Econometric Inference Using Hausman Instruments

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Abstract

We examine econometric inferential issues with Hausman instruments. The instrumental variable (IV) estimator based on Hausman instrument has a built-in correlation across observations, which may render the textbook-style standard error invalid. We develop a standard error that is robust to these problems. Clustered standard error is not always valid, but it can be a good pragmatic compromise to deal with the interlinkage problem if Hausman instrument is to be used in econometric models in the tradition of Berry, Levinsohn, and Pakes (1995).

JEL Classification: C14, C33, C36

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1 Introduction

Hausman instrument was first introduced by Hausman (Hausman, Leonard, and Zona, 1994; Hausman, 1996) as a way to address endogeneity of the (log of) price variable in linear demand equations. It was later adopted in the context of nonlinear specifications, following the tradition of Berry, Levinsohn, and Pakes (1995, BLP hereafter), for similar purposes (e.g., Nevo, 2001; Crawford and Yurukoglu, 2012). For a comprehensive discussion and documentation of the Hausman IV in the broader context of Industrial Organization (IO) models, refer to Aguirregabiria (2019). The Hausman IV is also one of the most

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popular instruments in quantitative marketing and financial market analyses (see, e.g., Crawford and Yurukoglu, 2012; Rossi, 2014; Egan, Hortaçsu, and Matvos, 2017; Scanlon, 2019).

All the existing literature applying the Hausman IV focuses on empirical applications, with no theoretical discussion on the asymptotic framework and distribution necessary for econometric inference using the Hausman IV. This paper addresses this gap by introducing various asymptotic frameworks to examine inferential issues associated with the Hausman IV. We utilize a pseudo-panel structure to study the Hausman IV in linear models, where n denotes the number of firms in a given market (market size), and T represents the number of times the market is observed (number of markets).

Our first finding is that the Hausman IV estimator has a built-in correlation across contemporary observations, which concurs with the endogeneity that requires an IV in the first place. We demonstrate that the textbook-style standard error formula is valid only under asymptotics where both n and T grow to infinity. If either n or T is fixed, this formula becomes invalid. Additionally, we consider a rescaled version of the textbook-style standard error and a clustered standard error, showing that the former is valid under large n asymptotics, while the latter is valid under large T asymptotics. To overcome these limitations, we develop a uniformly valid standard error that ensure correct asymptotic inference as long as either n or T increases to infinity. This standard error is constructed as an average of the rescaled textbook-style standard error and the clustered standard error.

Our asymptotic analysis is different from the typical results in the econometrics literature. It is because the weak convergence concept, which is a standard tool for asymptotic analysis in econometrics, was not adequate for our asymptotic analysis. We adopted the stable convergence concept, which is rarely used in econometrics.[1] In this sense, this paper makes a technical contribution as well.

Our results are based on the specification that the underlying model is linear, which superficially rules out the BLP specification. Because our analysis of the failure of the textbook-style standard error is based on the problems with the "numerator" of the standard error in linear models, and because the same issue arises in the " numerator" counterpart of the BLP, it is straightforward to conclude that the textbook-style standard errors are invalid in the BLP specification when Hausman IV is adopted.

Regarding the practical implications for the BLP model with Hausman IV, we concede that this paper does not extend the analysis of the uniformly consistent standard error, which was developed and justified for the linear case. The BLP model involves numerous other components, making it challenging to isolate and focus on the anomalies specifically related to the Hausman IV. While the large T asymptotic results

[1] Phillips and Ouliaris (1990), Phillips and Sul (2003), Kuersteiner and Prucha (2013), Hahn, Kuersteiner, and Mazzocco (2020), Barndorff-Nielsen, Hansen, Lunde, and Shephard (2008) are some small number of exceptions.

for linear models can be extended to the BLP model with Hausman IV in a straightforward manner—essentially requiring only an adjustment of the "numerator", we are uncertain if the same straightforward extension applies to the large n asymptotic results.[2] Since analyzing the uniformly consistent standard error requires characterizing the asymptotic distributions under conditions where either n or T grows to infinity, we are currently not in a position to establish a uniform inference method. Having said that, we speculate that, in practice, it may be reasonable to use the clustered standard error. Although the applied literature is not explicit on this point, it appears that large T asymptotics are implicitly adopted in many cases.[3]

The remainder of this paper is organized as follows. In Section 2, we introduce the Hausman IV estimator within a benchmark model and provide an intuitive overview of the main findings of this paper. Section 3 derives the asymptotic distribution of the IV estimator in the benchmark model and discusses the textbook-style standard error, clustered standard error, and provides the formula for a consistent standard error. Section 4 extends these results to cases where exogenous regressors are included in the structural equation. In Section 5, we demonstrate that the inference issues observed with the Hausman IV estimator also arise with other IV approaches, such as the judge IV and Bartik IV. Section 6 concludes the paper. The Appendix contains proofs of the main theoretical results, and the Supplemental Appendix provides auxiliary lemmas used in these proofs.

The following notation will be adopted throughout the paper. We use K to denote a generic strictly positive constant that may vary from one instance to another but remains independent of the panel dimensions n and T. We adopt the convention that a summation over an empty set equals zero. We use $a \equiv b$ to indicate that a is defined as b. For real numbers $a_1, \ldots, a_m$, $(a_j)_{j \leq m} \equiv (a_1, \ldots, a_m)^\top$. For any matrix A, $A^\top$ denotes the transpose of A, and $\|A\|$ denotes the Euclidean norm of A. For any doubly indexed sequence $a_{i,t}$ (where $i = 1, \ldots, n$ and $t = 1, \ldots, T$), we define $\bar{a}_{i,\cdot} \equiv T^{-1} \sum_{t \leq T} a_{i,t}$, $\bar{a}_{\cdot,t} \equiv n^{-1} \sum_{i \leq n} a_{i,t}$ and $\bar{a} \equiv (nT)^{-1} \sum_{t \leq T} \sum_{i \leq n} a_{i,t}$. The summation $\sum_{i' \neq i}$ is taken over all i' except i, which means $\sum_{i' \neq i} a_{i',t} = \sum_{i'=1}^{i-1} a_{i',t} + \sum_{i'=i+1}^{n} a_{i',t}$.

[2]Some of the subtle issues include the following. As in the linear case, the clustered standard error is invalid under the large n asymptotics as considered by Berry, Linton, and Pakes (2004). Having said that, the large n asymptotics is subject to the concern along the line of Armstrong (2016), based on economics (not statistics) consideration. It is not clear to us whether the same concern should be raised about the Hausman IV with large n asymptotics.

[3]For example, while Conlon and Gortmaker (2020) were not explicit about the asymptotics for standard error calculation, it is clear that they adopted the large T asymptotics implicitly due to their reference (p.1123) to Freyberger (2015) for bias correction and standard error adjustment since the latter considered the large T asymptotics.

2 Intuitive Overview of the Main Results

The class of models that the Hausman IV is applicable can be written in the linear simultaneous equations model of the form

$$y_{i,t} = \alpha_i + \beta x_{i,t} + u_{i,t}, \tag{1}$$

$$x_{i,t} = \eta_i + \gamma c_t + v_{i,t}, \tag{2}$$

for $i = 1, \ldots, n$ and $t = 1, \ldots, T$, where $y_{i,t}$ is some dependent variable (such as the quantity demanded), $x_{i,t}$ is some endogenous explanatory variable (such as the price), and the c_t can be understood to be the *latent* common shock (such as cost shocks), the residual terms $u_{i,t}$ and $v_{i,t}$ may be correlated, which leads to the endogeneity of $x_{i,t}$.[4] The i denotes a city and the t denotes the time, so the (i,t)-pair indexes the"market"as commonly understood in the IO literature. The model (1) - (2) can be viewed as a result of the more general model, such as (14) presented in Section 4 below, where other included exogenous variables are partialled out.

This linear model is also flexible enough to include the BLP as long as $y_{i,t}$ is understood to be some nonlinear transformation that may depend on some additional parameters. For instance, Nevo (2001)'s "full model" is similar to our "extended model" in (14) below with $J > 1$, where $y_{i,t}$ is the mean utility from the good, $x_{i,t}$ is the price, $w_{i,t}$ is a vector of product characteristics, and β and θ are the means of individual coefficients β_i and θ_i ($i = 1, \ldots, n$), which are assumed to follow a joint distribution up to additional parameters θ_2 (Nevo, 2001, eq. (3)). Given θ_2, all the $y_{i,t}$, $x_{i,t}$ and $w_{i,t}$, the market share can be calculated from the model by numerical integration, denoted as $s_{i,t}(y, x, w^\top \theta_2)$, so the "dependent variable" $y_{i,t}$ can be obtained by solving the system of equations that matches the model-predicted market share with the observed one from the data (Nevo, 2001, eq. (7)). The additional parameters as well as the transformation complicates notations without shedding any further light on the basic econometric problems, so we abstract away from the BLP-style complication.

Hausman (1996)'s insight is that under some conditions, the $x_{i',t}$ with $i' \neq i$, i.e., the contemporary endogenous regressor from a different city, can serve as the instrument for $x_{i,t}$.[5] Hausman IV can

[4] Here, we abstract from the possibility of multiple goods, so in terms of commonly adopted notations, we let $J = 1$. Our econometric analysis goes through even when $J > 1$ as long as it remains finite in the asymptotic framework.

[5] In this paper, we will adopt all his identifying assumptions, and focus on the inferential issues. While Bresnahan and Gordon (2008) have raised questions on the identifying assumptions in Hausman (1996)'s paper, which Nevo (2000) and Aguirregabiria (2019) have also noted, this paper will focus on addressing the inferential issues rather than revisiting the questions of identification.

be particularly useful because cost shocks, the preferred instruments for demand estimation, are often unavailable to researchers. When $n \geq 2$, a common practice is to use the average

$$z_{i,t} \equiv (n-1)^{-1} \sum_{i' \neq i} x_{i',t} \tag{3}$$

as the IV for $x_{i,t}$, which means that the IV estimator is numerically equal to

$$\hat{\beta}_{iv} \equiv \frac{\sum_{t \leq T} \sum_{i \leq n} \left(z_{i,t} - \bar{z}_{i,\cdot}\right)\left(y_{i,t} - \bar{y}_{i,\cdot}\right)}{\sum_{t \leq T} \sum_{i \leq n} \left(z_{i,t} - \bar{z}_{i,\cdot}\right)\left(x_{i,t} - \bar{x}_{i,\cdot}\right)} = \frac{\sum_{t \leq T} \sum_{i \leq n} z_{i,t}\left(y_{i,t} - \bar{y}_{i,\cdot}\right)}{\sum_{t \leq T} \sum_{i \leq n} z_{i,t}\left(x_{i,t} - \bar{x}_{i,\cdot}\right)}, \tag{4}$$

where we use the usual partialling out trick to eliminate the fixed effects. Since

$$y_{i,t} - \bar{y}_{i,\cdot} = \left(x_{i,t} - \bar{x}_{i,\cdot}\right)\beta + u_{i,t} - \bar{u}_{i,\cdot} \tag{5}$$

based on the expression for $y_{i,t}$ in (1), applying (5) to (4) gives:

$$\hat{\beta}_{iv} = \frac{\sum_{t \leq T} \sum_{i \leq n} z_{i,t}\left(\left(x_{i,t} - \bar{x}_{i,\cdot}\right)\beta + u_{i,t} - \bar{u}_{i,\cdot}\right)}{\sum_{t \leq T} \sum_{i \leq n} z_{i,t}\left(x_{i,t} - \bar{x}_{i,\cdot}\right)} = \beta + \frac{\sum_{t \leq T} \sum_{i \leq n} z_{i,t}\left(u_{i,t} - \bar{u}_{i,\cdot}\right)}{\sum_{t \leq T} \sum_{i \leq n} z_{i,t}\left(x_{i,t} - \bar{x}_{i,\cdot}\right)}.$$

It is straightforward to show that

$$\sum_{t \leq T} \sum_{i \leq n} z_{i,t}\left(u_{i,t} - \bar{u}_{i,\cdot}\right) = \sum_{i \leq n} \sum_{t \leq T} \gamma u_{i,t}\left(c_t - \bar{c}\right) + \frac{1}{n-1} \underline{\sum_{t \leq T} \sum_{i \leq n} \sum_{i' \neq i} u_{i,t} v_{i',t}} - \frac{T}{n-1} \sum_{i \leq n} \sum_{i' \neq i} \bar{u}_{i,\cdot} \bar{v}_{i',\cdot}, \tag{6}$$

where $\bar{c} \equiv T^{-1} \sum_{t \leq T} c_t$. Note that the second term on the right (underlined) is a sum over t of the U-statistics $\sum_{i \leq n} \sum_{i' \neq i} u_{i,t} v_{i',t}$. Elementary statistics suggests that the variance of such U-statistic would depend on the covariance between u and v.[6] The potential correlation between the u and v in the model (1) and (2) is the source of endogeneity that would make the OLS inconsistent, and it is the reason why the instrument is sought. Our contribution is to recognize that the U-statistic structure built-in as part of the Hausman IV brings back the endogeneity (i.e., the covariance between u and v) as part of the asymptotic variance.

Having described the intuition, we now summarize the basic theoretical results in the next section. In Theorem 1, we provide the asymptotic distribution of the IV estimator $\hat{\beta}_{iv}$. We consider the asymptotic framework where n and/or T can go to infinity, although we insist that at least one of them should go to infinity by requiring that $nT \to \infty$. It turns out that the asymptotic distribution of $\hat{\beta}_{iv}$ depends on the behavior of n and T in the limit. Theorem 1 is a general result that nests all possible limiting behaviors of n and T. The asymptotic variance may be "random" depending on the limiting behaviors, and in

[6]For intuition, consider the simple case with $n = 2$, where the U-statistic takes the form $u_{1,t}v_{2,t} + u_{2,t}v_{1,t}$. Obviously the variance of this quantity is equal to $\operatorname{Var}(u_{1,t}) \operatorname{Var}(v_{2,t}) + 2 \operatorname{Cov}(u_{1,t}, v_{1,t}) \operatorname{Cov}(u_{2,t}, v_{2,t}) + \operatorname{Var}(u_{2,t}) \operatorname{Var}(v_{1,t})$, assuming that the u's and the v's have zero means, as well as that $u_{i,t}$ and $v_{i',t'}$ are independent for $i \neq i'$ or $t \neq t'$.

order to accommodate such situations, Theorem 1 presents the asymptotic distribution using the stable convergence concept.

In Lemma 1, we consider the textbook-style standard error derived under the homoscedasticity and independence assumption. The lemma establishes that this standard error is consistent only when both n and T go to infinity; if n is fixed while $T \to \infty$, it ignores the covariance between u and v, and therefore is inconsistent; if T is fixed while $n \to \infty$, it is again inconsistent because it ignores a multiplicative factor that depends on the magnitude of T.

Recall that the concern about the covariance in the U-statistic arose in the decomposition (6) of the "numerator" of $\hat{\beta}_{iv}$. The assumption that $(u_{i,t}, v_{i,t})$ are i.i.d. across i and t implies that the U-statistic $\sum_{i \leq n} \sum_{i' \neq i} u_{i,t} v_{i',t}$ should be independent over t. This suggests that a standard error clustered at the time level might be consistent. However, Lemma 2 demonstrates that such a clustered standard error is consistent only when $T \to \infty$; when T is fixed and $n \to \infty$, it is inconsistent for a reason elaborated in Section 3.2 below.

To address these issues, we develop a new standard error. In Theorem 2, it is shown that the new standard error is consistent in general, and despite the relatively unusual stable convergence framework, it enables asymptotically valid statistical inference, similar to what would be achieved under the usual weak convergence.

