

Stable near-rational sunspot equilibria

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Abstract

A new class of near-rational sunspot equilibria is identified in economies expressed as non-linear forward-looking models. The new equilibria are natural extensions of the usual sunspot equilibria associated to the linearized version of the economy, and are near-rational in that agents use the optimal linear forecasting model when forming expectations. A generic existence result is established. Stability under learning is also examined: the near-rational sunspot equilibria are found to be E-stable provided that the linearized model's minimal state variable solution is E-stable.

1 Introduction

Dynamic macroeconomic models that include forward-looking agents may exhibit equilibrium multiplicity: there may exist rational expectations equilibria (REE) that depend upon extrinsic stochastic processes, that is, a sequence of shocks that influences the economy only because agents overlapping condition expectations on these shocks. Importantly, this dependency is self-fulfilling: it exists only because agents think it exists. Equilibria that depend upon such extrinsic shocks are called sunspot equilibria, with the shocks themselves referred to as the sunspots.

The possibility that competitive rational expectations models could have self-fulfilling solutions driven by extraneous stochastic processes was demonstrated by various authors, notably through the work of Shell (1977), Azariadis (1981), Cass and Shell (1983), Azariadis and Guesnerie (1986) and Guesnerie (1986).¹ These existence results were originally obtained in simple stylized models, such as the overlapping generations model of money, and the sunspot drivers were typically taken as finite state Markov process; but generic results providing criteria for local equilibrium uniqueness have also been established. Blanchard and Kahn (1980) present a practical technique for determining whether a linear model has a unique equilibrium, and Woodford (1984) shows that local equilibrium uniqueness in a non-linear model is implied by uniqueness in the linearized model.

The method of Blanchard and Kahn may also be used to establish the existence of sunspot equilibria in linear models. Importantly, this existence result is constructive: the equilibria present in an easily analyzed VAR form; and, the extrinsic processes – the sunspots – characterizing the sunspot equilibria in these linear models have (or, at least, can have) continuous support, and are thus more general than the finite state equilibria examined in the earlier literature. This observation has been exploited by many authors toward a number of ends. For example, Farmer and coauthors have developed an entire research program devoted to explaining business cycle co-movements through the incorporation of non-convexities into competitive DSGE models and through the analysis of the sunspot equilibria associated to the linearized versions of these models: see, for example, Farmer and Guo (1994) and Benhabib and Farmer (1994). Separately, a large literature has emerged warning of the dangers of sunspot equilibria resulting from poorly designed policy in DSGE models with price frictions. This literature too relies on the examination of sunspot equilibrium existence in the linearizations of the associated models.

Results establishing the existence of sunspot equilibria in non-linear models are available: Woodford (1984) showed that equilibrium multiplicity in the linearized model implies local equilibrium multiplicity in the non-linear model. Unlike their linearized counterparts, however, the sunspot equilibria associated to the non-linear models are not easily analyzed: the existence result relies on an implicit function theorem and is not constructive in nature; indeed, given a non-linear model, there is no general technique for establishing a closed-form representation, or even a numerical approximation of an equilibrium associated to a sunspot with continuous support.

The existence of sunspot equilibria raises the question of equilibrium selection. The simple presence of exotic equilibria does not justify their importance: why, after all, would we as modelers anticipate that agents would (choose to) coordinate

¹See the extensive survey in Guesnerie and Woodford (1992).

their expectations and actions on some extrinsic process that has no inherent economic immediacy? Woodford (1990) used adaptive learning to provide an answer to this question. Woodford showed, in a non-linear overlapping generations model, that if agents thought certain finite state Markov sunspot processes *might* be relevant for forecasting, these agents would learn that the sunspots *are* relevant: Woodford showed that the economy converged, in an appropriate sense, to the associated sunspot equilibrium.

Subsequent research on the stability under learning of constructible sunspot equilibria associated to linearized models has been less definitive. While certain linear(ized) models are known to have stable sunspot equilibria, Evans and Honkapohja (2001) showed that the sunspot equilibria associated to the model examined by Farmer and Guo (1994), at least for the particular calibration used, were not stable under learning. Evans and McGough (2005a) and Duffy and Xiao (????) extended this instability result to a host of non-convex RBC-type models.

Stability also depends upon the stochastic properties of the sunspot process associated to the equilibrium. For example, in a model previously thought to have no stable sunspot equilibria, Evans and McGough (2005c) found that the equilibria may be stable provided that the associated sunspot process exhibited the appropriate serial correlation, known as the “resonance frequency.” Using this insight, Evans and McGough (2005b) established the existence of stable sunspot equilibria in a variety of New Keynesian specifications.

The research on sunspot equilibria and their stability under learning has raised a number of concerns, a few of which we catalog here.

1. **No non-linear equilibrium recursions.** The challenge of constructing and analyzing continuous-support sunspot equilibria in non-linear models is problematic not only for the modeler, but also (indeed, even more so) for the model’s agents. If we, as theoretical economists, are unable to recursively represent a particular equilibrium and thereby capture the conditional distributions of the endogenous variables, how then do we imagine agents making optimal forecasts? And even if we wish to adopt a learning perspective, what forecasting model do we provide our agents?
2. **The knife-edge of resonance.** The discovery of resonance frequency sunspots has greatly expanded the literature’s catalog of models exhibiting stable sunspot equilibria; however, some researchers have questioned reliance on the existence of extrinsic processes meeting the knife-edge resonance frequency condition.
3. **No general stability results.** Woodford’s stability result has been extended to the general univariate, forward-looking case by Evans and Honkapohja (2003),

provided that the sunspots are finite state. No stability results are available for equilibria in non-linear models associated to sunspots with continuous support. In particular, it is not known whether sunspot stability in a linearized model is, in general, even related to stability of sunspot equilibria in the non-linear model.

In this paper, we develop a new equilibrium concept designed to simultaneously address the above questions and concerns. We take our cue from the literature on bounded rationality and embrace the possibility that our agents have insufficient information and/or cognitive capacity to resolve the economy's endogenous distributions. Instead, we assume agents use simple, linear forecasting models when forming expectations. If the linear forecasting model used by agents is optimal among all similarly specified linear models then the economy is in a near-rational equilibrium. If the linear model includes a conditional dependency upon a sunspot process then the economy is in a near-rational sunspot equilibrium (NRSE).

Because we propose to modify the adopted equilibrium paradigm, and because our modification impinges on the particulars of agent-level forecasting, it is important to provide context for our altered assumptions. Thus we begin by developing a simple overlapping generations (OLG) model, in which we take, as a behavioral primitive, that agents are not necessarily rational forecasters; we then develop the implied dynamic system and show how it reduces to the usual sequence of expectational difference equations when full rationality is imposed.

