Boundary Problems with the "Russell" Graph Measure of Technical Efficiency: A Refinement

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I. Introduction.

In an influential paper, Färe and Lovell [1978] proposed an (input based) technical efficiency index designed to correct two fundamental inadequacies of the Farrell [1956] index: its failure to satisfy (1) *indication* (the index is equal to 1 if and only if the input bundle is technically efficient in the sense of Koopmans [1951]) and (2) *monotonicity* (an increase in any one input quantity lowers the value of the index). In sharp contrast to the (maximal) radial-contraction construction of Farrell, the Färe-Lovell (FL) index is essentially a (maximal) average of coordinate-wise contractions of input quantities, with an adjustment to correct for potential violations of both indication and monotonicity at the boundary of input space. As it turns out the FL index also fails to satisfy monotonicity, but it does satisfy indication and *weak monotonicity* (an increase in any one input or a decrease in any one output can not increase the value of the index).[1]

In recent years, much more emphasis has been placed on technical efficiency measurement in the full space of input and output quantities—often referred to as "graph space"—as opposed to efficiency measurement in input (or output) space alone. A prominent index in graph space, proposed by Färe, Grosskopf, and Lovell [1985], is a straightforward extension of the FL index. It is a (maximal) average of coordinate-wise expansions of outputs and contractions of inputs, with an adjustment to correct for potential violations of indication and monotonicity at the boundary of graph space. Variations of this Färe-Grosskopf-Lovell (FGL) index have played a prominent role in the operations-research literature in recent years (see, *e.g.*, Cooper, Seiford, Tone, and Zhu [2007] and the references therein). These indexes are commonly referred to as "slacks-based measures" (and, for obscure historical reasons, as "Russell measures").

In this paper, we show that the the FGL adjustment to correct for problems at the boundary of graph space fails to do the job. In particular, the FGL index does not distinguish between efficient and inefficient points on the boundary of output space. Furthermore, an increase in an output quantity starting at an inefficient boundary output vector can *lower* the value of the FGL index. The FGL index, therefore, satisfies neither indication nor weak monotonicity.

In our view, this is a serious flaw, leaving the FGL index with no attractive properties whenever zero values of output quantities may occur. Of course, a zero output quantity would be highly improbable if there were a single output, but the FGL measure is designed to accommodate multiple outputs. Since a multiple-output firm might not produce some types of outputs—consider an agricultural unit that cultivates a subset of its crops in any given growing season—zeros in production vectors can be sensible choices.

[1] See Russell [1985].

The flaw in the FGL index is attributable to the boundary adjustment that cannot distinguish between efficient and inefficient production vectors when some output quantities are zero. We show, however, that a simple modification of the FGL index—in particular, a modification of the correction factor at the boundary—corrects these flaws, restoring the indication and weak-monotonicity properties.[2]

Zero outputs may arise naturally in empirical work for several reasons. A panel data application of agricultural firms may contain zeros in output space owing to seasonal crop rotation: outputs are only produced during specific growing seasons and thus will appear as zeros during off seasons. Another situation where problems may arise because of zero output values is in the comparison of relative efficiencies of firms that produce separate, but non-disjoint sets of outputs. For example one firm may produce goods A, B and C, while the other produces only goods B, C, and D.

Utilizing panel data with a rolling-window DEA analysis, Charnes, Clark, Cooper, and Golany [1984] examine the efficiency of maintenance units in the U.S. Air Force. While their data base contains no zero outputs, it is likely that zeros would occur if the frequency of data acquisition were to increase in a rolling-window or panel application.

To demonstrate the tractability of our proposal, we study an example in which zero outputs occur frequently. We measure a baseball player's batting performance by counting his singles, doubles, triples, and home runs during each game. If we measure these outputs for an entire season, zeros are likely to be rare, since most players produce all four types of hits over the course of an entire season. By increasing the frequency of observations to a per-game basis, we ensure that zero outputs are common. Our illustrative empirical application thus contributes to the DEA literature on efficiency of baseball players (*e.g.*, Anderson and Sharp [1997] and Mazur [1995]).

The paper unfolds as follows: Section II lays out the framework of our analysis and defines efficiency indexes and the indication and monotonicity axioms. Section III shows that the FGL index violates indication and weak monotonicity at the boundary. Section IV introduces our modified FGL index and proves that it satisfies indication and weak monotonicity. Section V discusses the implementation of the modified FGL and Section VI illustrates the practicality of the concept by applying it to two data sets, one synthetic and the other an actual data set on baseball performance. Section VII concludes.

II. Efficiency Indexes and Axioms.

The ⟨input, output⟩ production vector $\langle x, y\rangle \in \mathbf{R}_+^{n+m}$ is constrained to lie in a technology set $T \subset \mathbf{R}_+^{n+m}$. Denote the origin of this space by $\left\langle 0^{[n]}, 0^{[m]}\right\rangle$ and the unit vector by

[2] Our modification is not needed to maintain indication and weak monotonicity of the input-based FL index at the boundary of input space.

$\langle 1^{[n]}, 1^{[m]} \rangle$. The input requirement set for output y is $L(y) = \{x \in \mathbf{R}^n_+ \mid \langle x, y\rangle \in T\}$, and the output possibility set for input x is $P(x) = \{y \in \mathbf{R}^m_+ \mid \langle x, y\rangle \in T\}$.

We consider the collection of non-empty, closed technology sets that satisfy the following conditions:[3,4]

(i) $\langle x, y\rangle \in T$ and $\langle \bar{x}, -\bar{y}\rangle > \langle x, -y\rangle$ implies $\langle \bar{x}, \bar{y}\rangle \in T$ (free disposability of inputs and outputs),

(ii) $y > 0^{[m]} \implies \langle 0^{[n]}, y\rangle \notin T$ (no free lunch), and

(iii) $P(x)$ is non-empty and bounded for all $x \in \mathbf{R}^n_+$.

Denote by $\mathcal{T}$ the set of non-empty, closed technologies satisfying these conditions.

A production vector $\langle x, y\rangle \in T$ is technologically efficient if $\langle x, -y\rangle > \langle \bar{x}, -\bar{y}\rangle$ implies $\langle \bar{x}, \bar{y}\rangle \notin T$; denote the efficient subset of T by $\mathrm{Eff}(T)$. An efficiency index is a mapping, $E : \Xi \to (0, 1]$, with image $E(x, y, T)$, where

$$\Xi = \Big\{ \langle x, y, T\rangle \in T \times \mathcal{T} \mid \langle x, y\rangle \in T \ \wedge \ x \neq 0^{[n]} \Big\}. \tag{2.1}$$

Färe, Grosskopf, and Lovell [1985] proposed a "graph" efficiency index in the full space of inputs and outputs. This index is an extension of the Färe-Lovell index defined in the

[3] All but free disposability of these conditions are necessary to guarantee that our efficiency indexes are well defined. Free disposability could be dispensed with (theoretically); the only change that would be needed in what follows would be to redefine the inefficiency indexes on the free-disposal hull of T, $T + (\mathbf{R}^n_+ \times -\mathbf{R}^m_+)$, rather than on T itself (as in Russell [1987] for input-based efficiency indexes).