It is important to emphasize that Theorems 1 and 2, as well as Lemmas 1 and 2, are derived under the assumption that $(u_{i,t}, v_{i,t})$ are i.i.d. across i and t. Therefore, the inconsistency of both the textbook-style standard error and the clustered standard error is not attributable to any cluster structure among the pairs $(u_{i,t}, v_{i,t})$ over i or t.

3 Main Results in the Benchmark Model

In this section, we study the asymptotic properties of the IV estimator $\hat{\beta}_{iv}$ based on the model presented in (1) - (2). We refer to this as the benchmark model because the structural equation (1) does not include any exogenous regressors in equation (1). The IV estimator in an extended model, which includes additional regressors in (1), will be investigated in the next section. Throughout this paper, we consider an asymptotic framework where both n and T are indexed by $m = 1, 2, \ldots$ and both are non-decreasing in m, with $n_m T_m \to \infty$ as $m \to \infty$. For simplicity, the dependence of n_m and T_m on m is suppressed, provided there is no risk of confusion.

Assumption 1 *(i) $(u_{i,t}, v_{i,t})$ are i.i.d. across i and t with $\mathbb{E}[u_{i,t}] = 0$ and $\mathbb{E}[v_{i,t}] = 0$; (ii) c_{t_1} is independent of (u_{i,t_2}, v_{i,t_2}) for any i, and any t_1 and t_2; (iii) $\mathbb{E}[u_{i,t}^4] + \mathbb{E}[v_{i,t}^4] \leq K$ and $\max_t \mathbb{E}[c_t^4] \leq K$, (iv) $\hat{\sigma}_c^2 \equiv T^{-1} \sum_{t=1}^{T} (c_t - \bar{c})^2 \rightarrow_p \sigma_c^2$ where $\sigma_c^2 > 0$ almost surely; (v) $n \geq 2$, $T \geq 2$ and $(nT)^{-1} = o(1)$.*

Assumption 1 includes some regularity conditions used for studying the IV estimators. Conditions (i, ii) impose a dependence structure on the unobserved components, i.e., $u_{i,t}$, $v_{i,t}$, and c_t, allowing for correlation between $u_{i,t}$ and $v_{i,t}$. For the common factor c_t, we only require an upper bound on its fourth moment and a lower bound on its "sample variance". The factor c_t may exhibit time-varying distributions, making it non-stationary, and have a general dependence structure over time. Condition (v) requires that both n and T are strictly greater than 1, and nT diverges with the sample size. The restriction $(nT)^{-1} = o(1)$ allows for cases such as: (i) large n and small T; (ii) small n and large T; and (iii) large n and large T.

The independence assumption between the $\{c_t\}$ and $\{(u_{i,t}, v_{i,t})\}$ amounts to a homoscedasticity assumption as well as no apparent cluster structure in the error vector $\{(u_{i,t}, v_{i,t})\}$. We acknowledge that in a typical empirical question where the Hausman IV is applicable, $(u_{i,t}, v_{i,t})$ often has a cluster structure over i or t, or both, but an important and interesting feature of our result is that the asymptotic distribution of the Hausman IV estimator exhibits a clustering problem, introduced by the Hausman IV, even when there is no apparent cluster structure in the original model.

Theorem 1 *Let $\mathcal{F}_0$ denote the sigma-field generated by $\{c_t\}_{t=1}^{T_\infty}$. Under Assumption 1, we have*

$$(nT)^{1/2}(\hat{\beta}_{iv} - \beta) = \frac{(nT)^{-1/2} \sum_{t=1}^{T} \sum_{i=1}^{n} (\gamma u_{i,t}(c_t - \bar{c}) + \varepsilon_{i,t}) + O_p((nT)^{-1/2})}{\gamma^2 \hat{\sigma}_c^2 + O_p((nT)^{-1/2})}, \tag{7}$$

where $\varepsilon_{i,t} \equiv (n-1)^{-1} \sum_{i'=1}^{i-1} (u_{i,t} v_{i',t} + u_{i',t} v_{i,t})$. Moreover, as $m \rightarrow \infty$,

$$(nT)^{1/2}(\hat{\beta}_{iv} - \beta) \rightarrow \omega_\infty Z \qquad (\mathcal{F}_0\text{-stably}), \tag{8}$$

where $\omega_\infty^2 \equiv (\gamma^2 \sigma_u^2 \sigma_c^2 + (n_\infty - 1)^{-1} (\sigma_u^2 \sigma_v^2 + \sigma_{u,v}^2)) / (\gamma^4 \sigma_c^4)$ is independent of $Z \sim N(0,1)$, σ_u^2 and σ_v^2 denote the variances of $u_{i,t}$ and $v_{i,t}$, respectively, and $\sigma_{u,v}$ denotes the covariance between them.[7]

Theorem 1 derives the asymptotic distribution of the IV estimator $\hat{\beta}_{iv}$. The stable limit in (8) is required to address the case of large n and small T, where $\hat{\sigma}_c^2$ does not converge to a non-random constant; instead, its probability limit remains random in such a scenario. In the small n and large T case, it is evident that the covariance $\sigma_{u,v}$, in addition to the variances σ_u^2 and σ_v^2, appears in the

[7] The definitions of $\mathcal{G}$-stable convergence and $\mathcal{G}$-mixing convergence can be found in Section A of the Appendix.

"asymptotic variance" ω_∞^2. This arises from the cluster dependence of the Hausman IV $z_{i,t} - \bar{z}_{i,\cdot} = (n-1)^{-1}\sum_{i' \neq i}\left(x_{i',t} - \bar{x}_{i',\cdot}\right)$, which acts through $\varepsilon_{i,t}$, contributing to the U-statistic term in (6). Since $\sigma_{u,v}$ is the source of endogeneity, its appearance in ω_∞^2 highlights the critical importance of accounting for inherent cluster dependence when calculating the standard error for inference on the unknown parameter β.

3.1 Textbook-style Standard Error

In this subsection, we examine the textbook-style standard error for $\hat{\beta}_{iv}$. It turns out that such a standard error is consistent only when both n and T go to infinity, although a minor modification is consistent as long as $n \to \infty$.

Specifically, the textbook-style standard error formula for IV estimators with conditionally homoskedastic residuals is given by:

$$\widehat{\mathrm{SE}}_0(\hat{\beta}_{iv}) \equiv \sqrt{\frac{\left(\sum_{t \leq T}\sum_{i \leq n}\left(z_{i,t} - \bar{z}_{i,\cdot}\right)^2\right)\left(\sum_{t \leq T}\sum_{i \leq n}\hat{u}_{i,t}^2\right)}{nT\left(\sum_{t \leq T}\sum_{i \leq n}x_{i,t}\left(z_{i,t} - \bar{z}_{i,\cdot}\right)\right)^2}},$$

where $\hat{u}_{i,t} \equiv y_{i,t} - \bar{y}_{i,\cdot} - \hat{\beta}_{iv}(x_{i,t} - \bar{x}_{i,\cdot})$. The following lemma presents the asymptotic properties of $\widehat{\mathrm{SE}}_0(\hat{\beta}_{iv})$.

Lemma 1 *Under Assumption 1, we have:*

$$\sqrt{nT}\widehat{\mathrm{SE}}_0(\hat{\beta}_{iv}) \to_p \sqrt{\frac{\gamma^2\sigma_u^2\sigma_c^2 + (n_\infty - 1)^{-1}\sigma_u^2\sigma_v^2}{\gamma^4\sigma_c^4}(1 - T_\infty^{-1})}.$$

Lemma 1 shows that the textbook-style standard error is consistent only when both n and T go to infinity. In this scenario the asymptotic variance of the IV estimator in Theorem 1 simplifies to $\sigma_u^2/(\gamma^2\sigma_c^2)$, which is the same as the probability limit of $\widehat{\mathrm{SE}}_0(\hat{\beta}_{iv})$ after scaling.

In the large n and small T case, the textbook-style standard error is inconsistent due to the $1 - T_\infty^{-1}$ factor. However, we can apply a degrees of freedom adjustment to $\widehat{\mathrm{SE}}_0(\hat{\beta}_{iv})$, and show that the adjusted standard error, i.e., $\widehat{\mathrm{SE}}_0(\hat{\beta}_{iv})(1 - T^{-1})^{-1/2}$ is consistent in the large n scenarios.

In the small n and large T case, the covariance term $\sigma_{u,v}$ in the "asymptotic variance" ω_∞^2 is not captured in the probability limit of $\widehat{\mathrm{SE}}_0(\hat{\beta}_{iv})$. This indicates that the textbook-style standard error and its adjusted version are inconsistent as long as $\sigma_{u,v} \neq 0$. Since the endogeneity of $x_{i,t}$ arises from $\sigma_{u,v} \neq 0$, the inconsistency of the textbook-style standard error is a by-product of the necessity of using IV estimation.

3.2 Clustered Standard Error

It is natural to conjecture that the cluster structure induced by the Hausman IV ($\sigma^2_{u,v}$ in Theorem 1) might be intuitively handled by a clustered standard error, which has an additional bonus of providing a protection against potential heteroscedasticity. We now investigate the asymptotic properties of the conventional clustered standard error in this context. We show that the clustered standard error is consistent when $T \to \infty$ but not when T is bounded from above.

Under Assumptions 1(i, ii), it is clear that: $\{u_{i,t}(c_t - \bar{c})\}_{i \le n, t \le T}$ are uncorrelated across i and across t; and $u_{i,t}(c_t - \bar{c})$ and $\varepsilon_{i',t'}$ are uncorrelated for any $i, i' \le n$ and any $t, t' \le T$. Therefore, the cluster dependence in the estimation error of $\hat{\beta}_{iv}$ is introduced through $\varepsilon_{i,t}$. Indeed, for any $i_1, i_2 \le n$, $i'_1 \le i_1 - 1$, $i'_2 \le i'_2 - 1$ and any $t, t' \le T$ with $t \neq t'$, we have

$$\mathbb{E}[u_{i_1,t} v_{i'_1,t} u_{i_2,t'} v_{i'_2,t'}] = \mathbb{E}[u_{i_1,t} v_{i'_1,t}] \mathbb{E}[u_{i_2,t'} v_{i'_2,t'}] = 0,$$

which implies that

$$\mathbb{E}\left[\varepsilon_{i,t} \varepsilon_{i',t'}\right] = 0$$

as long as $t \neq t'$. Therefore, there is no clustering across t. On the other hand, we notice that

$$\sum_{t \le T} \sum_{i \le n} \varepsilon_{i,t} = (n-1)^{-1} \sum_{t \le T} \sum_{i \le n} \left(u_{i,t} \sum_{i' \neq i} v_{i',t} \right).$$

For any $t \le T$ and any $i_1, i_2 \le n$ with $i_1 \neq i_2$,

$$\mathbb{E}\left[u_{i_1,t} u_{i_2,t} \sum_{i'_1 \neq i_1} v_{i'_1,t} \sum_{i'_2 \neq i_2} v_{i'_2,t} \right] = \mathbb{E}\left[u_{i_1,t} u_{i_2,t} v_{i_2,t} v_{i_1,t}\right] = \mathbb{E}\left[u_{i_1,t} v_{i_1,t}\right] \mathbb{E}\left[u_{i_2,t} v_{i_2,t}\right] = \sigma^2_{u,v},$$

which shows that $\varepsilon_{i,t}$ has an equi-correlation across i, and it arises precisely due to the way the IV is constructed. This motivates the following clustered standard error

$$\widehat{\text{SE}}_1(\hat{\beta}_{iv}) \equiv \sqrt{\frac{\sum_{t \le T} \left(\sum_{i \le n} \hat{u}_{i,t} \left(z_{i,t} - \bar{z}_{i,\cdot} \right) \right)^2}{\left(\sum_{t \le T} \sum_{i \le n} x_{i,t} \left(z_{i,t} - \bar{z}_{i,\cdot} \right) \right)^2}}.$$

We next present the asymptotic properties of the clustered standard error.

Lemma 2 *Under Assumption 1, we have*

$$\widehat{\text{SE}}_1(\hat{\beta}_{iv})^2 = \frac{(nT)^{-1} \sum_{t \le T} \left(\sum_{i \le n} \gamma (c_t - \bar{c}) u_{i,t} - \xi_t \right)^2 + (n-1)^{-1} (\sigma^2_u \sigma^2_v + \sigma^2_{u,v})}{(nT) \gamma^4 \hat{\sigma}^4_c} + O_p((nT)^{-3/2}), \quad (9)$$

where $\xi_t \equiv n\gamma(c_t - \bar{c})(\bar{u} + \gamma(c_t - \bar{c})(\hat{\beta}_{iv} - \beta))$. *Moreover, if* $T \to \infty$ *as* $m \to \infty$, *then*

$$(nT)\widehat{\mathrm{SE}}_1(\hat{\beta}_{iv})^2 \to_p \frac{\gamma^2\sigma_u^2\sigma_c^2 + (n_\infty - 1)^{-1}(\sigma_u^2\sigma_v^2 + \sigma_{u,v}^2)}{\gamma^4\sigma_c^4}. \tag{10}$$

Lemma 2 provides the asymptotic approximation of the clustered standard error. The component denoted as ξ_t in (9) arises from estimating the unknown parameters α_i and β in the structural equation (1). When T approaches infinity, Lemma 2 shows that $(nT)\widehat{\mathrm{SE}}_1(\hat{\beta}_{iv})^2$ is a consistent estimator of the asymptotic variance of $\hat{\beta}_{iv}$. On the other hand, if T is bounded from above, the asymptotic approximation in (9) indicates that $\widehat{\mathrm{SE}}_1(\hat{\beta}_{iv})$ is an inconsistent estimator.

To illustrate this inconsistency, consider the simplified case where $u_{i,t}$ is known, and as a result ξ_t does not present in (9). In this case, the first term in the numerator of the faction on the right hand side of (9), i.e.,

$$(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}\gamma(c_t - \bar{c})u_{i,t}\right)^2 = \gamma^2 T^{-1}\sum_{t\leq T}\left(n^{-1/2}\sum_{i\leq n}(c_t - \bar{c})u_{i,t}\right)^2,$$

fails to approach to $\gamma^2\sigma_u^2\sigma_c^2$ in large sample, which causes the inconsistency of $\widehat{\mathrm{SE}}_1(\hat{\beta}_{iv})$.

The intuition underlying this failure is that the stable convergence of $(n^{-1/2}\sum_{i\leq n}(c_t - \bar{c})u_{i,t})_{t\leq T}$, combined with the Cramér-Wold device and continuous mapping theorem, would imply that as $m \to \infty$,

$$\gamma^2 T^{-1}\sum_{t\leq T}\left(n^{-1/2}\sum_{i\leq n}(c_t - \bar{c})u_{i,t}\right)^2 \to \gamma^2\sigma_u^2\sigma_c^2\left(T^{-1}\sum_{t\leq T}Z_t^2\right) \qquad (\mathcal{F}_0\text{-stably}),$$

where $(Z_t)_{t\leq T}$ is a vector of mutually independent standard normal random variables independent of σ_c^2. In other words, the term $(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}\gamma(c_t - \bar{c})u_{i,t}\right)^2$ converges to a scaled χ^2 random variable when T is small, not the desired non-stochastic component $\gamma^2\sigma_u^2\sigma_c^2$.

3.3 Averaging Textbook-style Standard Error and Clustered Standard Error

We now present a simple consistent standard error combining $\widehat{\mathrm{SE}}_0(\hat{\beta}_{iv})$ and $\widehat{\mathrm{SE}}_1(\hat{\beta}_{iv})$. From Lemma 1, it is clear that $(1 - T^{-1})^{-1/2}\widehat{\mathrm{SE}}_0(\hat{\beta}_{iv})$ is a consistent standard error as long as $n \to \infty$. On the other hand, Lemma 2 shows that $\widehat{\mathrm{SE}}_1(\hat{\beta}_{iv})$ is a consistent standard error whenever $T \to \infty$. This motivates the following averaging standard error:

$$\widehat{\mathrm{SE}}_{avg}(\hat{\beta}_{iv}) \equiv \frac{n}{T+n}(1 - T^{-1})^{-1/2}\widehat{\mathrm{SE}}_0(\hat{\beta}_{iv}) + \frac{T}{T+n}\widehat{\mathrm{SE}}_1(\hat{\beta}_{iv}). \tag{11}$$

We show that the averaging standard error is consistent under the general asymptotic framework with $nT \to \infty$ as $m \to \infty$.

