

Draft

***Axiomatic Foundations of Efficiency Measurement on
Convex Polyhedral (DEA) Technologies***

by

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May 2005

Any comments appreciated. We thank Sushama Murty for suggestions that significantly improved the exposition, which also benefitted from comments by participants in the joint UNSW / Sydney University Workshop in Economic Theory and the 2004 North American Productivity Workshop at Toronto University.

Printed on 27 May 2005

I. Introduction.

Axiomatization of (input) efficiency measurement was introduced by Färe and Lovell [1978], who proposed three axioms: *indication of efficient bundles* (the efficiency index equals one if and only if the input vector is Koopmans [1951] efficient); *monotonicity* (increasing input quantities reduces the value of the index; and *homogeneity* (*e.g.*, doubling all input quantities cuts the index in half). Bol [1986] showed that there does not exist an efficiency index that satisfies all three of these axioms for all technologies and suggested two approaches to axiomatization of technical efficiency measurement: (1) weakening the axioms and (2) restricting the set of technologies to which the index is to be applied. Russell [1985, 1987] suggested a weakening of monotonicity (increasing input quantities does not increase the value of the index) and a modification of the indication condition (indication of weakly efficient input vectors) and assessed several indexes in terms of the trade-offs among these axioms. The three indexes are the Debreu [1951] - Farrell [1957] index, the Färe-Lovell [1978] index, and the Zieschang [1984] index.

Dmitruk and Koshevoy [1991] is the only paper to take the alternative approach of restricting the set of technologies. They provide a complete characterization of the set of technologies for which an index satisfying the Färe-Lovell axioms exists. Interestingly, the technologies integral to the standard mathematical-programming (DEA) method of measuring technical efficiency belong to the class of technologies identified by the Dmitruk-Koshevoy theorem. Even more interesting is the fact that the Dmitruk-Koshevoy proof is constructive, suggesting a class of potentially programmable efficiency indexes that satisfy all three of the Färe-Lovell axioms on the most commonly constructed technologies. We identify, however, a fundamental flaw in the Dmitruk-Koshevoy index: unlike the indexes to which we have alluded above, it is not independent of units of measurement. In particular, it does not satisfy the commensurability axiom introduced by Russell [1987]. We believe this may be a more fundamentally desirable property of an index than any of the Färe-Lovell conditions.

We go on to formulate a modification of the Dmitruk-Koshevoy efficiency index that satisfies the commensurability axiom as well as the Färe-Lovell axioms on a well-defined subset of the set of closed, convex, polyhedral technologies (a subset that excludes, *e.g.*, Leontief technologies). To extend this efficiency index to the entire space of such technologies, we have to extend the notion of an efficiency index to map not just from input quantities and input-requirement sets but also from a space of unit transformations themselves. The commensurability axiom is then suitably reformulated in terms of this expanded notion of an efficiency index, which requires more information than does the standard notion. We then show that our modified Dmitruk-Koshevoy index satisfies the reformulated commensurability condition as well as the Färe-Lovell axioms. Finally, we show that this efficiency index has a nice interpretation in terms of shadow evaluations of inefficiencies.

Section II describes convex polyhedral technologies. Section III summarizes known results of efficiency measurement on general technologies, while Section IV extends these results to the restricted class of technologies introduced in Section II. Section V describes the Dmitruk-Koshevoy index, and Section VI shows that it fails to satisfy commensurability. Section VII develops our modified Dmitruk-Koshevoy index and shows that it satisfies commensurability as well as the Färe-Lovell axioms. Section VIII concludes.

II. Convex Polyhedral Technologies.

The theoretical literature on technical efficiency measurement has focused on a general class of technologies satisfying only very weak regularity conditions. In this section, we begin by describing these general technologies. Then we turn to the focus of the paper: the class of technologies that are represented by convex polyhedrons satisfying free disposability.

The input vector $x \in \mathbf{R}_+^n$ is constrained to lie in the input-requirement set L (the set of input vectors that can produce a stipulated vector of outputs).¹ Let L be the collection of non-empty, closed, input-requirement sets that exclude the origin of \mathbf{R}_+^n and satisfy the free-disposal condition, $L = L + \mathbf{R}_+^n$.² To simplify the language in the results that follow, we refer to “all technologies” when we mean “all input-requirement sets in L .”

An input vector $x \in L$ is efficient (in the sense of Koopmans [1951]) if $x > \bar{x}$ implies $\bar{x} \notin L$; it is weakly efficient if $x \gg \bar{x}$ implies $\bar{x} \notin L$.³ Under our free-disposability assumption, the set of weakly efficient input vectors is equivalent to the isoquant, defined by

$$\text{Isoq}(L) = \{ x \in L \mid \lambda x \notin L \forall \lambda \in [0, 1) \}. \quad (2.1)$$

The set of efficient points of L , which we denote by $\text{Eff}(L)$, is a subset of $\text{Isoq}(L)$.

Input requirement sets generated by nonparametric, mathematical programming, or data envelopment analysis (DEA) methods of measuring efficiency are convex polyhedral sets—*i.e.*, intersections of a finite number of closed half spaces. See, *e.g.*, Färe, Lovell and Grosskopf

¹ A complete characterization of the technology would be a correspondence mapping output vectors into subsets of input space. Since, however (in the tradition of axiomatic analysis of efficiency measure), we consider only input-based measures of efficiency for fixed output vectors, it is not necessary to formally incorporate output into our analysis.

