A New Family of Copulas, with Application to Estimation of a Production Frontier System

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Abstract

In this paper we propose a new family of copulas for which the copula arguments are uncorrelated but dependent. Specifically, if w_1 and w_2 are the uniform random variables in the copula, they are uncorrelated, but w_1 is correlated with $|w_2 - ½|$. We show how this family of copulas can be applied to the error structure in an econometric production frontier model. We also generalize the family of copulas to three or more dimensions, and we give an empirical application.

1. Introduction

Let $c(w_1, w_2)$ be a copula density. In this paper we will propose and use copulas that have the property that the correlation between the copula arguments is zero, but w_1 is correlated with $|w_2 - ½|$.

As a practical motivation for such copulas, suppose that we are interested in estimating a system of equations, where one equation is a production (or cost) function and the other equations are the first order conditions for cost minimization. In the production frontier literature that dates back to Aigner, Lovell and Schmidt (1977) and Meeusen and van den Broeck (1977), the production frontier gives the maximal output that can be produced from a vector of inputs. The equation representing the production function contains a one-sided error that represents technical inefficiency, that is, the failure to produce maximal output given the inputs. It is often assumed to be half-normal, though any one-sided distribution is possible. Also, because the first order conditions for cost minimization will not be satisfied exactly, the corresponding equations contain errors that represent allocative inefficiency, that is, the failure to use the inputs in the correct proportions given input prices. These errors are often assumed to be normal.

As a matter of generic notation, let $u \geq 0$ represent technical inefficiency and let ω (taken as a scalar for purposes of this discussion) represent the allocative error in the first order condition. So u represents the shortfall of output from the frontier, and ω represents the deviation of the actual from the optimal (log) input ratio. If technical inefficiency and allocative inefficiency are independent, there are no particular difficulties involved in deriving a likelihood for the model. See, e.g., Schmidt and Lovell (1979). However, if technical and allocative inefficiency are not independent, we need to model this dependence. Schmidt and Lovell (1980) did this in a specific way that will be discussed below, but which was tailored to the normal / half-normal case. More generally, given specific marginal distributions for u and ω, we need to specify a copula so that we can obtain their joint distribution.

We then encounter the issue that common copulas do not capture the type of dependence that the economic model implies. We do *not* want to model a non-zero covariance between u and ω. For example, a positive correlation between u and ω would imply that firms that are more technically inefficient (larger values of u) have, say, higher capital / labor ratios than more technically efficient firms, which is not what we have in mind. What we want is a positive correlation between u and $|\omega|$, which says that firms that are more technically inefficient have capital / labor ratios that are more in error (*either* too high *or* too low) than more technically efficient firms. That is, paraphrasing Schmidt and Lovell (1980, p. 96), we need to recognize that, as far as the extent of allocative inefficiency is concerned, what is relevant is not the size of ω, but the size of $|\omega|$.

The same argument can apply in a non-frontier setting. It does not hinge on $u \geq 0$. Even if u is a standard zero-mean error (e.g. normal), it may be reasonable to assume that u is correlated with $|\omega|$ rather than ω, reflecting the view that firms that are better at using the correct input ratios also on average produce more output from a given set of inputs.

In this paper, we propose a family of copulas that have the desired properties that $\text{cov}(w_1, w_2) = 0$ but $\text{cov}(w_1, |w_2 - ½|) > 0$. Here w_1 and w_2 are the uniformly distributed copula arguments, that is, the cdf values of u and ω respectively. If the distribution of ω is symmetric around zero, then $\omega = 0$ corresponds to $w_2 = ½$. We are not aware of any existing copulas, other than the one implicit in Schmidt and Lovell (1980), that have these properties. In the two-dimensional case, this is relatively straightforward. However, as is often the case in the copula literature, extending the two-dimensional results to three or more dimensions is non-trivial.

The plan of the paper is as follows. In Section 2 we give some more specific detail about the economic model we consider and discuss some related literature. In Section 3, we introduce our new family of copulas in the two-dimensional case. Section 4 gives a corresponding family of copulas for the three-dimensional case and discusses the difficulties in extending these results to four or more dimensions. Section 5 provides detail on the evaluation of the simulated likelihood that is used in estimation. Section 6 contains an empirical example, and Section 7 gives our concluding remarks.

2. A Specific Production Frontier System

Consider the stochastic frontier model

(1) $$y_i = \alpha + x_i'\beta + v_i - u_i = \alpha + x_i'\beta + \varepsilon_i \;\;,\;\; i = 1, \dots, N \;\;,$$

where v_i is distributed as $N(0, \sigma_v^2)$; u_i is distributed as $N^+(0, \sigma_u^2)$, i.e. "half normal"; and v_i and u_i are independent. In terms of the discussion of the previous section, v_i represents statistical noise and $u_i \geq 0$ represents technical inefficiency. When x_i is "exogenous" (independent of v_i and u_i), this is the model of Aigner, Lovell and Schmidt (1977) and Meeusen and van den Broeck (1977), which is commonly called the standard SFM.

We will consider specifically the Cobb-Douglas (log-linear) functional form, in which y_i is the natural log of the output of firm i, and x_i is a $k \times 1$ vector of the natural logs of the inputs. This leads us to the set of equations for the optimal input ratios:

(2) $$x_{i1} - x_{ij} = B_{ij} + \omega_{ij} \quad , \; j = 2, \dots, K \quad ,$$

where $B_{ij} = p_{ij} - p_{i1} + \ln(\beta_1) - \ln(\beta_j)$. Here p_{ij} is the natural log of the price of input j for firm i; β_j is the j^{th} element of β in (1); and ω_{ij} represents allocative inefficiency.

If we move to the non-statistical world by suppressing ε_i in (1) and the ω_{ij} in (2), then (2) is the set of first order conditions for the minimization (with respect to the choice of $x_{i1}, \dots, x_{ik}$) of the cost of producing output level y_i. Now we return to the statistical world by reintroducing the errors ε_i and ω_{ij}, and we assume that y_i (not x_i, as in the standard SFM) and the p_{ij} are exogenous. The inputs x_{ij} are the solution to equations (1) and (2) and are "endogenous" in the sense that they depend on the errors in the model. As described up to this point, this is the model of Schmidt and Lovell (1979).

We assume that the v_i are iid $N(0, \sigma_v^2)$; the u_i are $N^+(0, \sigma_u^2)$, i.e. "half normal"; the $\omega_i \equiv (\omega_{i2}, \dots, \omega_{ik})'$ are iid $N(\mu, \Sigma_{\omega\omega})$; and v is independent of u and ω. For the purposes of the current discussion we will take $\mu = 0$. The issue of this paper is the relationship of u and ω. In Schmidt and Lovell (1979), it was assumed that u and ω are independent. This implies the joint density of u, v and ω. The joint density of ε and ω (where $\varepsilon = v - u$) is calculated by an integral that is tractable. We then solve the system for x_i, calculate the Jacobian as equal to $r = \sum_{j=1}^{k} \beta_j$, and obtain the likelihood as given in equation (11), p. 357, of Schmidt and Lovell (1979).