The OLG laboratory results in a univariate, stochastic forward-looking system, which we then study in a general context, that is, without reference to the functional forms implied by the OLG structure. We establish a generic existence result: if the linearized model is indeterminate then near-rational sunspot equilibria (NRSE) exist. Importantly, while the existence result itself relies on a bifurcation argument and is thus not constructive in nature, NRSE are identified as fixed points to finite dimensional functions and thus easily computed; furthermore the associated equilibrium process has a VAR structure and so is amenable to detailed analysis. This addresses point one.

The sunspot processes associated to NRSE are found to be natural generalizations of the linearized model's resonance frequency sunspots: the processes are serially correlated with the required correlation converging to the associated resonance frequency as the model's curvature (non-linearity) vanishes. However, for given curvature there is an open set of serial correlations corresponding to NRSEs. We conclude that the knife-edge resonance frequency condition is an artifact of the linearization, and point two is addressed.

The linear structure of an NRSE make it amenable to stability analysis: simply provide agents with a linear perceived law of motion that precisely includes the conditioning variables in the NRSE. We find that if the minimal state variable (MSV) solution of the linearized model is stable under learning, and if an NRSE exists then that NRSE is also stable under learning. Noting that, by point two, NRSE exist if and only if resonance frequency sunspot equilibria exist, and recalling that a resonance frequency sunspot equilibrium is stable under learning exactly when the model's MSV solution is stable under learning, it follows that stable sunspot equilibria in the linear model imply stable NRSE in the non-linear model. This addresses point three.

The paper is organized as follows. In Section 2, we develop the overlapping generations laboratory, and use it to justify our assumptions over the bounded rationality of agents. In Section 3 we obtain generic existence results. Stability results are presented in Section 4, and Section 5 concludes.

2 An overlapping generations model

Our general analysis will be conducted within the context of a linear, univariate, a-theoretic, forward-looking model. To motivate the functional form of this model and to explore and expose the requirements of bounded rationality, we first develop a stylized overlapping generations model.

There is a continuum of agents born at each time t indexed by $\omega_t \in \Omega$. Each agent lives two periods, works when young and consumes when old. The population is constant at unit mass. Each agent owns a production technology that is linear in labor and produces a common, perishable consumption good. The agent can sell his produced good in a competitive market for a quantity of fiat currency, anticipating that he will be able to use this currency when old to purchase goods for consumption.

Let $\omega_t \in \Omega$ be the index of a representative agent born in time t . This agent's problem is given by

$$\begin{aligned} \max_{c_{t+1}(\omega_t), n_t(\omega_t), M_t(\omega_t)} \quad & \hat{E}(\omega_t) (u(c_{t+1}(\omega_t)) - \nu(n_t(\omega_t))) \\ \text{subject to} \quad & n_t(\omega_t) = q_t M_t(\omega_t) \quad \text{and} \quad c_{t+1}(\omega_t) = q_{t+1} M_t(\omega_t). \end{aligned} \tag{1}$$

Here, $n_t(\omega_t)$ is the agent's labor supply when young and $n_t(\omega_t)$ is his output. Also, q_t is the time t goods price of money and $c_{t+1}(\omega_t)$ is the agent's planned consumption when old. The expectations operator $\hat{E}(\omega_t)(\cdot)$ denotes the expectation of agent ω_t at time t , taken with respect to his subjective beliefs conditional on the information

available to him. This information includes $n_t(\omega_t)$, $M_t(\omega_t)$ and current and lagged values of q_t .

The first order condition is given by

$$\nu'(n_t(\omega_t)) = \hat{E}(\omega_t) \left(\frac{q_{t+1}}{q_t} u'(c_{t+1}(\omega_t)) \right), \quad (2)$$

and to make our model particularly tractable, we assume that $\nu' = 1$ and

$$u(c) = \frac{1}{1-\sigma} (c^{1-\sigma} - 1).$$

With simplification, we obtain agent ω_t 's decision rules:

$$\begin{aligned} n_t(\omega_t) &= \left(q_t^{\sigma-1} \hat{E}(\omega_t) (q_{t+1}^{1-\sigma}) \right)^{\frac{1}{\sigma}} \\ M_t(\omega_t) &= \left(\frac{1}{q_t} \hat{E}(\omega_t) (q_{t+1}^{1-\sigma}) \right)^{\frac{1}{\sigma}}; \end{aligned}$$

and we note that, as is natural, the quantity of money demanded by agent ω_t at time t , depends on, among other things, the price at time t .

Assuming a constant (unit) supply of money, we obtain the market-clearing condition

$$\int_{\Omega} M_t(\omega_t) d\omega_t = 1,$$

which yields

$$q_t = \left(\int_{\Omega} \left(\hat{E}(\omega_t) (q_{t+1}^{1-\sigma}) \right)^{\frac{1}{\sigma}} d\omega_t \right)^{\sigma}. \quad (3)$$

Equation (3) characterizes the equilibrium price path.

To close the model, we assume homogeneity of expectations so that $q_t = \hat{E}_t q_{t+1}^{1-\sigma}$. If agents hold rational expectations then $q = 1$ is the unique, perfect-foresight steady state. The system may be log-linearized around this steady state to yield $\log q_t = (1-\sigma)E_t \log q_{t+1}$. The steady state is indeterminate if $\sigma > 2$: in this case the expectational feedback parameter is negative and sunspot equilibria exist in both the linearized and non-linear models.

Now let's return to the boundedly rational case. Instead of assuming rationality, we provide agents with the following linear forecasting model for the endogenous variable q_t :

$$q_t = a + b\eta_t, \quad (4)$$

where η_t is an exogenous regressor (assumed Markov, with known conditional distribution) capturing whatever the agents think might be useful for forecasting. Using this forecasting model, agents form expectations as

$$\hat{E}_t q_{t+1}^{1-\sigma} = E \left((a + b\eta_{t+1})^{1-\sigma} \mid \eta_t \right).$$

It follows that the economy's data generating process (DGP) is given by

$$q_t = E \left((a + b\eta_{t+1})^{1-\sigma} \mid \eta_t \right) \equiv G(a, b, \eta_t), \quad (5)$$

where now the expectations operator is taken as rational, and the vector (a, b) indicates the dependence of the DGP on beliefs.

We will say that the economy is in equilibrium (more formally defined below) provided that agents beliefs are optimal in the following sense: the forecasting model given by $(a, b)'$ must be the orthogonal projection of $q_t = G(a, b, \eta_t)$ onto the linear span of $(1, \eta_t)$: in other words, (4) must be the best possible linear model (in $(1, \eta_t)$) for forecasting q_t . We note that this equilibrium notion is an example of the well-known concept of a restricted perceptions equilibrium: see Evans and Honkapohja (2001) and Branch (2004). We also note that the only required agent-level modification is in forecasting behavior: agents are still assumed to make fully optimal decisions given their forecasts.

3 Existence of near rational sunspot equilibria

The analysis of the OLG model suggests our general case, which we now proceed to examine with care. We take as primitive a five-times continuously differentiable function $F : \mathbb{R} \rightarrow \mathbb{R}$ so that $F(0) = 0$. Our model is written

$$y_t = \hat{E}_t F(y_{t+1}), \quad (6)$$

Letting $\beta = F'(0)$ we assume $|\beta| > 1$ so that, under rationality, the model is locally indeterminate at the origin.