² Nonemptiness, closedness, and exclusion of level sets containing the origin are necessary to guarantee that our efficiency indexes are well defined, but the free disposability assumption could be dispensed with. The only change that would be needed in what follows would be to redefine the Debreu-Farrell index on the free-disposal hull of L rather than on L itself (as in Russell [1987]).

³ Vector notation: $\bar{x} \geq x$ if $\bar{x}_i \geq x_i$ for all i ; $\bar{x} > x$ if $\bar{x}_i \geq x_i$ for all i and $\bar{x} \neq x$; and $\bar{x} \gg x$ if $\bar{x}_i > x_i$ for all i .

[1995] for a thorough discussion of these methods.⁴ Again to simplify the exposition, we refer below to “convex polyhedral technologies” when we mean “nonempty, closed, free-disposal, convex, polyhedral input-requirement sets that do not contain the zero vector.”

III. Efficiency Indexes on General Technologies.

Let Γ be the subset of $L \times L$ satisfying $x \in L$. An (input) efficiency index is a mapping, $E : \Gamma \rightarrow (0, 1]$, with the normalization that $E(x, L) = 1$ represents full efficiency.⁵ Three well-known efficiency indexes are as follows:

- The Debreu [1951] - Farrell [1957] index, defined by

$$E_{DF}(x, L) = \min\{ \lambda \mid \lambda x \in L \}. \quad (3.1)$$

- The Färe-Lovell [1978] index, defined by

$$E_{FL}(x, L) = \min_{\kappa} \left\{ \sum_i \kappa_i / \sum_i \delta(x_i) \mid Kx \in L \wedge \kappa_i \in [0, 1] \forall i \right\}, \quad (3.2)$$

where $\delta(x_i) = 1$ if $x_i > 0$, $\delta(x_i) = 0$ if $x_i = 0$, and K is the diagonal matrix with $(\kappa_1, \dots, \kappa_n)$ on the diagonal.

- The Zieschang [1984] index, defined by

$$E_Z(x, L) = E_{DF}(x, L) \cdot E_{FL}(E_{DF}(x, L) \cdot x, L). \quad (3.3)$$

Thus, the Debreu-Farrell index measures the maximal radial contraction of the input vector consistent with production feasibility, the Färe-Lovell index measures the maximal average of coordinatewise contractions, and the Zieschang index is a combination of the other two indexes, measuring the multiple of the maximal radial contraction to the isoquant and the average of coordinatewise contractions along the isoquant.

The Färe-Lovell axioms (for a given L) are as follows:

Indication of Efficient Input Bundles (I): For all $x \in L$, $E(x, L) = 1 \Leftrightarrow x \in (L)$.

⁴ Input requirement sets implicit in the free-disposal-hull approach to measuring efficiency, however, are not convex polyhedral sets (see, e.g., Tulkens [1993]).

⁵ Instead of restricting the domain of E , one could set $E(x, L) = \infty$ when $x \notin L$, but this is an unnecessary complication, especially since standard empirical methods of calculating efficiency indexes guarantee that the input vector and the reference technology match up, so that $x \in L$ is satisfied.

Monotonicity (M): For all $\langle x, \bar{x} \rangle \in L \times L$, $x > \bar{x} \Rightarrow E(x, L) < E(\bar{x}, L)$.

Homogeneity (H): For all $x \in L$, $E(\kappa x, L) = \kappa^{-1} E(x, L) \quad \forall \kappa > 0$.

The search for an efficiency index satisfying these conditions on all technologies was brought to a halt by an impossibility result:

Fact 1 (Bol [1986]): There does not exist an efficiency index satisfying (H), (M), and (I) for all technologies.⁶

As Bol pointed out, there exist two approaches to resolving this problem: (1) weakening the axioms and (2) restricting the set of technologies to which the index is to be applied. Russell [1985, 1987] suggested the following two axioms, the first a weakening of (M) and the second an alternative to (I):

Weak Monotonicity (WM): For all $\langle x, \bar{x} \rangle \in L \times L$, $x \geq \bar{x} \Rightarrow E(x, L) \leq E(\bar{x}, L)$.

Indication of Weakly Efficient Input Bundles (IW): For all $x \in L$, $E(x, L) = 1$ if and only if $x \in Isoq(L)$ (i.e., x is “weakly efficient.”)

The known results on the compatibility of the above indexes with all these axioms are encapsulated in the following:

Fact 2 (Färe and Lovell [1978], Färe, Lovell, and Zieschang [1983], Zieschang [1984], and Russell [1985, 1987]):

- E_{DF} satisfies (IW), (H), and (WM) and fails to satisfy (M) and (I) on all technologies.
- E_{FL} satisfies (I) and (WM) and fails to satisfy (M) and (H) on all technologies.
- E_Z satisfies (I) and (H) and fails to satisfy (WM) on all technologies.