Specifically, if both n and T go to infinity, then $(nT)^{1/2}(1-T^{-1})^{-1/2}\widehat{\mathrm{SE}}_0(\hat{\beta}_{iv})$ and $(nT)^{1/2}\widehat{\mathrm{SE}}_1(\hat{\beta}_{iv})$ converge to the same limit ω_∞, and so does $(nT)^{1/2}\widehat{\mathrm{SE}}_{avg}(\hat{\beta}_{iv})$. When n is bounded from above, $\widehat{\mathrm{SE}}_{avg}(\hat{\beta}_{iv})$ is dominated by the second term in (11), which as we have shown in (9) of Lemma 2, is a consistent estimator of ω_∞ after rescaled by $(nT)^{1/2}$. Finally, if T is bounded from above, $\widehat{\mathrm{SE}}_{avg}(\hat{\beta}_{iv})$ is dominated by the first term in (11), which is a consistent estimator of ω_∞ after rescaled by $(nT)^{1/2}$, as indicated by Lemma 1.

As a consequence, we arrive at the following theorem, showing that statistical inference based on the averaging standard error is valid when either n or T approaches infinity.

Theorem 2 *Under Assumption 1, we have* $(nT)^{1/2}\widehat{\mathrm{SE}}_{avg}(\hat{\beta}_{iv}) \to_p \omega_\infty$, *and*

$$\frac{\hat{\beta}_{iv}-\beta}{\widehat{\mathrm{SE}}_{avg}(\hat{\beta}_{iv})} \to N(0,1) \qquad (\mathcal{F}_0\text{-mixing}) \tag{12}$$

as $m \to \infty$.

Theorem 2 shows that asymptotically valid inference on β can be conducted using the stable limit stated in (12). For instance, the usual $(1-\alpha)$-confidence interval given by

$$CI_{1-\alpha} = \left[\hat{\beta}_{iv} - z_{\alpha/2}\widehat{\mathrm{SE}}_{avg}(\hat{\beta}_{iv}), \quad \hat{\beta}_{iv} + z_{\alpha/2}\widehat{\mathrm{SE}}_{avg}(\hat{\beta}_{iv})\right] \tag{13}$$

covers β with probability approaching $1-\alpha$ for any $\alpha \in (0,1)$, where $z_{\alpha/2}$ denotes the $(1-\alpha/2)$-quantile of the standard normal distribution.

4 Extended Model with Exogenous Regressors

The extended model incorporates several exogenous regressors, denoted as $w_{i,t}$ into the structural equation. Consequently, equation (1) becomes

$$y_{i,t} = \alpha_i + x_{i,t}\beta + w_{i,t}^\top\theta + u_{i,t}. \tag{14}$$

The additional d_w-dimensional regressors $w_{i,t}$ are allowed to be correlated with the common shock c_t and may exhibit both spatial and time series dependence. Ignoring these regressors could lead to omitted variable bias and/or incorrect standard error for the IV estimator and the inference procedures discussed in the previous section.[8]

[8] This is particularly important for the Hausman IV because its identification condition is likely to fail unless the advertising and promotional expenditure variables are included in the main regression (Rossi, 2014, p.666, footnote 8).

To define the IV estimator in the extended model, we introduce the following notation. Let $\hat{\lambda} \equiv \hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,x}$ and define $\hat{x}_{i,t}$ as $\hat{x}_{i,t} \equiv x_{i,t} - \bar{x}_{i,\cdot} - (w_{i,t} - \bar{w}_{i,\cdot})^\top \hat{\lambda}$, where

$$\hat{\Sigma}_w \equiv (nT)^{-1} \sum_{t \leq T} \sum_{i \leq n} (w_{i,t} - \bar{w}_{i,\cdot}) w_{i,t}^\top \text{ and } \hat{\Gamma}_{w,x} \equiv (nT)^{-1} \sum_{t \leq T} \sum_{i \leq n} (w_{i,t} - \bar{w}_{i,\cdot}) x_{i,t}.$$

Similarly, define $\hat{y}_{i,t}$ as $\hat{y}_{i,t} \equiv y_{i,t} - \bar{y}_{i,\cdot} - (w_{i,t} - \bar{w}_{i,\cdot})^\top \hat{\pi}$, where $\hat{\pi} \equiv \hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,y}$ and $\hat{\Gamma}_{w,y}$ is defined analogously to $\hat{\Gamma}_{w,x}$ with $x_{i,t}$ replaced by $y_{i,t}$. The IV estimator is then given by:

$$\hat{\beta}_{e,iv} \equiv \frac{\sum_{t \leq T} \sum_{i \leq n} z_{i,t} \hat{y}_{i,t}}{\sum_{t \leq T} \sum_{i \leq n} z_{i,t} \hat{x}_{i,t}}. \tag{15}$$

To study the properties of the IV estimator with additional regressors $w_{i,t}$, the following assumption is employed.

Assumption 2 *(i)* w_{i_1,t_1} *is independent of* $(u_{i_2,t_2}, v_{i_2,t_2})$ *for any* i_1 *and* i_2*, and any* t_1 *and* t_2*; (ii) there exist a matrix* Σ_w *such that* $\hat{\Sigma}_w \to_p \Sigma_w$*, where the eigenvalues of* Σ_w *are bounded away from zero almost surely; (iii) there exist a matrix* $\Gamma_{w,c}$ *such that* $\hat{\Gamma}_{w,c} \to_p \Gamma_{w,c}$*, where* $\hat{\Gamma}_{w,c}$ *is defined analogously to* $\hat{\Gamma}_{w,x}$ *with* $x_{i,t}$ *replaced by* c_t*; (iv)* $\max_i \max_t \mathbb{E}[||w_{i,t}||^4] \leq K$*; (v)* $\sigma_{e,c}^2 \equiv \sigma_c^2 - \Gamma_{w,c}^\top \Sigma_w^{-1} \Gamma_{w,c} > 0$ *almost surely.*

Condition (i) in Assumption 2 ensures that the regressors $w_{i,t}$ are strictly exogenous, while condition (iv) imposes a uniform upper bound on their fourth moment. Conditions (ii), (iii), and (iv) serve as regularity conditions to ensure that the IV estimator $\hat{\beta}_{e,iv}$ achieves a convergence rate of $(nT)^{-1/2}$. Assumption 2 permits both time series and spatial dependence in $w_{i,t}$, and allows for correlation between $w_{i,t}$ and the common shock c_t. In this scenario, the probability limit of $\hat{\Gamma}_{w,c}$, i.e., $\Gamma_{w,c}$ is a non-zero matrix.

Theorem 3 *Let* $\mathcal{F}_{e,0}$ *denote the sigma-field generated by* $\{\{c_t\}_{t \leq T_\infty}, \{w_{i,t}\}_{i \leq n_\infty, t \leq T_\infty}\}$*. Under Assumptions 1 and 2, we have as* $m \to \infty$

$$(nT)^{1/2}(\hat{\beta}_{e,iv} - \beta) \to \omega_{e,\infty} Z \qquad (\mathcal{F}_{e,0}\text{-stably}), \tag{16}$$

where $\omega_{e,\infty}^2 \equiv (\gamma^2 \sigma_u^2 \sigma_{e,c}^2 + (n_\infty - 1)^{-1}(\sigma_u^2 \sigma_v^2 + \sigma_{u,v}^2))/(\gamma^4 \sigma_{e,c}^4)$ *is independent of* $Z \sim N(0,1)$*.*

We next present the formula for the average standard error, which is based on both the textbook-style standard error and the clustered standard error, defined analogously to their counterparts in the benchmark model. Similar to the benchmark model, neither the textbook-style standard error nor the clustered standard error is consistent within the general asymptotic framework of $nT \to \infty$ employed in this paper. However, these standard errors can be combined to construct a consistent averaging standard error.

Let $\hat{z}_{i,t} \equiv (n-1)^{-1} \sum_{i' \neq i} \left(x_{i',t} - \bar{x}_{i',\cdot} \right) - (w_{i,t} - \bar{w}_{i,\cdot})^\top \hat{\varphi}$ denote the Hausman IV with α_i and $w_{i,t}$ partialled out, where

$$\hat{\varphi} \equiv \hat{\Sigma}_w^{-1} \hat{\Gamma}_{w,z} \quad \text{and} \quad \hat{\Gamma}_{w,z} \equiv (nT)^{-1} \sum_{t \leq T} \sum_{i \leq n} (w_{i,t} - \bar{w}_{i,\cdot}) z_{i,t}.$$

The textbook-style standard error is defined as

$$\widehat{\mathrm{SE}}_{e,0}(\hat{\beta}_{e,iv}) \equiv \sqrt{\frac{\left(\sum_{t \leq T} \sum_{i \leq n} \hat{z}_{i,t}^2 \right) \left(\sum_{t \leq T} \sum_{i \leq n} \hat{u}_{e,i,t}^2 \right)}{nT \left(\sum_{t \leq T} \sum_{i \leq n} z_{i,t} \hat{x}_{i,t} \right)^2}},$$

where $\hat{u}_{e,i,t} \equiv \hat{y}_{i,t} - \hat{x}_{i,t} \hat{\beta}_{e,iv}$. Under Assumptions 1 and 2, we can show that

$$(nT)\widehat{\mathrm{SE}}_{e,0}(\hat{\beta}_{e,iv})^2 \to_p \frac{\gamma^2 \sigma_u^2 \sigma_{e,c}^2 + (n_\infty - 1)^{-1} \sigma_u^2 \sigma_v^2}{\gamma^4 \sigma_{e,c}^4} (1 - T_\infty^{-1}), \tag{17}$$

which indicates that the textbook-style standard error does not account for the covariance term $\sigma_{u,v}$, and therefore is inconsistent when n is bounded from above.[9]

Lemma 3 *Under Assumptions 1 and 2, we have*

$$\sqrt{nT}\widehat{\mathrm{SE}}_{e,0}(\hat{\beta}_{iv}) \to_p \sqrt{\frac{\gamma^2 \sigma_u^2 \sigma_c^2 + (n_\infty - 1)^{-1} \sigma_u^2 \sigma_v^2}{\gamma^4 \sigma_c^4} (1 - T_\infty^{-1})}.$$

The clustered standard error in the extended model is defined as

$$\widehat{\mathrm{SE}}_{e,1}(\hat{\beta}_{e,iv}) \equiv \sqrt{\frac{\sum_{t \leq T} \left(\sum_{i \leq n} \hat{z}_{i,t} \hat{u}_{e,i,t} \right)^2}{\left(\sum_{t \leq T} \sum_{i \leq n} z_{i,t} \hat{x}_{i,t} \right)^2}}.$$

Similar to its counterpart in the benchmark model, the clustered standard error is consistent only when T goes to infinity. Therefore, we can combine the textbook-style standard error and the clustering "robust" standard error to obtain an averaging standard error defined as

$$\widehat{\mathrm{SE}}_{e,avg}(\hat{\beta}_{e,iv}) \equiv \frac{n}{T+n} (1 - T^{-1})^{-1/2} \widehat{\mathrm{SE}}_{e,0}(\hat{\beta}_{e,iv}) + \frac{T}{T+n} \widehat{\mathrm{SE}}_{e,1}(\hat{\beta}_{e,iv}), \tag{18}$$

which is consistent, as shown in the lemma below.

Lemma 4 *Under Assumptions 1 and 2, we have* $(nT)^{1/2} \widehat{\mathrm{SE}}_{e,avg}(\hat{\beta}_{e,iv}) \to_p \omega_{e,\infty}$.

By Theorem 3 and Lemmas 3 and 4, $\widehat{\mathrm{SE}}_{e,avg}(\hat{\beta}_{e,iv})$ can be used to construct confidence intervals, as in (13), and to perform statistical inference for the unknown parameter β.

[9] See the proof of Lemma 3 in the Appendix for the derivation of (17).

5 Potential Inferential Issues with Other Instruments

The inferential issue attributable to the U-statistic structure can arise in other contexts as well. For example, we suspect that the judge instrument would be subject to the similar problem depending on how it is used. Like equations (1) and (2) in Kling (2006), we consider a linear regression model $Y_i = S_i\gamma + \varepsilon_i$, where Y_i and S_i respectively denote the outcome and the treatment variable in case i. In Kling (2006), the judge assigned to the case is subscripted by j (and the assignment is supposed to be random), and the instrument is $Z_j\pi$ based on $S_j = Z_j\pi + Q_j\theta + \eta_j$, where the additional variable Q_j denotes the indicator of the district office.

In order to understand the connection between the judge IV to the Hausman IV, we replace the j subscript with t, and consider a pseudo-panel representation of the model $Y_{i,t} = S_{i,t}\gamma + \varepsilon_{i,t}$, where i denotes the i-th case (defendant) handled by judge t, and we will abstract away from Q_t in this section. Now noting that Z_t is just a judge's identity, we finally obtain the model in panel data notation:

$$Y_{i,t} = S_{i,t}\gamma + \varepsilon_{i,t},$$
$$S_{i,t} = \pi_t + \eta_{i,t}.$$

There can be many variants of the judge IV estimator. One possible variant uses an estimate of π_t for the (i,t) observation by using an estimate deleting the i-th case, i.e., $z_{i,t} = (n-1)^{-1}\sum_{i'\neq i} S_{i',t}$. If so, the judge IV estimate of the treatment effect is

$$\hat{\gamma}_{iv} = \frac{\sum_{t\leq T}\sum_{i\leq n} z_{i,t}Y_{i,t}}{\sum_{t\leq T}\sum_{i\leq n} z_{i,t}S_{i,t}} = \gamma + \frac{\sum_{t\leq T}\sum_{i\leq n} z_{i,t}\varepsilon_{i,t}}{\sum_{t\leq T}\sum_{i\leq n} z_{i,t}S_{i,t}},$$

where

$$\sum_{t\leq T}\sum_{i\leq n} z_{i,t}\varepsilon_{i,t} = \sum_{t\leq T}\sum_{i\leq n} \varepsilon_{i,t}\pi_t + (n-1)^{-1}\sum_{t\leq T}\sum_{i\leq n}\sum_{i'\neq i} \varepsilon_{i,t}\eta_{i',t}.$$

It is evident that the second term after the equality of the above expression shares the same sum-of-U-statistic structure as the second component in (6). Therefore, clustering will be an issue for econometric inference if n is small. This problem can be solved by using a standard error clustered at the judge level only when T, the number of judges, is large.

The U-statistic creates the problem that the observations are unintentionally interlinked. Such a problem may arise even in Bartik instrument context. For example, in Diamond (2016, equation (23)), we see that the instrument is computed by using the average log wage in cities within 25 mile radius of a given city, excluding the log wage of the given city itself. This naturally creates the inter-linkage problem. Unlike the Hausman IV or the judge IV, where the inter-linkage is confined to a cluster and we could resort to an asymptotic analysis where the number of clusters goes to infinity, the circles with 25-mile radius may overlap with each other, so from an asymptotic analysis perspective, the clustering here takes a more complicated form . We leave the analysis of such a complicated problem as a topic for future research.

6 Conclusion

In this paper, we address econometric inferential issues related to Hausman instruments. The IV estimator based on Hausman instruments has a "numerator" that involves U-statistics, naturally introducing a clustering problem even when the errors are independent of each other. The clustering issue can be important depending on the size of the clusters relative to the total sample size. We develop a standard error that is robust to these problems. While clustered standard errors are not always valid, they can serve as a pragmatic compromise for addressing the inter-linkage issue when using Hausman IV in BLP.

