmatrix} 0_{1\bullet} \\ 0_{2\bullet} \\ \vdots \\ 0_{n\bullet} \end{pmatrix}$$

Therefore,

$$h\begin{pmatrix} c_{1\bullet} \\ c_{2\bullet} \\ \vdots \\ c_{n\bullet} \end{pmatrix} = h\begin{pmatrix} 0_{1\bullet} \\ c_{2\bullet} \\ \vdots \\ c_{n\bullet} \end{pmatrix} + h\begin{pmatrix} c_{1\bullet} \\ 0_{2\bullet} \\ \vdots \\ 0_{n\bullet} \end{pmatrix}.$$

Similarly, it can be shown that, from Independence and Normalization,

$$h\begin{pmatrix}0_{1\bullet}\\c_{2\bullet}\\\vdots\\c_{n\bullet}\end{pmatrix}=h\begin{pmatrix}0_{1\bullet}\\0_{2\bullet}\\c_{3\bullet}\\\vdots\\c_{n\bullet}\end{pmatrix}+h\begin{pmatrix}0_{1\bullet}\\c_{2\bullet}\\0_{3\bullet}\\\vdots\\0_{n\bullet}\end{pmatrix}.$$

By the repeated use of the above method and from Independence and Normalization, we have the following

$$h\begin{pmatrix}c_{1\bullet}\\c_{2\bullet}\\\vdots\\c_{n\bullet}\end{pmatrix}=h\begin{pmatrix}c_{1\bullet}\\0_{2\bullet}\\0_{3\bullet}\\\vdots\\0_{n\bullet}\end{pmatrix}+h\begin{pmatrix}0_{1\bullet}\\c_{2\bullet}\\0_{3\bullet}\\\vdots\\0_{n\bullet}\end{pmatrix}+\cdots+h\begin{pmatrix}0_{1\bullet}\\0_{2\bullet}\\\vdots\\0_{n-1\bullet}\\c_{n\bullet}\end{pmatrix}$$

For each $i \in N$, let $\varphi_i(c_{i\bullet}) = nh(C_i)$ where C_i is the deprivation matrix in which the deprivation vector of individual i is given by $c_{i\bullet}$ and all other individuals' deprivation vectors are zero vectors. Since the choice of $c_{i\bullet}$ is arbitrary, each φ_i is thus a function, given that h is a function. From Normalization, it is clear that $0 \leq \varphi_i(c_{i\bullet}) \leq 1$ and $\varphi_i(c_{i\bullet}) = 0$ when $c_{i\bullet}$ is the 0 vector. By Anonymity, φ_i is the same for all $i \in N$. Let $\varphi_i = \varphi$ for all $i \in N$. By Monotonicity, φ is increasing. By Normalization, $\varphi(1_{i\bullet}) = 1$.

We have therefore shown that, for the above defined function φ, for all $C = (c_{ij})_{n\times m} \in \mathcal{D}$, $h(C) = \frac{1}{n}\sum_{i\in N}\varphi(c_{i\bullet})$. □

B. Proof of Proposition 2

Proof. Suppose $m \leq 4$. Let a deprivation measure h satisfy Normalization, Monotonicity, Anonymity, Independence and Additivity (I). From Proposition 1, there exists an increasing function $\varphi : \{0,1\}^m \to [0,1]$ with $\varphi(0,\cdots,0) = 0$ and $\varphi(1,\cdots,1) = 1$ such that,

$$\text{for all } C = (c_{ij})_{n\times m} \in \mathcal{D},\ h(C) = \frac{1}{n}\sum_{i\in N}\varphi(c_{i\bullet})$$

Therefore, it suffices to show that φ is given by $\varphi(c_{i\bullet}) = g(\sum_{j=1}^{m}\omega_j c_{ij})$ for some increasing function $g : [0,1] \to [0,1]$ with $g(0) = 0$ and $g(1) = 1$, and constants $\omega_1 > 0, \cdots, \omega_m > 0$ such that, $\omega_1 + \cdots + \omega_m = 1$.