These results underscore the trade-offs among the three efficiency indexes. The choice between E_{DF} and E_{FL} reflects the trade-off between homogeneity and the strong-efficiency

⁶ Bol’s three-dimensional example purporting to show that convexity is not relevant contains a minor error: his input-requirement set,

$$L = \left\{ \langle x_1, x_2, x_3 \rangle \mid x_1 = 1 \wedge x_2 = 2 \wedge x_3 = e^{(-x_1-1)(x_2-1)} \right\}, \quad (3.4)$$

is not convex. A minor change to

$$L = \left\{ \langle x_1, x_2, x_3 \rangle \mid x_1 = 1 \wedge x_2 = 2 \wedge x_3 = e^{(-x_1-1)^{1/2}(x_2-1)^{1/2}} \right\}, \quad (3.5)$$

however, results in an input-requirement set that establishes his point.

form of the indication condition. The choice between E_{DF} and E_Z reflects the trade-off between weak monotonicity and the strong-efficiency form of the indication condition. Choosing between E_{FL} and E_Z reflects the trade-off between strong monotonicity and homogeneity.

IV. Efficiency Indexes on Convex Polyhedral Technologies.

In this section, we investigate the axioms for efficiency measures with technologies restricted to be convex polyhedral input-requirement sets.

Our first theorem re-examines the trade-offs implicit in Fact 2 in the context of convex polyhedral technologies.

Theorem 1: *For all convex polyhedral technologies,*

- E_{DF} satisfies (IW), (H), and (WM) and fails to satisfy (M) and (I).
- E_{FL} satisfies (I) and (WM) and fails to satisfy (M) and (H);
- E_Z satisfies (I) and (H) and fails to satisfy (M).

Proof: As the claimed properties of each of the indexes are implied by Fact 2, the theorem is established by providing counterexamples for each of the properties not satisfied by the relevant index.

To show that all three indexes fail to satisfy (M), consider an input-requirement set L in which, for some i , $\kappa x_i \in \text{Isoq}(L)$ for all $\kappa \geq 0$. A two-dimensional example is shown in Figure 1. Consider two vectors, x and x' , with $x_j = \hat{x}_j$ for all $j \neq i$ and $\hat{x}_i > x_i > 0$. It is easy to see that $E_{FL}(x, L) = (n - 1)/n = E_{FL}(\hat{x}, L)$, $E_Z(x, L) = (n - 1)/n = E_Z(\hat{x}, L)$, and $E_{DF}(x, L) = 1 = E_{DF}(\hat{x}, L)$. This last equality also shows that E_{DF} violates (I).

Consider now the two-dimensional, closed, convex, polyhedral input-requirement set in Figure 2. Clearly,

$$E_{FL}(\lambda \bar{x}, L) < \lambda^{-1} = \lambda^{-1} E_{FL}(\bar{x}, L) \quad (4.1)$$

if $1 < \lambda < 2$, a violation of (H). The counterexample is extended to n dimensions by any closed, convex, polyhedral extension of L to \mathbf{R}_+^n . ■

Note that this theorem is silent on the question of whether the Zieschang index satisfies (WM) on convex polyhedral technologies. Our conjecture is that it does, but we have not been able to prove this; nor have we been able to construct a counterexample. If our conjecture is correct, an interesting implication would be the apparent domination by the Zieschang efficiency index of the other two indexes when the technologies are restricted to the convex polyhedral input-requirement sets generated by standard DEA algorithms. In any event, the Zieschang index certainly does violate monotonicity, and the question that arises is whether

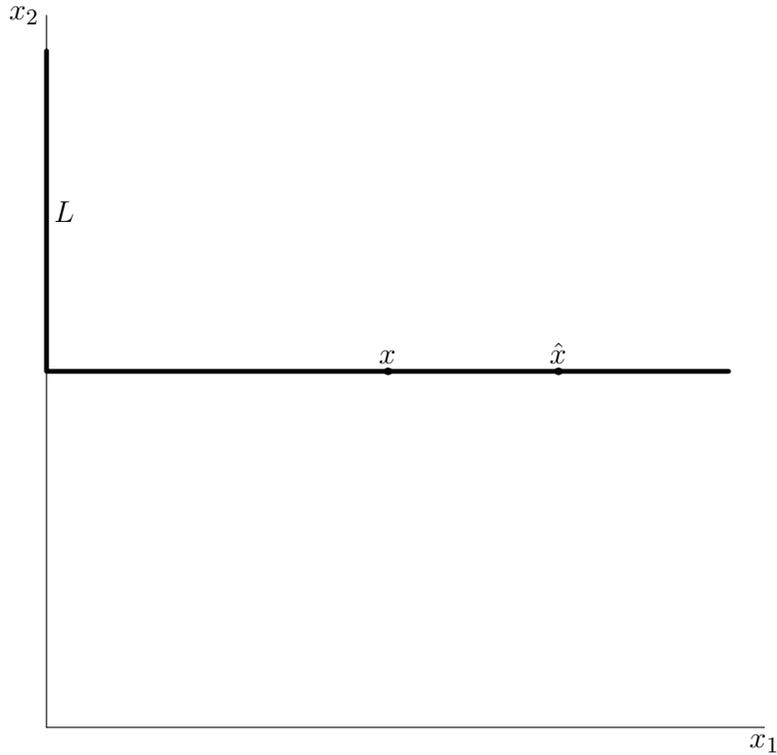


Figure 1.

there exists an alternative efficiency index that satisfies all three of the Färe-Lovell axioms. The answer, provided by Dmitruk and Koshevoy [1991], is “yes,” and we turn to that now.

V. The Dmitruk-Koshevoy Efficiency Index.

Dmitruk and Koshevoy introduced the idea of measuring inefficiency relative to reference sets that may differ in specific ways from the input requirement set. The existence of the reference set provides the restriction on technologies needed for the Färe-Lovell axioms to hold.