A rational expectations equilibrium (REE) of the model is any appropriately bounded stochastic process y_t satisfying (6) when $\hat{E}_t = E_t$.² By assumption, $y_t = 0$ is an REE, often referred to as the minimal state variable (MSV) solution. Because

²The appropriate notion of boundedness is model dependent. In a linear model, typically the process y_t is assumed to be uniformly bounded in the L^∞ norm so that the linearization remains reasonable; in a more general setting alternatives include asymptotic stationarity or that the divergence rate is almost everywhere geometrically bounded.

we have assumed $|\beta| > 1$, we know from Woodford (1984) that the model is locally indeterminate: given any open neighborhood V of the origin, there is a non-MSV equilibrium (a sunspot equilibrium) with support in V ; however, as noted in the Introduction, these sunspot equilibria are, in general, difficult to characterize or even numerically approximate.

3.1 The linearized model

Our construction of near-rational sunspot processes for the nonlinear model (6) is motivated by the corresponding sunspots in the rational linear model. The linearized model associated to (6) is given by

$$y_t = \beta E_t y_{t+1}. \quad (7)$$

We define an REE of this model to be any stationary process y_t satisfying (7). Now let ξ_t be a zero-mean iid process, and with $\lambda = \beta^{-1}$, set

$$\eta_t = \lambda \eta_{t-1} + \xi_t. \quad (8)$$

Then η_t is stationary provided that $|\beta| > 1$. Further, if $\hat{y}_t = \eta_t$ then $E_t \hat{y}_{t+1} = \lambda \eta_t$, so that \hat{y}_t is a solution to (6). In this case, \hat{y}_t is an REE associated to the serially correlated sunspot process η_t . We conclude with the well-known result that if $|\beta| > 1$ then sunspot equilibria exist.

3.2 Near-rational sunspot equilibria

In this section, we define a new equilibrium notion couched in the language and paradigms of bounded rationality. Similar to rational sunspot equilibria, the equilibrium processes we identify will also depend upon extrinsic noise in a self-fulfilling manner: the dependence exists only if agents believe it exists. Unlike sunspot equilibria, however, the new equilibria are easily characterized, and amenable to both numerical and analytical examination.

To develop the notion of NRSE, we embrace bounded rationality as we did in Section 2: we assume agents form expectations using linear forecasting models; and to impart discipline, we require in an NRSE that the agent's forecasting model is optimal among similarly specified linear models.

We consider sunspot processes of the form (8). Let $\lambda(\mu) = \beta^{-1} + \mu$ have modulus less than unity, and let $\{\xi_t\}$ be any zero mean, iid process with continuous density

f_ξ so that the stochastic process

$$\eta_t = \sum_{n \geq 0} \lambda(\mu)^n \xi_{t-n}$$

has unconditional density $f_\eta(\cdot, \mu)$ that is four-times continuously differentiable in μ . We may, for example, assume the ξ_t are iid, normal, with zero mean and variance σ_ξ^2 , so that η_t is normal with zero mean and variance $\sigma_\xi^2/(1 - \lambda(\mu)^2)$.

As suggested in the OLG example above, we provide agents with a linear forecasting model, or perceived law of motion (PLM) in η :

$$y_t = a + b\eta_t. \tag{9}$$

We assume that agents observe η_t and know $\lambda(\mu)$. The data generating process, or actual law of motion (ALM) implied by the agents beliefs (a, b) (together with the associated PLM (9)) is given by

$$y_t = \int F(a + b(\lambda(\mu)\eta_t + \xi)) f_\xi(\xi) d\xi = 0. \tag{10}$$

Writing $E_{\eta\xi}\Xi(\eta, \xi) = \int \int \Xi(\eta, \xi) f_\xi(\xi) f_\eta(\eta) d\xi d\eta$, we may now define a map $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ as follows:

$$\begin{aligned} a &\rightarrow E_{\eta\xi} F(a + b(\lambda(\mu)\eta + \xi)) \\ b &\rightarrow \frac{1}{\sigma_\eta^2} E_{\eta\xi} \eta F(a + b(\lambda(\mu)\eta + \xi)), \end{aligned}$$

that is, T is the projection of $y_t = E_\xi F(a + b(\lambda(\mu)\eta_t + \xi))$ onto the span of $(1, \eta_t)$. Defined in this way, $T(a, b)$ identifies the the MSE-minimizing forecast model conditional on the agent's beliefs (a, b) . In an NRSE, these beliefs correspond to the MSE-minimizing forecast model within the linear class considered. Thus we have the following definition.

Definition. A non-trivial fixed point of the T-map is a *near-rational sunspot equilibrium*.

3.2.1 The simple cubic

To establish existence of NRSE, we must study the fixed points of T , which requires a somewhat tedious two-dimensional bifurcation analysis. Before tackling the general specification of F , we first restrict attention to the case that F is cubic and symmetric about the origin:

$$F(y) = \beta y + \phi_3 y^3.$$

We also assume here that $\sigma_\xi^3 = \sigma_\eta^3 = 0$. All of these assumptions will be relaxed in the general case. The assumed symmetry implies (abusing notation) that $T(0, b) = (0, T(b))$. This reduces the dimension of the problem to one, greatly simplifying the analysis.

Write $\lambda(\mu) = \beta^{-1} + \mu$. Then, emphasizing the dependence on μ , we easily compute

$$T(b, \mu) = \beta b \lambda(\mu) + \phi_3 \theta(\mu) b^3, \quad (11)$$

where

$$\theta(\mu) = \frac{\lambda(\mu)^3 \sigma_\eta^4}{\sigma_\eta^2} + 3\lambda(\mu) \sigma_\xi^2.$$

Given μ , the function T is a cubic in b and so has either one or three fixed points, and in case there are three fixed points, it follows that NRSE exist.

The solutions to $T(b, \mu) = b$ are given by $b = 0$ and

$$b = \pm \left(\frac{-\beta\mu}{\phi_3\theta(\mu)} \right)^{\frac{1}{2}}.$$

Since, for μ near zero, $\theta(\mu)$ is always positive it follows that NRSE exist when $\phi_3 < 0$ and $\mu > 0$, or when $\phi_3 > 0$ and $\mu < 0$. Importantly, there is an open set of “resonance frequencies” near β^{-1} for which NRSE exist: the “knife-edge of resonance” is indeed an artifact of the linearization. Of course our work allows us to conclude much more. We know exactly what the associated sunspots look like, and given the map F , we know how to compute the NRSE.

To prepare ourselves for the work of the next section, it is helpful to revisit the existence question using bifurcation theory. To this end, we interpret the T-map as identifying a dynamic system with rest points corresponding to NRSE. While we could envision this interpretation quite naturally as a discrete time system, for reasons that will become apparent later it will be helpful to work in continuous time. Thus we consider the dynamic system

$$\dot{b} = H(b, \mu) \equiv T(b, \mu) - b, \quad (12)$$

and note that b corresponds to an NRSE provided that $H(b, \mu) = 0$.