[4] Vector notation: $\bar{x} \geq x$ if $\bar{x}_i \geq x_i$ for all i; $\bar{x} > x$ if $\bar{x}_i \geq x_i$ for all i and $\bar{x} \neq x$; and $\bar{x} \gg x$ if $\bar{x}_i > x_i$ for all i.

input space. We formulate their index as follows:[5]

$$E_{FGL}(x,y,T) = \min_{\alpha,\beta} \left\{ \frac{\sum_i \delta(x_i)\alpha_i + \sum_j \delta(y_j)\beta_j}{\sum_i \delta(x_i) + \sum_j \delta(y_j)} \;\Big|\; \langle \alpha, \beta \rangle \in \Omega(x,y,T) \right\}, \tag{2.2}$$

where

$$\Omega(x,y,T) = \left\{ \langle \alpha, \beta \rangle \mid \langle\, \alpha \otimes x, y \oslash \beta \rangle \in T \;\wedge\; 0^{[n]} \leq \alpha \leq 1^{[n]} \;\wedge\; 0^{[m]} \ll \beta \leq 1^{[m]} \right\}, \tag{2.3}$$

$\alpha \otimes x = \langle \alpha_1 x_1, \ldots, \alpha_n x_n \rangle$, $y \oslash \beta = \langle y_1/\beta_1, \ldots, y_m/\beta_m \rangle$, and

$$\delta(z) = \begin{cases} 1 & \text{if } z > 0 \\ 0 & \text{if } z = 0 \end{cases} \tag{2.4}$$

for $\alpha = \langle \alpha_1, \ldots, \alpha_n \rangle$ and $\beta = \langle \beta_1, \ldots, \beta_m \rangle$.

The use of the indicator function δ is a fundamentally important aspect of their formulation, aimed at correcting serious problems at the boundary. If $\langle x, y \rangle \gg 0^{[n+m]}$ so that $\sum_i \delta(x_i) + \sum_j \delta(y_j) = n + m$, the objective function in (2.2) is a simple average of the proportional, coordinate-wise, input-contraction and output-expansion factors. If $\sum_i \delta(x_i) + \sum_j \delta(y_j) < n + m$, the objective function is a simple average of proportional, coordinate-wise, input-contraction and output-expansion factors for *positive* input and output quantities, in which case inputs and outputs with zero quantity values are simply ignored in the efficiency calculation.

The axioms relevant to our analysis are as follows:[6]

[5] FGL order coordinates so that the first k input quantities and first l output quantities are positive and then minimize only over the sum of these $k+l$ coordinates, all of which have positive values. Our characterization does not require a permutation of the coordinates whenever a production vector (or a technology) is changed.

For the sake of symmetry, we weight each input-contraction factor by $\delta(x_i)$, $i = 1, \ldots n$, but the index would be unaffected by omitting these weights, since the value of α_i at the minimum is zero if $x_i = 0$.

In the initial specification of the FGL index (page 154), α is restricted to be strictly positive, in which case their minimization problem has no solution if, for some i, $x_i > 0$ and the minimum value of α_i is zero; this case is illustrated in Figure 2, where x' would be contracted to $\hat{x}$. In their characterization of the domain of the index (page 153), however, they restrict α to be non-negative. We choose the latter formulation to ensure that the index is well-defined on the domain, which includes input and output quantities with zero values.

Note that we are able to use the min operator instead of inf even though the constraint set $\Omega(x,y,T)$ is not closed because, using inf, $\overset{*}{\beta}_j = 0$ only if $\delta(y_j) = 0$, in which case β_j does not appear in the objective function in (2.2).

[6] Neither of the indexes we consider satisfies the stronger property of (strict) monotonicity. Nor does either satisfy continuity. See Russell and Schworm [2008a] for details.

Indication of Efficiency (I): For all $\langle x, y, T\rangle \in \Xi$, $E(x, y, T) = 1$ if and only if $\langle x, y\rangle \in \mathrm{Eff}(T)$.

Weak Monotonicity (WM): For all pairs $\langle x, y, T\rangle \in \Xi$ and $\langle \bar{x}, \bar{y}, T\rangle \in \Xi$ satisfying $\langle \bar{x}, -\bar{y}\rangle > \langle x, -y\rangle$, $E(\bar{x}, \bar{y}, T) \leq E(x, y, T)$.

III. The Failure of the Färe-Grosskopf-Lovell Index on the Boundary.

The inability of the FGL Index to satisfy the indication and weak monotonicity axioms is demonstrated by the simple example with $m = 2$ displayed in Figure 1. Assume that $\langle x, y''\rangle$ is efficient with $x \gg 0$ and note that $\langle x, y^o\rangle$ is inefficient. Calculate the FGL Index at $\langle x, y^o\rangle$ as follows:

$$E_{FGL}(x, y^o, T) = \max_{\alpha,\beta} \left\{ \frac{\sum_{i=1}^{n} \alpha_i + \beta_1}{n+1} \;\middle|\; \langle \alpha, \beta\rangle \in \Omega(x, y^o, T) \right\} \tag{3.1}$$

where

$$\Omega(x, y^o, T) = \left\{ \langle 1^{[n]}, 1, \beta_2^o\rangle \;\middle|\; 0 < \beta_2^o \leq 1 \right\}. \tag{3.2}$$

The objective function is simplified since $\delta(x_i) = 1$ for all $i = 1, \ldots, n$, $\delta(y_1^o) = 1$, and $\delta(y_2^o) = 0$. The constraint set is reduced to the line between $\langle 1^{[n]}, 1, 0\rangle$ and $\langle 1^{[n]}, 1, 1\rangle$. Since β_2^o does not affect the objective function, all points in $\Omega(x, y^o, T)$ are minimizing vectors and the efficiency index is

$$E_{FGL}(x, y^o, T) = \frac{n+1}{n+1} = 1. \tag{3.3}$$

Therefore, indication (I) is violated.

Next consider a point like y' in Figure 1 and note that

$$E_{FGL}(x, y', T) = \frac{n + 1 + \overset{*}{\beta}_2}{n+2} < 1, \tag{3.4}$$

where $\overset{*}{\beta}_2 = y_2'/y_2'' < 1$. Therefore, $y^0 < y'$ and $E_{FGL}(x, y^o, T) = 1$, so that (WM) is violated.

We summarize with the following theorem:

Theorem 1: E_{FGL} violates (I) and (WM).

While the above example places the output vector at a "corner" of the production possibility set, the violation of weak monotonicity occurs for any feasible production vector in this diagram with a zero value of y_2. To see this, consider the points $\hat{y}$ and $\tilde{y}$ in Figure 1. So long as the slope of the frontier segment $[y^e y'']$ is low enough (so that y'' is the reference