Dmitruk and Koshevoy show that there exists an index satisfying the Färe-Lovell axioms for a technology $L \in \mathcal{L}$ if and only if there exists a set Q satisfying the following conditions:

- (Q1) Q is closed.
- (Q2) $Q + \mathbf{R}_+^n \subseteq Q$.
- (Q3) $\text{Isoq}(Q) = \text{Eff}(Q)$.
- (Q4) $L \subseteq Q$.
- (Q5) $\text{Eff}(L) \subseteq \text{Eff}(Q)$.

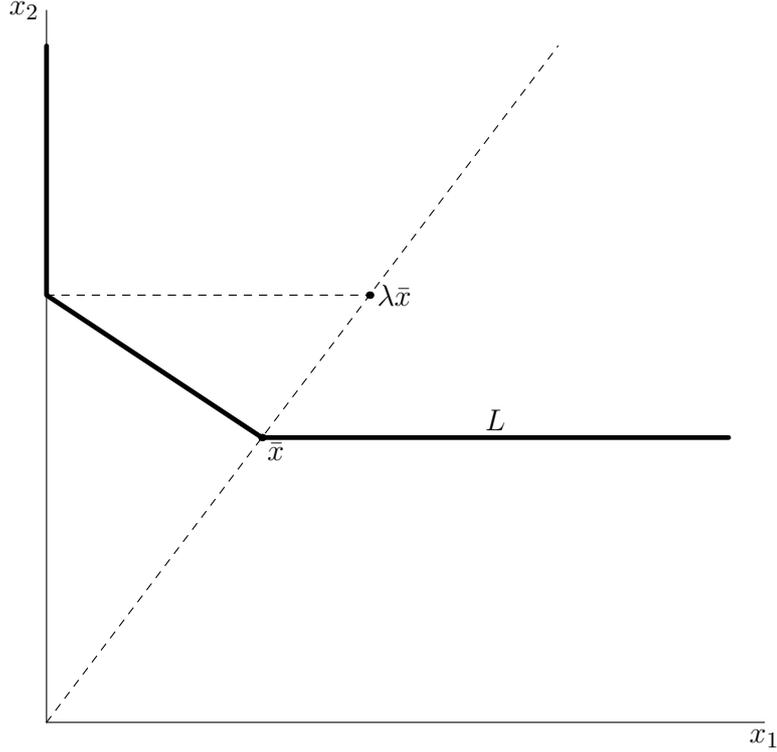


Figure 2.

Let $\bar{L} \subset L$ denote the technologies for which there exists a reference set Q satisfying these properties. Let \mathcal{Q} denote the set of reference technologies for $L \in \bar{L}$ and define a function $\Gamma : \bar{L} \rightarrow \mathcal{Q}$ that associates a reference technology $Q \in \mathcal{Q}$ with any technology $L \in \bar{L}$. The Dmitruk-Koshevoy efficiency index is defined as the Debreu-Farrell efficiency measure on the reference set $Q = \Gamma(L)$:

$$E_{DK}(x, L) = E_{DF}(x, \Gamma(L)) \quad (5.1)$$

for $x \in L$ and $L \in \bar{L}$.

Dmitruk and Koshevoy establish the following result:

Fact 3 (Dmitruk and Koshevoy [1991]): E_{DK} satisfies (I), (M), and (H) on \bar{L} .

As noted in the introduction, the Dmitruk-Koshevoy sufficiency proof of their main theorem explicitly constructs the reference set Q when one exists. We sketch the procedure here for convex polyhedral technologies (which facilitates simplification of their more general construction).

First, let $K(\theta)$ be the closed (symmetric, convex) cone generated by the n vectors $e^i + \theta \mathbf{1}$, where $\mathbf{1}$ is the n -dimensional unit vector; for each i , e^i is the (basis) vector satisfying $x_i = -1$ and $x_j = 0$ for all $j \neq i$; and $\theta \in [0, 1/n]$. For convex polyhedral technologies, the set, $A =$

$(\text{Eff}(L) - \mathbf{R}_+^n) \cap \mathbf{R}_+^n$, is closed. For all $a \in A$, define $Z_a = a + K(\theta_a)$, where

$$\theta_a = \max \{ \theta \in [0, 1/n] \mid L \cap \text{int}(Z_a) = \emptyset \} . \quad (5.2)$$

Let $Q_a = \mathbf{R}_+^n \setminus \text{int}(Z_a)$ and $Q = \bigcap_{a \in A} Q_a$. Dmitruk and Koshevoy then show that the Q , constructed in this way, satisfies (Q1)–Q(5). This construction of Q is a bit complicated, but essentially it is the intersection of the complements (relative to \mathbf{R}^n) of all symmetric, convex, polyhedral cones that “support” the efficient subset. The D-K efficiency index is then defined by

$$E_{DK}(x, L) = E_{DF}(x, Q). \quad (5.3)$$

The immediate implication of the foregoing, along with Fact 3, is the following:

Fact 4 (Dmitruk and Koshevoy [1991]): E_{DK} satisfies (I), (M), and (H) on all convex polyhedral technologies.

Figure 3 illustrates the construction of Q for a particular two-dimensional, convex polyhedral input-requirement set L . In the diagram, $E_{DK}(x, L) = \|\bar{x}\|/\|x\|$. This two-dimensional construction illustrates plainly the satisfaction of (I), (M), and (H) by the D-K index.