Let $i \in N$, and $c_{i\bullet}, c'_{i\bullet}, d_{i\bullet}, d'_{i\bullet} \in \{0,1\}^m$ be such that $c_{i\bullet} - c'_{i\bullet} = d_{i\bullet} - d'_{i\bullet} = (1_j; 0_{-j}) \in \{0,1\}^m$ for some $j \in M$; that is, $c_{i\bullet}$ and $c'_{i\bullet}$ are identical except at the jth component, and $d_{i\bullet}$ and $d'_{i\bullet}$ are identical except at the jth component. Consider the following simple deprivation matrices, $C, C', D, D' \in \mathcal{D}$ such as, the ith rows of C, C', D, D' are given respectively by $c_{i\bullet}, c'_{i\bullet}, d_{i\bullet}, d'_{i\bullet}$, and each of the other rows of each deprivation matrix is the zero vector. Then C and C' are (ij)-variant, and D and D' are (ij)-variant. By Additivity (I), we have

$$h(C) \geq h(D) \Leftrightarrow h(C') \geq h(D') \tag{A1}$$

Then, from (A1), we have

$$\varphi(c_{i\bullet}) \geq \varphi(d_{i\bullet}) \text{ iff } \varphi(c'_{i\bullet}) \geq \varphi(d'_{i\bullet}) \tag{A2}$$

In other words, we have shown that the function φ satisfies the following property:

$$x, y, a, b \in \{0,1\}^m \text{ and for } j \in M = \{1, \cdots, m\},$$

$$\text{if } x - a = y - b = (1_j, 0_{-j}) \in \{0,1\}^m, \text{ then } \varphi(a) \geq \varphi(b) \text{ iff } \varphi(x) \geq \varphi(y)$$

Note that $m \leq 4$. Then, there exist $\alpha_1, \cdots, \alpha_m$, and for all $c_{i\bullet}, d_{i\bullet} \in \{0,1\}^m$, $\varphi(c_{i\bullet}) \geq \varphi(d_{i\bullet}) \Leftrightarrow \sum_{j=1}^m \alpha_j c_{ij} \geq \sum_{j \in M} \alpha_j d_{ij}$ (see Kraft, Pratt and Seidenberg (1959), or Fishburn (1996)). Since $\varphi(c_{i\bullet})$ is increasing in each of its argument, $\alpha_j > 0$ for all $j \in M$. Let $\omega_j = \alpha_j / \sum_{k \in M} \alpha_k$. Then, $\omega_1 > 0, \cdots, \omega_m > 0$ and $\omega_1 + \cdots + \omega_m = 1$. Since φ is a function and given that $\varphi(c_{i\bullet}) \geq \varphi(d_{i\bullet}) \Leftrightarrow \sum_{j=1}^m \omega_j c_{ij} \geq \sum_{j=1}^m \omega_j d_{ij}$ for all $c_{i\bullet}, d_{i\bullet} \in \{0,1\}^m$, there exists an increasing function $g : [0,1] \to [0,1]$ such that $\varphi(c_{i\bullet}) = g(\sum_{j=1}^m \omega_j c_{ij})$. Note that $\varphi(0_{i\bullet}) = 0$ and $\varphi(1_{i\bullet}) = 1$ by Normalization. We then have $g(0) = 0$ and $g(1) = 1$. □

Example 1. Recall that the result in Proposition 2 holds for $m \leq 4$. When $m \geq 5$, Normalization, Anonymity, Monotonicity, Independence, and Additivity (I), together, do not guarantee that $h \in H$. The following example, adapted from Kraft, Pratt and Seidenberg (1959), illustrates the point. For simplicity, we focus on the φ function figuring in Proposition 1. Consider $m = 5$ and a φ function given below:

$$\begin{aligned}
\varphi(0,1,1,0,1) &> \varphi(1,0,0,1,0), \\
\varphi(1,0,0,0,1) &> \varphi(0,1,1,0,0), \\
\varphi(0,0,1,1,0) &> \varphi(0,1,0,0,1), \\
\varphi(0,1,0,0,0) &> \varphi(0,0,1,0,1).
\end{aligned}$$