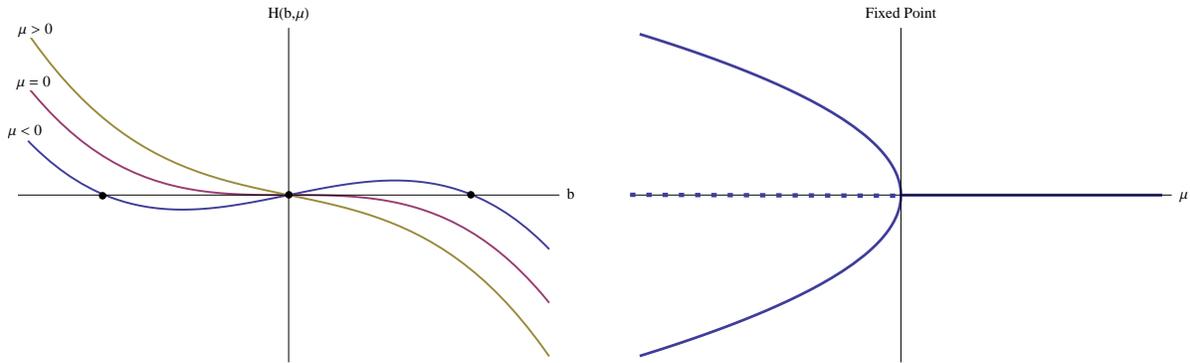
Observe that $H(0, \mu) = 0$ and that $H_b(0, 0) = 0$, indicating that there is a steady state at $b = 0$ and that the system bifurcates at $\mu = 0$. Also, $H_\mu(0, 0) = H_{bb}(0, 0) = 0$, and

$$H_{b\mu}(0, 0) = \beta \neq 0, \quad \text{and} \quad H_{bbb}(0, 0) = 6\phi_3\theta(0) \neq 0, \quad (13)$$

which identifies the occurrence of a pitchfork bifurcation.

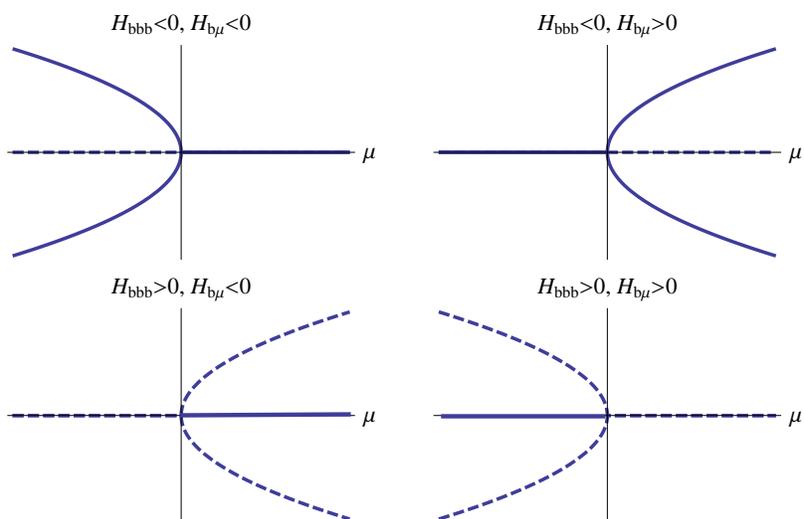
As we have noted, $b = 0$ is always a rest point of (12). A pitchfork bifurcation is characterized by the emergence of two additional rest points as μ crosses the origin from the appropriate direction. Geometrically, as μ crosses zero, the graph of the cubic H morphs to intersect the horizontal axis at two additional points. This phenomenon is witnessed in Figure 1. Here, we have chosen $H_{bbb} < 0$ and $H_{b\mu} < 0$. The left panel shows the graph of $H(b, \mu)$ for μ greater than, equal to, and less than zero, and the right panel identifies the fixed points of (12) for given μ . Notice that for $\mu \geq 0$ there is a unique fixed point, and for $\mu < 0$ there are three fixed points, indicating the existence of NRSE.

Figure 1: Subcritical Pitchfork Bifurcation: $H_{b\mu} < 0$, $H_{bbb} < 0$.



Stability information, which will be useful in the sequel, can also be gleaned from Figure 1. The ode (12) is Lyapunov stable if H_b is negative. By observing the left panel, we see as μ crosses the origin from above, the zero steady state destabilizes and the two emergent steady states – the NRSE – are stable. This example illustrates a more general phenomenon: at a pitchfork bifurcation the stability of the origin flips and the stability of the non-trivial fixed points are opposite the stability of the origin. The exact pattern of emergence and stability depend on the relative signs of $H_{b\mu}$ and H_{bbb} , as indicated in Figure 2. For the cubic case, using (13), these signs are easily translated into conditions on λ and ϕ_3 ; however, and importantly, we note that Figure 2 is general: it holds for any univariate system $\dot{b} = H(b, \mu)$ that undergoes a pitchfork bifurcation at the origin. We will use this fact in Section 4 when studying the stability of NRSE under learning.

Figure 2: Pitchfork Bifurcations: Dashed curves indicate unstable fixed points.



3.2.2 The general case

In this section, we allow F to be general and the odd moments of the sunspot to be non-zero. The essential difference is that we can no longer rely on the existence of NRSE with $a = 0$: the T-map is generically two dimensional. Establishing existence of NRSE proceeds as above, but the bifurcation analysis is more tedious because a center manifold reduction must first be performed. We have the following result, which is stated to emphasize the open set of resonance frequencies consistent with NRSE.

Theorem 1 *Assume that $|\beta| > 1$ and that either of the following two regularity conditions is met:*

1. $\sigma_\xi^3 \neq 0$ and $F''(0) \neq 0$;
2. $\sigma_\xi^3 = 0$ or $F''(0) = 0$, and $F'''(0) \neq 0$.

Then there exists a neighborhood V of β^{-1} so that given any open set $W \subset V$ containing β^{-1} there is a $\lambda(\mu) = \beta^{-1} + \mu \in W$ and a point $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ with the following property: if $\eta_t = \lambda(\mu)\eta_{t-1} + \xi_t$ then $y_t = a + b\eta_t$ is an NRSE.

It is interesting to us that the generic existence conditions turn on the third moment of the sunspot's conditional shock, but brief reflection provides the intuition: if $\sigma_\xi^3 \neq 0$

and $F''(0) \neq 0$ then H^b is $\mathcal{O}(\|(a, b)\|^2)$ so that the associated bifurcation is transcritical; in case $\sigma_\xi^3 = 0$ or $F''(0) = 0$, and $F'''(0) \neq 0$, it follows that H^b is $\mathcal{O}(\|(a, b)\|^3)$ so that the associated bifurcation is pitchfork. The nature of the bifurcation does not impinge on existence; however, there is an interesting implication for stability: see discussion following Theorem 2.