However, as argued above, independence of u and ω is not an attractive assumption. Schmidt and Lovell (1980) proposed a model with the desired properties that u is uncorrelated with ω, but u is positively correlated with the absolute value of each element of ω. They assumed that $u = |u^*|$, where $\begin{bmatrix} u^* \\ \omega \end{bmatrix} \sim N(0, \Sigma)$ and where $\Sigma = \begin{bmatrix} \sigma_u^2 & \Sigma_{\omega u}{}' \\ \Sigma_{\omega u} & \Sigma_{\omega\omega} \end{bmatrix}$. This is consistent with the marginal distributions given above, since u is half normal and ω is multivariate normal. Now u and ω are uncorrelated, and the correlation between u and $|\omega_j|$ is

$$(2/\pi)[\sqrt{1-\rho_j^2} + \rho_j \arcsin(\rho_j) - 1] \geq 0 ,$$

where ρ_j is the correlation between u^* and ω_j. They give the likelihood for this model in equation (6), p. 88.

Clearly there must be a copula implicit in this construction. It is not hard to see that this copula is the mixture (with weights equal to ½) of the Gaussian copula with variance matrix Σ and the Gaussian copula with variance matrix $\Sigma^* = \begin{bmatrix} \sigma_u^2 & -\Sigma_{\omega u}{}' \\ -\Sigma_{\omega u} & \Sigma_{\omega\omega} \end{bmatrix}$. For lack of a better name, we will call this the SL copula. Some details about it are given in Appendix 1.

While this construction depends on the half-normal / normal assumption, the SL copula is a valid copula that can be used regardless of the marginal distributions chosen for u and ω. However, it would be desirable to have alternative copulas to accomplish the same objectives as the SL copula. We can observe the following. Suppose that $c(w_1, w_2; \theta)$ is a copula and that $c(w_1, w_2; -\theta)$ is also a copula Then $c_*(w_1, w_2; \theta) = \frac{1}{2}c(w_1, w_2; \theta) + \frac{1}{2}c(w_1, w_2; -\theta)$ is also a copula. We will call it a *folded copula*. The SL copula is the folded normal copula. More generally, if the value of Spearman's rho for the copula $c(w_1, w_2; \theta)$ is an odd function of θ, then Spearman's rho equals zero for the folded copula. The normal copula has this property, and that is why the SL copula generates uncorrelated copula arguments. The Student-t copula with fixed number of degrees of freedom also has this property. Some copulas that have this property may not yield a useful folded copula. For example, the folded Farlie-Gumbel-Morgenstern (FGH) copula is just the independence copula. However, most common copulas do not have this property. We will therefore construct and propose some alternatives in the next two sections of the paper. These will not be folded copulas but they will be constructed so that they have Spearman's rho equal to zero.

Although our discussion has closely followed the specific models of Schmidt and Lovell (1979, 1980) there are many papers, both theoretical and applied, that consider systems consisting of a production or cost function and a set of first-order conditions for maximization or minimization of a criterion function (e.g. cost minimization or profit maximization). Examples include Christensen and Greene (1976), Greene (1980), Kumbhakar (1987, 1991, 1997), Ferrier and Lovell (1990) and Atkinson and Cornwell (1994).

3. The APS-2 Copulas

In this section we consider the two-dimensional case in which we specify a copula density $c(w_1, w_2)$, where w_1 and w_2 are scalars. In terms of our economic model, this corresponds to the case of two inputs, and correspondingly the relevant random variables are u and scalar ω; then the copula arguments are $w_1 = F_u(u)$ and $w_2 = F_\omega(\omega)$.

The well-known FGM copula is of the form $c(w_1, w_2) = 1 + \theta(1 - 2w_1)(1 - 2w_2)$ with $|\theta| < 1$. This generalizes to the Sarmanov (1966) family of copulas, which are of the form $c(w_1, w_2) = 1 + \theta g(w_1)h(w_2)$, where $\int_0^1 g(s)ds = \int_0^1 h(s)ds = 0$, and where the restrictions on θ that are necessary for c to be a density are the restrictions that guarantee that $c(w_1, w_2) \geq 0$ for all w_1, w_2 in the unit cube. These depend on the specific forms of the functions g and h.

DEFINITION 1. An APS-2 copula is a two-dimensional Sarmanov copula with $g(w_1) = 1 - 2w_1$ (as in the FGM copula) and $h(w_2) = 1 - k_q^{-1}q(w_2)$, where $q(s)$ is integrable on [0,1]; $q(s)$ is symmetric around $s = ½$, that is, $q(s) = q(1 - s)$; $q(s)$ is monotonically decreasing on $[0, ½]$ and therefore monotonically increasing on $[½, 1]$; and $k_q = \int_0^1 q(s)\, ds$ so that $\int_0^1 h(s) ds = 0$.

Therefore an APS-2 copula is of the form $c(w_1, w_2) = 1 + \theta(1 - 2w_1)(1 - k_q^{-1}q(w_2))$ where the function q has the properties given in Definition 1. Some restriction on θ will be necessary for this to actually be a copula. This restriction will depend on the form of q.

RESULT 1. For any APS-2 copula and any value of θ, $\mathrm{cov}(w_1, w_2) = 0$.

The proofs of the results in this section are given in Appendix 2.

Result 1 depends only on the symmetry of $q(s)$ around $s = ½$. It holds not just for $g(w_1) = 1 - 2w_1$, but for any $g(w_1)$ such that $\int_0^1 g(s) ds = 0$, that is, for a larger class of copulas than the APS-2 family.

RESULT 2. For any APS-2 copula, $\mathrm{cov}\big(w_1, q(w_2)\big) = \frac{1}{6}\theta k_q^{-1}\mathrm{var}\big(q(w_2)\big)$.

Result 2 implies that, in an APS-2 copula, θ is proportional to $\mathrm{cov}\big(w_1, q(w_2)\big)/\,\mathrm{var}\big(q(w_2)\big)$, and therefore to $\sqrt{\mathrm{var}\big(q(w_2)\big)} \cdot \mathrm{corr}\big(w_1, q(w_2)\big)$.

Results 1 and 2 are for the copula arguments w_1 and w_2. As is true throughout the copula literature, if we consider instead the original variables $u = F_u^{-1}(w_1)$ and $\omega = F_\omega^{-1}(w_2)$, there is little that can be said because the transformation from the copula arguments to the original random variables is nonlinear and it depends on the marginal distributions of these variables. However, we can show that the variables u and ω are uncorrelated if their marginal distributions are symmetric and they are linked by an APS-2 copula.

RESULT 3. Suppose that u and ω have symmetric marginal distributions with finite variance, and that they are linked by an APS-2 copula. Then cov(u, ω) = 0.

To proceed beyond Results 1 and 2, we will consider two specific members of the APS-2 family, as follows.