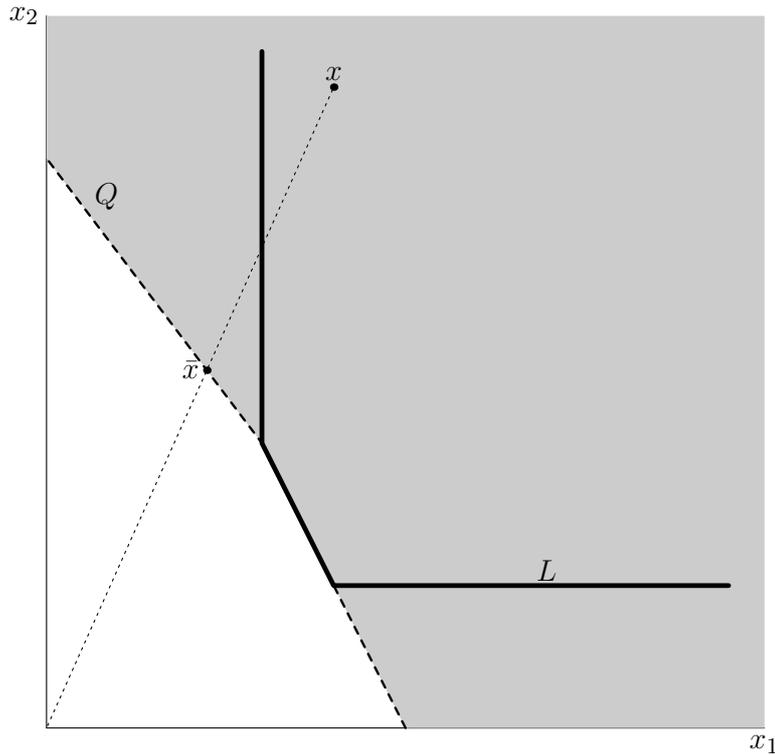


Figure 3.

VI. Commensurability

A fundamental criterion for an efficiency index is invariance to changes in the units of measurement. One would not want the efficiency of an input vector to depend on whether inputs were measured in pounds or kilograms. Russell [1987] introduced the following axiom to require this invariance.

Commensurability (C): For all $x \in L$, if $x' = \Omega x$, where Ω is a positive diagonal matrix, and $L' = \{ x' \mid \Omega^{-1} x' \in L \}$, then $E(x, L) = E(x', L')$. Commensurability is satisfied by the most commonly used efficiency measures.

Fact 5 (Russell [1987]): E_{DF} , E_{FL} , and E_Z satisfy (C) on all technologies.

The following theorem demonstrates an important weakness of the E_{DK} index.

Theorem 2: E_{DK} fails to satisfy (C) on convex polyhedral technologies.

Proof: In Figure 4, consider a change in the units in which input 1 is measured, so that, in the new units, $x'_1 = \omega x_1$ (and, of course, $x'_2 = x_2$). In addition, the level set in the transformed units of input 1 is L' . After the change in units, the value of the efficiency index is $E_{DK}(x', L') = \|\bar{x}'\|/\|x'\| \neq E_{DK}(x, L)$. The reference vectors, relative to Q and Q' , \bar{x} and \bar{x}' , would have to lie on a horizontal line (*i.e.*, satisfy $\bar{x}_2 = \bar{x}'_2$) in order for the index to be unaffected by a change in the units of input 1. As is demonstrated in the graph, $\bar{x}_2 \neq \bar{x}'_2$ so that $E_{DK}(x, L) \neq E_{DK}(x', L')$.⁷ ■

This outcome raises questions about the primacy of the D-K index, as compared to the more commonly employed indexes.

The violation of (C) by the D-K index should not have been surprising, given the following result:

Fact 6 (Russell [1987]): There does not exist an efficiency index satisfying (M) and (C) on all technologies.

As the counterexample establishing Fact 6 entails a convex polyhedral technology, the impossibility goes through for this restricted class of technologies as well. The counterexample is given by Figure 1, when x' is reinterpreted as $\langle \omega x_1, x_2 \rangle$, where ω is the unit change. Commensurability requires $E(x, L) = E(x', L)$ but monotonicity requires $E(x, L) > E(x', L)$. For this example, $L = L'$ and the efficiency index cannot tell the difference between a unit transformation for input 1 and an increase in the quantity of input 1.

⁷ The unit changes in Figure 4 have actually been programmed, thus establishing the unit dependence numerically. Note that adjustment of the horizontal axis to the change in units would result in \bar{x} and \bar{x}' on the same horizontal line, but the slope of the efficient facet would change, resulting in a change in the value of the D-K index.