The proof of this theorem is contained in the Appendix. While the details of the proof are somewhat tedious, a discussion of the argument is useful. Letting $\Theta = (a, b)'$, write $H(\Theta, \mu) = T(\Theta, \mu) - \Theta$. Direct computation allows for the following decomposition:

$$H(\Theta, \mu) = \begin{pmatrix} \beta - 1 & 0 \\ 0 & \beta\mu \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} f(a, b, \mu) \\ g(a, b, \mu) \end{pmatrix}, \quad (14)$$

where f and g are $\mathcal{O}(\|(a, b, \mu)\|^2)$.

It is evident that a bifurcation of the system $\dot{\Theta} = H(\Theta)$ occurs at $\mu = 0$. To assess the nature of this bifurcation, we appeal to the center manifold theorem. This theorem guarantees the existence of a sufficiently smooth function $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ characterizing an invariant, parameter-dependent manifold, that is, a differentiable subset $W_c(\mu)$ of \mathbb{R}^2 , tangent to the b -axis, so that

- For μ and b near zero, $W_c(\mu)$ is the graph of $a = h(b, \mu)$.
- $W_c(\mu)$ is invariant under the action of H : $H(W_c(\mu)) \subset W_c(\mu)$.

The invariance of the center manifold may be used to specify a functional equation characterizing h . Specifically, by definition, $\dot{a} = (\beta - 1)a + f(a, b, \mu)$; and, on $W_c(\mu)$, $a = h(b, \mu)$, so that

$$\dot{a} = h_b(b, \mu)\dot{b} = h_b(b, \mu)(\beta\mu b + g(a, b, \mu)).$$

We conclude that h must satisfy the functional equation

$$(\beta - 1)h(b, \mu) + f(h(b, \mu), b, \mu) = h_b(b, \mu)(\beta\mu b + g(h(b, \mu), b, \mu)).$$

Using this equation together with the implicit function theorem allows for the computation of the Taylor expansion of h to arbitrary order.

The importance of the manifold $W_c(\mu)$ follows from a corollary to the center manifold theorem which states that the dynamic behavior of the two-dimensional

system $\dot{\Theta} = H(\Theta)$ is locally equivalent in a natural sense to its behavior on $W_c(\mu)$; and, using h , this behavior is captured by the univariate system

$$\dot{b} = \beta\mu b + g(h(b, \mu), b, \mu). \quad (15)$$

Finally, because this system is univariate, bifurcation analysis proceeds just as in the cubic case. The proof in the Appendix simply involves fleshing out the details of this analysis.

Theorem 1 generically addresses the first two concerns raised in the introduction and identified as motivating this effort. We now know when NRSE exist and what they look like. Further, we know that the resonance frequency restriction is an artifact the linearization procedure: in fact, the sunspot's serial correlation acts a bifurcation parameter in the general case. Finally, and perhaps most interestingly, existence of NRSE obtains *if and only if* rational sunspot equilibria exist. This observation is particularly important from a practical perspective: assessing whether a given model may exhibit NRSE requires no new analytic tools.

4 Stability

Having established the generic existence of NRSE in case $|\beta| > 1$, we now turn to the question of stability under learning. As is standard in the literature and natural given our assumptions regarding the forecasting behavior of agents, we have agents update their beliefs over time using recursive least squares: see Marcet and Sargent (1989) and Evans and Honkapohja (2001). Let $\gamma_t = (a_t, b_t)'$ represent agents' beliefs conditional on information dated t and earlier. These beliefs evolve according to the following recursions:

$$\gamma_t = \gamma_{t-1} + \frac{1}{t} R_t^{-1} \begin{pmatrix} 1 \\ \eta_t \end{pmatrix} \left(E_\xi F(a_{t-1} + b_{t-1}(\lambda\eta_t + \xi)) - \gamma'_{t-1} \begin{pmatrix} 1 \\ \eta_t \end{pmatrix} \right) \quad (16)$$

$$R_t = R_{t-1} + \frac{1}{t} \left(\begin{pmatrix} 1 \\ \eta_t \end{pmatrix} \begin{pmatrix} 1 & \eta_t \end{pmatrix} - R_{t-1} \right),$$

where R_t captures the sample second-moments matrix. The asymptotic behavior of this system may be analyzed by considering the differential system

$$\begin{aligned} \dot{\gamma} &= R^{-1} E_\eta \left(\begin{pmatrix} 1 \\ \eta \end{pmatrix} E_\xi F(a + b(\lambda\eta + \xi)) \right) - R^{-1} M \gamma \\ \dot{R} &= M - R, \end{aligned}$$

where

$$M = E_\eta \left(\begin{pmatrix} 1 \\ \eta \end{pmatrix} \begin{pmatrix} 1 & \eta \end{pmatrix} \right)$$

is the a.e. limit of R_t by the law of large numbers. The stability of this system at a given rest point (γ^*, M) is determined by the stability of

$$\dot{\gamma} = T(\gamma) - \gamma \tag{17}$$

at γ^* . Since γ^* corresponds to a fixed point of the T-map, it identifies an NRSE. The theory of stochastic recursive algorithms tells us that if this fixed point is a Lyapunov stable rest point of (17), then an appropriately modified version of (16) will converge to it:³ the associated NRSE is stable under learning. We note that the ode (17) corresponds to the usual E-stability differential equation, and thus in the sequel, we will rely on E-stability when assessing the stability NRSE under learning.

4.1 Linear analysis

If the model is linear then, as noted above, NRSE correspond to resonance frequency sunspot equilibria: $\lambda = \beta^{-1}$. Assuming agents know λ , it follows that $E_t y_{t+1} = a + b\lambda\eta_t$, so that the actual law of motion is given by

$$y_t = \beta a + b\eta_t.$$

We find that $T(a, b) = (\beta a, b)'$, so that the eigenvalues of DT are β and 1. We conclude that sunspot stability obtains provided that $\beta < -1$.⁴

4.2 The simple cubic

To gain intuition for harder work in the next section, we restrict attention here to PLM's of the form $y_t = b\eta_t$, which is consistent with NRSE in the cubic case. The T-map is given by (11), so that the E-stability ode may thus be written

$$\dot{b} = \beta\mu b + \phi_3\theta(\mu)b^3. \tag{18}$$

³To guarantee almost sure convergence, the learning algorithm must be modified to include a projection facility: see Evans and Honkapohja (2001) for details.

⁴Is it standard, in the stability analysis of sunspot equilibria associated to linear models, for the T-map to have at least one unit eigenvalue. This neutral stability reflects the (artificial) fact that, in a linear environment, any scalar multiple of a sunspot is again a sunspot. For a discussion, see Evans and McGough (2005a).