(3A) APS-2-A $c(w_1, w_2) = 1 + \theta(1 - 2w_1)[1 - 12(w_2 - ½)^2]$, $|\theta| \leq ½$

(3B) APS-2-B $c(w_1, w_2) = 1 + \theta(1 - 2w_1)(1 - 4|w_2 - ½|)$, $|\theta| \leq 1$

These are the copula densities. For a Sarmanov copula of the form $c(w_1, w_2) = 1 + \theta g(w_1)h(w_2)$, the copula cdf is $C(w_1, w_2) = w_1 w_2 + \theta G(w_1)H(w_2)$, where $G(w_1) = \int_0^{w_1} g(s)ds$ and $H(w_2) = \int_0^{w_2} h(s)ds$. So, for the APS-2 copula with copula density $c(w_1, w_2) = 1 + \theta(1 - 2w_1)\left(1 - k_q^{-1}q(w_2)\right)$, the copula cdf is $C(w_1, w_2) = w_1 w_2 + \theta w_1(1 - w_1)\left(w_2 - k_q^{-1}Q(w_2)\right)$, where $Q(w_2) = \int_0^{w_2} q(s)ds$. Specifically, for the APS-2-A copula, k_q^{-1}= 12 and $Q(w_2) = \frac{1}{3}w_2^3 - \frac{1}{2}w_2^2 + \frac{1}{4}w_2$, so that $C(w_1, w_2) = w_1 w_2 + \theta w_1 w_2$ $(1 - w_1)[1 - (4w_2^2 - 6w_2 + 3)]$. Similarly, for the APS-2-B copula, k_q^{-1}= 4 and

(4) $$Q(w_2) = \begin{cases} \frac{1}{2}w_2(1-w_2)\ , & w_2 \le ½ \\ \frac{1}{4} - \frac{1}{2}w_2(1-w_2)\ , & w_2 > ½ \end{cases}$$

RESULT 4. (i) The APS-2-A copula in (3A) above is a copula for $|\theta| \le ½$. (ii) $\mathrm{cov}[w_1, (w_2 - ½)^2] = \frac{1}{90}\theta$. (iii) $\mathrm{var}[(w_2 - ½)^2] = \frac{1}{180}$. (iv) $\mathrm{corr}[w_1, (w_2 - ½)^2] = \frac{2}{\sqrt{15}}\theta \cong 0.516\,\theta$.

RESULT 5. (i) The APS-2-B copula in (3B) above is a copula for $|\theta| \le 1$. (ii) $\mathrm{cov}[w_1, |w_2 - ½|] = \frac{1}{72}\theta$. (iii) $\mathrm{var}(|w_2 - ½|) = \frac{1}{48}$. (iv) $\mathrm{corr}[w_1, |w_2 - ½|] = \frac{1}{3}\theta$.

4. The Three Dimensional Case

4.1 Some General Comments

We now consider the three-dimensional case. In terms of our economic model, this would correspond to the case of three inputs, and therefore two equations for the optimal input ratios, as in equation (2) above. We have three random variables u, ω_2 and ω_3, and correspondingly three copula arguments, $w_1 = F_u(u)$, $w_2 = F_{\omega_2}(\omega_2)$ and $w_3 = F_{\omega_3}(\omega_3)$. We want w_2 and w_3 to follow any standard bivariate copula, such as bivariate normal, and we want w_1 to be linked to w_2 and w_3 as in the APS-2 copulas. That is, as before, we want w_1 to be uncorrelated with w_2 (and w_3) but correlated with $|w_2 - ½|$.

Most of the copula literature covers the two-dimensional case. Moving from two dimensional copulas to copulas of three or more dimensions is non-trivial. As Nelsen (2006, p. 105) notes, “Constructing n-copulas is difficult. Few of the procedures discussed earlier … have *n*-dimensional analogs." The problem is that there is inevitably an infinity of possibilities.

To illustrate this issue, start with the two-dimensional FGM copula $c_{12} \equiv c(w_1, w_2) = 1 + \theta(1 - 2w_1)(1 - 2w_2)$. Now consider the following three-dimensional copulas:

(5A) $$c_{123}^A = 1 + \theta_{123}(1 - 2w_1)(1 - 2w_2)(1 - 2w_3)$$

(5B) $$c_{123}^B = 1 + \theta_{12}(1 - 2w_1)(1 - 2w_2) + \theta_{13}(1 - 2w_1)(1 - 2w_3) + \theta_{23}(1 - 2w_2)(1 - 2w_3)$$

(5C) $$c_{123}^C = 1 + \theta_{12}(1 - 2w_1)(1 - 2w_2) + \theta_{13}(1 - 2w_1)(1 - 2w_3) + \theta_{23}(1 - 2w_2)(1 - 2w_3) + \theta_{123}(1 - 2w_1)(1 - 2w_2)(1 - 2w_3)$$

The last of these, c_{123}^C, is given in Nelsen (2006, p. 108). So far as we are aware, the other two are new. In any case, for suitable values of the θ's, these are all copulas; they are densities, and their two-dimensional marginals are two-dimensional copulas ("2-copulas"). For c_{123}^A, the implied 2-copulas are uniform, e.g. $\int c_{123}^A \, dw_1 = 1$. So c_{123}^A is a distribution in which the three w's are pairwise independent but not jointly independent. For c_{123}^B and c_{123}^C, the implied 2-copulas are FGM, e.g. $\int c_{123}^B \, dw_1 = \int c_{123}^C \, dw_1 = 1 + \theta_{23}(1 - 2w_2)(1 - 2w_3)$. So c_{123}^B and c_{123}^C are different joint distributions that have the same marginals of order two and one. The problem is that it is not clear which of these is in some sense more natural.

4.2 Some General Results

Suppose very generally that we wish to extend a 2-copula to a 3-copula. An intuitively reasonable possibility is to use a copula as an argument in a copula. More specifically, suppose that c^o and c are 2-copulas, and we define

(6) $$c^*(w_1, w_2, w_3) = c^o(c(w_1, w_2), w_3).$$

That is, we use the copula c^o to link the copula c to a third random variable w_3 (which could be another copula). This may be intuitively reasonable, but unfortunately it does not generally yield a 3-copula. This is the so-called compatibility problem, discussed by Nelsen (2006, pp. 105-107), for which there are quite a few results, most of them negative. That discussion is in terms of copula cdf's, not densities, but the same negative conclusion holds for densities as in (6). For example, suppose that c is an arbitrary copula and c^o is FGM. So

$$c^*(w_1, w_2, w_3) = 1 + \theta[1 - 2c(w_1, w_2)](1 - 2w_3).$$

Then $\int c^*(w_1, w_2, w_3)\, dw_3 = 1$ but

$$\int c^*(w_1, w_2, w_3)\, dw_1 = 1 + \theta(1 - 2w_3)[1 - 2\int c(w_1, w_2) dw_1] = 1 - \theta(1 - 2w_3)$$

which is not a 2-copula. (And a similar argument applies to the integral with respect to w_2.)

An apparent solution is to remove the factor of 2 in the term $2c(w_1, w_2)$.

RESULT 6. Suppose that $c(w_1, w_2)$ is a 2-copula and define

(7) $$c^*(w_1, w_2, w_3) = 1 + \theta[1 - c(w_1, w_2)](1 - 2w_3).$$

Then, for values of θ such that $c^*(w_1, w_2, w_3) \geq 0$ for all w_1, w_2, w_3, c^* is a 3-copula.

The proof of Result 6 is simply to calculate that $\int c^*(w_1, w_2, w_3)\, dw_1 = \int c^*(w_1, w_2, w_3)\, dw_2 = \int c^*(w_1, w_2, w_3)\, dw_3 = 1$, so that all three implied 2-copulas are the uniform (independence) copula. So we have joint dependence but pairwise independence, which is not what we want. The copula c_{123}^A in equation (5A) is of the form of (7) and suffers from this same problem, as noted above.

Another possible extension of a 2-copula to a 3-copula is given by the following result.