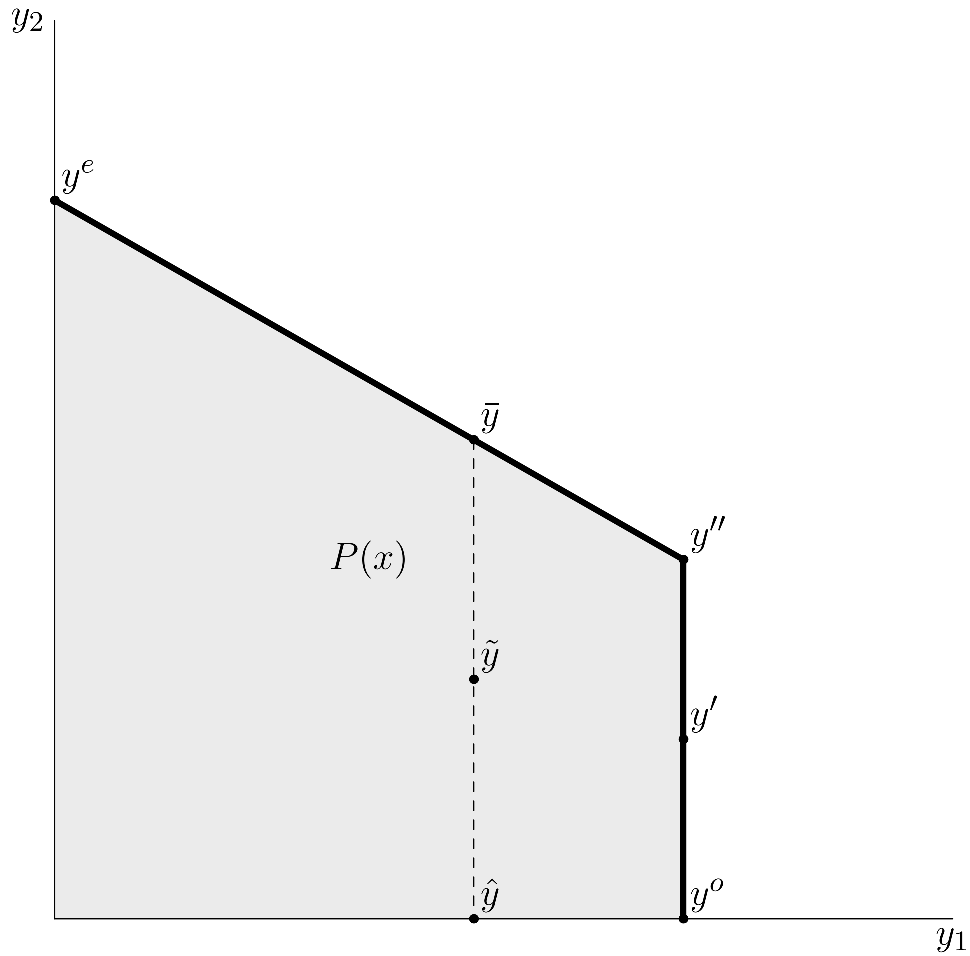

Figure 1: P(x) with Efficient and Inefficient Boundary Point

point for $\tilde{y}$), the FGL efficiency index values (again assuming efficiency in input space) is given by

$$E_{FGL}(x, \tilde{y}, T) = \frac{n + \hat{y}_1/y_1^o + \tilde{y}_2/y_2^{''}}{n+2}. \tag{3.5}$$

The last term in the numerator of (3.5) can be made arbitrarily close to zero by shifting the cusp y'' vertically until the following inequality is established

$$E_{FGL}(x, \tilde{y}, T) = \frac{n + \hat{y}_1/y_1^o + \tilde{y}_2/y_2^{''}}{n+2} < \frac{n + \hat{y}_1/y_1^o}{n+1} = E_{FGL}(x, \hat{y}, T). \tag{3.6}$$

This inequality shows that weak monotonicity is violated.

The example suggests that the problem is caused by the boundary adjustment when some element of the output vector is zero. Eliminating this adjustment for output and defining the objective function for a modified FGL Index by[7]

$$\frac{\sum_i \delta(x_i)\alpha_i + \sum_j \beta_j}{\sum_i \delta(x_i) + m} \tag{3.7}$$

would ensure that the index is less than one at $\langle x, y^o \rangle$. This alteration, however, would fail at the point y^e in Figure 1 where $\langle x, y^e \rangle$ an efficient point. In this case, the boundary adjustment is needed to ensure that $E_{FGL}(x, y^e, T) = 1$.

The difficulty with the FGL Index is that it is unable to distinguish between inefficient boundary points like y^0 and efficient boundary points like y^e in Figure 1. In the next section, we propose a modification that allows this distinction.

A natural question is whether an analogous problem arises for inputs on the boundary that would also affect the (input oriented) Färe-Lovell Index, defined as follows:

$$E_{FL}(x, y, T) = \min_{\alpha} \left\{ \frac{\sum_i \delta(x_i)\alpha_i}{\sum_i \delta(x_i)} \;\middle|\; \langle \alpha \otimes x, y \rangle \in T \ \wedge \ 0^{[n]} \le \alpha \le 1^{[n]} \right\}. \tag{3.8}$$

Figure 2 displays an example in which $\hat{x}$ is efficient but the simple average of minimal contraction factors is 1/2; on the other hand, $E_{FL}(\hat{x}, y, T) = 1$. Therefore, the boundary adjustment works correctly for input vectors with zero components.

These examples show that boundary points for inputs and outputs require different treatments if an index is to satisfy the indication and weak monotonicity axioms. Any input vector can be contracted by each component (with zero elements remaining unchanged) until an efficient point is reached. An output vector cannot always be expanded until an efficient point is reached, because the zero elements remain unchanged.[8]

IV. The Modified Färe-Grosskopf-Lovell Index.

To eliminate these boundary problems for the FGL index, we need to enable the index to distinguish between the two boundary points, y^0 and y^e, in Figure 1. First define $\epsilon^j \in \mathbf{R}^m_+$ as the vector with the j^{th} coordinate equal to $\epsilon \in \mathbf{R}_{++}$ and all other coordinates equal to zero. For inputs, we use the indicator for interior points $\delta(x_i)$ as defined in (2.4). For outputs, we use indicators for interior points or inefficient boundary points:

$$\psi_j(x, y, T) = \begin{cases} 1 & \text{if } y_j > 0 \text{ or } \left[y_j = 0 \ \wedge \ \langle x, y + \epsilon^j \rangle \in T \text{ for some } \epsilon > 0 \right] \\ 0 & \text{if } y_j = 0 \ \wedge \ \langle x, y + \epsilon^j \rangle \notin T \text{ for all } \epsilon > 0 \end{cases} \tag{4.1}$$

[7] And replacing min with inf.

[8] These output boundary problems also arise for the FGL output-oriented index formulated by Färe, Grosskopf, and Lovell [1985, pp. 148–149] (and further analyzed by Färe, Grosskopf, and Lovell [1994, pp. 115–118]).