We learned above that as μ crosses zero from the appropriate side, a pitchfork bifurcation indicates the emergence of three fixed points. Importantly, the bifurcation also switches the stability of the original fixed point, and the new fixed points inherit the stability previously afforded the original fixed point. Thus if before the bifurcation the origin was stable then after the bifurcation the origin is unstable and the two new fixed points are stable. These stability transfers are evident in Figure 1, where we remember that a sufficient condition for Lyapunov stability in the univariate case is that the derivative be negative.

Applying these observations to the system (18), we conclude that if $\text{sign}(\beta) \neq \text{sign}(\phi_3)$ then the cubic NRSE is stable under learning mechanisms consistent with (18). We note that the cubic assumption is not innocuous: as we will see in the next section, requiring that agents include a constant in their regression imparts additional restrictions.

4.3 The general case

Finally, we return to the general case so that the relevant ode is given by (17). The analysis here again proceeds as it did with the cubic, and again, the principal distinction and difficulty is the center manifold analysis. Fortunately, we can rely on all of the hard work already done in the existence proof. We have the following result:

Theorem 2 *If $\beta < -1$ and either*

1. $\sigma_\xi^3 \neq 0$ and $F''(0) \neq 0$; or
2. $\sigma_\xi^3 = 0$ or $F''(0) = 0$, and $F'''(0) \neq 0$ and the following non-generic condition holds:

$$F'''(0) \left(\frac{3\sigma_\xi^2}{\beta} + \frac{\sigma_\eta^4}{\beta^3\sigma_\eta^2} \right) + \frac{3(F''(0))^2 \left(\frac{\sigma_\eta^2}{\beta^2} + \sigma_\xi^2 \right)}{(1-\beta)\beta} < 0;$$

then E-stable NRSE exist.

The proof of this proposition is contained in the Appendix. We note also that the E-stability conditions obtained in the linear model, that $\beta < -1$, are necessary and sufficient for the existence of stable NSRE, provided $F''(0) \neq 0$.

Theorems 1 and 2 provide vindication for resonance frequency sunspot equilibria: the knife-edge requirement needed in linear models is an artefact of the linearization and the tendency of resonance frequency sunspot equilibria to inherit the stability of

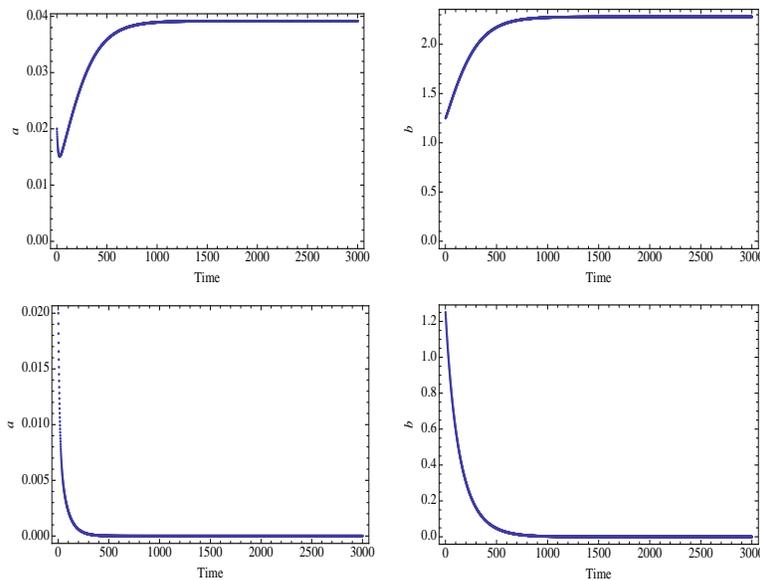
the MSV solutions prevails in the non-linear world. Put differently, by Theorem 2, E-stability of resonance frequency sunspot equilibria in the linear model guarantees the existence of stable NRSE in the non-linear model (provided $F''(0) \neq 0$), which is a striking demonstration of the deep and broad reach of the E-stability principle.

4.4 The OLG model revisited

To see our results in action, recall the reduced form system corresponding to the overlapping generations model of Section 2: $q_t = \hat{E}_t q_{t+1}^{1-\sigma}$. Letting $F(y) = (y+1)^{1-\sigma} - 1$ recovers the system $y_t = E_t F(y_{t+1})$ as studied in this section. We compute $F'(0) = 1 - \sigma$ and $F''(0) = \sigma(\sigma - 1)$, so that stable NRSE exist provided $\sigma > 2$.

To assess this claim numerically, we calibrate the model, setting $\sigma = 5$. Then, choosing an asymmetric martingale difference sequence ξ_t and perturbations $\pm\mu$, we compute the dynamics of the E-stability differential equation: see Figure 3. As predicted by our bifurcation theory, for small perturbations on one side of the critical value ($\mu = 0$), only the fundamental equilibrium ($a = b = 0$) is stable (bottom panel), and for small perturbations on the other side of the critical value, the dynamics converge to an NRSE (top panel).

Figure 3: Learning Dynamics



5 Conclusion

According to Blanchard, “. . . the world economy is pregnant with multiple equilibria – self-fulfilling outcomes of pessimism or optimism, with major macroeconomic implications.”⁵ This conclusion, and others like it, makes imperative understanding when and how these sunspot equilibria are consistent with the DSGE modeling paradigm of the macroeconomic literature.

Investigations of sunspot equilibria in mainstream models has met with a variety of obstacles. Most notably, and as indicated in the Introduction, sunspot equilibria in non-linear models have complicated stochastic structure, making them difficult for researchers and economic agents to model, and thus rendering stability analysis impossible.

Our embrace of a linear-forecasting framework allows us to circumvent this obstacle while preserving natural, agent-level behavior. We establish the existence of (near-rational) sunspot equilibria that have simple recursive stochastic structure. By providing agents an understanding of this structure, we are then able to assess stability under adaptive learning, and indeed establish generic stability results.

It is important to emphasize the link between the existence and stability of NRSE, and the existence and stability of sunspot equilibria in the corresponding rational model. We find that if sunspot equilibria exist in the rational (linearized, and thus the non-linear) model then NRSE exist; and if sunspot equilibria are stable in the linearized model then the NRSE are stable. In fact, an even deeper connection prevails: the extension of an observed phenomenon, which we call *The MSV Principle*, to a non-linear environment. The MSV Principle states that in a linear(ized) model, if the steady state is indeterminate and the MSV solution is stable under learning then there exist stable sunspot equilibria. While we have not formally established this in a completely general setting, in all of our work we know of no counterexample. The power of this principle lies in its computational simplicity: it is often quite easy to identify and analyze the stability of the MSV solution to a linear model. Our work here generalizes the principle as follows: If the steady state is indeterminate and the MSV solution to the linearized model is stable under learning then there exist stable NRSE associated to the non-linear model.