It can be checked that this φ is consistent with the properties specified in Proposition 1, and satisfies the corresponding property of Additivity (I) figuring in Proposition 2:

$$x, y, a, b \in \{0,1\}^m \text{ and for } j \in M = \{1, \cdots, m\},$$
$$\text{if } x - a = y - b = (1_j, 0_{-j}) \in \{0,1\}^m, \text{ then } \varphi(a) \geq \varphi(b) \text{ iff } \varphi(x) \geq \varphi(y)$$

If the result of Proposition 2 holds, then, we would have, for some $\omega_1 > 0, \omega_2 > 0, \omega_3 > 0, \omega_4 > 0, \omega_5 > 0$:

$$\begin{aligned} \omega_2 + \omega_3 + \omega_5 &> \omega_1 + \omega_4, \\ \omega_1 + \omega_5 &> \omega_2 + \omega_3, \\ \omega_3 + \omega_4 &> \omega_2 + \omega_5, \\ \omega_2 &> \omega_3 + \omega_5. \end{aligned}$$

Note that, from the above inequalities, we have

$$\omega_2 + \omega_3 + \omega_5 + \omega_1 + \omega_5 + \omega_3 + \omega_4 + \omega_2 > \omega_1 + \omega_4 + \omega_2 + \omega_3 + \omega_2 + \omega_5 + \omega_3 + \omega_5$$

or

$$\omega_1 + 2\omega_2 + 2\omega_3 + \omega_4 + 2\omega_5 > \omega_1 + 2\omega_2 + 2\omega_3 + \omega_4 + 2\omega_5$$

a contradiction.

C. Proof of Proposition 3

Proof. Suppose n is relatively large relative to m. Let a deprivation measure h satisfy Normalization, Monotonicity, Anonymity, Independence and Additivity (II). From Proposition 1, there exists an increasing function $\varphi : \{0,1\}^m \to [0,1]$ such that,

$$\text{for all } C = (c_{ij})_{n \times m} \in \mathcal{D},\ h(C) = \frac{1}{n} \sum_{i \in N} \varphi(c_{i\bullet}) \text{ with } \varphi(0, \cdots, 0) = 0 \text{ and } \varphi(1, \cdots, 1) = 1 \quad \text{(A3)}$$

Therefore, it suffices to show that φ is given by $\varphi(c_{i\bullet}) = g(\sum_{j=1}^m \omega_j c_{ij})$ for some increasing function $g : [0,1] \to [0,1]$ with $g(0) = 0$ and $g(1) = 1$, and constants $\omega_1 > 0, \cdots, \omega_m > 0$ such that, $\omega_1 + \cdots + \omega_m = 1$.

We first show the following:

$$\text{For all integer } k \geq 2, \text{ and all } x^p, y^p \in \{0,1\}^m \text{ with } p = 1, \cdots, k, \tag{A4}$$
$$\text{and } (\forall j \in M : |\{j : x_j^p = 1, p = 1, \cdots, k\}| = |\{j : y_j^p = 1, p = 1, \cdots, k\}|),$$
$$\text{if } \varphi(x^p) \geq \varphi(y^p) \text{ for all } p < k, \text{ then } \varphi(y^k) \geq \varphi(x^k)$$

Suppose to the contrary that (A4) does not hold; that is,

$$\text{for some } k \geq 2, \text{ there are } x^p, y^p \in \{0,1\}^m \text{ with } p = 1, \cdots, k \text{ such that} \tag{A5}$$
$$(\forall j \in M : |\{j : x_j^p = 1, p = 1, \cdots, k\}| = |\{j : y_j^p = 1, p = 1, \cdots, k\}|),$$
$$\varphi(x^p) \geq \varphi(y^p) \text{ for all } p \in \{1, \cdots, k\}, \text{ and } \varphi(x^p) > \varphi(y^p) \text{ for some } p \in \{1, \cdots, k\}$$