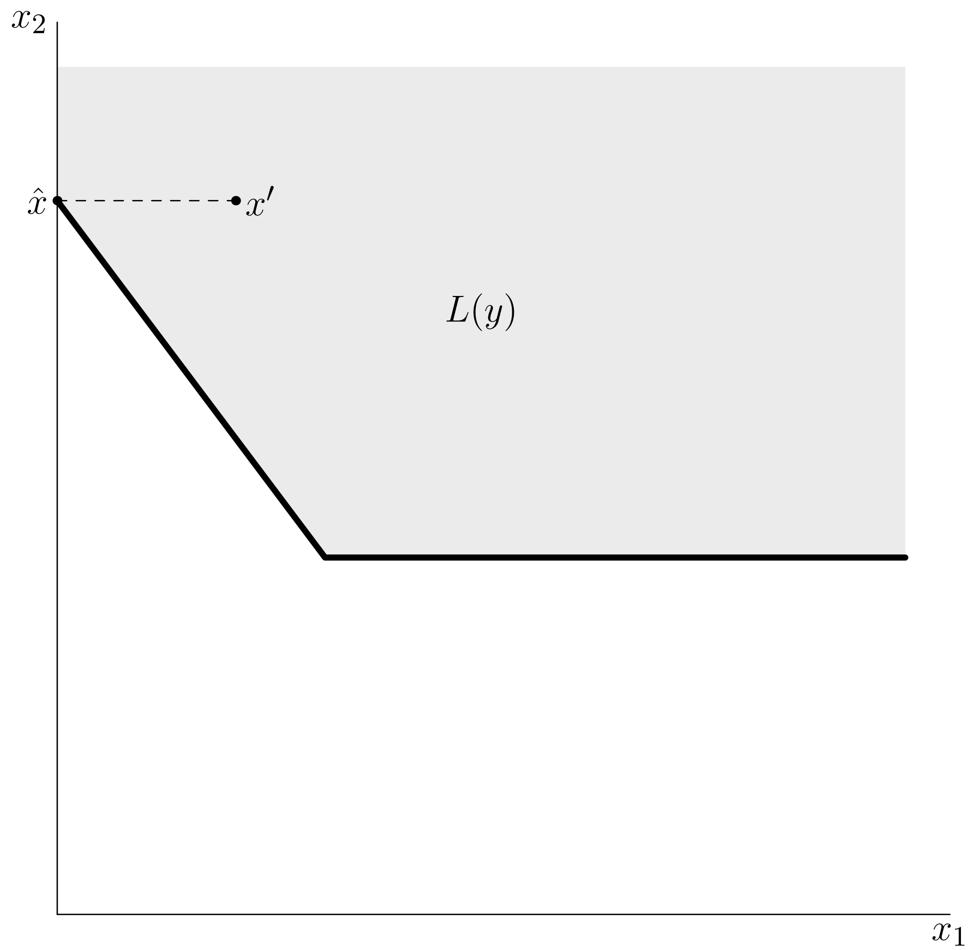

Figure 2: L(y) with Efficient Boundary Point

for $j = 1, \ldots, m$.

The indicator functions for output variables are output specific and are enhanced to distinguish between efficient and inefficient zero values for output variables. This modification requires that the output indicator functions, $\psi_j,\ j = 1, \ldots m$, depend on technologies and all input and output quantities rather than on y_j alone. Define the modified Färe-Grosskopf-Lovell index as follows:[9]

$$\bar{E}_{FGL}(x, y, T) = \inf_{\alpha,\beta} \left\{ \frac{\sum_i \delta(x_i)\alpha_i + \sum_j \psi_j(x, y, T)\beta_j}{\sum_i \delta(x_i) + \sum_j \psi_j(x, y, T)} \;\middle|\; \langle \alpha, \beta \rangle \in \Omega(x, y, T) \right\}, \qquad (4.2)$$

where Ω is defined in (2.3). Let $\langle \overset{*}{\alpha}, \overset{*}{\beta} \rangle$ be values of $\langle \alpha, \beta \rangle$ that yield the infimum of the objective function in (4.2) and note that $\overset{*}{\beta}_j$ is an arbitrary selection from $(0, 1]$ if $\psi_j(x, y, T) =$

[9] Note that we must use the infimum here, because $\psi_j(x, y, T)$ can be non-zero when $y_j = 0$.

0.[10]

Let us begin by seeing how the modified index works for the example in Figure 1. For the inefficient point $\langle x, y^o \rangle$ in Figure 1, $\psi_2(x, y^o, T) = 1$ and $\overset{*}{\beta}_2 = 0$, so that the infimum is

$$\bar{E}_{FGL}(x, y^0, T) = \frac{n+1+0}{n+1+1} < 1, \tag{4.3}$$

and indication is no longer violated. Also, note that $\psi_2(x, y', T) = 1$ and $\beta_2 = y_2'/y_2'' < 1$, so that

$$\bar{E}_{FGL}(x, y', T) = \frac{n+1+\beta_2}{n+1+1} > \frac{n+1}{n+2} = \bar{E}_{FGL}(x, y^0, T) \tag{4.4}$$

and weak monotonicity is satisfied for the change from *ynot* to y'.

For the efficient point $\langle x, y^e \rangle$ in Figure 1, $\psi_1(x, y^e, T) = 0$ and $\langle \overset{*}{\alpha}, \overset{*}{\beta} \rangle = \langle 1^{[n]}, \beta_1, 1 \rangle$ for any $\beta_1 \in (0, 1]$, so that

$$\bar{E}_{FGL}(x, y^e, T) = \frac{n+0+1}{n+0+1} = 1, \tag{4.5}$$

and indication is satisfied. As there is no feasible variation in y_1^e alone, there is no violation of weak monotonicity.

Now, we turn to the general result.

Theorem 2: *$\bar{E}_{FGL}$ satisfies (I) and (WM).*

We begin with a lemma regarding the indicator functions.

Lemma 1: *If $\langle x, y \rangle \in T$, $\langle \bar{x}, \bar{y} \rangle \in T$, and $\langle x, -y \rangle < \langle \bar{x}, -\bar{y} \rangle$, then (i) $\delta(x_i) \leq \delta(\bar{x}_i)$ for all $i = 1, \ldots, n$ and (ii) $\psi_j(x, y, T) \leq \psi_j(\bar{x}, \bar{y}, T)$ for all $j = 1, \ldots, m$.*

Proof: As (i) is obvious, we need only prove (ii). It suffices to consider two cases: (a) $x < \bar{x}$ (and $y = \bar{y}$) and (b) $y > \bar{y}$ (and $x = \bar{x}$).

In case (a), $x < \bar{x}$ and free disposability imply $P(x) \subseteq P(\bar{x})$. Since $y = \bar{y}$ by assumption, $\psi_j(x, y, T) \neq \psi_j(\bar{x}, \bar{y}, T)$ for some j only if $y_j = \bar{y}_j = 0$, $y + \epsilon^j \notin P(x)$ for all $\epsilon > 0$, and $y + \epsilon^j \in P(\bar{x})$ for some $\epsilon > 0$ (refer to definition (4.1)). Figure 3 displays this situation. In this case, $\psi_j(x, y, T) = 0$ and $\psi_j(\bar{x}, \bar{y}, T) = 1$ so that the inequality in (ii) is satisfied.

In case (b), it suffices to establish the result for an arbitrary coordinate; suppose, therefore, that $y_{j'} > \bar{y}_{j'}$ for some j' and that $y_j = \bar{y}_j$ for all $j \neq j'$. Suppose that

[10] And, of course, in this formulation $\overset{*}{\alpha}_i$ is an arbitrary selection from [0,1] if $\delta(x_i) = 0$. Note that, in addition to this arbitrariness, the "solution" values for α_i or β_j not associated with zero values of x_i or y_j need not be unique, since there can be ties in the optimization problem.