⁵IMF blog, <http://blog-imfdirect.imf.org/2011/12/21/2011-in-review-four-hard-truths/>

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6 Appendix

Proof of Theorem 1. To employ the center manifold theorem, write the continuous time system associated to the map T , where, as before, we set $\mu = \lambda + \mu$ and this time we include the bifurcation parameter as an introduced, trivial state variable:

$$\begin{pmatrix} \dot{a} \\ \dot{b} \\ \dot{\mu} \end{pmatrix} = \begin{pmatrix} T(a, b, \mu) \\ 0 \end{pmatrix} - \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \equiv H(a, b, \mu). \quad (19)$$

Next, decompose H into linear and higher-order terms:

$$H(a, b, \mu) = \begin{pmatrix} \beta - 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ \mu \end{pmatrix} + \begin{pmatrix} f(a, b, \mu) \\ g(a, b, \mu) \\ 0 \end{pmatrix},$$

where f and g are $\mathcal{O}(\|(a, b, \mu)\|^2)$. This step is analogous to equation to the decomposition (14), except here, since μ is a state variable, the term $\beta\mu b$ is second-order and incorporated into g .

The system (19) is already linearly decoupled: f and g have no first order terms. Thus, by the center manifold theorem, the orbit structure of the dynamic system determined by (19) is topologically equivalent to the projection of the system on to the parameter-dependent center manifold, which may be expressed by a $C^4(\mathbb{R}^2)$ function: $a = h(b, \mu)$. The remainder of the proof involves two steps: computing the center manifold; and conducting bifurcation analysis of the projected system.

Computing the center

A closed form representation of h is not available, but we may use the invariance of the center manifold together with a Taylor expansion of h to establish a sufficient approximation. By (19), we have that

$$\dot{a} = (\beta - 1)h(b, \mu) + f(h(b, \mu), b, \mu).$$

Differentiating h with respect to time, we get $\dot{a} = h_b \dot{b} + h_\mu \dot{\mu}$. Using (19) and that $\dot{\mu} = 0$, we also have

$$\dot{a} = h_b(b, \mu)g(h(b, \mu), b, \mu).$$

Thus h is characterized by the functional equation

$$L(b, \mu) \equiv (\beta - 1)h(b, \mu) + f(h(b, \mu), b, \mu) = h_b(b, \mu)g(h(b, \mu), b, \mu) \equiv R(b, \mu)$$

This functional equation, together with the implicit function theorem, may be used to approximate h : simply compute the Taylor expansions of L and R , equate like terms, and solve the coefficients in the Taylor expansion of h .

Since the center manifold is tangent to the eigenspaces of the linear component of H , it follows that $h_b(0,0) = h_\mu(0,0) = 0$. Also, the origin is a steady state: $h(0,0) = 0$. Thus, we may write

$$h(b, \mu) = 1/2 \cdot (h_{bb} \cdot b^2 + h_{\mu\mu} \cdot \mu^2) + h_{b\mu} \cdot \mu \cdot b + \mathcal{O}(\|(b, \mu)\|^3).$$

Here, all derivatives are evaluated at $(0,0)$. As notation, also write

$$L(b, \mu) = L_b \cdot b + L_\mu \cdot \mu + 1/2 \cdot (L_{bb} \cdot b^2 + L_{\mu\mu} \cdot \mu^2) + L_{b\mu} \cdot b \cdot \mu + \mathcal{O}(\|(b, \mu)\|^3),$$

$$R(b, \mu) = R_b \cdot b + R_\mu \cdot \mu + 1/2 \cdot (R_{bb} \cdot b^2 + R_{\mu\mu} \cdot \mu^2) + R_{b\mu} \cdot b \cdot \mu + \mathcal{O}(\|(b, \mu)\|^3).$$

Noting that, for example, $\frac{\partial}{\partial b} f = f_a \cdot h_b + f_b$, we compute

$$\begin{aligned} L_b &= (\beta - 1)h_b + f_a \cdot h_b + f_b \\ L_{bb} &= (\beta - 1)h_{bb} + h_{bb} \cdot f_a + h_b \cdot \frac{\partial}{\partial b} f_a + h_b \cdot f_{ab} + f_{bb} \\ R_b &= h_{bb} \cdot g + h_b \cdot (g_a \cdot h_b + g_b) \\ R_{bb} &= h_{bbb} \cdot g + 2h_{bb} \cdot (g_a \cdot h_b + g_b) + h_b \cdot \frac{\partial}{\partial b} (g_a \cdot h_b + g_b). \end{aligned}$$

Since f, g , and h are zero at the origin and have no first order terms, we see $h_{bb} = \frac{f_{bb}}{1-\beta}$. Writing $f(a, b, \mu) = E_{\eta\xi} F(a + b(\lambda(\mu)\eta + \xi)) - (\beta - 1)b$, we see

$$f_{bb}(a, b, \mu) = E_{\eta\xi} ((\lambda(\mu)\eta + \xi)^2 F''(a + b(\lambda(\mu)\eta + \xi))),$$

so that

$$h_{bb} = \left(\frac{\sigma_\eta^2 \beta^{-2} + \sigma_\xi^2}{1 - \beta} \right) F''(0).$$

As we will determine below, other second-order terms of h are not needed for the bifurcation analysis, and so our computation of the center manifold approximation is complete.

Bifurcation analysis

The local dynamics of (19) are topologically equivalent to the suspension of the projected system by the associated saddle. In particular, if the projected system undergoes a particular bifurcation then so too does the system (19). The projected system is given by

$$\dot{b} = g(h(b, \mu), b, \mu) \equiv G(b, \mu). \quad (20)$$

To conduct bifurcation analysis, the higher-order derivatives of G are needed. That $G(0,0) = 0$ is immediate. Writing

$$g(a, b, \mu) = \sigma_\eta^{-2} E_{\eta\xi} \eta F(a + b(\lambda(\mu)\eta + \xi)) - b,$$

we have that

$$\begin{aligned} g_{aa} &= \frac{1}{\sigma_\eta^2} E_{\eta\xi} \eta F''(0) = 0 \\ g_b(a, b, \mu) &= \frac{1}{\sigma_\eta^2} E_{\eta\xi} \eta (\lambda(\mu)\eta + \xi) F'(a + b(\lambda(\mu)\eta + \xi)) - 1 \\ g_{ab} &= \frac{F''(0)}{F'(0)} \\ g_{bb} &= \frac{1}{\sigma_\eta^2} E_{\eta\xi} \eta (\lambda(0)\eta + \xi)^2 F''(0) = \frac{\sigma_\eta^3}{\beta \sigma_\eta^2} F''(0) \\ g_{b\mu} &= \beta + \sigma_\eta^2 \cdot \frac{\partial}{\partial \mu} \sigma_\eta^2 \\ g_{bbb} &= \frac{1}{\sigma_\eta^2} E_{\eta\xi} \eta (\lambda(0)\eta + \xi)^3 F'''(0) = \left(\frac{\sigma_\eta^4}{\beta^3 \sigma_\eta^2} + \frac{3\sigma_\xi^2}{\beta} \right) F'''(0). \end{aligned}$$