References

Aguirregabiria, V. (2019): "Empirical industrial organization: models, methods, and applications," *University of Toronto*.

Armstrong, T. B. (2016): "Large market asymptotics for differentiated product demand estimators with economic models of supply," *Econometrica*, 84(5), 1961–1980.

Barndorff-Nielsen, O. E., P. R. Hansen, A. Lunde, and N. Shephard (2008): "Designing realized kernels to measure the ex post variation of equity prices in the presence of noise," *Econometrica*, 76(6), 1481–1536.

Berry, S., J. Levinsohn, and A. Pakes (1995): "Automobile prices in market equilibrium," *Econometrica*, pp. 841–890.

Berry, S., O. B. Linton, and A. Pakes (2004): "Limit theorems for estimating the parameters of differentiated product demand systems," *Review of Economic Studies*, 71(3), 613–654.

Bresnahan, T. F., and R. J. Gordon (2008): *The economics of new goods*, vol. 58. University of Chicago Press.

Conlon, C., and J. Gortmaker (2020): "Best practices for differentiated products demand estimation with PyBLP," *The RAND Journal of Economics*, 51(4), 1108–1161.

Crawford, G. S., and A. Yurukoglu (2012): "The welfare effects of bundling in multichannel television markets," *American Economic Review*, 102(2), 643–685.

Diamond, R. (2016): "The determinants and welfare implications of US workers' diverging location choices by skill: 1980-2000," *The American Economic Review*, 106(3), 479–524.

Egan, M., A. Hortaçsu, and G. Matvos (2017): "Deposit competition and financial fragility: Evidence from the US banking sector," *American Economic Review*, 107(1), 169–216.

Freyberger, J. (2015): "Asymptotic theory for differentiated products demand models with many markets," *Journal of Econometrics*, 185(1), 162–181.

Hahn, J., G. Kuersteiner, and M. Mazzocco (2020): "Estimation with aggregate shocks," *The Review of Economic Studies*, 87(3), 1365–1398.

Häusler, E., and H. Luschgy (2015): *Stable Convergence and Stable Limit Theorems*, vol. 74. Springer.

Hausman, J., G. Leonard, and J. D. Zona (1994): "Competitive analysis with differenciated products," *Annales d'Economie et de Statistique*, pp. 159–180.

Hausman, J. A. (1996): "Valuation of new goods under perfect and imperfect competition," in *The Economics of New Goods*, pp. 207–248. University of Chicago Press.

Kling, J. R. (2006): "Incarceration length, employment, and earnings," *American Economic Review*, 96(3), 863–876.

Kuersteiner, G. M., and I. R. Prucha (2013): "Limit theory for panel data models with cross sectional dependence and sequential exogeneity," *Journal of Econometrics*, 174(2), 107–126.

Nevo, A. (2000): "A practitioner's guide to estimation of random-coefficients logit models of demand," *Journal of Economics & Management Strategy*, 9(4), 513–548.

——— (2001): "Measuring market power in the ready-to-eat cereal industry," *Econometrica*, 69(2), 307–342.

Phillips, P. C. B., and S. Ouliaris (1990): "Asymptotic properties of residual based tests for cointegration," *Econometrica*, 58, 165–193.

Phillips, P. C. B., and D. Sul (2003): "Dynamic panel estimation and homogeneity testing under cross section dependence," *The Econometrics Journal*, 6(1), 217–259.

Rossi, P. E. (2014): "Even the rich can make themselves poor: A critical examination of IV methods in marketing applications," *Marketing Science*, 33(5), 655–672.

Scanlon, P. (2019): "New goods and asset prices," *Journal of Financial Economics*, 132(3), 140–157.

Appendix

A Notations and Definitions

We begin by introducing some concepts related to the stable convergence of random variables from the literature, see, e.g., Häusler and Luschgy (2015). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the underlying probability space, and $\mathcal{X}$ be a Polish topological space equipped with its Borel sigma-field $\mathcal{B}(\mathcal{X})$. For a sub-sigma-field $\mathcal{G} \subset \mathcal{F}$, and a $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$-valued random variable X, let $\mathbb{P}^X$ and $\mathbb{P}^{X|\mathcal{G}}$ denote the marginal distribution of X and the conditional distribution of X given $\mathcal{G}$, respectively. Let $\mathcal{C}_b(\mathcal{X})$ denote the space of all continuous, bounded functions $h : \mathcal{X} \longmapsto \mathbb{R}$ equipped with the sup-norm $\|h\|_\infty \equiv \sup_{x \in \mathcal{X}} |h(x)|$.

Definition 1 *Let $\mathcal{G} \subset \mathcal{F}$ be a sub-sigma-field. A sequence $(X_m)_{m \geq 1}$ of $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$-valued random variables is said to converge stably to an $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$-valued random variable X, denoted as*

$$X_m \to X \qquad (\mathcal{G}\text{-stably}),$$

if $\mathbb{P}^{X_m|\mathcal{G}} \to \mathbb{P}^{X|\mathcal{G}}$ weakly as $m \to \infty$. That is

$$\lim_{m \to \infty} \mathbb{E}\left[g \mathbb{E}\left[h(X_m)|\mathcal{G}\right]\right] = \int g \int h(x) \mathbb{P}^{X|\mathcal{G}}(\cdot, dx) d\mathbb{P},$$

for every $\mathcal{G}$-measurable function g with $\mathbb{E}\left[|g|\right] < \infty$ and every $h \in \mathcal{C}_b(\mathcal{X})$. In case that $\mathbb{P}^{X|\mathcal{G}}$ equals $\mathbb{P}^X$ almost surely, then $(X_m)_{m \geq 1}$ is said to converge $\mathcal{G}$-mixing to X, denoted as

$$X_m \to X \qquad (\mathcal{G}\text{-mixing}).$$

The limit $\mathbb{P}^{X|\mathcal{G}}$ in the $\mathcal{G}$-stable convergence is a Markov kernel from $(\Omega, \mathcal{F})$ to $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ such that $\mathbb{P}^{X|\mathcal{G}}(\omega, \cdot)$ is a probability measure on $\mathcal{B}(\mathcal{X})$ for every $\omega \in \Omega$, and $\mathbb{P}^{X|\mathcal{G}}(\cdot, B)$ is $\mathcal{F}$-measurable for every $B \in \mathcal{B}(\mathcal{X})$. For presenting the main results of this paper, we have $\mathcal{X} = \mathbb{R}$ and $\mathbb{P}^{X|\mathcal{G}} = \mathbb{P}^{\eta Z|\mathcal{G}} = N(0, \eta^2)$ for a $\mathcal{G}$-measurable non-negative random variable η, and a standard normal random variable Z which is independent of $\mathcal{G}$, in the stable martingale central limit theorem.

B Proofs of the Main Results

In this section, we provide the proofs of the main results, including Lemmas 1 and 2, and Theorems 1 and 2 from Section 3, as well as Theorem 3, and Lemmas 3 and 4 from Section 4. The auxiliary lemmas used in these proofs are presented in the Supplemental Appendix (hereafter referred to as SA).

Proof of Theorem 1. The expression in (7) follows directly from (4) and Lemmas SA.2 and SA.3 in SA. By Lemma SA.4 in SA,

$$(nT)^{-1/2}\sum_{i\leq n}\sum_{t\leq T}(\gamma u_{i,t}(c_t-\bar{c})+\varepsilon_{i,t})\to\tilde{\omega}_\infty Z \qquad (\mathcal{F}_0\text{-stably}), \tag{19}$$

which together with Assumption 1(iv) and (7) implies that

$$(nT)^{1/2}(\hat{\beta}_{iv}-\beta)=\sum_{i\leq n}\sum_{t\leq T}\frac{\gamma u_{i,t}(c_t-\bar{c})+\varepsilon_{i,t}}{(nT)^{1/2}\gamma^2\hat{\sigma}_c^2}+O_p((nT)^{-1/2}). \tag{20}$$

Since σ_c^2 is $\mathcal{F}_0$-measurable, by Assumption 1(iv) and (19), we can apply Theorem 3.18(b) in Häusler and Luschgy (2015) to get:

$$\left(\sum_{i\leq n}\sum_{t\leq T}\frac{\gamma u_{i,t}(c_t-\bar{c})+\varepsilon_{i,t}}{(nT)^{1/2}},\gamma^2\hat{\sigma}_c^2\right)\to\left(\tilde{\omega}_\infty Z,\gamma^2\sigma_c^2\right) \qquad (\mathcal{F}_0\text{-stably}). \tag{21}$$

For any $(x,y)\in\mathbb{R}\times\mathbb{R}$, let

$$g(x,y)\equiv\begin{cases} x/y, & y>0 \\ 0, & y\leq 0\end{cases}.$$

Then $g(x,y)$ is Borel-measurable and $\mathbb{P}^{(\tilde{\omega}_\infty Z,\gamma^2\sigma_c^2)}$-continuous almost surely. Therefore by Theorem 3.18(c) in Häusler and Luschgy (2015),

$$g\left(\sum_{i\leq n}\sum_{t\leq T}\frac{\gamma u_{i,t}(c_t-\bar{c})+\varepsilon_{i,t}}{(nT)^{1/2}},\gamma^2\hat{\sigma}_c^2\right)\to g(\tilde{\omega}_\infty Z,\gamma^2\sigma_c^2) \qquad (\mathcal{F}_0\text{-stably}). \tag{22}$$

The claim of the Theorem follows from (22) and the definition of $g(x,y)$. ■

Proof of Lemma 1. By the definition of $\widehat{\mathrm{SE}}_0(\hat{\beta}_{iv})$, we can write

$$\sqrt{nT}\widehat{\mathrm{SE}}_0(\hat{\beta}_{iv})=\sqrt{\frac{(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\left(\sum_{i'\neq i}\left(x_{i',t}-\bar{x}_{i',\cdot}\right)\right)^2}{\left((nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\sum_{i'\neq i}x_{i,t}\left(x_{i',t}-\bar{x}_{i',\cdot}\right)\right)^2}(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\hat{u}_{i,t}^2}.$$

The claim of the lemma follows from Assumption 1(iv), and Lemmas SA.2, SA.5 and SA.6 in SA. ■

Proof of Lemma 2. We begin by expressing

$$(nT)(\widehat{\mathrm{SE}}_1(\hat{\beta}_{iv}))^2=\frac{(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}\hat{u}_{i,t}\left(z_{i,t}-\bar{z}_{i,\cdot}\right)\right)^2}{\left((nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}x_{i,t}\left(z_{i,t}-\bar{z}_{i,\cdot}\right)\right)^2}. \tag{23}$$

By Lemma SA.2 in SA, the denominator on the right-hand side of (23) satisfies:

$$\left((nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}x_{i,t}\left(z_{i,t}-\bar{z}_{i,\cdot}\right)\right)^2=\gamma^4\hat{\sigma}_c^4+O_p((nT)^{-1/2}). \tag{24}$$

Using Lemmas SA.8 and SA.9 in SA, we approximate the numerator on the right-hand side of (23) as

$$\begin{aligned}&(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}\hat{u}_{i,t}\left(z_{i,t}-\bar{z}_{i,\cdot}\right)\right)^2\\&=(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}\gamma(c_t-\bar{c})u_{i,t}-\xi_t\right)^2+\frac{\sigma_u^2\sigma_v^2+\sigma_{u,v}^2}{n-1}+O_p((nT)^{-1/2}).\end{aligned} \tag{25}$$

By Lemmas SA.10 and SA.11 in SA, we have

$$(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}\gamma(c_t-\bar{c})u_{i,t}-\xi_t\right)^2=O_p(1). \tag{26}$$

The claim in (9) follows from Assumption 1(iv) and (23)-(26). In view of (9), to prove (10), it is sufficient to show that as $T\to\infty$,

$$(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}\gamma(c_t-\bar{c})u_{i,t}-\xi_t\right)^2\to_p\gamma^2\sigma_u^2\sigma_c^2. \tag{27}$$

Applying Lemmas SA.10 and SA.11 in SA, and applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}\gamma(c_t-\bar{c})u_{i,t}-\xi_t\right)^2&=(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}\gamma(c_t-\bar{c})u_{i,t}\right)^2+(nT)^{-1}\sum_{t\leq T}\xi_t^2\\&\quad-2\gamma(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\gamma(c_t-\bar{c})u_{i,t}\xi_t\\&=(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}\gamma(c_t-\bar{c})u_{i,t}\right)^2+O_p(T^{-1/2})\\&=\gamma^2\sigma_u^2\hat{\sigma}_c^2+2\gamma(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})^2\mu_{i,t}+O_p(T^{-1/2}).\end{aligned} \tag{28}$$

By Assumptions 1(i, ii, iii), we have

$$\mathbb{E}\left[\left|(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})^2\mu_{i,t}\right|^2\right]=(nT)^{-2}\sum_{t\leq T}\sum_{i\leq n}\mathbb{E}[(c_t-\bar{c})^4]\mathbb{E}[\mu_{i,t}^2]\leq KT^{-1},$$

which, together with Markov's inequality, implies that

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})^2\mu_{i,t}=O_p(T^{-1/2}). \tag{29}$$

The desired result in (27) follows from Assumption 1(iv), along with (28), (29) and Slutsky's Theorem. ■

Proof of Theorem 2. First consider the case where both n and T go to infinity. By Lemmas 1 and 2, we have

$$(nT)^{1/2}(1-T^{-1})^{-1/2}\widehat{\mathrm{SE}}_0(\hat{\beta}_{iv}) = \frac{\sigma_u}{\gamma\sigma_c} + o_p(1) \quad \text{and} \quad (nT)^{1/2}\widehat{\mathrm{SE}}_1(\hat{\beta}_{iv}) = \frac{\sigma_u}{\gamma\sigma_c} + o_p(1),$$

which implies that

$$\begin{aligned}(nT)^{1/2}\widehat{\mathrm{SE}}_{avg}(\hat{\beta}_{iv}) &= \frac{\sigma_u}{\gamma\sigma_c} + \frac{n}{T+n}\left((nT)^{1/2}(1-T^{-1})^{-1/2}\widehat{\mathrm{SE}}_0(\hat{\beta}_{iv}) - \frac{\sigma_u}{\gamma\sigma_c}\right) \\ &\quad + \frac{T}{T+n}\left((nT)^{1/2}\widehat{\mathrm{SE}}_1(\hat{\beta}_{iv}) - \frac{\sigma_u}{\gamma\sigma_c}\right) = \frac{\sigma_u}{\gamma\sigma_c} + o_p(1). \end{aligned} \tag{30}$$

Since $\omega_\infty = \sigma_u/(\gamma\sigma_c)$ in this case, from (30) we have $(nT)^{1/2}\widehat{\mathrm{SE}}_{avg}(\hat{\beta}_{iv}) \to_p \omega_\infty$. Second, consider the case where n is bounded from above and T approaches infinity. In this scenario,

$$\frac{n}{T+n} = o(1) \quad \text{and} \quad \frac{T}{T+n} = 1 + o(1). \tag{31}$$

Moreover, Lemma 1 shows that

$$(nT)^{1/2}(1-T^{-1})^{-1/2}\widehat{\mathrm{SE}}_0(\hat{\beta}_{iv}) = O_p(1),$$

which together with (10) of Lemma 2 and (31) implies that

$$(nT)^{1/2}\widehat{\mathrm{SE}}_{avg}(\hat{\beta}_{iv}) = \frac{T}{T+n}(nT)^{1/2}\widehat{\mathrm{SE}}_1(\hat{\beta}_{iv}) + o_p(1) = \omega_\infty + o_p(1).$$