RESULT 7. Suppose that $c(w_1, w_2)$ is a 2-copula and define

(8) $$c^*(w_1, w_2, w_3) = c(w_1, w_2) + \theta[1 - c(w_1, w_2)](1 - 2w_3).$$

Then, for values of θ such that $c^*(w_1, w_2, w_3) \geq 0$ for all w_1, w_2, w_3, c^* is a copula.

It is easy to calculate that $\int c^*(w_1, w_2, w_3)\, dw_1 = \int c^*(w_1, w_2, w_3)\, dw_2 = 1$ and that $\int c^*(w_1, w_2, w_3)\, dw_3 = c(w_1, w_2)$, all of which are 2-copulas. So the 2-copula for w_1, w_2 that we started with is preserved, but the other two 2-copulas are the independence copula, which is restrictive, and in our case not what we want.

The purpose of the last two examples is to stress that it is not hard to extend a 2-copula to a 3-copula, but the resulting 3-copula may not have the properties that we want. However, we are now ready to give a positive and (we hope) useful result.

RESULT 8. Let $c_{12}(w_1, w_2)$, $c_{13}(w_1, w_3)$ and $c_{23}(w_2, w_3)$ be 2-copulas. Define

(9) $$c^*(w_1, w_2, w_3) = 1 + (c_{12} - 1) + (c_{13} - 1) + (c_{23} - 1).$$

Then if c^* is a density, it is a 3-copula, and the implied 2-copulas are c_{12}, c_{13} and c_{23}.

The proof is trivial. Simply calculate, e.g., $\int c^*(w_1, w_2, w_3) dw_1 = 1 + 0 + 0 + (c_{23} - 1) = c_{23}$. This is a very simple construction, but so far as we are aware it is original. Because the integral of c^* equals one, the requirement that c^* be a density is just the requirement that $c^*(w_1, w_2, w_3) \geq 0$ for all w_1, w_2, w_3 in the unit cube.

The result is important because it shows how, if we start with 2-copulas that capture the bivariate dependence between any two of w_1, w_2, w_3, we can construct a 3-copula that gives their joint distribution, and does so in such a way that the form of the bivariate dependence is preserved.

The FGM 3-copula c_{123}^B in equation (5B) above is of this form.

The construction in Result 8 generalizes to higher dimensions. For example, in the four-dimensional case, we could construct $c^*(w_1, w_2, w_3, w_4) = 1 + (c_{12} - 1) + (c_{13} - 1) + (c_{14} - 1) + (c_{23} - 1) + (c_{24} - 1) + (c_{34} - 1)$. If this is a density, it is a copula, its 3-copulas are of the form given in Result 8, and its 2-copulas are the c_{ij} with which we started. However, this is not the only option for extending Result 8 to four dimensions. We discuss this issue in Appendix 3.

4.3 The APS-3 Copulas

We now return to the special case of our economic model with three inputs, and therefore two equations that give the optimal input ratios. We have three random variables u, ω_2 and ω_3, and correspondingly three copula arguments, $w_1 = F_u(u)$, $w_2 = F_{\omega_2}(\omega_2)$ and $w_3 = F_{\omega_3}(\omega_3)$. We want w_2 and w_3 to follow any standard bivariate copula, such as bivariate normal, and we want w_1 to be linked to w_2 and w_3 as in the APS-2 copulas. We can use Result 8 to accomplish this.

Specifically, we define the APS-3-A and APS-3-B copulas as follows.

(10A) APS-3-A $c^*(w_1, w_2, w_3) = 1 + (c_{12} - 1) + (c_{13} - 1) + (c_{23} - 1)$ where

$$c_{12}(w_1, w_2) = 1 + \theta_{12}(1 - 2w_1)[1 - 12(w_2 - \tfrac{1}{2})^2]$$

$$c_{13}(w_1, w_3) = 1 + \theta_{13}(1 - 2w_1)[1 - 12(w_3 - \tfrac{1}{2})^2]$$

$c_{23}(w_2, w_3)$ = bivariate normal copula

(10B) APS-3-B $c^*(w_1, w_2, w_3) = 1 + (c_{12} - 1) + (c_{13} - 1) + (c_{23} - 1)$ where

$$c_{12}(w_1, w_2) = 1 + \theta_{12}(1 - 2w_1)(1 - 4\left|w_2 - \tfrac{1}{2}\right|)$$

$$c_{13}(w_1, w_3) = 1 + \theta_{13}(1 - 2w_1)(1 - 4\left|w_3 - \tfrac{1}{2}\right|)$$

$$c_{23}(w_2, w_3) = \text{bivariate normal copula}$$

For these to be copulas, they must be densities, that is, we must have $c^*(w_1, w_2, w_3) \geq 0$ for all w_1, w_2, w_3. This will require restrictions on θ_{12}, θ_{13} and the correlation parameter ρ in the bivariate normal copula. For example, in the APS-3-A case, relevant bounds for the various terms in the copula are: $-2 \leq (1 - 2w_1)[1 - 12\left(w_2 - \frac{1}{2}\right)^2] \leq 1$ (and similarly for w_3 in place of w_2), and $0 \leq \text{bivariate normal copula} \leq (1 - \rho^2)^{-\frac{1}{2}}$. However, it is not easy to convert these into explicit restrictions on θ_{12}, θ_{13} and ρ so that $c^*(w_1, w_2, w_3) \geq 0$. It is easy to come up with sufficient conditions but not to see that these restrictions are tight. See Nelsen (2006, p. 108) for an analysis of the (simpler) FGM case. It will generally be easier to just check positivity numerically in the course of the maximization of the likelihood that the copula leads to.

5. Some Remarks on Simulation of the Likelihood

We now return to the problem of the estimation of the model of Section 2. To form a likelihood we need the joint density of ε, ω_2 and ω_3, which we will denote as $f_{\varepsilon,\omega_2,\omega_3}(\varepsilon, \omega_2, \omega_3)$.

We can obtain the joint density of u, ω_2 and ω_3 by specifying their marginal densities and a copula. That is,

(11) $$f_{u,\omega_2,\omega_3}(u, \omega_2, \omega_3) = c^*(w_1, w_2, w_3) \cdot f_u(u) \cdot f_{\omega_2}(\omega_2) \cdot f_{\omega_3}(\omega_3) \,,$$

where as before $w_1 = F_u(u)$, $w_2 = F_{\omega_2}(\omega_2)$ and $w_3 = F_{\omega_3}(\omega_3)$. Here $c^*(w_1, w_2, w_3)$ could be any copula, for example, the APS-3-A or APS-3-B copula as given in equations (10A) and (10B) above. The marginal densities $f_u(u), f_{\omega_2}(\omega_2)$ and $f_{\omega_3}(\omega_3)$ could be anything, though what we have in mind for our model is half-normal, normal and normal.