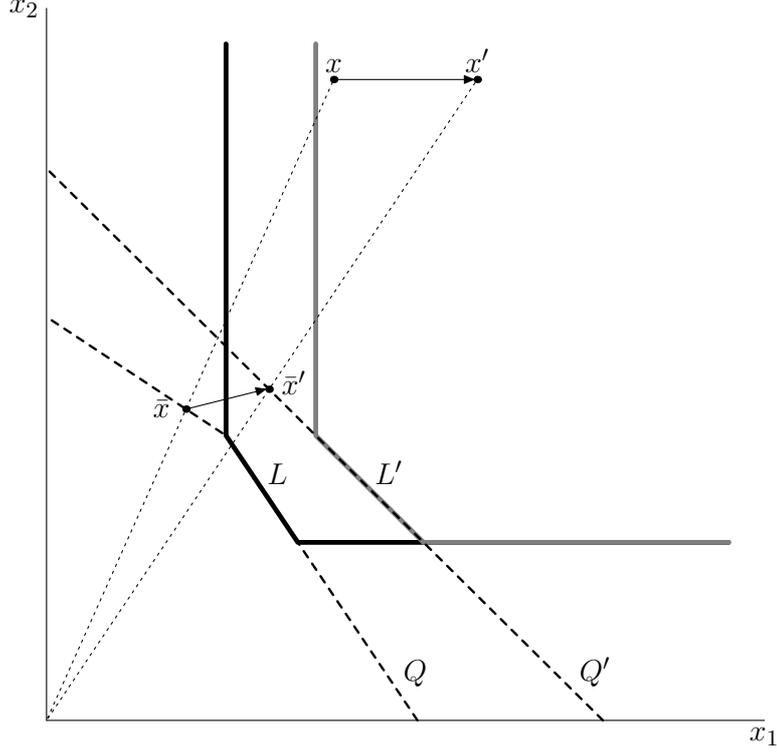


Figure 4.

The example suggests that we may want to weaken the commensurability axiom by extending the definition of an efficiency index to incorporate unit-of-measurement information.⁸ To this end, define the extended efficiency index \tilde{E} as the mapping, $\tilde{E} : \Gamma \times \mathcal{W} \rightarrow (0, 1]$, where \mathcal{W} is the space of positive $n \times n$ diagonal matrices, with image

$$\tilde{E}(x, L, \Omega) := E(x'(x, \Omega), L'(L, \Omega)), \quad (6.1)$$

where

$$x'(x, \Omega) = \Omega x \quad (6.2)$$

and

$$L'(L, \Omega) = \{ x' \mid \Omega^{-1} x' \in L \}. \quad (6.3)$$

The commensurability axiom (C) can be similarly extended as follows:

Weak Commensurability (WC): For all $x \in L$, if $x' = \Omega x$ and $L' = \{ x' \mid \Omega^{-1} x' \in L \}$, where Ω is a positive diagonal matrix, then $\tilde{E}(x, L, \Omega) = \tilde{E}(x', L', I)$.

⁸ After all, unit-of-measurement transformations are known to the investigator, and often such transformations are required to avoid serious rounding errors in calculation algorithms.

VII. An Efficiency Index Satisfying Commensurability and the Färe-Lovell Axioms on Convex Polyhedral Technologies.

In this section, we formulate a modified D-K efficiency index that satisfies (C), as well as the Färe-Lovell axioms, on a robust subset of convex polyhedral technologies. We show, further, that this index satisfies the slightly weaker commensurability condition (WC) on the full set of convex polyhedral technologies.

Any non-empty, closed, convex, free-disposal, polyhedral input requirement set can be written as

$$L = \{ x \mid \rho^j \cdot x \geq \alpha_j \ \forall j \in J \}, \quad (7.1)$$

where $\rho^j \in \mathbf{R}_+^n$, $\alpha_j \in \mathbf{R}_+$, and J is a (finite) set of indexes representing the hyperplanes supporting the facets of L . Denote by L_+ the subset of non-empty, closed, convex, free-disposal, polyhedral input-requirement sets containing at least one facet for which the supporting hyperplane has a positive normal. This restriction rules out Leontief input requirement sets in two-space, Leontief technologies and technologies with the efficient subset spanned by \mathbf{R}^2 (*i.e.*, an “edge” of the input requirement set parallel to one of the axes) in three-space, and so forth for higher dimensions.

For input-requirement sets in L_+ , define the modified D-K reference level set,

$$Q = \{ x \mid \rho^j \cdot x \geq \alpha_j \ \forall j \in J_+ \}, \quad (7.2)$$

where J_+ is the set of indexes representing supporting hyperplanes with strictly positive normals. This modified Q set is illustrated in Figure 5 for the same input-requirement set that was used in Figure 4 to illustrate the failure of E_{DK} to satisfy (C). The modified D-K index, E_M , restricted to L_+ , is given by

$$E_M(x, L) = \min \{ \lambda \mid \lambda x \in Q \}. \quad (7.3)$$

Theorem 3: E_M satisfies (C), (I), (M), and (H) on all input-requirement sets in L_+ .

Proof: We first show that Q satisfies the D-K conditions. As the intersection of closed sets, Q is closed. (Q2) is similarly immediate, since $\rho^j \cdot x \geq \alpha_j$ for all $j \in J_+$ and $\bar{x} > x$ implies $\rho^j \cdot \bar{x} > \alpha_j$ for all $j \in J_+$. To establish the $\text{Isoq}(Q) \subseteq \text{Eff}(Q)$ part of (Q3), it suffices to note that, for all $x \in \text{Isoq}(Q)$, the vector of marginal products is colinear with the positive normal ρ^j of the supporting hyperplane(s) at x ; hence, x is efficient. Conversely, any $x \notin \text{Isoq}(Q)$ is clearly inefficient. (Q4) follows immediately from a comparison of (7.1) and (7.2). Finally, $x \in \text{Eff}(L)$ only if at x there exists a hyperplane with positive normal supporting L at x , which implies that $x \in Q$.