To finish the proof of the first claim of the theorem, consider the last case where T is bounded from above and n approaches infinity. In this scenario,

$$\frac{n}{T+n} = 1 + o(1) \quad \text{and} \quad \frac{T}{T+n} = o(1). \tag{32}$$

Moreover, by Lemmas SA.10 and SA.11 in SA,

$$(nT)^{1/2}\widehat{\mathrm{SE}}_1(\hat{\beta}_{iv}) = O_p(1),$$

which together with Lemma 1 and (32) implies that

$$(nT)^{1/2}\widehat{\mathrm{SE}}_{avg}(\hat{\beta}_{iv}) = \frac{n}{T+n}(nT)^{1/2}(1-T^{-1})^{-1/2}\widehat{\mathrm{SE}}_0(\hat{\beta}_{iv}) + o_p(1) = \omega_\infty + o_p(1).$$

In sum, we have shown that $(nT)^{1/2}\widehat{\mathrm{SE}}_{avg}(\hat{\beta}_{iv}) \to_p \omega_\infty$ in the asymptotic framework with $nT \to \infty$. It is evident that ω_∞^2 is $\mathcal{F}_0$-measurable. Therefore by the first claim of the lemma, Lemma SA.4 in SA and similar arguments in the proof of Theorem 1, we can show that

$$\frac{\hat{\beta}_{iv} - \beta}{\widehat{\mathrm{SE}}_{avg}(\hat{\beta}_{iv})} \to Z \qquad (\mathcal{F}_0\text{-stably}).$$

Since Z and $\mathcal{F}_0$ are independent, the convergence above is $\mathcal{F}_0$-mixing. ■

Proof of Theorem 3. By Assumptions 1(iv), and 2(ii, iii, v), and Lemmas SB.14 and SB.15 in SA, we have

$$(nT)^{1/2}(\hat{\beta}_{e,iv} - \beta) = (nT)^{-1/2} \sum_{i \le n} \sum_{t \le T} \frac{\gamma(c_t - \bar{c} - \hat{\Gamma}_{w,c}^\top \hat{\Sigma}_w^{-1}(w_{i,t} - \bar{w}_{i,\cdot}))u_{i,t} + \varepsilon_{i,t}}{\gamma^2(\hat{\sigma}_c^2 - \hat{\Gamma}_{w,c}^\top \hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c})} + O_p((nT)^{-1/2}). \tag{33}$$

Since $\sigma_{e,c}^2$ is $\mathcal{F}_{e,0}$-measurable, by Assumptions 1(iv) and 2(ii, iii, v) and (33), we can apply Theorem 3.18(b) in Häusler and Luschgy (2015) to obtain:

$$\begin{pmatrix} \sum_{i \le n} \sum_{t \le T} \frac{\gamma(c_t - \bar{c} - \hat{\Gamma}_{w,c}^\top \hat{\Sigma}_w^{-1}(w_{i,t} - \bar{w}_{i,\cdot}))u_{i,t} + \varepsilon_{i,t}}{(nT)^{1/2}} \\ \gamma^2(\hat{\sigma}_c^2 - \hat{\Gamma}_{w,c}^\top \hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c}) \end{pmatrix} \to \begin{pmatrix} \tilde{\omega}_{e,\infty} Z \\ \gamma^2 \sigma_{e,c}^2 \end{pmatrix} \qquad (\mathcal{F}_{e,0}\text{-stably}). \tag{34}$$

The claim of the theorem follows from (34) and the same arguments used in the proof of (22). ■

Proof of Lemma 3. By Lemmas SB.13, SB.16 and SB.17 in SA, we have

$$\begin{aligned} (nT)\widehat{\mathrm{SE}}_{e,0}(\hat{\beta}_{e,iv})^2 &= \frac{\left((nT)^{-1}\sum_{t \le T}\sum_{i \le n} \hat{z}_{i,t}^2\right) \times \left((nT)^{-1}\sum_{t \le T}\sum_{i \le n} \hat{u}_{e,i,t}^2\right)}{\left((nT)^{-1}\sum_{t \le T}\sum_{i \le n} \hat{x}_{i,t} z_{i,t}\right)^2} \\ &= \frac{\gamma^2 \hat{\sigma}_{e,c}^2 + \sigma_v^2 (n-1)^{-1} + O_p((nT)^{-1/2})}{(\gamma^2 \hat{\sigma}_{e,c}^2 + O_p((nT)^{-1/2}))^2} (\sigma_u^2(1 - T^{-1}) + O_p((nT)^{-1/2})), \end{aligned}$$

which, together with Assumptions 1(iv) and 2(ii, iii, v), shows that

$$\frac{(nT)\widehat{\mathrm{SE}}_{e,0}(\hat{\beta}_{e,iv})^2}{1 - T^{-1}} \to_p \frac{\gamma^2 \sigma_{e,c}^2 + \sigma_v^2 (n_\infty - 1)^{-1}}{\gamma^4 \sigma_{e,c}^4} \sigma_u^2. \tag{35}$$

This implies the claim of the lemma. ■

Proof of Lemma 4. The proof follows from (35), Lemmas SB.13 and SB.24 in SA, and similar arguments to those used in the proof of Theorem 2. Therefore, it is omitted. ■

Supplemental Appendix to
"Econometric Inference Using Hausman Instruments"

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Abstract

This supplement consists of two appendices. Appendix SA presents the lemmas used in proving the results from Section 3 of the paper. Appendix SB contains auxiliary lemmas utilized in the proofs of the results in Section 4.

SA Auxiliary Lemmas for Results in Section 3

Lemma SA.1 *Under Assumption 1, we have:*

$$(nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}(x_{i,t}-\bar{x}_{i,\cdot})^2=\gamma^2\hat{\sigma}_c^2+\sigma_v^2(1-T^{-1})+O_p((nT)^{-1/2}).$$

Proof. Using (2), we can write

$$x_{i,t}-\bar{x}_{i,\cdot}=\gamma(c_t-\bar{c})+v_{i,t}-\bar{v}_{i,\cdot} \tag{SA.1}$$

for any $i\leq n$ and $t\leq T$. Therefore

$$(nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}(x_{i,t}-\bar{x}_{i,\cdot})^2=\gamma^2T^{-1}\sum_{t\leq T}(c_t-\bar{c})^2+(nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}(v_{i,t}-\bar{v}_{i,\cdot})^2$$
$$+2\gamma(nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}(c_t-\bar{c})(v_{i,t}-\bar{v}_{i,\cdot}). \tag{SA.2}$$

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Since $T^{-1}\sum_{t\leq T}(v_{i,t}-\bar{v}_{i,\cdot})^2 = T^{-1}\sum_{t\leq T}v_{i,t}^2-\bar{v}_{i,\cdot}^2$, the second summation after the equality in (SA.2) can be expressed as

$$(nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}(v_{i,t}-\bar{v}_{i,\cdot})^2=(nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}v_{i,t}^2-n^{-1}\sum_{i\leq n}\bar{v}_{i,\cdot}^2. \tag{SA.3}$$

By Assumptions 1(i, iii) and Markov's inequality,

$$(nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}v_{i,t}^2=\sigma_v^2+O_p((nT)^{-1/2}). \tag{SA.4}$$

The second summation after the equality in (SA.3) can be further written as

$$n^{-1}\sum_{i\leq n}\bar{v}_{i,\cdot}^2=(nT^2)^{-1}\sum_{i\leq n}\sum_{t\leq T}v_{i,t}^2+2(nT^2)^{-1}\sum_{i\leq n}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}v_{i,t}v_{i,t'}. \tag{SA.5}$$

By Assumptions 1(i, iii),

$$\mathbb{E}\left[\left|(nT^2)^{-1}\sum_{i\leq n}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}v_{i,t}v_{i,t'}\right|^2\right]\leq K((nT^2)^{-1}),$$

which together with (SA.4) and Markov's inequality implies that

$$n^{-1}\sum_{i\leq n}\bar{v}_{i,\cdot}^2-T^{-1}\sigma_v^2=O_p((nT^2)^{-1/2}). \tag{SA.6}$$

Collecting the results in (SA.3), (SA.4) and (SA.6) obtains

$$(nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}(v_{i,t}-\bar{v}_{i,\cdot})^2=\sigma_v^2(1-T^{-1})+O_p((nT)^{-1/2}). \tag{SA.7}$$

For the third item after the equality in (SA.2), we have

$$\begin{aligned}\mathbb{E}\left[\left|(nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}(c_t-\bar{c})(v_{i,t}-\bar{v}_{i,\cdot})\right|^2\right]&=\mathbb{E}\left[\left|(nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}(c_t-\bar{c})v_{i,t}\right|^2\right]\\&=\sigma_v^2(nT^2)^{-1}\sum_{t\leq T}\mathbb{E}\left[(c_t-\bar{c})^2\right]\leq K(nT)^{-1},\end{aligned}$$

where the second equality is by Assumptions 1(i, ii) and the law of iterated expectations, and the inequality is by Assumptions 1(ii, iii). Therefore by Markov's inequality,

$$(nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}(c_t-\bar{c})(v_{i,t}-\bar{v}_{i,\cdot})=O_p((nT)^{-1/2}). \tag{SA.8}$$

The claim of the lemma follows from (SA.2), (SA.7) and (SA.8). ■

Lemma SA.2 *Under Assumption 1, we have:*

$$(nT)^{-1} \sum_{i \le n} \sum_{t \le T} x_{i,t} \left(z_{i,t} - \bar{z}_{i,\cdot} \right) = \gamma^2 \hat{\sigma}_c^2 + O_p((nT)^{-1/2}). \tag{SA.9}$$

Proof. Applying (SA.1) to the summand before the equality in (SA.9), in view of (3), results in:

$$\begin{aligned}
(nT)^{-1} \sum_{i \le n} \sum_{t \le T} x_{i,t} \left(z_{i,t} - \bar{z}_{i,\cdot} \right) &= ((n-1)nT)^{-1} \sum_{i \le n} \sum_{t \le T} \sum_{i' \ne i} x_{i,t} \left(x_{i',t} - \bar{x}_{i',\cdot} \right) \\
&= (nT)^{-1} \gamma \sum_{i \le n} \sum_{t \le T} x_{i,t} (c_t - \bar{c}) \\
&\quad + ((n-1)nT)^{-1} \sum_{i \le n} \sum_{t \le T} \sum_{i' \ne i} x_{i,t} (v_{i',t} - \bar{v}_{i',\cdot}),
\end{aligned} \tag{SA.10}$$

where the first summation after the equality, in view of (2), can be further decomposed as

$$(nT)^{-1} \gamma \sum_{i \le n} \sum_{t \le T} x_{i,t} (c_t - \bar{c}) = \gamma^2 T^{-1} \sum_{t \le T} c_t (c_t - \bar{c}) + (nT)^{-1} \gamma \sum_{i \le n} \sum_{t \le T} v_{i,t} (c_t - \bar{c}). \tag{SA.11}$$

By Assumptions 1(i, ii, iii),

$$\mathbb{E}\left[\left| \sum_{i \le n} \sum_{t \le T} v_{i,t} (c_t - \bar{c}) \right|^2 \right] = \sum_{i \le n} \sum_{t \le T} \mathbb{E}\left[v_{i,t}^2 (c_t - \bar{c})^2 \right] = \sigma_v^2 \sum_{i \le n} \sum_{t \le T} \mathbb{E}\left[(c_t - \bar{c})^2 \right] \le KnT,$$

which together with (SA.11) and Markov's inequality implies that

$$(nT)^{-1} \gamma \sum_{i \le n} \sum_{t \le T} x_{i,t} (c_t - \bar{c}) = \gamma^2 \hat{\sigma}_c^2 + O_p((nT)^{-1/2}). \tag{SA.12}$$

Applying (2) to the second summation in the right hand side of (SA.10) leads to

$$\begin{aligned}
((n-1)nT)^{-1} \sum_{i \le n} \sum_{t \le T} \sum_{i' \ne i} x_{i,t} (v_{i',t} - \bar{v}_{i',\cdot}) &= \gamma ((n-1)nT)^{-1} \sum_{t \le T} c_t \sum_{i \le n} \sum_{i' \ne i} (v_{i',t} - \bar{v}_{i'\cdot}) \\
&\quad + ((n-1)nT)^{-1} \sum_{i \le n} \sum_{t \le T} v_{i,t} \sum_{i' \ne i} (v_{i',t} - \bar{v}_{i'\cdot}).
\end{aligned} \tag{SA.13}$$

Since

$$\sum_{i \le n} \sum_{i' \ne i} (v_{i',t} - \bar{v}_{i'\cdot}) = \sum_{i \le n} \sum_{i' \ne i} v_{i',t} - \sum_{i \le n} \sum_{i' \ne i} \bar{v}_{i'\cdot} = (n-1) \sum_{i \le n} v_{i,t} - n(n-1)\bar{v},$$

it is evident that

$$((n-1)nT)^{-1} \sum_{t \le T} c_t \sum_{i \le n} \sum_{i' \ne i} (v_{i',t} - \bar{v}_{i'\cdot}) = (nT)^{-1} \sum_{i \le n} \sum_{t \le T} c_t v_{i,t} - \bar{c}\bar{v} = O_p((nT)^{-1/2}), \tag{SA.14}$$

where the second equality is by Assumptions 1(i, ii, iii) and Markov's inequality. For the second summation term on the right-hand side of (SA.13), we can write

$$\sum_{i\leq n}\sum_{t\leq T}v_{i,t}\sum_{i'\neq i}(v_{i',t}-\bar{v}_{i'\cdot}) = \sum_{t\leq T}\sum_{i\leq n}\sum_{i'\neq i}v_{i,t}v_{i',t} - \sum_{i\leq n}\sum_{t\leq T}v_{i,t}\sum_{i'\neq i}\bar{v}_{i'\cdot}$$
$$= 2\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}v_{i,t}v_{i',t} - 2T\sum_{i=2}^{n}\sum_{i'=1}^{i-1}\bar{v}_{i\cdot}\bar{v}_{i'\cdot}. \tag{SA.15}$$

By Assumptions 1(i, iii),

$$\mathbb{E}\left[\left|\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}v_{i,t}v_{i',t}\right|^2\right] = \sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}\mathbb{E}\left[\left|v_{i,t}v_{i',t}\right|^2\right] \leq Kn^2T$$

and

$$\mathbb{E}\left[\left|\sum_{i=2}^{n}\sum_{i'=1}^{i-1}\bar{v}_{i\cdot}\bar{v}_{i'\cdot}\right|^2\right] = \sum_{i=2}^{n}\sum_{i'=1}^{i-1}\mathbb{E}\left[\left|\bar{v}_{i\cdot}\bar{v}_{i'\cdot}\right|^2\right] \leq Kn^2T^{-2},$$

which, along with (SA.15), show that

$$((n-1)nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}v_{i,t}\sum_{i'\neq i}(v_{i',t}-\bar{v}_{i'\cdot}) = O_p((n^2T)^{-1/2}). \tag{SA.16}$$

Collecting the results in (SA.13), (SA.14) and (SA.16) yields

$$((n-1)nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}\sum_{i'\neq i}x_{i,t}(v_{i',t}-\bar{v}_{i',\cdot}) = O_p((nT)^{-1/2}). \tag{SA.17}$$

The claim of the lemma follows from (SA.10), (SA.12) and (SA.17). ■

Lemma SA.3 *Under Assumption 1, we have:*

$$\sum_{t\leq T}\sum_{i\leq n}u_{i,t}\left(z_{i,t}-\bar{z}_{i,\cdot}\right) = \sum_{t\leq T}\sum_{i\leq n}(\gamma u_{i,t}(c_t-\bar{c})+\varepsilon_{i,t}) + O_p(1). \tag{SA.18}$$