Since v is independent of u, ω_2 and ω_3, the joint density of v, u, ω_2 and ω_3 is

(12) $$f_{v,u,\omega_2,\omega_3}(v, u, \omega_2, \omega_3) = f_v(v) \cdot c^*(w_1, w_2, w_3) \cdot f_u(u) \cdot f_{\omega_2}(\omega_2) \cdot f_{\omega_3}(\omega_3).$$

Then

(13) $$\begin{aligned} f_{\varepsilon,\omega_2,\omega_3}(\varepsilon, \omega_2, \omega_3) &= \int f_{v,u,\omega_2,\omega_3}(u + \varepsilon, u, \omega_2, \omega_3) du \\ &= \int c^*(w_1, w_2, w_3) \cdot f_u(u) \cdot f_{\omega_2}(\omega_2) \cdot f_{\omega_3}(\omega_3) \cdot f_v(u + \varepsilon)\, du \\ &= f_{\omega_3}(\omega_3) \cdot f_{\omega_2}(\omega_2) \cdot \int c^*(w_1, w_2, w_3) \cdot f_v(u + \varepsilon) \cdot f_u(u)\, du \end{aligned}$$

(Note that c^* remains inside the integral sign because w_1 is a function of u.)

The integral in (13) is generally intractable. However, we can write this as

(14) $$f_{\varepsilon,\omega_2,\omega_3}(\varepsilon, \omega_2, \omega_3) = f_{\omega_2}(\omega_2) \cdot f_{\omega_3}(\omega_3) \cdot E_u[c^*(w_1, w_2, w_3) \cdot f_v(u + \varepsilon)]$$

where E_u represents the expectation with respect to the distribution of u. This expectation can be evaluated (approximated) by taking the average over a large number of draws from the distribution of u. The log likelihood for the model can then be obtained by summing (over observations) the log of the simulated densities in (14). This leads to the method of simulated likelihood, for which a standard reference is Greene (2003).

In the special case of the APS-3 copulas, the expression in (14) can be rewritten as follows. Let $g(w) = (1 - 2w)$ and $h(w) = 1 - 12(w - \frac{1}{2})^2$ [for the APS-3-A copula] or $h(w) = \left(1 - 4\left|w - \frac{1}{2}\right|\right)$ [for the APS-3-B copula]. Note that $c_{12}(w_1, w_2) = 1 +$

$\theta_{12}g(w_1)h(w_2)$, $c_{13}(w_1, w_3) = 1 + \theta_{13}g(w_1)h(w_3)$ and $c^*(w_1, w_2, w_3) = c_{12} + c_{13} + c_{23} - 2 = \theta_{12}g(w_1)h(w_2) + \theta_{13}g(w_1)h(w_3) + c_{23} = g(w_1)[\theta_{12}h(w_2) + \theta_{13}h(w_3)] + c_{23}$. Inserting this expression for c^* into (14), we obtain $f_{\varepsilon,\omega_2,\omega_3}(\varepsilon, \omega_2, \omega_3) = f_{\omega_2}(\omega_2) \cdot f_{\omega_3}(\omega_3) \cdot [\theta_{12}h(w_2) + \theta_{13}h(w_3)] \cdot E_u[g(w_1) \cdot f_v(u + \varepsilon)] + c_{23} \cdot f_{\omega_3}(\omega_3) \cdot f_{\omega_2}(\omega_2) \cdot E_u f_v(u + \varepsilon)$. But $c_{23} \cdot f_{\omega_3}(\omega_3) \cdot f_{\omega_2}(\omega_2)$ = the bivariate normal density $\varphi(\omega_2, \omega_3)$, and $E_u f_v(u + \varepsilon) = f_\varepsilon(\varepsilon)$. So

$$f_{\varepsilon,\omega_2,\omega_3}(\varepsilon, \omega_2, \omega_3) = f_\varepsilon(\varepsilon) \cdot \varphi(\omega_2, \omega_3) + f_{\omega_2}(\omega_2) \cdot f_{\omega_3}(\omega_3) \cdot [\theta_{12}h(w_2) + \theta_{13}h(w_3)] \cdot E_u[g(w_1) \cdot f_v(u + \varepsilon)] \tag{15}$$

This expectation is simpler and may be easier to simulate than the expectation in (14) above.

6. Empirical Example

We now present the results of an empirical example, which is intended to illustrate the applicability of the APS methods that we have suggested.

The data that we use are the same as the data that were used by Schmidt and Lovell (1979) and Schmidt and Lovell (1980). Briefly, our sample consists of 111 privately-owned steam electric generating plants constructed in the US between 1947 and 1965. We have data on output, total cost, and prices and quantities of three inputs (capital, fuel and labor), for the first year of operation of the plant. For more detail on the data, see Schmidt and Lovell (1979). For a lot more detail on the data, see Cowing (1970).

The model that we will estimate is as given in Section 2. We have the production function (1) and the first-order conditions for cost minimization (2), where the v_i are iid $N(0, \sigma_v^2)$; the u_i are $N^+(0, \sigma_u^2)$, i.e. "half normal"; the $\omega_i \equiv (\omega_{i2}, \ldots, \omega_{ik})'$ are iid $N(\mu, \Sigma_{\omega\omega})$; and v is independent of u and ω. Our model will be the same as the model of Schmidt and

Lovell (1980) except for the copula used to model dependence between u_i, ω_{i2} and ω_{i3}

Our estimates for the various models are given in Table 1. The first two columns give the results from Schmidt and Lovell (1980) and our attempt at the replication of these results. (This was an adventure in intellectual archeology, since the old FORTRAN programs and printouts were discarded long ago, and all that remained was what was in the published paper, plus a paper copy of the data that had to be excavated from the bottom of a large pile of more recent artifacts.) The first set of results is from Schmidt and Lovell (1980), Table 1, column 1, and the second set of results is our attempt at replication. The two sets of results are somewhat similar but not as similar as one might hope. For most of the parameters there is not too much difference between the two sets of results, but there are substantial differences in the results for the parameters $\mu_2, \mu_3, \Sigma_{2u}$ and Σ_{3u}. The most likely explanation for these differences is that the Schmidt and Lovell (1980) results were inaccurate. There are three reasons to believe that. The first is simply that the log likelihood value for the current estimation (-73.6978) is considerably larger than the log likelihood value for the model evaluated at the old estimates (-95.2385). The second reason is that numerical optimization of a complicated likelihood was a much less familiar task 40 years ago than it is now. The old estimates were calculated in FORTRAN using the GQOPT optimization subroutines written by S. Goldfeld and R. Quandt, which were not nearly as sophisticated as the MATLAB routines used in our replication attempt. The author of this paper who was involved in both sets of calculations (P. Schmidt) has no doubt that the more recent calculations are the more trustworthy. The third reason, discussed in the next paragraph, is that our estimates for the Schmidt and Lovell (1980) model are similar to those using the APS-3 copulas whereas the old estimates are not.

The next two sets of results are for the MLE's of the models that use the APS-3-A and

APS-3-B copulas. In each case the likelihood was evaluated using a simulation based on the expression in equation (14), although using the expression in equation (15) yielded almost identical results. The number of replications for the simulation of the likelihood was 1000.

These results are similar to each other and to our current estimates of the Schmidt and Lovell (1980) model. The log likelihoods are also similar, with the model that uses the APS-3-A copula having a very slightly higher log likelihood value than the other two. So the choice of copula (SL versus APS-3-A versus APS-3-B) does not make much difference in the results, and the main interest in the application is that it demonstrates the feasibility of estimating the models based on the APS-3 copulas by simulated MLE.

7. Concluding Remarks

In this paper we propose a new family of copulas for which the copula arguments are uncorrelated but dependent. We want to use this copula to construct random variables that are uncorrelated, but where the first random variable is correlated with the absolute value of the second. We show how this family of copulas can be applied to the error structure in an econometric production frontier model, and we give an empirical application.