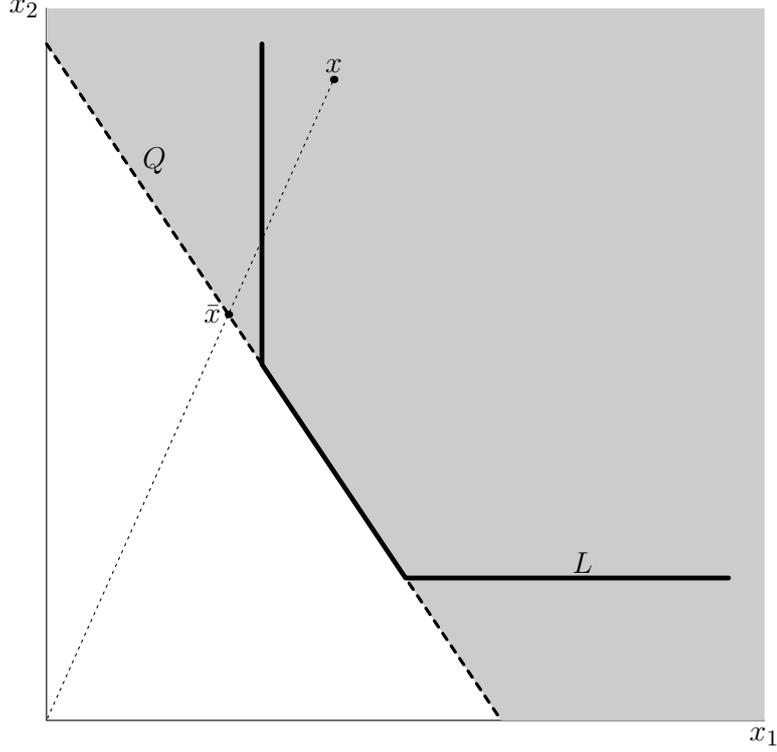


Figure 5.

It remains to show that the efficiency score,

$$E_M(x, L) = \min \{ \lambda \mid \rho^j \cdot (\lambda x) \geq \alpha_j \forall j \in J_+ \} \quad (7.4)$$

satisfies (C). Consider a change in units: $x' = \Omega x$, with the commensurate level set,

$$\begin{aligned} L' &= \{ x' \mid \rho^j \cdot \Omega^{-1} x' \geq \alpha_j \forall j \in J_+ \} \\ &= \{ x' \mid \hat{\rho}^j \cdot x' \geq \alpha_j \forall j \in J_+ \}, \end{aligned} \quad (7.5)$$

where $\hat{\rho}^j = \Omega^{-1} \rho^j$ for all $j \in J_+$.

The new reference level set is

$$Q' = \{ x \mid \hat{\rho}^j \cdot x' \geq \alpha_j \forall j \in J_+ \} \quad (7.6)$$

and the new efficiency score is

$$\begin{aligned} E_M(x', L') &= \min \{ \lambda \mid \hat{\rho}^j \cdot (x'/\lambda) \geq \alpha_j \forall j \in J_+ \} \\ &= \min \{ \lambda \mid \Omega^{-1} \rho^j \cdot (\Omega \lambda x) \geq \alpha_j \forall j \in J_+ \} \\ &= \min \{ \lambda \mid \rho^j \cdot (\lambda x) \geq \alpha_j \forall j \in J_+ \} \\ &= E_M(x, L). \quad \blacksquare \end{aligned} \quad (7.7)$$

The satisfaction of the commensurability condition is illustrated in Figure 6. Again, $x'_1 = \omega x_1$, $x'_2 = x_2$, and L' is the input-requirement set after the transformation of the units of input 1. In this case, as contrasted to Figure 4, the reference point on the reference set Q' corresponding to L' is on the horizontal line through \bar{x} , the reference point on Q , and $E_M(x, L) = E_M(x', L')$.

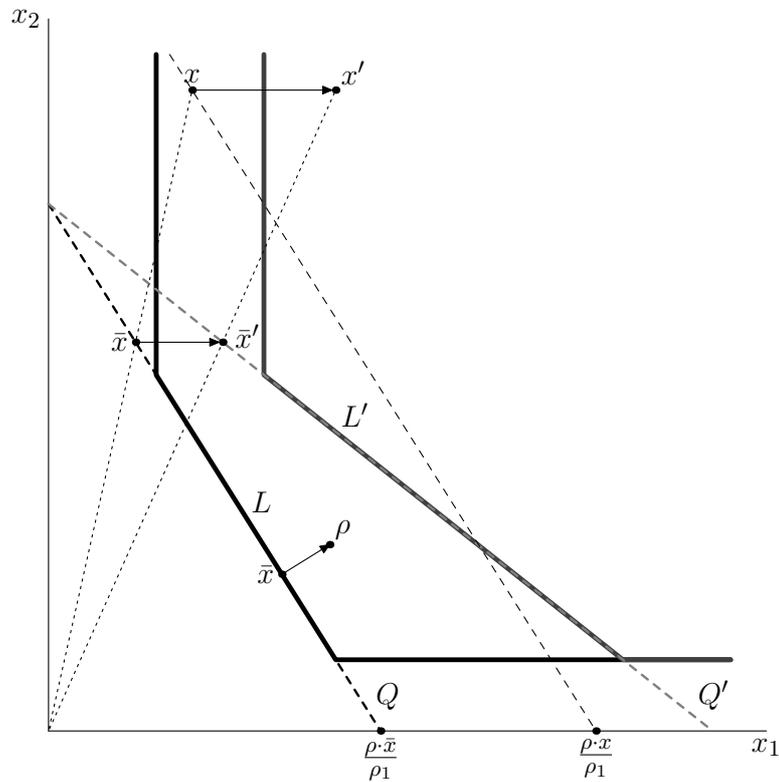


Figure 6.