Proof. In view of (SA.1), the summation term before the equality in (SA.18) can be written as:

$$\sum_{t\leq T}\sum_{i\leq n}u_{i,t}\left(z_{i,t}-\bar{z}_{i,\cdot}\right) = (n-1)^{-1}\sum_{i\leq n}\sum_{t\leq T}\sum_{i'\neq i}u_{i,t}\left(x_{i',t}-\bar{x}_{i',\cdot}\right)$$
$$= \sum_{i\leq n}\sum_{t\leq T}\gamma u_{i,t}(c_t-\bar{c}) + (n-1)^{-1}\sum_{i\leq n}\sum_{t\leq T}\sum_{i'\neq i}u_{i,t}(v_{i',t}-\bar{v}_{i',\cdot}). \tag{SA.19}$$

We next show that

$$(n-1)^{-1}\sum_{i\leq n}\sum_{t\leq T}\sum_{i'\neq i}u_{i,t}(v_{i',t}-\bar{v}_{i',\cdot}) = \sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{i,t} + O_p(1), \tag{SA.20}$$

which together with (SA.19) proves the claim of the lemma. Some elementary algebra leads to

$$\sum_{i\leq n}\sum_{t\leq T}\sum_{i'\neq i} u_{i,t}(v_{i',t}-\bar{v}_{i',\cdot}) = \sum_{t\leq T}\sum_{i\leq n}\left(u_{i,t}\sum_{i'\neq i} v_{i',t}\right) - \sum_{i\leq n}\sum_{t\leq T} u_{i,t}\sum_{i'\neq i}\bar{v}_{i'\cdot}$$

$$= \sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(u_{i,t}v_{i',t}+u_{i',t}v_{i,t}) - T\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(\bar{u}_{i\cdot}\bar{v}_{i'\cdot}+\bar{u}_{i'\cdot}\bar{v}_{i\cdot}). \tag{SA.21}$$

By Assumptions 1(i, iii),

$$\mathbb{E}\left[\left|\sum_{i=2}^{n}\sum_{i'=1}^{i-1}\bar{u}_{i\cdot}\bar{v}_{i'\cdot}\right|^2\right] = \sum_{i=2}^{n}\sum_{i'=1}^{i-1}\mathbb{E}[|\bar{u}_{i\cdot}|^2]\mathbb{E}[|\bar{v}_{i'\cdot}|^2] \leq Kn^2T^{-2},$$

and similarly,

$$\mathbb{E}\left[\left|\sum_{i=2}^{n}\sum_{i'=1}^{i-1}\bar{u}_{i'\cdot}\bar{v}_{i\cdot}\right|^2\right] \leq Kn^2T^{-2},$$

which together with Markov's inequality imply that

$$(n-1)^{-1}T\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(\bar{u}_{i\cdot}\bar{v}_{i'\cdot}+\bar{u}_{i'\cdot}\bar{v}_{i\cdot}) = O_p(1). \tag{SA.22}$$

The desired result in (SA.20) follows from (SA.21), (SA.22) and the definition of $\varepsilon_{i,t}$ in Theorem 1. ■

Lemma SA.4 *Under Assumption 1, we have*

$$(nT)^{-1/2}\sum_{t\leq T}\sum_{i\leq n}(\gamma u_{i,t}(c_t-\bar{c})+\varepsilon_{i,t}) \to \tilde{\omega}_\infty Z \qquad (\mathcal{F}_0\text{-stably}),$$

where $\tilde{\omega}^2_\infty \equiv \gamma^2\sigma_u^2\sigma_c^2 + (n_\infty-1)^{-1}(\sigma_u^2\sigma_v^2+\sigma_{u,v}^2)$ *is independent of* $Z\sim N(0,1)$.

Proof. We shall apply the stable Martingale Central Limit Theorem (MGCLT) to prove the claim of the lemma. Some notation is needed. For any $k=1,\ldots,nT$, we define $t_k\equiv\lceil k/n\rceil$ and $i_k\equiv k-n(t_k-1)$, where $\lceil k/n\rceil$ denotes the smallest integer which is larger than or equal to k/n. Let $\mathcal{F}_{0,m}$ denote the sigma-field generated by $\{c_1,\ldots,c_{T_m}\}$. For $k=1,\ldots,nT$, let $\mathcal{F}_{k,m}$ denote the sigma field generated by $\{c_1,\ldots,c_{T_m},\{u_{i_l,t_l}\}_{l\leq k},\{v_{i_l,t_l}\}_{l\leq k}\}$. Using such notation, we can write

$$(nT)^{-1/2}\sum_{t\leq T}\sum_{i\leq n}(\gamma u_{i,t}(c_t-\bar{c})+\varepsilon_{i,t}) = \sum_{k=1}^{nT}\overbrace{\frac{\gamma u_{i_k,t_k}(c_{t_k}-\bar{c})+\varepsilon_{i_k,t_k}}{(nT)^{1/2}}}^{\equiv\tilde{\eta}_k}. \tag{SA.23}$$

We first show that $\{\tilde{\eta}_k\}_{k\leq nT}$ is a martingale difference array (MDA) adapted to $\mathcal{F}_{k,m}$. By the definitions of $\mathcal{F}_{k,m}$, t_k and i_k, it is evident that $\tilde{\eta}_k$ is $\mathcal{F}_{k,m}$-measurable. It remains to show that

$$\mathbb{E}[\tilde{\eta}_k|\mathcal{F}_{k-1,m}] = 0, \quad \text{for any } k=1,\ldots,nT. \tag{SA.24}$$

For any $k = 1, \ldots, nT$, we have either: (i) $t_k = t_{k-1}$ and $i_k = i_{k-1} + 1$; or (ii) $t_k = t_{k-1} + 1$, $i_{k-1} = n$ and $i_k = 1$. In the first scenario, we can apply Assumptions 1(i, ii) to show that:

$$\mathbb{E}\left[u_{i_k,t_k}(c_{t_k} - \bar{c})|\mathcal{F}_{k-1,m}\right] = (c_{t_k} - \bar{c})\mathbb{E}\left[u_{i_{k-1}+1,t_{k-1}}|\mathcal{F}_{k-1,m}\right] = (c_{t_k} - \bar{c})\mathbb{E}\left[u_{i_{k-1}+1,t_{k-1}}\right] = 0,$$

and

$$\begin{aligned}
\mathbb{E}\left[\varepsilon_{i_k,t_k}|\mathcal{F}_{k-1,m}\right] &= \mathbb{E}\left[(n-1)^{-1}\sum_{i'=1}^{i_k-1}(u_{i_k,t_k}v_{i',t_k} + u_{i',t_k}v_{i_k,t_k})|\mathcal{F}_{k-1,m}\right] \\
&= (n-1)^{-1}\sum_{i'=1}^{i_{k-1}}\left(\mathbb{E}\left[v_{i',t_{k-1}}u_{i_{k-1}+1,t_{k-1}}|\mathcal{F}_{k-1,m}\right] + \mathbb{E}\left[u_{i',t_{k-1}}v_{i_{k-1}+1,t_{k-1}}|\mathcal{F}_{k-1,m}\right]\right) \\
&= (n-1)^{-1}\sum_{i'=1}^{i_{k-1}}\left(v_{i',t_{k-1}}\mathbb{E}\left[u_{i_{k-1}+1,t_{k-1}}\right] + u_{i',t_{k-1}}\mathbb{E}\left[v_{i_{k-1}+1,t_{k-1}}\right]\right) = 0,
\end{aligned}$$

which, together with the definition of $\tilde{\eta}_k$, shows that (SA.24) holds. Similarly, in the second scenario, we have $\varepsilon_{i_k,t_k} = 0$, and

$$\mathbb{E}\left[u_{i_k,t_k}(c_{t_k} - \bar{c})|\mathcal{F}_{k-1,m}\right] = (c_{t_k} - \bar{c})\mathbb{E}\left[u_{1,t_{k-1}+1}|\mathcal{F}_{k-1,m}\right] = (c_{t_k} - \bar{c})\mathbb{E}\left[u_{1,t_{k-1}+1}\right] = 0,$$

which again verifies (SA.24). We next show that

$$\sum_{k=1}^{nT}\mathbb{E}\left[\tilde{\eta}_k^2\middle|\mathcal{F}_{k-1,m}\right] \to_p \tilde{\omega}_\infty^2, \tag{SA.25}$$

and for any $\varepsilon > 0$,

$$\sum_{k=1}^{nT}\mathbb{E}[\tilde{\eta}_k^2 I\{|\tilde{\eta}_k| > \varepsilon\}|\mathcal{F}_{k-1,m}] \to_p 0. \tag{SA.26}$$

Let $\mathcal{G}_{k,m} \equiv \bigcap_{m' \geq m}\mathcal{F}_{k,m'}$ for any $m \geq 1$ and any $k = 0, 1, \ldots, n_m T_m$. Under (SA.23)-(SA.26), we can apply the stable MGCLT, e.g., Theorem 6.1 in Häusler and Luschgy (2015), to show that

$$(nT)^{-1/2}\sum_{t\leq T}\sum_{i\leq n}(\gamma u_{i,t}(c_t - \bar{c}) + \varepsilon_{i,t}) \to \tilde{\omega}_\infty^2 Z \qquad (\mathcal{G}\text{-stably}), \tag{SA.27}$$

where $\mathcal{G}$ denotes the sigma-field generated by $\bigcup_{m=1}^{\infty}\mathcal{G}_{n_m T_m,m}$. Since $\mathcal{F}_0 \subset \mathcal{G}$, the claim of the lemma follows from (SA.27).[1] To verify (SA.25), we first apply Assumptions 1(i, ii) to obtain

$$(nT)\sum_{k=1}^{nT}\mathbb{E}\left[\tilde{\eta}_k^2\middle|\mathcal{F}_{k-1,m}\right] = \gamma^2\sum_{k=1}^{nT}\mathbb{E}\left[u_{i_k,t_k}^2(c_{t_k} - \bar{c})^2\middle|\mathcal{F}_{k-1,m}\right]$$

[1]We could have used the results in Kuersteiner and Prucha (2013) to establish the stable convergence, but we found it easier to work with the regularity conditions in Häusler and Luschgy (2015) in the setup of this paper.

$$
\begin{aligned}
&+ 2\gamma \sum_{k=1}^{nT} \mathbb{E}\left[(c_{t_k} - \bar{c}) u_{i_k,t_k} \varepsilon_{i_k,t_k} \middle| \mathcal{F}_{k-1,m} \right] + \sum_{k=1}^{nT} \mathbb{E}\left[\varepsilon_{i_k,t_k}^2 \middle| \mathcal{F}_{k-1,m} \right] \\
&= (nT)\gamma^2 \sigma_u^2 \hat{\sigma}_c^2 + \frac{2\gamma}{n-1} \sum_{k=1}^{nT} \sum_{i'=1}^{i_k-1} (c_{t_k} - \bar{c})(v_{i',t_k} \sigma_u^2 + u_{i',t_k} \sigma_{u,v}) \\
&+ \frac{\sigma_u^2 \sum_{k=1}^{nT} \left(\sum_{i'=1}^{i_k-1} v_{i',t_k} \right)^2 + \sigma_v^2 \sum_{k=1}^{nT} \left(\sum_{i'=1}^{i_k-1} u_{i',t_k} \right)^2}{(n-1)^2} \\
&+ \frac{2\sigma_{u,v}^2 \sum_{k=1}^{nT} \sum_{i_1'=1}^{i_k-1} \sum_{i_2'=1}^{i_k-1} u_{i_1',t_k} v_{i_2',t_k}}{(n-1)^2}. \qquad \text{(SA.28)}
\end{aligned}
$$

The second term after the second equality of (SA.28) can be written as

$$
\frac{2\gamma}{n-1} \sum_{k=1}^{nT} \sum_{i'=1}^{i_k-1} (c_{t_k} - \bar{c})(v_{i',t_k} \sigma_u^2 + u_{i',t_k} \sigma_{u,v}) = \frac{2\gamma}{n-1} \sum_{t \le T} \sum_{i=2}^{n} \sum_{i'=1}^{i-1} (c_t - \bar{c})(\sigma_u^2 v_{i',t} + \sigma_{u,v} u_{i',t}). \qquad \text{(SA.29)}
$$

By Assumptions 1(i, ii, iii), we have

$$
\begin{aligned}
&\mathbb{E}\left[\left| \sum_{t \le T} \sum_{i=2}^{n} \sum_{i'=1}^{i-1} (c_t - \bar{c})(v_{i',t} \sigma_u^2 + u_{i',t} \sigma_{u,v}) \right|^2 \middle| \mathcal{F}_{0,m} \right] \\
&= \sum_{t \le T} (c_t - \bar{c})^2 \mathbb{E}\left[\left| \sum_{i=1}^{n-1} (n-i)(v_{i,t} \sigma_u^2 + u_{i,t} \sigma_{u,v}) \right|^2 \right] \\
&\le K \sum_{t \le T} (c_t - \bar{c})^2 \sum_{i=1}^{n-1} (n-i)^2 \le K n^3 T \hat{\sigma}_c^2
\end{aligned}
$$

and

$$
\mathbb{E}\left[\hat{\sigma}_c^2 \right] = T^{-1} \sum_{t \le T} \mathbb{E}\left[(c_t - \bar{c})^2 \right] \le T^{-1} \sum_{t \le T} \mathbb{E}\left[c_t^2 \right] \le K.
$$

Therefore, by Markov's inequality and (SA.29),

$$
\frac{2\gamma}{(n-1)nT} \sum_{k=1}^{nT} \sum_{i'=1}^{i_k-1} (c_{t_k} - \bar{c})(v_{i',t_k} \sigma_u^2 + u_{i',t_k} \sigma_{u,v}) = O_p((nT)^{-1/2}). \qquad \text{(SA.30)}
$$

We next study the third term after the second equality of (SA.28). By the definitions of i_k and t_k, we can write

$$
\begin{aligned}
&\sum_{k=1}^{nT} \left(\sum_{i'=1}^{i_k-1} v_{i',t_k} \right)^2 - \frac{Tn(n-1)\sigma_v^2}{2} = \sum_{t \le T} \sum_{i=2}^{n} \left[\left(\sum_{i'=1}^{i-1} v_{i',t} \right)^2 - \sum_{i'=1}^{i-1} \sigma_v^2 \right] \\
&= \sum_{t \le T} \sum_{i=2}^{n} \left(\sum_{i'=1}^{i-1} (v_{i',t}^2 - \sigma_v^2) + 2 \sum_{i_1'=2}^{i-1} \sum_{i_2'=1}^{i_1'-1} v_{i_1',t} v_{i_2',t} \right)
\end{aligned}
$$

$$= \sum_{t \leq T} \sum_{i=1}^{n-1} (n-i)(v_{i,t}^2 - \sigma_v^2) + 2 \sum_{t \leq T} \sum_{i=2}^{n-1} (n-i) \sum_{i'=1}^{i-1} v_{i,t} v_{i',t}. \tag{SA.31}$$

Applying Assumptions 1(i, ii) leads to

$$\mathbb{E}\left[\left|\sum_{t \leq T} \sum_{i=1}^{n-1} (n-i)(v_{i,t}^2 - \sigma_v^2)\right|^2\right] \leq \sum_{t \leq T} \sum_{i=1}^{n-1} (n-i)^2 \mathbb{E}\left[v_{i,t}^4\right] \leq Kn^3 T$$

and

$$\mathbb{E}\left[\left|\sum_{t \leq T} \sum_{i=2}^{n-1} (n-i) \sum_{i'=1}^{i-1} v_{i,t} v_{i',t}\right|^2\right] = \sum_{t \leq T} \sum_{i=2}^{n-1} (n-i)^2 \sum_{i'=1}^{i-1} \sigma_v^4 \leq Kn^4 T,$$

which together with Markov's inequality and (SA.31) shows that

$$\frac{\sigma_u^2 \sum_{k=1}^{nT} \left(\sum_{i'=1}^{i_k-1} v_{i',t_k}\right)^2}{n(n-1)^2 T} - \frac{\sigma_u^2 \sigma_v^2}{2(n-1)} = O_p((n^2 T)^{-1/2}). \tag{SA.32}$$