From $h(C) = \frac{1}{n}\sum_{i\in N}\varphi(c_{i\bullet})$ with $\varphi(0, \cdots, 0) = 0$ and $\varphi(1, \cdots, 1) = 1$ and the definition of $\geq_{dom}$, we have $x^p \geq_{dom} y^p$ for all $p = 1, \cdots, k$ and $x^p >_{dom} y^p$ for some $p \in \{1, \cdots, k\}$. Consider two deprivation matrices C and D defined as follows: the first k rows of C consist of $x^1, \cdots, x^k$, and each of the remaining rows of C is the zero vector, and the first k rows of D consist of $y^1, \cdots, y^k$, and each of the remaining rows of D is the zero vector. Note that $(\forall j \in M : |\{j : x_j^p = 1, p = 1, \cdots, k\}| = |\{j : y_j^p = 1, p = 1, \cdots, k\}|)$. It then follows that, for each $j \in M$, there is a permutation σ^j over N such that $c_{ij} \geq d_{\sigma^j(i)j}$ for all $i \in N$. Then, $C \bowtie^{dom} D$ follows easily. Let $i \in N$ and $j \in M$ be such that $c_{ij} = 1$. Since $\forall j \in M : |\{j : x_j^p = 1, p = 1, \cdots, k\}| = |\{j : y_j^p = 1, p = 1, \cdots, k\}|$, and by Anonymity, we can arrange $y^1, \cdots, y^k$ so that $d_{ij} = 1$. Consider C^1, which is obtained from C by changing its ij-th element from 1 to 0 while keeping all other elements of C unchanged, and D^1, which is obtained from D by changing its ij-th element from 1 to 0 while keeping all other elements of D unchanged. By Additivity (II), we then obtain $C^1 \bowtie^{dom} D^1$. By repeating the above argument and procedure $\sum_{j\in M}|\{j : x_j^p = 1, p = 1, \cdots, k\}| - 1 = s - 1$ times, we finally arrive at the conclusion that $C^s \bowtie^{dom} D^s$ where C^s and D^s are both the zero matrix, a contradiction. Therefore, (A4) holds.

Then, there exist $\alpha_1, \cdots, \alpha_m$, and for all $c_{i\bullet}, d_{i\bullet} \in \{0,1\}^m$, $\varphi(c_{i\bullet}) \geq \varphi(d_{i\bullet}) \Leftrightarrow \sum_{j=1}^m \alpha_j c_{ij} \geq \sum_{j=1}^m \alpha_j d_{ij}$ (see Kraft, Pratt and Seidenberg (1959), or Fishburn (1996)). Since $\varphi(c_{i\bullet})$ is increasing in each of its argument, $\alpha_j > 0$ for all $j \in M$. Let $\omega_j = \alpha_j / \sum_{k\in M}\alpha_k$. Then, $\omega_1 > 0, \cdots, \omega_m > 0$

and $\omega_1 + \cdots + \omega_m = 1$. Since φ is a function and given that $\varphi(c_{i\bullet}) \geq \varphi(d_{i\bullet}) \Leftrightarrow \sum_{j=1}^m \omega_j c_{ij} \geq \sum_{j=1}^m \omega_j d_{ij}$ for all $c_{i\bullet}, d_{i\bullet} \in \{0,1\}^m$, there exists an increasing function $g : [0,1] \to [0,1]$ such that $\varphi(c_{i\bullet}) = g(\sum_{j=1}^m \omega_j c_{ij})$. Noting that $\varphi(0_{i\bullet}) = 0$ and $\varphi(1_{i\bullet}) = 1$, we then have $g(0) = 0$ and $g(1) = 1$. □

D. Proof of Proposition 4

Proof. It is clear that, if $h \in H'$, then h satisfies Normalization, Monotonicity, Anonymity, Independence and Strong Additivity (I). So we shall only prove that, if h satisfies Normalization, Monotonicity, Anonymity, Independence and Strong Additivity (I), then $h \in H'$,i.e., there exist positive constants $\omega_1, \cdots, \omega_m$, with $\omega_1 + \cdots + \omega_m = 1$, such that, for all $C = (c_{ij})_{n \times m} \in \mathcal{D},\ h(C) = \frac{1}{n} \sum_{i \in N} \sum_{j \in M} \omega_j c_{ij}$.