The construction of the modified DM efficiency index in (7.3) fails when the technology does not contain at least one facet for which the supporting hyperplane has a positive normal. In this case, there exists a collection of hyperplanes supporting the efficient subset (a Leontief cusp or, more generally, a subset spanned by a lower-dimensional subspace), and a halfspace defined by an arbitrary selection from this collection intersected with \mathbf{R}_+^2 constitutes a Q set that would work. To preserve invariance with respect to units of measurement, however, the normal of the supporting hyperplane would have to be adjusted. The proof of the Lemma and Figure 4 illustrate the way in which the construction of the Q set automatically adjusts the slope of the reference hyperplane to accommodate the commensurability condition (C) when units are changed. In particular, the unit transformation Ω leads to a change in the normal of the supporting hyperplane given by the transformation Ω^{-1} . This transformation

suggests the following formulation of the Q set when the set of facets with strictly positive normals of supporting hyperplanes is empty:

$$Q(\Omega) = \{x' \in \mathbf{R}_+^n \mid \Omega^{-1}\mathbf{b} \cdot x' \geq \Omega^{-1}\mathbf{b} \cdot \bar{x}'\}, \quad (7.8)$$

where $b_i = \rho_i$ if $\rho_i > 0$, $b_i = 1$ (arbitrarily) if $\rho_i = 0$, $x' = \Omega x$, and $\bar{x}' = \Omega \bar{x}$. The (arbitrary) base Q is given by $Q(I)$, where I is the identity matrix.

The set Q is thus defined as follows:

$$Q(\Omega) = \begin{cases} \{x' \in \mathbf{R}_+^n \mid \rho^j \cdot \Omega^{-1}x' \geq \alpha_j, \forall j \in J_+\} & \text{if } J_+ \neq \emptyset \\ \{x' \in \mathbf{R}_+^n \mid \Omega^{-1}\mathbf{b} \cdot x' \geq \Omega^{-1}\mathbf{b} \cdot \bar{x}'\} & \text{if } J_+ = \emptyset. \end{cases} \quad (7.9)$$

The efficiency scores are then given by

$$\tilde{E}_M(x, L, \Omega) := E_M(x'(x, \Omega), L'(L, \Omega)) = \min \{ \lambda \mid \lambda x \in Q(\Omega) \}, \quad (7.10)$$

For the base case,

$$\tilde{E}_M(x, L, I) = E_M(x, L) = \min \{ \lambda \mid \lambda x \in Q(I) \}. \quad (7.11)$$

Theorem 4: \tilde{E}_M satisfies (WC), (I), (M), and (H) on all convex polynomial technologies.

Proof: The case where $J_+ \neq \emptyset$ is proved in the Lemma. When $J_+ = \emptyset$, it is clear that $Q(\Omega)$ satisfies (Q1)-(Q5), implying that E_M satisfies (I), (M), and (H) on all convex polyhedral technologies. At the base set of units,

$$\tilde{E}(x, L, I) = \min \{ \lambda \mid \mathbf{1} \cdot \lambda x \geq \mathbf{1} \cdot \bar{x} \}. \quad (7.12)$$

After the unit change given by $x' = \Omega x$, the efficiency measure becomes

$$\begin{aligned} \tilde{E}(x', L', \Omega) &= \min \left\{ \lambda \mid \Omega^{-1}\mathbf{1} \cdot x' / \lambda \geq \Omega^{-1}\mathbf{1} \cdot \bar{x}' \right\} \\ &= \min \left\{ \lambda \mid \Omega^{-1}\mathbf{1} \cdot \Omega \lambda x \geq \Omega^{-1}\mathbf{1} \cdot \Omega \bar{x} \right\} \\ &= \min \{ \lambda \mid \mathbf{1} \cdot \lambda x \geq \mathbf{1} \cdot \bar{x} \} \\ &= E_M(x, L, I). \quad \blacksquare \end{aligned} \quad (7.13)$$

VIII. Conclusion.

We conclude with two observations.

First, the extended D-K efficiency index is not only potentially, but indeed pragmatically, programmable. There exist algorithms for identifying the facets of a polyhedral set and the normals of the hyperplanes supporting those facets. We are currently exploring these possibilities.

Second, we believe the reference sets, Q , at least for technologies in L_+ , are not completely arbitrary constructions that yield an index satisfying the requisite axioms. Rather, the normals of the supporting hyperplanes that bound the Q set are shadow prices evaluated, for example, at the point on the isoquant where an input just becomes redundant. Thus, the modified D-K efficiency index in (7.4) reflects the cost of redundancy of some inputs, evaluated at the appropriate shadow prices ρ^j . In Figure 6, the shadow price vector has the direction ρ , and the relative shadow value of the redundancy of input vector x is $\rho \cdot x / \rho \cdot \bar{x}$.⁹ In the case of a Leontief technology, these prices are undefined or arbitrary, depending on one's interpretation. It is this arbitrariness that leads to the need for the extension of the efficiency index to incorporate information about units of measurement.

⁹ Of course, only relative shadow prices are determined by the normal of the efficient facet.

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