Similar result can be established for $(nT(n-1)^2)^{-1} \sigma_v^2 \sum_{k=1}^{nT} \left(\sum_{i'=1}^{i_k-1} u_{i',t_k}\right)^2$, which along with (SA.32) implies that

$$\frac{\sigma_u^2 \sum_{k=1}^{nT} \left(\sum_{i'=1}^{i_k-1} v_{i',t_k}\right)^2 + \sigma_v^2 \sum_{k=1}^{nT} \left(\sum_{i'=1}^{i_k-1} u_{i',t_k}\right)^2}{nT(n-1)^2} - \frac{\sigma_u^2 \sigma_v^2}{n-1} = O_p((n^2 T)^{-1/2}). \tag{SA.33}$$

For the last term after the second equality of (SA.28), we can write the triple summation as

$$\begin{aligned}
\sum_{k=1}^{nT} \sum_{i_1'=1}^{i_k-1} \sum_{i_2'=1}^{i_k-1} u_{i_1',t_k} v_{i_2',t_k} &= \sum_{t \leq T} \sum_{i=2}^{n} \left(\sum_{i_1'=1}^{i-1} \sum_{i_2'=1}^{i-1} u_{i_1',t} v_{i_2',t}\right) \\
&= \sum_{t \leq T} \sum_{i=1}^{n-1} (n-i) u_{i,t} v_{i,t} + \sum_{t \leq T} \sum_{i=2}^{n-1} (n-i) \sum_{i'=1}^{i-1} (u_{i,t} v_{i',t} + v_{i,t} u_{i',t}).
\end{aligned}$$

Applying the similar arguments for showing (SA.33) obtains

$$\frac{2\sigma_{u,v}^2 \sum_{k=1}^{nT} \sum_{i_1'=1}^{i_k-1} \sum_{i_2'=1}^{i_k-1} u_{i_1',t_k} v_{i_2',t_k}}{nT(n-1)^2} - \frac{\sigma_{u,v}^2}{n-1} = O_p((nT)^{-1/2}). \tag{SA.34}$$

Collecting the results in (SA.28), (SA.30), (SA.33) and (SA.34) obtains

$$\sum_{k=1}^{nT} \mathbb{E}\left[\tilde{\eta}_k^2 \middle| \mathcal{F}_{k-1,m}\right] = \omega_{nT}^2 + O_p((nT)^{-1/2}), \tag{SA.35}$$

where $\omega_{nT}^2 \equiv \gamma^2 \sigma_u^2 \hat{\sigma}_c^2 + (n-1)^{-1}(\sigma_u^2 \sigma_v^2 + \sigma_{u,v}^2)$. Since $\omega_{nT}^2 \to_p \tilde{\omega}_\infty^2$ by Assumption 1(iv), (SA.25) follows from (SA.35). We proceed to demonstrate (SA.26). Beginning with the definition of $\tilde{\eta}_k$, we have

$$\sum_{k=1}^{nT} \mathbb{E}[\tilde{\eta}_k^2 I\{|\tilde{\eta}_k| > \varepsilon\} | \mathcal{F}_{k-1,m}] \leq \varepsilon^{-2} \sum_{k=1}^{nT} \mathbb{E}[\tilde{\eta}_k^4 | \mathcal{F}_{k-1,m}]$$

$$\leq \sum_{k=1}^{nT} \frac{K\gamma^4 \mathbb{E}[u_{i_k,t_k}^4 (c_{t_k} - \bar{c})^4 | \mathcal{F}_{k-1,m}]}{\varepsilon^2 (nT)^2} + \sum_{k=1}^{nT} \frac{K\mathbb{E}[\varepsilon_{i_k,t_k}^4 | \mathcal{F}_{k-1,m}]}{\varepsilon^2 (nT)^2}. \tag{SA.36}$$

By Assumptions 1(i, ii, iii), we observe that the first summation after the second inequality in (SA.36) is bounded:

$$\begin{aligned}\sum_{k=1}^{nT} \frac{\mathbb{E}[u_{i_k,t_k}^4 (c_{t_k} - \bar{c})^4 | \mathcal{F}_{k-1,m}]}{(nT)^2} &\leq K(nT)^{-2} \sum_{k=1}^{nT} (c_{t_k} - \bar{c})^4 \\ &= K(nT^2)^{-1} \sum_{t \leq T} (c_t - \bar{c})^4 = O_p((nT)^{-1}),\end{aligned} \tag{SA.37}$$

where the second equality in (SA.37) follows by

$$\mathbb{E}\left[T^{-1} \sum_{t \leq T} (c_t - \bar{c})^4\right] \leq K\mathbb{E}\left[T^{-1} \sum_{t \leq T} c_t^4 + \bar{c}^4\right] \leq KT^{-1} \sum_{t \leq T} \mathbb{E}[c_t^4] \leq K \tag{SA.38}$$

and Markov's inequality. To bound the second summation after the second inequality of (SA.36), we observe that by Assumptions 1(i, iii),

$$\begin{aligned}(n-1)^4 \sum_{k=1}^{nT} \mathbb{E}[\varepsilon_{i_k,t_k}^4 | \mathcal{F}_{k-1,m}] &\leq K \sum_{k=1}^{nT} \mathbb{E}\left[u_{i_k,t_k}^4 \left(\sum_{i'=1}^{i_k-1} v_{i',t_k} \right)^4 + v_{i_k,t_k}^4 \left(\sum_{i'=1}^{i_k-1} u_{i',t_k} \right)^4 \middle| \mathcal{F}_{k-1,m} \right] \\ &\leq K \sum_{k=1}^{nT} \left(\left(\sum_{i'=1}^{i_k-1} v_{i',t_k} \right)^4 + \left(\sum_{i'=1}^{i_k-1} u_{i',t_k} \right)^4 \right).\end{aligned} \tag{SA.39}$$

Under Assumption 1(i), we can apply Rosenthal's inequality (see, e.g., Theorem 2.12 in Hall and Heyde (1980)) to obtain:

$$\begin{aligned}\mathbb{E}\left[\sum_{k=1}^{nT} \left(\sum_{i'=1}^{i_k-1} v_{i',t} \right)^4 \right] &= \sum_{t \leq T} \sum_{i=2}^{n} \mathbb{E}\left[\left(\sum_{i'=1}^{i-1} v_{i',t} \right)^4 \right] \\ &\leq K \sum_{t \leq T} \sum_{i=2}^{n} \left[\left(\sum_{i'=1}^{i-1} \mathbb{E}[v_{i',t}^2] \right)^2 + \sum_{i'=1}^{i-1} \mathbb{E}[v_{i',t}^4] \right] \\ &\leq K \sum_{t \leq T} \sum_{i=2}^{n} (i-1)^2 \leq Kn^3 T,\end{aligned} \tag{SA.40}$$

where the second inequality is by Assumption 1(iii). Similarly, we can show that

$$\mathbb{E}\left[\sum_{k=1}^{nT} \left(\sum_{i'=1}^{i_k-1} u_{i',t_k} \right)^4 \right] \leq Kn^3 T,$$

which, along with (SA.39) and (SA.40) and Markov's inequality, implies that

$$(nT)^{-2} \sum_{k=1}^{nT} \mathbb{E}[\varepsilon_{i_k,t_k}^4 | \mathcal{F}_{k-1,m}] = O_p((n^3 T)^{-1}). \tag{SA.41}$$

By combining the results in (SA.36), (SA.37) and (SA.41), we derive (SA.26). ■

Lemma SA.5 *Under Assumption 1, we have*

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\hat{u}_{i,t}^2=\sigma_u^2(1-T^{-1})+O_p((nT)^{-1/2}).$$

Proof. Since $\hat{u}_{i,t}=u_{i,t}-\bar{u}_{i,\cdot}-(\hat{\beta}_{iv}-\beta)(x_{i,t}-\bar{x}_{i,\cdot})$, we can write

$$\begin{aligned}(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\hat{u}_{i,t}^2 &= (nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(u_{i,t}-\bar{u}_{i,\cdot})^2\\ &\quad+(\hat{\beta}_{iv}-\beta)^2(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(x_{i,t}-\bar{x}_{i,\cdot})^2\\ &\quad-2(\hat{\beta}_{iv}-\beta)(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(x_{i,t}-\bar{x}_{i,\cdot})(u_{i,t}-\bar{u}_{i,\cdot}).\end{aligned}\tag{SA.42}$$

Some elementary algebra leads to

$$\begin{aligned}&(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(u_{i,t}-\bar{u}_{i,\cdot})^2-\sigma_u^2(1-T^{-1})\\ &=(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(u_{i,t}^2-\sigma_u^2)-n^{-1}\sum_{i\leq n}\bar{u}_{i,\cdot}^2+\sigma_u^2T^{-1}\\ &=(nT)^{-1}(1-T^{-1})\sum_{i\leq n}\sum_{t\leq T}(u_{i,t}^2-\sigma_u^2)-2(nT^2)^{-1}\sum_{i\leq n}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}u_{i,t}u_{i,t'}.\end{aligned}\tag{SA.43}$$

By Assumptions 1(i, iii), we have

$$\mathbb{E}\left[\left|(nT)^{-1}(1-T^{-1})\sum_{t\leq T}\sum_{i\leq n}(u_{i,t}^2-\sigma_u^2)\right|^2\right]\leq(nT)^{-1}\mathbb{E}[u_{i,t}^4]\leq K(nT)^{-1}$$

and

$$\mathbb{E}\left[\left|(nT^2)^{-1}\sum_{i\leq n}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}u_{i,t}u_{i,t'}\right|^2\right]=(nT^2)^{-2}\sum_{i\leq n}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sigma_u^4\leq K(nT^2)^{-1},$$

which together with (SA.43) and Markov's inequality shows that

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(u_{i,t}-\bar{u}_{i,\cdot})^2=\sigma_u^2(1-T^{-1})+O_p((nT)^{-1/2}).\tag{SA.44}$$

Lemma SA.1 and (8) in Theorem 1 together yield

$$(\hat{\beta}_{iv}-\beta)^2(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(x_{i,t}-\bar{x}_{i,\cdot})^2=O_p((nT)^{-1}).\tag{SA.45}$$

For the third term after the equality in (SA.42), we can use the Cauchy-Schwarz inequality to get

$$\begin{aligned}
&\left|(\hat{\beta}_{iv}-\beta)(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(x_{i,t}-\bar{x}_{i,\cdot})(u_{i,t}-\bar{u}_{i,\cdot})\right| \\
&\leq \sqrt{(\hat{\beta}_{iv}-\beta)^2(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(x_{i,t}-\bar{x}_{i,\cdot})^2}\times\sqrt{(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(u_{i,t}-\bar{u}_{i,\cdot})^2} \\
&= O_p((nT)^{-1/2}), \qquad\qquad \text{(SA.46)}
\end{aligned}$$

where the equality follows by (SA.44) and (SA.45). The claim of the lemma follows from (SA.42), (SA.44), (SA.45) and (SA.46). ∎

Lemma SA.6 *Under Assumption 1, we have*

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(z_{i,t}-\bar{z}_{i,\cdot})^2=\gamma^2\hat{\sigma}_c^2+\sigma_v^2(n-1)^{-1}+O_p((nT)^{-1/2}). \qquad \text{(SA.47)}$$

Proof. Applying (SA.1) to the term before the equality in (SA.47) leads to:

$$\begin{aligned}
&(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(z_{i,t}-\bar{z}_{i,\cdot})^2=(n(n-1)^2T)^{-1}\sum_{t\leq T}\sum_{i\leq n}\left(\sum_{i'\neq i}(x_{i',t}-\bar{x}_{i',\cdot})\right)^2 \\
&=(n(n-1)^2T)^{-1}\sum_{t\leq T}\sum_{i\leq n}\left(\gamma(n-1)(c_t-\bar{c})+\sum_{i'\neq i}(v_{i',t}-\bar{v}_{i',\cdot})\right)^2 \\
&=\gamma^2T^{-1}\sum_{t\leq T}(c_t-\bar{c})^2+(n(n-1)^2T)^{-1}\sum_{t\leq T}\sum_{i\leq n}\left(\sum_{i'\neq i}(v_{i',t}-\bar{v}_{i',\cdot})\right)^2 \\
&\quad+2\gamma(n(n-1)T)^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})\sum_{i'\neq i}(v_{i',t}-\bar{v}_{i',\cdot}) \\
&=\gamma^2\hat{\sigma}_c^2+(n(n-1)^2T)^{-1}\sum_{t\leq T}\sum_{i\leq n}\left(\sum_{i'\neq i}(v_{i',t}-\bar{v}_{i',\cdot})\right)^2+O_p((nT)^{-1/2}),
\end{aligned}$$

where the last equality is by the definition of $\hat{\sigma}_c^2$, and (SA.14). It remains to show that

$$(n(n-1)^2T)^{-1}\sum_{t\leq T}\sum_{i\leq n}\left(\sum_{i'\neq i}(v_{i',t}-\bar{v}_{i',\cdot})\right)^2=\sigma_v^2(n-1)^{-1}+O_p((nT)^{-1/2}). \qquad \text{(SA.48)}$$

The term before the equality in (SA.48) can be decomposed as:

$$(n(n-1)^2T)^{-1}\sum_{t\leq T}\sum_{i\leq n}\left(\sum_{i'\neq i}(v_{i',t}-\bar{v}_{i',\cdot})\right)^2$$

$$= (n(n-1)T)^{-1} \sum_{t \le T} \sum_{i \le n} (v_{i,t} - \bar{v}_{i,\cdot})^2 + 2(n-2)(n(n-1)^2 T)^{-1} \sum_{t \le T} \sum_{i=2}^{n} \sum_{i'=1}^{i-1} (v_{i,t} - \bar{v}_{i,\cdot}) (v_{i',t} - \bar{v}_{i',\cdot}) . \tag{SA.49}$$

Therefore, (SA.48) follows by Assumption 1(v), (SA.7) and (SA.16). ■

Lemma SA.7 *Under Assumption 1, we have (i)* $(nT)^{-1} \sum_{t \le T} \left(\sum_{i \le n} \varepsilon_{i,t} \right)^2 - (n-1)^{-1}(\sigma_u^2 \sigma_v^2 + \sigma_{u,v}^2) = O_p((n^2 T)^{-1/2})$*; (ii)* $(nT)^{-1} \sum_{t \le T} \sum_{i \le n} \sum_{i' \le n} (c_t - \bar{c}) u_{i,t} \varepsilon_{i',t} = O_p((nT)^{-1/2})$.