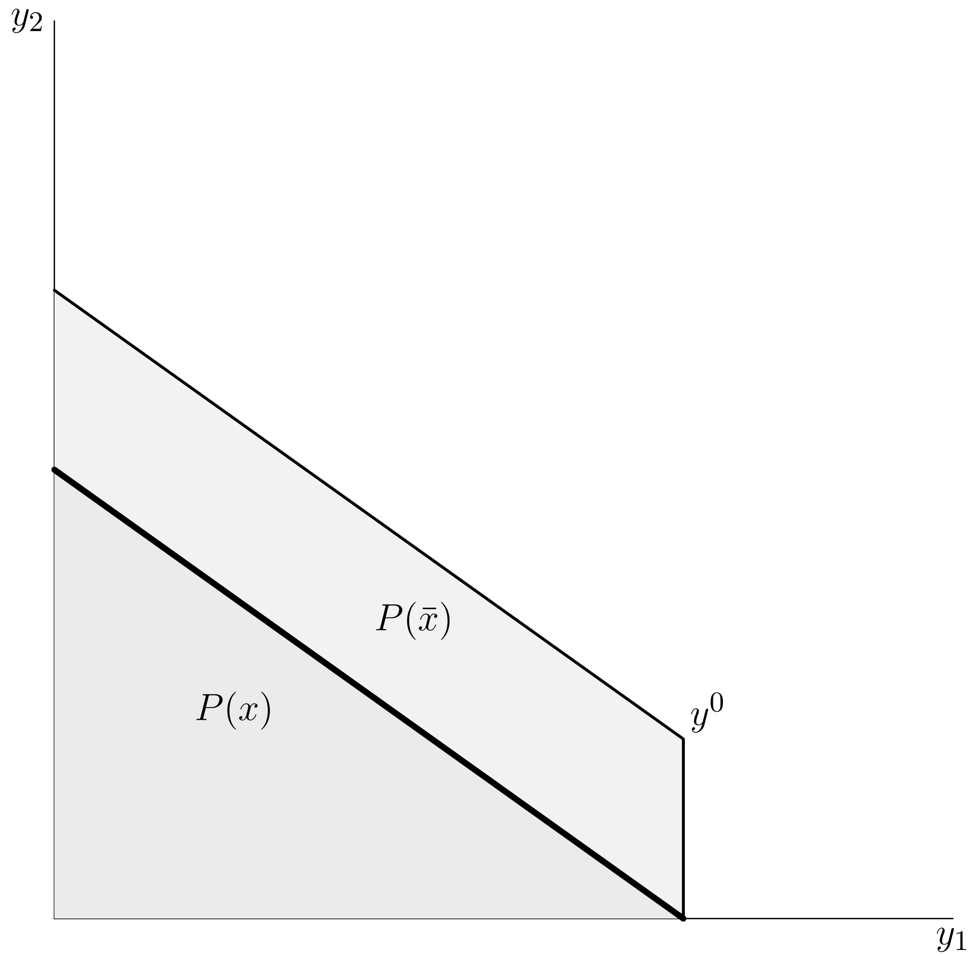

Figure 3: $\psi_2(x, y) < \psi_2(\bar{x}, y)$

$\psi_{j'}(x, y, T) \neq \psi_{j'}(\bar{x}, \bar{y}, T)$. Then $y_{j'} > 0$ and $\bar{y}_{j'} = 0$ (refer again to definition (4.1)). It follows from $\langle x, y \rangle \in T$ that $\langle \bar{x}, \bar{y} + \epsilon^{j'} \rangle \in T$, where $\epsilon = y_{j'} - \bar{y}_{j'}$. Thus, in fact, $\psi_{j'}(x, y, T) = \psi_{j'}(\bar{x}, \bar{y}, T) = 1$.

Consider now $\psi_j(x, y, T)$ where $j \neq j'$. Since $y_j = \bar{y}_j$ by assumption, either $y_j > 0$ and $\bar{y}_j > 0$, in which case $\psi_j(x, y, T) = \psi_j(x, \bar{y}, T) = 1$ or $y_j = \bar{y}_j = 0$. In the latter case, $\psi_j(x, y, T) > \psi_j(\bar{x}, \bar{y}, T)$ only if $\langle x, y + \epsilon^j \rangle \in T$ for some $\epsilon > 0$ and $\langle x, \bar{y} + \epsilon^j \rangle \notin T$ for all $\epsilon > 0$, in which case $\psi_j(x, y, T) = 1$ and $\psi_j(x, \bar{y}, T) = 0$. Writing out these vectors more explicitly (assuming, without loss of generality, that $1 < j < j' < m$), we have, for some $\epsilon > 0$,

$$\langle y_1, \ldots, 0_j + \epsilon, \ldots, y_{j'}, \ldots, y_m \rangle \in P(x) \tag{4.6}$$

and

$$\langle y_1, \ldots, 0_j + \epsilon, \ldots, \bar{y}_{j'}, \ldots, y_m \rangle \notin P(x) \tag{4.7}$$

where 0_j is the placeholder for the zero value of $y_j = \bar{y}_j$. But since $y_{j'} > \bar{y}_{j'}$, this violates free disposability. ■

We now prove the theorem.

Proof of Theorem:

(i) Indication.

To show that $\bar{E}_{FGL}$ satisfies (I), suppose first that $\bar{E}_{FGL}(x, y, T) < 1$. There are three (nonexclusive) possibilities: (a) there exists at least one i' with $\delta(x_i) = 1$ and $0 \leq \overset{*}{\alpha}_{i'} < 1$; (b) there exists at least one j' with $\psi_{j'}(x, y, T) = 1$ and $0 < \overset{*}{\beta}_{j'} < 1$; and (c) there exists at least one j' with $\psi_{j'}(x, y, T) = 1$ and $\overset{*}{\beta}_{j'} = 0$.

Case (a) implies that there is a vector $\bar{x} = \overset{*}{\alpha} \otimes x$ with $\bar{x} < x$ and $\langle \bar{x}, y \rangle \in T$. Case (b) implies that either there is a vector $\bar{y} = y \oslash \overset{*}{\beta}$ with $\bar{y} > y$ and $\langle x, \bar{y} \rangle \in T$. In both cases, $\langle x, y \rangle$ is inefficient. Case (c) implies that there exists a vector $\bar{y}$, with $\bar{y}_{j'} > y_{j'} = 0$ for some j' and $\bar{y}_j = y_j$ for all $j \neq j'$, satisfying $\langle x, \bar{y} \rangle \in T$, so that $\langle x, y \rangle$ is inefficient.

Now suppose that $\langle x, y \rangle$ is inefficient in T, implying the existence of a vector $\langle \bar{x}, \bar{y} \rangle \in T$ satisfying $\langle \bar{x}, -\bar{y} \rangle < \langle x, -y \rangle$. There are again three (nonexclusive) possibilities: (a) there exists at least one i' such that $0 \leq \bar{x}_{i'} < x_{i'}$; (b) there exists at least one j' such that $0 < y_{j'} < \bar{y}_{j'}$; or (c) there exists at least one j' such that $0 = y_{j'} < \bar{y}_{j'}$.

In cases (a) and (b), there exists a vector $\langle \alpha, \beta \rangle \in \Omega(x, y, T)$ with $\alpha_{i'} < \overset{*}{\alpha}_{i'} \leq 1$ in case (a) and $\beta_{j'} < \overset{*}{\beta}_{j'} \leq 1$ in case (b); moreover, $\delta(\bar{x}_{i'}) = \delta(x_{i'}) = 1$ in case (a) and $\psi_{j'}(\bar{x}, \bar{y}, T) = \psi_{j'}(x, y, T) = 1$ in case (b), so that, *ceterus paribus*, $\sum_i \delta(x_i) + \sum_j \psi_j(x, y, T) = \sum_i \delta(\bar{x}_i) + \sum_j \psi_j(\bar{x}, \bar{y}, T)$ in either case. In case (c), we have $\overset{*}{\beta}_{j'} = 0$ and $\psi_{j'}(x, y, T) = 1$. In each case, we obtain $\bar{E}_{FGL}(x, y, T) < 1$.

(ii) Weak Monotonicity.