Let h satisfy Normalization, Monotonicity, Anonymity, Independence and Strong Additivity (I). We first show that

$$\text{for all } C, D \in \mathcal{D},\ \text{if, for all } j \in M, \qquad \text{(A6)}$$
$$\#\{i \in N : c_{ij} = 1\} = \#\{i \in N : d_{ij} = 1\},\ \text{then } h(C) = h(D).$$

Suppose to the contrary that (A6) does not hold. Then, for some $C, D \in \mathcal{D}$, we have [for all $j \in M, \#\{i \in N : c_{ij} = 1\} = \#\{i \in N : d_{ij} = 1\}$] and $h(C) \neq h(D)$. Without loss of generality, let $h(C) > h(D)$. Since [for all $j \in M$, $\#\{i \in N : c_{ij} = 1\} = \#\{i \in N : d_{ij} = 1\}$], by Anonymity we can arrange the rows of D so that, for some $i \in N$ and some $j \in M$, $c_{ij} = d_{ij} = 1$. Consider C^1 and D^1 defined as follows: C and C^1 are ij-variant and D and D^1 are ij-variant with $c^1_{ij} = d^1_{ij} = 0$. Then, by Strong Additivty (I), we have $h(C^1) > h(D^1)$. Note that, for all $j \in M, \#\{i \in N : c^1_{ij} = 1\} = \#\{i \in N : d^1_{ij} = 1\}$. By the repeated use of the above argument p times, we have $h(C^p) > h(D^p)$ where both C^p and D^p are the zero matrix, a contradiction. Therefore, (A6) holds.

By Proposition 1, there exists an increasing function $\varphi : \{0,1\}^m \to [0,1]$, such that:

$$\varphi(0, \cdots, 0) = 0, \varphi(1, \cdots, 1) = 1 \text{ and,for all } C = (c_{ij}) \in \mathcal{D},\ h(C) = \frac{1}{n} \sum_{i \in N} \varphi(c_{i\bullet}) \qquad \text{(A7)}$$

Consider $C, D \in \mathcal{D}$, distinct $i, i' \in N$, and $r \in M$, such that:

$$\text{[for all } k \in N - \{i\}, c_{kj} = 0 \text{ for all } j \in M] \text{ and } D \text{ is derived from } C \text{ by replacing,} \qquad \text{(A8)}$$

in C, c_{ir} by $c_{i'r} = 0$ and $c_{i'r} = 0$ by $c_{ir},$ while leaving all other entries of C unchanged.

[Note that, if c_{ir} happens to be 0, then, by our specification, $D = C$, but if $c_{ir} = 1$, then $D \neq C$.]

Given our specfication of C and D, noting (A7), we have

$$h(C) = \frac{1}{n}\varphi(c_{ir}) \text{ and } h(D) = \frac{1}{n}[\varphi(d_i) + \varphi(d_{i'})]. \tag{A9}$$

But, by (A8), for all $j \in M, \#\{i \in N : c_{ij} = 1\} = \#\{i \in N : d_{ij} = 1\}$. Hence, noting (A6), $h(C) = h(D)$. Given (A9), it follows that: $\varphi(c_{ir}) - \varphi(d_i) = \varphi(d_i)$. Recall that, c_{ir} could be any element of $\{0,1\}^m$ and r could be any element of M. Thus, what we have shown is this:

$$\text{for all } x, y, x' \in \{0,1\}^m \text{ and for all } r \in M, \text{ if } y_r = 0, \tag{A10}$$

$$[\text{for all } j \in M - \{r\}, x_j = y_j], \text{ and } [x'_r = x_r \text{ and, for all } j \in M - \{r\}, x^{'}_j = 0].$$