Our family of copulas can be two or three dimensional. As in much of the copula literature, the most difficult remaining problem is how to properly extend these result to higher dimensions. The problem is not that it is hard to find an extension, but rather that there are multiple possible extensions and it is hard to judge which is useful.

APPENDIX 1

The SL Copula

For notational simplicity only, we will consider the case that ω is a scalar.

In the SL model, $\begin{bmatrix} u^* \\ \omega \end{bmatrix} \sim N(0, \Sigma)$ where $= \begin{bmatrix} \sigma_u^2 & \Sigma_{\omega u} \\ \Sigma_{\omega u} & \sigma_\omega^2 \end{bmatrix}$. Then $u = |u^*|$.

The joint density of $u^*$and ω, say $g(u^*, \omega)$, is the bivariate normal density of $N(0, \Sigma)$. The joint density of u and ω is then

$$h(u, \omega) = g(u, \omega) + g(-u, \omega)$$

$$= \frac{1}{2\pi} |\Sigma|^{-1/2} \exp\left[-\frac{1}{2}(u, \omega)\Sigma^{-1}\begin{pmatrix} u \\ \omega \end{pmatrix}\right] + \frac{1}{2\pi} |\Sigma|^{-1/2} \exp\left[-\frac{1}{2}(-u, \omega)\Sigma^{-1}\begin{pmatrix} -u \\ \omega \end{pmatrix}\right],$$

as given in Schmidt and Lovell (1980, equation (A.12)).

Now define $\Sigma_* = \begin{bmatrix} \sigma_u^2 & -\Sigma_{\omega u} \\ -\Sigma_{\omega u} & \sigma_\omega^2 \end{bmatrix}$. It is easy to verify that $|\Sigma_*| = |\Sigma|$ and that $(-u, \omega)\Sigma^{-1}\begin{pmatrix} -u \\ \omega \end{pmatrix} = (u, \omega)\Sigma_*^{-1}\begin{pmatrix} u \\ \omega \end{pmatrix}$. Therefore

$$h(u, \omega) = \frac{1}{2\pi} |\Sigma|^{-1/2} \exp\left[-\frac{1}{2}(u, \omega)\Sigma^{-1}\begin{pmatrix} u \\ \omega \end{pmatrix}\right] \qquad \text{“term 1”}$$

$$+ \frac{1}{2\pi} |\Sigma_*|^{-1/2} \exp\left[-\frac{1}{2}(u, \omega)\Sigma_*^{-1}\begin{pmatrix} u \\ \omega \end{pmatrix}\right] \qquad \text{“term 2”}$$

To calculate the copula, we now need to divide $h(u, \omega)$ by the product of the marginal densities of u and ω, that is, by

$$\frac{2}{\sqrt{2\pi}} \frac{1}{\sigma_u} \exp\left(-\frac{1}{2\sigma_u^2} u^2\right) \cdot \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma_\omega} \exp\left(-\frac{1}{2\sigma_\omega^2} \omega^2\right).$$

Carrying out this division, the first term above (“term 1”) becomes, by standard algrebra used in the derivation of the normal copula,

$$\tfrac{1}{2}|R|^{-1/2}\exp\left[-\tfrac{1}{2}(u,\omega)(R^{-1}-I)\begin{pmatrix}u\\ \omega\end{pmatrix}\right] \qquad \text{where } R=\begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}$$

$$=\tfrac{1}{2}(1-\rho^2)^{-\frac{1}{2}}\exp\left[-\tfrac{1}{2}(1-\rho^2)^{-1}(\rho^2u^2+\rho^2\omega^2-2\rho u\omega)\right]$$

which is one-half times the normal copula with parameter ρ. Similarly, the second term above ("term 2") becomes one-half times the normal copula with parameter $-\rho$.

APPENDIX 2

Properties of the APS-2 Copulas

Proof of Result 1

We have a copula of the form $c(w_1,w_2)=1+g(w_1)h(w_2)$, where $\int_0^1 g(s)ds = \int_0^1 h(s)ds=0$, and specifically where $h(w_2)=1-k_q^{-1}q(w_2)$ with $k_q=\int_0^1 q(s)\,ds$. Define $G(w_1)=\int_0^{w_1} g(s)ds$, $H(w_2)=\int_0^{w_2} h(s)ds$, $Q(w_2)=\int_0^{w_2} q(s)ds$, $G^*=\int_0^1 G(w_1)dw_1$ $H^*=\int_0^1 H(w_2)dw_2$ and $Q^*=\int_0^1 Q(w_2)dw_2$, and note that $H(w_2)=w_2-Q(w_2)$ and $k_q=Q(1)$. A general result for Sarmanov copulas (Rodriguez-Lallena and Ubeda-Flores (2004)) is that cov(w_1,w_2) = G^*H^*. The value of G^* is $\theta/6$ when $g(w_1)=\theta(1-2w_2)$, but this does not feature in the proof, which simply establishes that $H^*=0$.

To show that $H^*=0$, we use the symmetry of $q(s)$ around $s=½$, which implies that $Q(s)=Q(1)-Q(1-s)$ for $s>½$. Therefore $Q^*=\int_0^1 Q(w_2)dw_2 = \int_0^{½} Q(w_2)dw_2 + \int_{½}^1[Q(1)-Q(1-w_2)]dw_2 = \int_0^{½} Q(w_2)dw_2+\frac{1}{2}Q(1)-\int_{½}^1 Q(1-w_2)dw_2=\frac{1}{2}Q(1)=\frac{1}{2}k_q$. Then $H^*=\frac{1}{2}-k_q^{-1}Q^*=\frac{1}{2}-k_q^{-1}(\frac{1}{2}k_q)$ = 0, which implies that cov$(w_1,w_2)=0$.

Proof of Result 2

$$E[w_1 q(w_2)] = \int\int w_1\, q(w_2) dw_1 dw_2$$

$$+ [\theta \int w_1(1-2w_1)dw_1] \cdot \left[\int q(w_2)\left(1-k_q^{-1}q(w_2)\right)dw_2\right]$$

Here all integrals are from zero to one.

The first term on the r.h.s. of this equation is $\int w_1\, dw_1 \cdot \int q(w_2)dw_2 = \frac{1}{2}Q(1) = \frac{1}{2}k_q$.

The first term in brackets following the "+" sign equals $\theta\left(\frac{1}{2}-\frac{2}{3}\right) = -\frac{1}{6}\theta$.