Consider two production vectors satisfying $\langle \bar{x}, -\bar{y} \rangle > \langle x, -y \rangle$, which implies

$$\Omega(x, y, T) \subset \Omega(\bar{x}, \bar{y}, T). \tag{4.8}$$

This implies that

$$\begin{aligned} \bar{E}_{FGL}(x, y, T) &= \inf_{\alpha, \beta} \left\{ \frac{\sum_i \delta(x_i)\alpha_i + \sum_j \psi_j(x, y, T)\beta_j}{\sum_i \delta(x_i) + \sum_j \psi_j(x, y, T)} \;\middle|\; \langle \alpha, \beta \rangle \in \Omega(x, y, T) \right\} \\ &\geq \inf_{\alpha, \beta} \left\{ \frac{\sum_i \delta(x_i)\alpha_i + \sum_j \psi_j(x, y, T)\beta_j}{\sum_i \delta(x_i) + \sum_j \psi_j(x, y, T)} \;\middle|\; \langle \alpha, \beta \rangle \in \Omega(\bar{x}, \bar{y}, T) \right\}. \end{aligned} \tag{4.9}$$

(Note that, while the constraints in these two optimization problem differ, the objective functions are identical.) Our task now is to show that replacing the weights in the right-hand-side (RHS) optimization problem, $\delta(x_i),\ i = 1,\dots n$, and $\psi_j(x, y, T),\ j = 1,\dots,m$, with $\delta(\bar{x}_i),\ i = 1,\dots n$, and $\psi_j(\bar{x}, \bar{y}, T),\ j = 1,\dots,m$, does not affect the inequality. To this end, denote the solution to the RHS optimization problem by $\langle \bar{\alpha}, \bar{\beta} \rangle$.

By Lemma 1, $x_{i'} < \bar{x}_{i'}$ and $\delta(x_{i'}) \neq \delta(\bar{x}_{i'})$ imply $\delta(x_{i'}) = 0$ and $\delta(\bar{x}_{i'}) = 1$, hence $x_{i'} = 0$ and $\bar{x}_{i'} > 0$, in which case replacing $\delta(x_{i'})$ with $\delta(\bar{x}_{i'})$ replaces zeros in both the numerator and denominator with $\alpha_{i'}$ in the numerator and 1 in the denominator. Clearly, $\langle \bar{\alpha}, \bar{\beta} \rangle$ remains feasible in the new optimization problem, so that the infimum is at least as small as before the substitution.[11]

By Lemma 1, $\psi_{j'}(x, y, T) \neq \psi_{j'}(\bar{x}, \bar{y}, T)$ implies $\psi_{j'}(x, y, T) = 0$ and $\psi_{j'}(\bar{x}, \bar{y}, T) = 1$. From the definition of $\psi_{j'}$, this implies that $y_{j'} = 0$ and either (i) $\bar{y}_{j'} > 0$ or (ii) $\bar{y}_{j'} = 0$ and there exists an ϵ such that $\langle \bar{x}, \bar{y} + \epsilon^{j'} \rangle \in T$. Alternative (i) is ruled out, since $y > \bar{y}$ by assumption. Under alternative (ii), replacing $\psi_{j'}(x, y, T)$ with $\psi_{j'}(\bar{x}, \bar{y}, T)$ replace zeros in the numerator and denominator of the RHS objective function with $\beta_{j'}$ and 1 in the numerator and denominator, respectively. Again, $\langle \bar{\alpha}, \bar{\beta} \rangle$ remains feasible in the new optimization problem, so that the infimum is at least as small as before the substitution.

This completes the proof. ■

V. Empirical Implementation.

Implementation of the modified FGL index is not straightforward, since the functions $\psi_j, j = 1,\dots,m$, depend on infinitesimal comparisons, the constraint set is not closed, and the use of the infimum is salient (as the minimum does not always exist). In this section, we discuss possible solutions to these problems and suggest methods for calculating $\bar{E}_{FGL}(x, y, T)$.

If the technology is known, then a frequently employed method of dealing with zero inputs or outputs can be used: simply replace the zeros with small positive numbers. Since the FGL index handles zero elements of the input vector correctly, we need only perturb the output vector.

Consider $\langle x, y \rangle \in T$ with some elements of the input and output vectors possibly zero. Define y^ϵ by replacing all zero components of y with a small number $\epsilon > 0$ and consider the perturbed production vector $\langle x, y^\epsilon \rangle$, which may or may not be feasible.

[11] In fact, it will be lower, since the zeros in the optimal values of the numerator and denominator will be replaced by a zero in the numerator and a 1 in the denominator.

Define the functions ψ_j^ϵ for $j = 1, \ldots, m$ as follows:

$$\psi_j^\epsilon(x, y, T) = \begin{cases} 1 & \text{if } y_j > 0 \text{ or } \big[\, y_j = 0 \ \wedge \ \langle x, y + \epsilon^j \rangle \in T \,\big] \\ 0 & \text{if } y_j = 0 \ \wedge \ \langle x, y + \epsilon^j \rangle \notin T \end{cases} \tag{5.1}$$

Then replace $\bar{E}_{FGL}$ with a modification $\bar{E}_{FGL}^\epsilon$ defined by

$$\bar{E}_{FGL}^\epsilon(x, y, T) = \min_{\alpha, \beta} \left\{ \frac{\sum_i \delta(x_i)\alpha_i + \sum_j \psi_j^\epsilon(x, y, T)\beta_j}{\sum_i \delta(x_i) + \sum_j \psi_j^\epsilon(x, y, T)} \;\Big|\; \langle \alpha, \beta \rangle \in \Omega(x, y^\epsilon, T) \right\} \tag{5.2}$$

which is well defined, since the minimum is attained at $\overset{*}{\beta} \gg 0^{[m]}$.

The formulation in (5.2) provides a good approximation for small ϵ only if $\bar{E}_{FGL}^\epsilon$ is continuous at the boundary of output space for sequences approaching the boundary from the interior. Although Russell and Schworm [2008b] have shown that the FGL index itself is not in general continuous in input or output quantities, we can show that $\bar{E}_{FGL}^\epsilon$ converges to $\bar{E}_{FGL}$ at the boundary for the restricted paths $y^\epsilon \to y$ as $\epsilon \to 0$.

For any $\langle x, y, T \rangle$ define the reference point $\langle \bar{x}, \bar{y}$ by $\bar{x} = \overset{*}{\alpha} \otimes x$ and $\bar{y} = y \oslash \overset{*}{\beta}$. Then we can reformulate (5.2) as

$$\bar{E}_{FGL}^\epsilon(x, y, T) = \min_{\alpha, \bar{y}} \left\{ \frac{\sum_i \delta(x_i)\alpha_i + \sum_j \psi_j^\epsilon(x, y, T) y_j^\epsilon / \bar{y}_j}{\sum_i \delta(x_i) + \sum_j \psi_j^\epsilon(x, y, T)} \;\Big|\; \langle \alpha, \beta \rangle \in \Omega(x, y^\epsilon, T) \right\}. \tag{5.3}$$

From this formulation, it is clear that $\bar{E}_{FGL}^\epsilon(x, y, T) \to \bar{E}_{FGL}(x, y, T)$ as $\epsilon \to 0$.

If the technology is known and convex, we can sketch a alternative method for calculating $\psi_j(x, y, T)$. If $y_j > 0$, set $\psi_j(x, y, T) = 1$. If $y_j = 0$, calculate the shadow prices for $\langle x, y \rangle$. If the shadow price of the j^{th} output is positive for any shadow price vector supporting $\langle x, y \rangle$, set $\psi_j(x, y, T) = 0$. Otherwise, set $\psi_j(x, y, T) = 1$. If the technology is not convex, shadow prices may not exist and it is necessary to compute $\psi_j(x, y, T)$ directly by checking feasibility of small changes in output as above.