Using our information about h , we compute

$$\begin{aligned} G_b &= g_a \cdot h_b + g_b = 0 \\ G_\mu &= g_a \cdot h_\mu + g_\mu = 0 \\ G_{bb} &= g_a \cdot h_{bb} + h_b \cdot (g_{aa} \cdot h_b + g_{ab}) + g_{ab} \cdot h_b + g_{bb} = g_{bb} \\ G_{b\mu} &= h_b \cdot \frac{\partial}{\partial \mu} g_a + g_a \cdot h_{b\mu} + g_{ba} \cdot h_\mu + g_{b\mu} = g_{b\mu} \\ G_{bbb} &= g_a h_{bbb} + 2h_{bb} \cdot (g_{aa} \cdot h_b + g_{ab}) + h_b \cdot \frac{\partial}{\partial b} (g_{aa} \cdot h_b + g_{ab}) \\ &\quad + g_{ab} \cdot h_{bb} + h_b \cdot \frac{\partial}{\partial b} g_{ab} + h_b \cdot g_{bba} + g_{bbb} = 3h_{bb} \cdot g_{ab} + g_{bbb}, \end{aligned}$$

where, in each computation, the second equality follows from the work just above and that h and g have no first order terms.

Since $G = G_b = G_\mu = 0$, and $G_{b\mu}$ is generically non-zero, we can assess the type of bifurcation by looking at the higher order terms in b . In particular, the type of bifurcation experienced by the system (20) depends on whether $G_{bb} = 0$. Noting g_{bb} is proportional to $\sigma_\eta^3 F''(0)$, assuming non-trivial second order curvature in F , we see whether $g_{bb} = 0$ depends, generically, on whether $E(\xi_t^3) = 0$.

Case 1: $E(\xi_t^3) = 0$.

Since G_b , G_μ and $G_{bb} = 0$, and $G_{b\mu} \neq 0$ the system undergoes a pitchfork bifurcation provided that $G_{bbb} \neq 0$. Simplifying G_{bbb} , we get the following regularity condition:

$$G_{bbb} = F'''(0) \left(\frac{3\sigma_\xi^2}{\beta} + \frac{\sigma_\eta^4}{\beta^3\sigma_\eta^2} \right) + \frac{3(F''(0))^2 \left(\frac{\sigma_\eta^2}{\beta^2} + \sigma_\xi^2 \right)}{(1-\beta)\beta} \neq 0. \quad (21)$$

where we note that under the assumptions of the proposition, G_{bbb} is generically non-zero in that the set of all such parameters for which the condition (21) is not satisfied has Lebesgue measure zero in parameter space. We conclude that if $E(\xi_t^3) = 0$ then, generically, the system undergoes a pitchfork bifurcation, indicating the emergence of two additional fixed points to the T-map: these fixed points corresponding to NRSE. The bifurcation of the system is topologically equivalent to the bifurcation indicated in Figures 1 and 2. We note that, as in the cubic case, NRSE exist only for perturbations μ of the appropriate sign, as determined by the sign of $G_{b\mu}$.

Case 2: $E(\xi_t^3) \neq 0$.

In this case we have $G_b = 0$, $G_\mu = 0$, and $G_{b\mu} \neq 0$. Therefore the system undergoes a transcritical bifurcation provided that $G_{bb} \neq 0$. Simplifying, we get

$$G_{bb} = \frac{\sigma_\eta^3}{\beta\sigma_\eta^2} F''(0).$$

Again, we note that under the assumptions of the proposition, G_{bb} is generically non-zero.

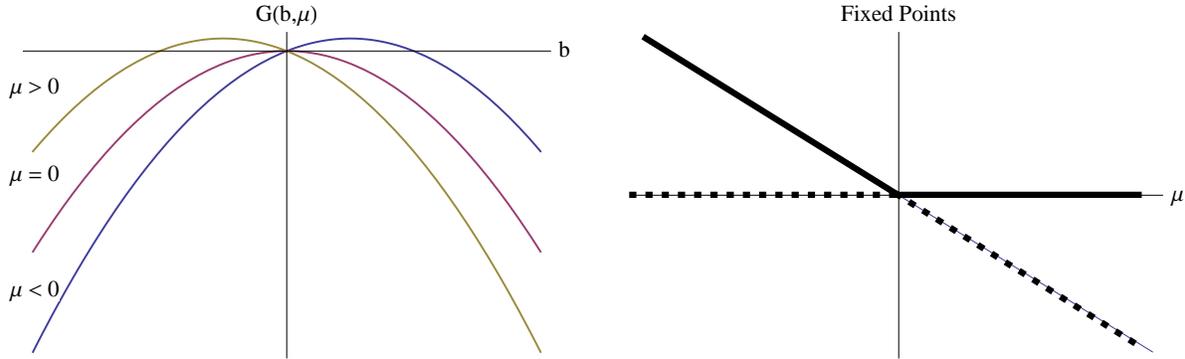
As we have noted, $b = 0$ is a fixed point of the projected system (20). Since the system undergoes a transcritical bifurcation at μ it follows that (20) is locally topologically equivalent to the system

$$\dot{b} = s(\mu)b + \frac{1}{2}G_{bb} \cdot b^2,$$

where, $s(0) = 0$ and $\text{sign}(s_\mu) = \text{sign}(G_{b\mu})$. We conclude that for *any* $\mu \neq 0$, the system (20) has a non-trivial fixed point. See Figure 4 for a graphical interpretation of this bifurcation. The function G in this figure corresponds to the function H in Figures 1 and 2.

We have established that at $\mu = 0$, the projected system experiences either a pitchfork or transcritical bifurcation, and in either case, as μ crosses zero (and in case of a pitchfork bifurcation, the crossing is from the appropriate side) non-zero steady states emerge. Because for μ near zero, the orbits of the suspension of the projected

Figure 4: Transcritical Bifurcation



system are topologically equivalent to the orbits of the original system, it follows that the original system must also undergo a bifurcation with emergent steady states. By using this equivalence, we may choose our neighborhood V . ■

Proof of Theorem 2. As noted in the text, the hard work has been done. Using the notation from the previous proof, we are interesting in knowing when the bifurcation results in two new fixed points of (19), at least one of which is Lyapunov stable. Because, locally, the dynamics of (19) are topologically equivalent to suspension of the projected system by the associated saddle, stability of the post-bifurcation fixed points entails two requirements: first, the associated saddle must be stable, that is, $\beta - 1 < 0$; and second, the emergent fixed points of the projected system (20) must be Lyapunov stable. In case $E(\xi_t^3) \neq 0$, the bifurcation is transcritical in nature, so that we may simply choose an appropriate perturbation μ to obtain a stable fixed point. In case $E(\xi_t^3) = 0$, additional restrictions are required: the new fixed points inherit the stability of the origin. Thus stability of the new fixed points – the NRSE – requires in this case that $G_{bbb} < 0$, which yields the additional non-generic condition identified in the theorem. Note that we may still conclude that if $\beta < -1$ and $F''(0) \neq 0$ then stable NSRE exist. ■