Let s^0 denote any given $\{s^0_1, ..., s^0_m\} \in \{0,1\}^m$. For all $j \in M$,let s^j denote $(0, ..., 0, s^0_{j+1}, s^0_{j+2}, ..., s^0_m)$, and, for all $j \in M$, let z^j denote the vector in $\{0,1\}^m$ such that $[z^j_j = s^0_j$ and, for all $r \in M - \{j\}, z^j_r = 0]$.Then, by (A10), we have:

$$[\varphi(s^0) - \varphi(s^1) = \varphi(z^1_1)];$$
$$[\varphi(s^1) - \varphi(s^2) = \varphi(z^2_2)];$$
$$\cdots \qquad \cdots$$
$$[\varphi(s^{m-1}) - \varphi(0, 0, ..., 0)] = \varphi(z^m_m)].$$

Summing up, and noting that, by our specification, for all $j \in M$, $\varphi(z^j_j) = w_j s^0_j$,we have $\varphi(s^0) - \varphi(0, 0, ..., 0) = \sum_{j \in M} w_j s^0_j$. Since $\varphi(0, 0, ..., 0) = 0$, it follows that $\varphi(s^0) = \sum_{j \in M} w_j s^0_j$. Given that, for all $C \in \mathcal{D}$, $h(C) = \frac{1}{n}\sum_{i \in N} \varphi(c_{i\bullet})$, we have $[h(C) = \frac{1}{n}\sum_{i \in N}\sum_{j \in M} \omega_j c_{ij}$ for all $C \in \mathcal{D}]$. □

E. Proof of Proposition 5

Proof. Let $h \in H$. We omit the straightforward proof of the fact that, if , for all $(g, \omega_1, \cdots, \omega_m) \in E_h$, $\min\{\omega_j : j \in M_B\} > \sum_{j' \in M_{NB}} \omega_{j'}$,then h satisfies WPBA. We only prove that, if h satisfies WPBA, then, for all $(g, \omega_1, \cdots, \omega_m) \in E_h$, $\min\{\omega_j : j \in M_B\} > \sum_{j' \in M_{NB}} \omega_{j'}$.

Suppose h satisfies WPBA. Let $(g, , \omega_1, \cdots, \omega_m) \in E_h$. Consider any two matrices $C, D \in \mathcal{D}$ such that, all $i \in N$, all $j \in M_B$, with $[c_{k\bullet} = d_{k\bullet}$ for all $k \in N \setminus \{i\}]$, $[c_{ij} = 1, (c_{ij'} = 0$ for all

$j' \in M_{NB})]$ and $[d_{ij} = 0, (d_{ij'} = 1$ for all $j' \in M_{NB}), (d_{ip} = c_{ip}$ for all $p \in M_B \setminus \{j\})]$. Given $h \in H$ and h satisfies WPBA, we have

$$h(C) - h(D) = \frac{1}{n}[\sum_{i' \in N\setminus\{i\}} g(\sum_{j\in M} \omega_j c_{i'j}) + g(\sum_{j\in M} \omega_j c_{ij})] - \frac{1}{n}[\sum_{i' \in N\setminus\{i\}} g(\sum_{j\in M} \omega_j d_{i'j}) + g(\sum_{j\in M} \omega_j d_{ij})]$$

$$= \frac{1}{n}[g(\sum_{j\in M} \omega_j c_{ij}) - g(\sum_{j\in M} \omega_j d_{ij})] = \frac{1}{n}[g(\sum_{j\in M_B} \omega_j c_{ij} + \sum_{j'\in M_{NB}} \omega_{j'} c_{ij'}) - g(\sum_{j\in M_B} \omega_j d_{ij} + \sum_{j'\in M_{NB}} \omega_{j'} d_{ij'})]$$

$$= \frac{1}{n}[g(\omega_j + \sum_{p\in M_B\setminus\{j\}} \omega_p c_{ip}) - g(\sum_{p\in M_B\setminus\{j\}} \omega_p d_{ip} + \sum_{j'\in M_{NB}} \omega_{j'})] > 0$$

Since g is increasing, we know that

$$(\omega_j + \sum_{p\in M_B\setminus\{j\}} \omega_p c_{ip}) - (\sum_{p\in M_B\setminus\{j\}} \omega_p d_{ip} + \sum_{j'\in M_{NB}} \omega_{j'}) = \omega_j - \sum_{j'\in M_{NB}} \omega_{j'} > 0$$