If the technology is not known, it is necessary to use the data to estimate the technology. We sketch here a method of calculating—or at least approximating arbitrarily closely—the modified FLG index using DEA methods.

Assume we have data on inputs and outputs for D decision making units (DMUs): $\langle x^d, y^d \rangle, \ d = 1, \ldots, D$. Define $I_+^d = \{i \mid x_i^{d'} > 0\}$ and $J_+^d = \{j \mid y_j^{d'} > 0\}$.

The original FGL index for a specific DMU, d', is calculated by first running the following program:[12]

$$
\begin{aligned}
&\min_{\alpha,\beta,\lambda} \quad \sum_{i \in I_+^{d'}} \alpha_i + \sum_{j \in J_+^{d'}} \beta_j \quad \text{s.t.} \\
&\alpha_i x_i^{d'} \geq \sum_{d=1}^{D} \lambda_d \, x_i^d \quad \wedge \quad 0 \leq \alpha_i \leq 1 \quad \forall\, i \in I_+^{d'}, \\
&y_j^{d'}/\beta_j \leq \sum_{d=1}^{D} \lambda_d \, y_j^d \quad \wedge \quad 0 < \beta_j \leq 1 \quad \forall\, j \in J_+^{d'}, \\
&\lambda_d \geq 0, \quad d = 1, \ldots, D.
\end{aligned}
\tag{5.4}
$$

Let A be the solution to this program. Then

$$
E_{FGL} = \frac{A}{S\left(x^{d'}, y^{d'}\right)}, \tag{5.5}
$$

where

$$
S(x^{d'}, y^{d'}) = \sum_i \delta\left(x_i^{d'}\right) + \sum_j \delta_j\left(y_j^{d'}\right) = |I_+^{d'}| + |J_+^{d'}|. \tag{5.6}
$$

The proposed method of calculating the data-based, modified FGL index has three steps:

Step 1: For a selected DMU, d', solve

$$
\begin{aligned}
&\min_{\alpha,\beta,\lambda} \quad \sum_{i \in I_+^{d'}} \alpha_i + \sum_{j=1}^{m} \beta_j \quad \text{s.t.} \\
&\alpha_i x_i^{d'} \geq \sum_{d=1}^{D} \lambda_d \, x_i^d \quad \wedge \quad 0 \leq \alpha_i \leq 1 \quad \forall\, i \in I_+^{d'}, \\
&\left(y_j^{\epsilon}\right)^{d'}/\beta_j \leq \sum_{d=1}^{D} \lambda_d \, \left(y_j^{\epsilon}\right)^d \quad \wedge, \quad 0 < \beta_j \leq 1, \quad j = 1, \ldots, m, \\
&\lambda_d \geq 0, \quad d = 1, \ldots, D.
\end{aligned}
\tag{5.7}
$$

Step 2: Denote the minimizing values of β by $\overset{*}{\beta}$ and set $J_\epsilon^{d'} = \{j \mid y_j = 0 \ \wedge \ \overset{*}{\beta}_j < 1\}$.

[12] As noted above, the minimum is well defined (no attempt to divide by zero): since $y_j^{d'}$ is non-zero and the output constraint set is bounded, the solution value for each β_j will be non-zero.

Now solve

$$
\begin{aligned}
\min_{\alpha,\beta,\lambda} \quad & \sum_{i\in I_+^{d'}} \alpha_i + \sum_{j\notin J_\epsilon^{d'}} \beta_j \quad \text{s.t.} \\
& \alpha_i x_i^{d'} \geq \sum_{d=1}^{D} \lambda_d \; x_i^d \quad \wedge \quad 0 \leq \alpha_i \leq 1 \quad \forall\, i \in I_+^{d'}, \\
& (y_j^\epsilon)^{d'}/\beta_j \leq \sum_{d=1}^{D} \lambda_d \; \left(y_j^\epsilon\right)^d, \quad \wedge \quad 0 < \beta_j \leq 1 \qquad \forall j \notin J_\epsilon^{d'}, \\
& \lambda_d \geq 0, \quad d = 1, \ldots, D.
\end{aligned}
\tag{5.8}
$$

Step 3: Denote the solution to (5.8) by A. The (approximate) modified FGL index is then given by

$$
E_{FGL}^{\epsilon} = \frac{A}{|I_+^{d'}| + |J_+^{d'}| + |J_\epsilon^{d'}|}. \tag{5.9}
$$

Step 1 distinguishes between zero values of outputs with $\overset{*}{\beta}_j = 1$ and those with $\overset{*}{\beta}_j < 1$ (*i.e.*, between efficient points with zero outputs and inefficient points with zero outputs). Step 2 calculates the minimal value of the numerator of the objective function under the constraint that $\beta_j = 0$ for all zero outputs that are coordinate-wise inefficient. In step 3, divide the minimal value of the numerator by the sum of the coordinates with positive values of input or output quantities plus the number of outputs with zero values that are coordinate-wise inefficient.

This algorithm yields some values that are only approximately correct (if zero output values exist), owing largely to the use of the ϵ perturbation of the data. In fact, the method could treat a point with zero output values as efficient if y were, in some sense, less than ϵ below the frontier. But such a point would be approximately efficient. Moreover the chance of such incorrect identification of efficient points is extremely remote if ϵ is chosen small enough.

VI. Empirical Example.

In this section, we employ two data sets to illustrate the practicality of our proposed approach to implementing the modified FGL. The first, a synthetic data set with two inputs and two outputs, illustrates the practical restoration of the indication and monotonicity properties at the boundary of output space. The second is an actual data base on an athlete's relative (self) performance over the course of a season.

The synthetic data base contains 10 observations on the two outputs and two inputs. Because the problems with the FGL index arise only at the boundary of output space, we set input quantities to unity, allowing us to represent the technology in output space. The output data and efficiency scores of the two indexes are displayed in Table 1, and the production possibility set (the DEA hull in output space) is depicted in Figure 4.

Table 1: $\bar{E}_{FLG}$ versus E_{FGL} for the Synthetic Data

Observation	y_1	y_2	$E_{FGL}(x, y, T)$	$\bar{E}_{FGL}(x, y, T)$
1	0	6	1	.81
2	3	6	1	1
3	4	5	1	1
4	5	4	1	1
5	9	0	1	1
6	0	3	.67	.50
7	1	5	.77	.77
8	3	4	.88	.88
9	5	3	.94	.94
10	5	0	.70	.54

As required by theory, the two indexes agree at all points except inefficient points containing a zero value (observations 1, 6, and 10). Observation 1 is identified as output efficient by the original FGL index, whereas the modified index corrects for inefficiency at this boundary point. Observation 5 is properly identified as an output-efficient point despite the zero value of output 2.

As discussed in the Introduction, zero outputs may arise naturally in empirical work for several reasons. To impose a demanding test on our proposed modification of the FGL index, we select a problem with frequent observations with zeros in the outputs. In particular, we measure a baseball player's batting performance in each game by counting the singles, doubles, triples, and home runs during the game.