The second term in brackets equals

$$\int q(w_2)dw_2 - k_q^{-1}\int q(w_2)^2 dw_2 = k_q - k_q^{-1}Eq(w_2)^2$$

$$= k_q - k_q^{-1}\left[\text{var}\left(q(w_2)\right) + k_q^2\right] = -k_q^{-1}\text{var}(q(w_2))$$

Combining terms, $E[w_1 q(w_2)] = \frac{1}{2}k_q + \frac{1}{6}\theta k_q^{-1}\text{var}(q(w_2))$ and therefore

$$\text{cov}[w_1 q(w_2)] = \frac{1}{6}\theta k_q^{-1}\text{var}(q(w_2)) .$$

Proof of Result 3

We have the APS-2 copula $c(w_1, w_2) = 1 + \theta(1-2w_1)(1-k_q^{-1}q(w_2))$ where $w_1 = F_u(u)$ and $w_2 = F_\omega(\omega)$. For notational simplicity only, suppose that $E(u) = E(\omega) = 0$. (Otherwise we just have to do the analysis below in terms of deviations from means.) Then

$$\text{cov}(u,\omega) = E(u\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u\omega\, f_u(u)f_\omega(\omega)\, c(F_u(u), F_\omega(\omega))\, du\, d\omega$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u\omega\, f_u(u)f_\omega(\omega)\, du\, d\omega \qquad \text{[term 1]}$$

$$+ \theta\int_0^{\infty}\int_0^{\infty} u\omega\, f_u(u)f_\omega(\omega)\,(1-2w_1)(1-k_q^{-1}q(w_2))\, du\, d\omega \qquad \text{[term 2]}$$

$$+ \theta\int_{-\infty}^{0}\int_{-\infty}^{0} u\omega\, f_u(u)f_\omega(\omega)\,(1-2w_1)(1-k_q^{-1}q(w_2))\, du\, d\omega \qquad \text{[term 3]}$$

$$+ \theta\int_{-\infty}^{0}\int_0^{\infty} u\omega\, f_u(u)f_\omega(\omega)\,(1-2w_1)(1-k_q^{-1}q(w_2))\, du\, d\omega \qquad \text{[term 4]}$$

$$+ \theta\int_0^{\infty}\int_{-\infty}^{0} u\omega\, f_u(u)f_\omega(\omega)\,(1-2w_1)(1-k_q^{-1}q(w_2))\, du\, d\omega \qquad \text{[term 5]}$$

where again, for visual simplicity, $w_1 = F_u(u)$ and $w_2 = F_\omega(\omega)$.

Term 1 equals zero because $E(u) = E(\omega) = 0$.

Term 2 equals the negative of term 3 (i.e. they sum to zero). Because u and ω are symmetric, $f_u(u) = f_u(-u)$ and $f_\omega(\omega) = f_\omega(-\omega)$. Also $F_u(u) = 1 - F_u(-u)$ so that $1 - 2F_u(-u) = -[1 - 2F_u(u)]$. Finally, $q(s)$ is symmetric around $= \frac{1}{2}$, so that $q(s) = q(1-s)$ and therefore $q\big(F_\omega(-\omega)\big) = q(1 - F_\omega(\omega)) = q\big(F_\omega(\omega)\big)$. This implies that the value of the integrand in term 2 for any u, ω pair (e.g., (0.3, 0.4)) is the negative of the value of the integrand in term 3 of the corresponding pair (e.g., (-0.3, -0.4)), and thus the two terms sum to zero.

Similarly term 4 and term 5 sum to zero, and then $\text{cov}(u, \omega) = 0$.

Proof of Result 4

(i) It is easy to verify that the marginals of $c(w_1, w_2)$ are uniform, so we just need to verify that $c(w_1, w_2) \geq 0$ for all w_1, w_2 in the unit cube. We have $-1 \leq 1 - 2w_1 \leq 1$ and $-2 \leq 1 - 12(w_2 - ½)^2 \leq 1$, so $-2 \leq (1 - 2w_1)[1 - 12(w_2 - ½)^2] \leq 2$. Therefore $c(w_1, w_2) \geq 0$ if $-½ \leq \theta \leq ½$.

(ii), (iii) Some useful integrals (integrals are from zero to one):

(a) $\int (w_2 - ½)^2 dw_2 = \frac{1}{12}$.

(b) $\int (w_2 - ½)^4 dw_2 = \frac{1}{80}$

Therefore $\text{var}(w_2 - ½)^2 = \frac{1}{12} - (\frac{1}{80})^2 = \frac{1}{180}$ which is (iii). To establish (ii), use Result 2 to obtain $\text{cov}(w_1, q(w_2)) = \frac{1}{6}\theta \cdot 12 \cdot \frac{1}{180} = \frac{1}{90}\theta$.

(iv) $\text{corr}(w_1, q(w_2)) = \dfrac{\frac{1}{90}\theta}{\sqrt{1/12}\sqrt{1/180}} = \frac{2}{\sqrt{15}}\theta$.

Proof of Result 5

(i) Once again is easy to verify that the marginals of $c(w_1, w_2)$ are uniform, so we just need to verify that $c(w_1, w_2) \geq 0$ for all w_1, w_2 in the unit cube. We have $-1 \leq 1 - 2w_1 \leq 1$ and $-1 \leq 1 - 4|w_2 - ½| \leq 1$, so $-1 \leq (1 - 2w_1)[1 - 4|w_2 - ½|] \leq 1$. Therefore $c(w_1, w_2) \geq 0$ if $-1 \leq \theta \leq 1$.

(ii), (iii) Some useful integrals (integrals are from zero to one):

(a) $\int |w_2 - ½| dw_2 = \frac{1}{4}$.

(b) $\int |w_2 - ½|^2 dw_2 = \frac{1}{12}$

Therefore var($|w_2 - ½|$) $= \frac{1}{12} - (\frac{1}{4})^2 = \frac{1}{48}$ which is (iii). Then use Result 2 to obtain

cov$(w_1, q(w_2)) = \frac{1}{6}\theta \cdot 4 \cdot \frac{1}{48} = \frac{1}{72}\theta$.

(iv) corr$(w_1, q(w_2)) = \frac{\frac{1}{72}\theta}{\sqrt{1/12}\sqrt{1/48}} = \frac{1}{3}\theta$.

APPENDIX 3

Generalization of Result 8 to Higher Dimensions

Consider the four-dimensional case. We start with two-dimensional copulas $c_{12}, c_{13}, c_{14}, c_{23}, c_{24}$ and c_{34}. We can construct three-dimensional copulas as in Result 8. We can construct a four-dimensional copula as

$$c^*(w_1, w_2, w_3, w_4) = 1 + (c_{12} - 1) + (c_{13} - 1) + (c_{14} - 1) + (c_{23} - 1) + (c_{24} - 1) + (c_{34} - 1).$$

The implied 3-copulas are as given in Result 8, for example,

$$\int c^*(w_1, w_2, w_3, w_4) dw_1 = 1 + 0 + 0 + 0 + (c_{23} - 1) + (c_{24} - 1) + (c_{34} - 1) = c^*(w_2, w_3, w_4).$$

The implied 2-copulas are therefore the two-copulas with which we started, e.g. c_{12}, c_{13}, etc.

This extends to arbitrary dimensionality d. We can define a d-copula $c^*(w_1, \dots, w_d)$ using $\binom{d}{2}$ bivariate copulas c_{ij} as follows:

$$c^*(w_1, \dots, w_d) = 1 + \sum_{1 \le i < j \le d}(c_{ij} - 1) .$$

Returning for purposes of discussion to the four-dimensional case, the copula $c^*(w_1, w_2, w_3, w_4)$ is a copula if it is a density (i.e. it is non-negative) and there is no issue if we are satisfied with the lower dimensional copulas that it implies. But this may not always be the case. For example, suppose that we have a production frontier system as in equations (1) and (2) but now we have four inputs, so that we have four random errors (u, ω_2, ω_3 and ω_4) instead of three. It might be natural to want $(\omega_2, \omega_3, \omega_4)$ to be trivariate normal, that is, to be marginally normal and to have the trivariate normal copula. But $c^*(w_1, w_2, w_3, w_4)$ as defined above does not imply a trivariate normal 3-copula, even if c_{23}, c_{24} and c_{34} are all bivariate normal copulas.