Such inequality is true for all $j \in M_B$. Therefore, $\min\{\omega_j : j \in M_B\} > \sum_{j'\in M_{NB}} \omega_{j'}$. This completes the proof of Proposition 5. □

F. Proof of Proposition 6

Proof. Let $h \in H$. Let $(g, \omega_1, \cdots, \omega_m) \in E_h$ be such that g is convex, and differentiable function and $\sum_{j\in M_{NB}} \omega_j < \frac{\sigma}{2ng'(1)}$. Note that, since $h \in H$ and $(g, \omega_1, \cdots, \omega_m) \in E_h$, g is increasing, and $g(0) = 0$. Let

$$\sigma = \min\{|\sum_{i\in N} g(\sum_{j\in M_B} \omega_j c_{ij}) - \sum_{i\in N} g(\sum_{j\in M_B} \omega_j d_{ij})| \neq 0 : \text{ for all } C, D \in \mathcal{D}\}.$$

Consider any $C, D \in \mathcal{D}$. Suppose $\sum_{i\in N} g(\sum_{j\in M_B} \omega_j c_{ij}) > \sum_{i\in N} g(\sum_{j\in M_B} \omega_j d_{ij})$. Let $\delta = \sum_{j\in M_{NB}} \omega_j > 0$. Then, by the Mean-Value theorem, $g(\sum_{j\in M_B} \omega_j d_{ij} + \delta) = g(\sum_{j\in M_B} \omega_j d_{ij}) + g'(t_{d_i})\delta$ for some $t_{d_i} \in [\sum_{j\in M_B} \omega_j d_{ij}, \sum_{j\in M_B} \omega_j d_{ij} + \delta]$. Note that $g'(t_{d_i}) > 0$ because $\delta > 0$ and g is increasing.

Choose $\delta < \frac{\sigma}{2ng'(1)}$.

From the definition of σ, we have

$$\sum_{i\in N} g(\sum_{j\in M_B} \omega_j c_{ij}) > \sum_{i\in N} g(\sum_{j\in M_B} \omega_j d_{ij}) + \frac{\sigma}{2}$$

Note that $\delta < \frac{\sigma}{2ng'(1)}$, or $\sigma/2 > n\delta g'(1)$. From the above, we then have

$$\begin{aligned}\sum_{i\in N} g(\sum_{j\in M_B} \omega_j c_{ij}) &> \sum_{i\in N} g(\sum_{j\in M_B} \omega_j d_{ij}) + \frac{\sigma}{2} \\ &> \sum_{i\in N} g(\sum_{j\in M_B} \omega_j d_{ij}) + n\delta g'(1) \\ &= \sum_{i\in N}(g(\sum_{j\in M_B} \omega_j d_{ij}) + \delta g'(1))\end{aligned}$$

Since g is convex, we have $g'(1) \geq g'(t)$ for any $t \in [0, 1]$. Therefore, from the above,

$$\begin{aligned}\sum_{i\in N} g(\sum_{j\in M_B} \omega_j c_{ij}) &> \sum_{i\in N}(g(\sum_{j\in M_B} \omega_j d_{ij}) + \delta g'(t_{d_i})) \\ &= \sum_{i\in N} g(\sum_{j\in M_B} \omega_j d_{ij} + \delta)\end{aligned}$$

Noting that g is increasing, therefore, $h(C) > h(D)$.

Since g is increasing, when $\sum_{i\in N} g(\sum_{j\in M_B} \omega_j c_{ij}) = \sum_{i\in N} g(\sum_{j\in M_B} \omega_j d_{ij})$, $h(C) \geq h(D) \Leftrightarrow \sum_{i\in N} g(\sum_{j\in M_{NB}} \omega_j c_{ij}) \geq \sum_{i\in N} g(\sum_{j\in M_{NB}} \omega_j d_{ij})$.

When $\sum_{i\in N} g(\sum_{j\in M_B} \omega_j c_{ij}) < \sum_{i\in N} g(\sum_{j\in M_B} \omega_j d_{ij})$, a similar proof shows that $h(C) < h(D)$. $\square$