Mazur [1995] evaluates players' relative efficiencies by utilizing a zero-dimensional input space and a three-dimensional output space composed of a player's batting average, runs batted in (RBIs), and home runs. Many baseball statisticians, however, have rejected this formulation, because RBI's are a function of batting average as well as the number of players currently on base, which may not be directly related to a particular batter's performance. Anderson and Sharp [1997] utilize the composite batter index (CBI) as a performance mea-

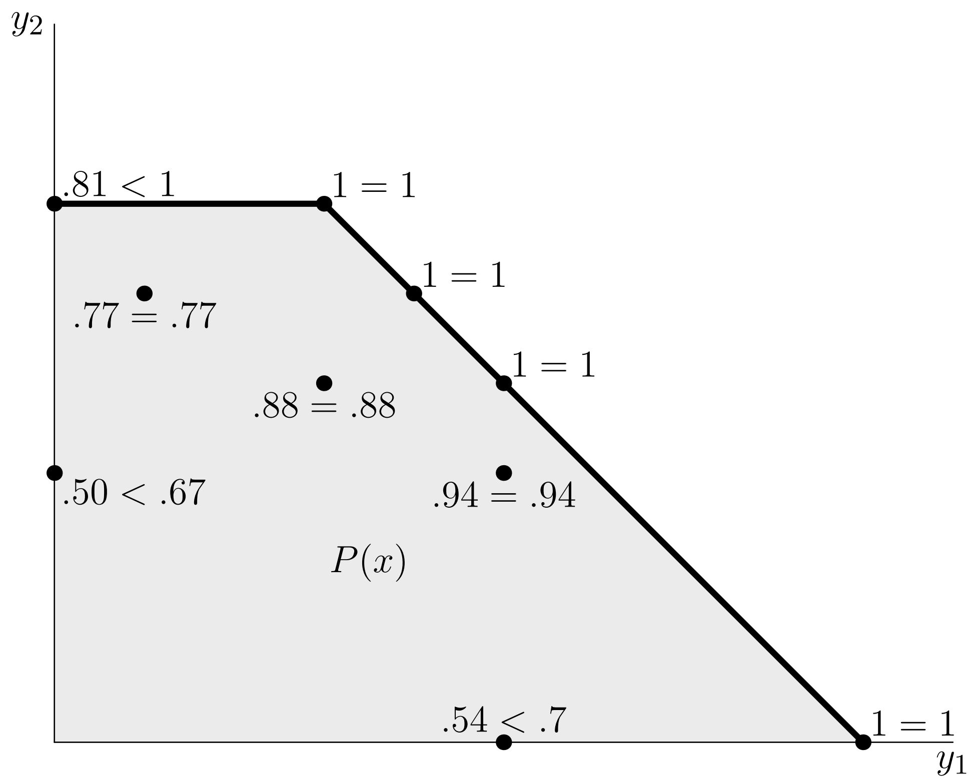

Figure 4: The Technology for the Synthetic Data

sure of a baseball players batting prowess. They adopt plate appearances—the number of at bats plus the number of walks—as the one-dimensional input. The five outputs are walks, singles, doubles, triples, and home runs. Their analysis, however, uses inputs and outputs aggregated over an entire season, in which case zero values in output space are unlikely.

We proceed in a fashion similar to Anderson and Sharp [1997], adopting a modification of their CBI. We drop from the output space the number of walks a player accumulates and look only at active hit performance, singles, doubles, triples, and home runs, since walks are mainly a function of the pitcher's performance, not the batter's. Specifically, we track the batting performance of Babe Ruth, one of history's most renowned baseball figures, on a per-game basis, using both the FGL and the modified FGL for the 1923 season with the New York Yankees.

We are unaware of any studies that use such a high-frequency statistic as batting performance on a per game basis. The flexibility of the modified FGL index in handling zeros in the output data allows us to increase the frequency at which we are able to calculate relative efficiencies. The data used can be found at http://baseball-reference.com. We track the performance of Babe Ruth over the duration of the 1923 regular season, in which he won the MVP award. The season consisted of 152 games total, and we treat Babe Ruth's performance in each game as a separate DMU to assess performance relative to his performance in other games played during that season.

We compute, for each of the 152 games, the original FGL index and the modified FGL index numerically, by solving the non-linear programs laid out in the previous section using a combination of active-set, trust-region, and line-search algorithms.[13]

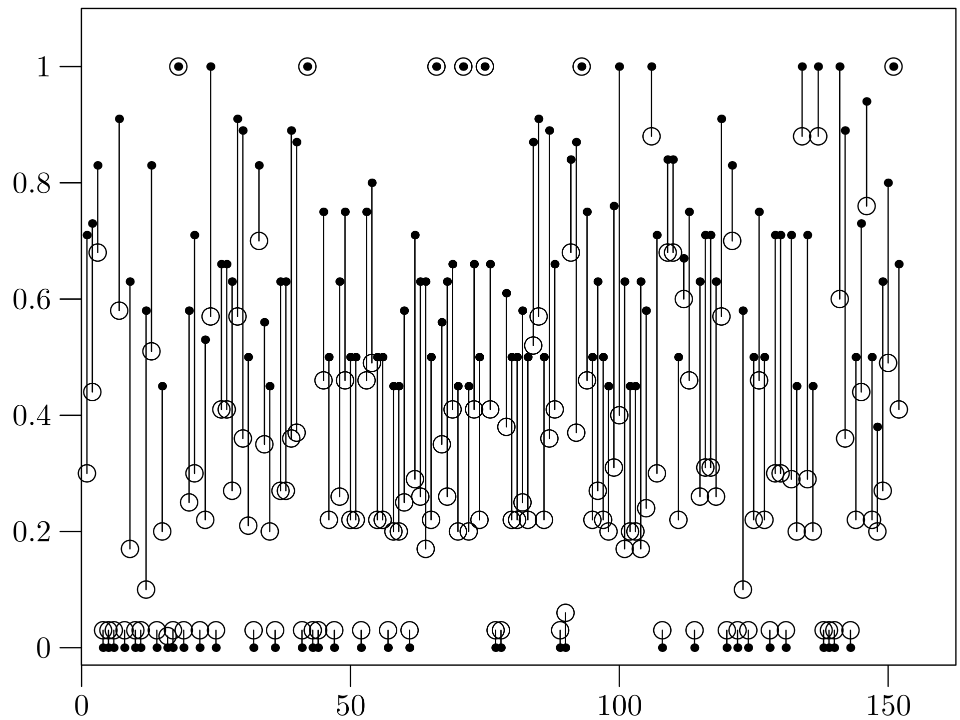

Figure 5: $\bar{E}_{FLG}$ versus E_{FGL}

As can be seen from Figure 5, the modified FGL index corrects for inefficient boundary points where there are zero outputs. The Spearman rank and Pearson product moments between the FGL and the modified FGL indexes for this exercise are 0.7073 and 0.7102 respectively. Thus, the modified FGL index, by adjusting for inefficient boundary points, does in fact significantly alter the relative rankings of performance across games.

VII. Conclusion.

Satisfaction of the indication condition is the principal (putative) advantage of the FGL index (as well as related "slacks based" indexes that contract inputs and expand outputs in coordinate-wise directions) over indexes that contract inputs and expand outputs in a radial or arbitrary direction—*e.g.*, the hyperbolic index (Färe, Grosskopf, and Lovell [1985]),

[13] The programming codes are available at http://economics.ucr.edu/people/russell/index.html.

the Briec [1997] index, and the directional-distance index (Luenberger [1992], Chung, Färe, and Grosskopf [2000], and Färe and Grosskopf [2000]). Failure to satisfy indication at the boundary is therefore a serious inadequacy of the FGL index, as is the failure to satisfy weak monotonicity, which *is* satisfied by the radial and directional indexes. We therefore believe that the modification of the FGL index we propose to restore indication and weak monotonicity is essential. In addition, we provide some illustrative empirical evidence, using baseball performance data, that the modified index can be practicably implemented and can yield results that are markedly different from those generated by the original FGL index.

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