An alternative construction is as follows. Let $c^o(w_2, w_3, w_4)$ be a trivariate normal copula, and c_{12}, c_{13} and c_{14} be the desired 2-copulas linking w_1 to w_2, w_3 and w_4. Then define the 4-copula

$$c^o(w_1, w_2, w_3, w_4) = 1 + (c_{12} - 1) + (c_{13} - 1) + (c_{14} - 1) + (c^o(w_2, w_3, w_4) - 1).$$

If c_{23}, c_{24} and c_{34} are all bivariate normal copulas, then the 2-copulas implied by c^o are the same as those implied by c^*, and so are the 3-copulas that involve w_1. But the 3-copula for w_2, w_3 and w_4 is different, because $c^o(w_1, w_2, w_3, w_4)$ implies the 3-copula $c^o(w_2, w_3, w_4)$, whereas $c^*(w_1, w_2, w_3, w_4)$ implies the 3-copula $1 + (c_{23} - 1) + (c_{24} - 1) + (c_{34} - 1)$, which is not a trivariate normal copula even if its constituent 2-copulas are all bivariate normal.

Another way to construct a four-copula from lower dimensional copulas is to use a vine copula, as in Joe (1996) and Aas et al. (2009). These require specification of two-dimensional marginal and conditional copulas. In general there are many such vine copulas because they depend on the vine structure and the numbering of the variables. However, in our problem there is arguably a natural structure where the two dimensional marginals are APS-2 and the conditional copulas are Gaussian. Thus the first variable is the half-normal error, and the remaining three variables are multivariate normal. Because of the multivariate normal assumption, the ordering of the last three variables does not matter. The benefit is that this representation results in a somewhat simpler functional form of the density than other vine representations.

TABLE 1

Estimates of the System of Equations (1) and (2)

	SL80 Table 1 Col 1		SL80 our calc		APS-3-A		APS-3-B	
	Est	St.Er.	Est	St.Er.	Est	St.Er.	Est	St.Er.
α	-11.6849	0.3848	-11.2700	0.2510	-11.3455	0.2473	-11.3209	0.2470
β_1	0.1290	0.0251	0.0428	0.0246	0.0463	0.0234	0.0467	0.0230
β_2	0.9743	0.0238	1.0754	0.0272	1.0791	0.0237	1.0771	0.0241
β_3	0.0631	0.0296	0.0137	0.0319	0.0090	0.0277	0.0091	0.0275
σ_u^2	0.0144	0.0085	0.0119	0.0036	0.0098	0.0040	0.0102	0.0039
σ_v^2	0.0032	0.0005	0.0020	0.0009	0.0035	0.0012	0.0034	0.0012
Σ_{22}	0.3372	0.0458	0.3366	0.0435	0.3408	0.0343	0.3425	0.0349
Σ_{23}	0.2052	0.0469	0.2100	0.0577	0.2360	0.0601	0.2360	0.0602
Σ_{33}	0.5918	0.0811	0.5901	0.1010	0.6005	0.0999	0.5998	0.0995
μ_2	0.8852	0.2088	1.9861	0.6052	1.9227	0.5487	1.8916	0.5283
μ_3	0.4501	0.3285	-0.0527	2.4745	-0.5673	3.1476	-0.5998	3.0863
Σ_{2u}	0.0365	0.0119	0.0148	0.0242				
Σ_{3u}	0.0051	0.0140	-0.0138	0.0383				
θ_{12}					0.5132	0.3282	0.7527	0.4722
θ_{13}					-0.3324	0.3648	-0.4912	0.5199
LL	-95.2385		-73.6973		-72.9994		-72.9496	

REFERENCES

Aas, K., C. Czado, A. Frigessi and H. Bakken (2009), "Pair-Copula Constructions of Multiple Dependence," *Insurance: Mathematics and Economics*, 44, 182-198.

Aigner, D.J., C.A.K. Lovell and P. Schmidt (1977), "Formulation and Estimation of Stochastic Frontier Production Function Models," *Journal of Econometrics,* 6, 21-37.

Atkinson, S.E. and C. Cornwell (1994), "Parametric Estimation of Technical and Allocative Inefficiency with Panel Data," *International Economic Review*, 35, 231-244.

Christensen, L.R. and W.H. Greene (1976), "Economies of Scale in U.S. Electric Power Generation," *Journal of Political Economy*, 84, 655-676.

Cowing, T.G. (1970), *Technical Change in Steam-Electric Generation: An Engineering Approach,* unpublished Ph.D. dissertation, University of California, Berkeley.

Ferrier, G.D. and C.A.K. Lovell (1990), "Measuring Cost Efficiency in Banking: Econometric and Linear Programming Evidence," *Journal of Econometrics*, 46, 229-245.

Greene, W.H. (1980), "On the Estimation of a Flexible Frontier Production Model," *Journal of Econometrics*, 13, 101-115.

Greene, W.H. (2003), "Simulated Likelihood Estimation of the Normal-Gamma Stochastic Frontier Function," *Journal of Productivity Analysis,* 19, 179-190.

Joe, H. (1996), "Families of m-variate Distributions with Given Margins and m(m-1)/2 Bivariate Dependence Parameters," in L. Rüschendorf, B. Schweizer and M.D. Taylor (eds.), *Distributions with Fixed Marginals and Related Topics,* IMS Lecture Notes Monograph Series, Institute of Mathematical Statistics.

Kumbhakar, S.C. (1987), "The Specification of Technical and Allocative Inefficiency in Stochastic Production and Profit Frontiers," *Journal of Econometrics*, 34, 335-348.

Kumbhakar, S.C. (1991), "The Measurement and Decomposition of Cost Efficiency: The Translog Cost System," *Oxford Economic Papers*, 43, 667-683.

Kumbhakar, S.C. (1997), "Modelling Allocative Inefficiency in a Translog Cost Function and Cost Share Equations: An Exact Relationship," *Journal of Econometrics*, 76, 351-356.

Meeusen, W. and J. van den Broeck (1977), "Efficiency Estimation from Cobb-Douglas Production Functions with Composed Error," *International Economic Review*, 18, 435-444.

Nelsen, R.B. (2006), *An Introduction to Copulas,* 2nd edition, Springer.

Rodriguez-Lallena, J.A. and M. Ubeda-Flores (2004), "A New Class of Bivariate Copulas," *Statistics & Probability Letters*, 66, 315-325

Sarmanov, O.V. (1966), "Generalized Normal Correlation and Two-Dimensional Frechet Classes," *Doklady (Soviet Mathematics), 168*, 596-599.

Schmidt, P. and C.A.K. Lovell (1979), "Estimating Technical and Allocative Inefficiency Relative to Stochastic Production and Cost Frontiers," *Journal of Econometrics,* 9. 343-366.

Schmidt, P. and C.A.K. Lovell (1980), "Estimating Stochastic Production and Cost Frontiers When Technical and Allocative Inefficiency Are Correlated," *Journal of Econometrics,* 13, 83-100.