Proof. (i) By the definition of $\varepsilon_{i,t}$ and Assumption 1(i),

$$\mathbb{E}\left[\varepsilon_{i,t}^2\right] = \frac{\sum_{i'=1}^{i-1} \mathbb{E}\left[(u_{i,t} v_{i',t} + u_{i',t} v_{i,t})^2\right]}{(n-1)^2} = \frac{2 \sum_{i'=1}^{i-1} (\sigma_u^2 \sigma_v^2 + \sigma_{u,v}^2)}{(n-1)^2} = \frac{2(i-1)(\sigma_u^2 \sigma_v^2 + \sigma_{u,v}^2)}{(n-1)^2}. \tag{SA.50}$$

Therefore

$$(nT)^{-1} \sum_{t \le T} \mathbb{E}\left[\left(\sum_{i \le n} \varepsilon_{i,t} \right)^2 \right] = (nT)^{-1} \sum_{t \le T} \sum_{i \le n} \mathbb{E}\left[\varepsilon_{i,t}^2\right] = (n-1)^{-1}(\sigma_u^2 \sigma_v^2 + \sigma_{u,v}^2). \tag{SA.51}$$

By Assumptions 1(i, iii) and (SA.51),

$$\begin{aligned} &\mathbb{E}\left[\left| (nT)^{-1} \sum_{t \le T} \left(\sum_{i \le n} \varepsilon_{i,t} \right)^2 - (n-1)^{-1}(\sigma_u^2 \sigma_v^2 + \sigma_{u,v}^2) \right|^2 \right] \\ &= \mathbb{E}\left[\left| (nT)^{-1} \sum_{t \le T} \left(\left(\sum_{i \le n} \varepsilon_{i,t} \right)^2 - \mathbb{E}\left[\left(\sum_{i \le n} \varepsilon_{i,t} \right)^2 \right] \right) \right|^2 \right] \le (nT)^{-2} \sum_{t \le T} \mathbb{E}\left[\left(\sum_{i \le n} \varepsilon_{i,t} \right)^4 \right]. \end{aligned} \tag{SA.52}$$

Let $\mathcal{F}_{i,t}$ denote the sigma-field generated by $\{\{u_{i',t}\}_{i' \le i}, \{v_{i',t}\}_{i' \le i}\}$. Then by Assumptions 1(i, ii), $\{\varepsilon_{i,t}\}_{i \le n}$ is a MDA adapted to $\{\mathcal{F}_{i,t}\}_{i \le n}$. By Rosenthal's inequality,

$$\mathbb{E}\left[\left(\sum_{i \le n} \varepsilon_{i,t} \right)^4 \right] \le K \left(\sum_{i \le n} \mathbb{E}\left[\varepsilon_{i,t}^4\right] + \mathbb{E}\left[\left(\sum_{i \le n} \mathbb{E}\left[\varepsilon_{i,t}^2 | \mathcal{F}_{i-1,t}\right] \right)^2 \right] \right). \tag{SA.53}$$

By Assumptions 1(i, iii) and Rosenthal's inequality,

$$\sum_{i \le n} \mathbb{E}\left[\varepsilon_{i,t}^4\right] \le \frac{K}{(n-1)^4} \sum_{i \le n} \mathbb{E}\left[\left(\sum_{i'=1}^{i-1} v_{i',t} \right)^4 + \left(\sum_{i'=1}^{i-1} u_{i',t} \right)^4 \right] \le \frac{K}{(n-1)^4} \sum_{i \le n} (i-1)^2 \le K n^{-1}. \tag{SA.54}$$

For the conditional variance of $\varepsilon_{i,t}$, we apply its definition and obtain the following upper bound:

$$\sum_{i \le n} \mathbb{E}\left[\varepsilon_{i,t}^2 | \mathcal{F}_{i-1,t}\right] \le \frac{K}{(n-1)^2} \left[\sum_{i \le n} \left(\sum_{i'=1}^{i-1} u_{i',t} \right)^2 + \sum_{i \le n} \left(\sum_{i'=1}^{i-1} v_{i',t} \right)^2 \right]. \tag{SA.55}$$

Since

$$\sum_{i\leq n}\left(\sum_{i'=1}^{i-1}u_{i',t}\right)^2 = \sum_{i=1}^{n-1}(n-i)u_{i,t}^2 + \sum_{i=2}^{n-1}(n-i)\sum_{i'=1}^{i-1}u_{i,t}u_{i',t}$$
$$= \frac{n(n-1)}{2}\sigma_u^2 + \sum_{i=1}^{n-1}(n-i)(u_{i,t}^2-\sigma_u^2) + \sum_{i=2}^{n-1}(n-i)\sum_{i'=1}^{i-1}u_{i,t}u_{i',t},$$

we can use Assumptions 1(i, iii) to get

$$\mathbb{E}\left[\left|\sum_{i\leq n}\left(\sum_{i'=1}^{i-1}u_{i',t}\right)^2\right|^2\right] \leq K\left(n^2(n-1)^2 + \sum_{i=1}^{n-1}(n-i)^2(i-1)\right) \leq Kn^4. \tag{SA.56}$$

Similarly, we can show that

$$\mathbb{E}\left[\left|\sum_{i\leq n}\left(\sum_{i'=1}^{i-1}v_{i',t}\right)^2\right|^2\right] \leq Kn^4,$$

which, along with (SA.55) and (SA.56), shows that

$$\mathbb{E}\left[\left(\sum_{i\leq n}\mathbb{E}\left[\varepsilon_{i,t}^2|\mathcal{F}_{i-1,t}\right]\right)^2\right] \leq K. \tag{SA.57}$$

Collecting the results in (SA.52), (SA.53), (SA.54) and (SA.57) leads to

$$\mathbb{E}\left[\left|(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}\varepsilon_{i,t}\right)^2 - (n-1)^{-1}(\sigma_u^2\sigma_v^2+\sigma_{u,v}^2)\right|^2\right] \leq K(n^2T)^{-1},$$

which together with Markov's inequality proves the first claim of the lemma. (ii) To show the second claim of the lemma, we begin by writing

$$\sum_{t\leq T}\sum_{i\leq n}\sum_{i'\leq n}(c_t-\bar{c})u_{i,t}\varepsilon_{i',t} = \sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})u_{i,t}\varepsilon_{i,t} + \sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(c_t-\bar{c})(u_{i,t}\varepsilon_{i',t}+u_{i',t}\varepsilon_{i,t}). \tag{SA.58}$$

The first term after the equality in the above equation can be decomposed as

$$\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})u_{i,t}\varepsilon_{i,t} = (n-1)^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})\left(u_{i,t}^2\sum_{i'=1}^{i-1}v_{i',t} + u_{i,t}v_{i,t}\sum_{i'=1}^{i-1}u_{i',t}\right). \tag{SA.59}$$

By Assumptions 1(i, ii, iii), we have

$$\mathbb{E}\left[\left|((n-1)nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})u_{i,t}^2\sum_{i'=1}^{i-1}v_{i',t}\right|^2\right]$$
$$\leq K\mathbb{E}\left[\left|((n-1)nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})\sigma_u^2\sum_{i'=1}^{i-1}v_{i',t}\right|^2\right]$$

$$+K\mathbb{E}\left[\left|((n-1)nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})(u_{i,t}^2-\sigma_u^2)\sum_{i'=1}^{i-1}v_{i',t}\right|^2\right]$$
$$\leq K\frac{\sum_{t\leq T}\sum_{i\leq n}(n-i)^2+\sum_{t\leq T}\sum_{i\leq n}(i-1)}{((n-1)nT)^2}\leq K(nT)^{-1},$$

where $\sum_{i\leq n}\sigma_u^2\sum_{i'=1}^{i-1}v_{i',t}=\sigma_u^2\sum_{i\leq n}(n-i)v_{i,t}$ is used in deriving the second inequality. Therefore by Markov's inequality,

$$((n-1)nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})u_{i,t}^2\sum_{i'=1}^{i-1}v_{i',t}=O_p((nT)^{-1/2}). \quad \text{(SA.60)}$$

Similarly, we can show that

$$((n-1)nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})u_{i,t}v_{i,t}\sum_{i'=1}^{i-1}u_{i',t}=O_p((nT)^{-1/2}),$$

which along with (SA.59) and (SA.60) leads to

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})u_{i,t}\varepsilon_{i,t}=O_p((nT)^{-1/2}). \quad \text{(SA.61)}$$

To bound the second term after the equality in (SA.58), we begin by observing that by Assumptions 1(i, ii, iii),

$$\mathbb{E}\left[\left((nT)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}(c_t-\bar{c})u_{i,t}\sum_{i'=1}^{i-1}\varepsilon_{i',t}\right)^2\right]\leq K(nT)^{-2}\sum_{t\leq T}\mathbb{E}\left[\left(\sum_{i=2}^{n}u_{i,t}\sum_{i'=1}^{i-1}\varepsilon_{i',t}\right)^2\right]$$
$$\leq K(nT)^{-2}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}\mathbb{E}\left[\varepsilon_{i',t}^2\right]$$
$$\leq K((n-1)nT)^{-2}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(i'-1)\leq K(nT)^{-1}.$$

Hence by Markov's inequality,

$$(nT)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}(c_t-\bar{c})u_{i,t}\sum_{i'=1}^{i-1}\varepsilon_{i',t}=O_p((nT)^{-1/2}). \quad \text{(SA.62)}$$

Next note that

$$(n-1)\sum_{i=2}^{n}\varepsilon_{i,t}\sum_{i'=1}^{i-1}u_{i',t}=\sum_{i=2}^{n}v_{i,t}\sum_{i_1'=1}^{i-1}\sum_{i_2'=1}^{i-1}u_{i_1',t}u_{i_2',t}+\sum_{i=2}^{n}u_{i,t}\sum_{i_1'=1}^{i-1}\sum_{i_2'=1}^{i-1}v_{i_1',t}u_{i_2',t}. \quad \text{(SA.63)}$$

The first term after the equality in (SA.63) can be further written as

$$\sum_{i=2}^{n}v_{i,t}\sum_{i_1'=1}^{i-1}\sum_{i_2'=1}^{i-1}u_{i_1',t}u_{i_2',t}=\sigma_u^2\sum_{i=2}^{n}(i-1)v_{i,t}+\sum_{i=2}^{n}v_{i,t}\sum_{i'=1}^{i-1}(u_{i',t}^2-\sigma_u^2)+2\sum_{i=2}^{n}v_{i,t}\sum_{i_1'=2}^{i-1}\sum_{i_2'=1}^{i_1'-1}u_{i_1',t}u_{i_2',t}. \quad \text{(SA.64)}$$

By Assumptions 1(i, ii, iii), we obtain the following moment bounds

$$\mathbb{E}\left[\left(((n-1)nT)^{-1}\sigma_u^2\sum_{t\leq T}\sum_{i=2}^{n}(c_t-\bar{c})(i-1)v_{i,t}\right)^2\right]\leq K(nT)^{-1},$$

$$\mathbb{E}\left[\left(((n-1)nT)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}(c_t-\bar{c})v_{i,t}\sum_{i'=1}^{i-1}(u_{i',t}^2-\sigma_u^2)\right)^2\right]\leq K(n^2T)^{-1},$$

$$\mathbb{E}\left[\left(((n-1)nT)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}(c_t-\bar{c})v_{i,t}\sum_{i_1'=2}^{i-1}\sum_{i_2'=1}^{i_1'-1}u_{i_1',t}u_{i_2',t}\right)^2\right]\leq K(nT)^{-1},$$

which together with (SA.64) and Markov's inequality implies that

$$((n-1)nT)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}(c_t-\bar{c})v_{i,t}\sum_{i_1'=1}^{i-1}\sum_{i_2'=1}^{i-1}u_{i_1',t}u_{i_2',t}=O_p((nT)^{-1/2}). \tag{SA.65}$$

By the same arguments, we can show that

$$((n-1)nT)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}(c_t-\bar{c})u_{i,t}\sum_{i_1'=1}^{i-1}\sum_{i_2'=1}^{i-1}v_{i_1',t}u_{i_2',t}=O_p((nT)^{-1/2})$$

which combined with (SA.63) and (SA.65) yields

$$(nT)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}(c_t-\bar{c})\varepsilon_{i,t}\sum_{i'=1}^{i-1}u_{i',t}=O_p((nT)^{-1/2}). \tag{SA.66}$$

Collecting the results in (SA.62) and (SA.66) leads to

$$(nT)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(c_t-\bar{c})(u_{i,t}\varepsilon_{i',t}+u_{i',t}\varepsilon_{i,t})=O_p((nT)^{-1/2}). \tag{SA.67}$$

The desired result in part (ii) of the lemma follows from (SA.58), (SA.61) and (SA.67). ■

Lemma SA.8 *Under Assumption 1, we have*

$$(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}\hat{u}_{i,t}\left(z_{i,t}-\bar{z}_{i,\cdot}\right)\right)^2$$
$$=(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(\gamma(c_t-\bar{c})(u_{i,t}-\bar{u}_{i,\cdot})+\varepsilon_{i,t})-(\hat{\beta}_{iv}-\beta)\gamma^2 n(c_t-\bar{c})^2\right)^2+O_p((nT)^{-1}).$$

Proof. Since $\hat{u}_{i,t}=u_{i,t}-\bar{u}_{i,\cdot}-(\hat{\beta}_{iv}-\beta)(x_{i,t}-\bar{x}_{i,\cdot})$, we can write

$$\sum_{i\leq n}\hat{u}_{i,t}\left(z_{i,t}-\bar{z}_{i,\cdot}\right)=(n-1)^{-1}\sum_{i\leq n}\hat{u}_{i,t}\sum_{i'\neq i}\left(x_{i',t}-\bar{x}_{i',\cdot}\right)$$

$$= (n-1)^{-1} \sum_{i \le n} (u_{i,t} - \bar{u}_{i,\cdot}) \sum_{i' \ne i} \left(x_{i',t} - \bar{x}_{i',\cdot} \right)$$

$$- (n-1)^{-1} (\hat{\beta}_{iv} - \beta) \sum_{i \le n} (x_{i,t} - \bar{x}_{i,\cdot}) \sum_{i' \ne i} \left(x_{i',t} - \bar{x}_{i',\cdot} \right). \tag{SA.68}$$

Using (SA.1), the two terms after the equality in (SA.68) can be further written as

$$(n-1)^{-1} \sum_{i \le n} (u_{i,t} - \bar{u}_{i,\cdot}) \sum_{i' \ne i} \left(x_{i',t} - \bar{x}_{i',\cdot} \right) = \sum_{i \le n} (\gamma (c_t - \bar{c})(u_{i,t} - \bar{u}_{i,\cdot}) + \varepsilon_{i,t}) + I_{1,t} \tag{SA.69}$$

and

$$(n-1)^{-1} (\hat{\beta}_{iv} - \beta) \sum_{i \le n} (x_{i,t} - \bar{x}_{i,\cdot}) \sum_{i' \ne i} \left(x_{i',t} - \bar{x}_{i',\cdot} \right) = (\hat{\beta}_{iv} - \beta) \gamma^2 n (c_t - \bar{c})^2 + (\hat{\beta}_{iv} - \beta) I_{2,t}, \tag{SA.70}$$

respectively, where

$$I_{1,t} \equiv (n-1)^{-1} \left(\sum_{i \le n} \bar{u}_{i,\cdot} \sum_{i' \ne i} \bar{v}_{i',\cdot} - \sum_{i \le n} u_{i,t} \sum_{i' \ne i} \bar{v}_{i',\cdot} - \sum_{i \le n} \bar{u}_{i,\cdot} \sum_{i' \ne i} v_{i',t} \right) \tag{SA.71}$$

and

$$I_{2,t} \equiv 2 \left(\gamma (c_t - \bar{c}) \left(\sum_{i \le n} v_{i,t} - n \bar{v} \right) + (n-1)^{-1} \sum_{i=2}^{n} \sum_{i'=1}^{i-1} (v_{i,t} - \bar{v}_{i,\cdot})(v_{i',t} - \bar{v}_{i',\cdot}) \right). \tag{SA.72}$$

Therefore,

$$\frac{\sum_{i \le n} \hat{u}_{i,t} \sum_{i' \ne i} \left(x_{i',t} - \bar{x}_{i',\cdot} \right)}{n-1} = \sum_{i \le n} (\gamma (c_t - \bar{c})(u_{i,t} - \bar{u}_{i,\cdot}) + \varepsilon_{i,t})$$

$$- (\hat{\beta}_{iv} - \beta) \gamma^2 n (c_t - \bar{c})^2 + I_{1,t} + (\hat{\beta}_{iv} - \beta) I_{2,t}. \tag{SA.73}$$

By Lemma SA.7, we can deduce that

$$(nT)^{-1} \sum_{t \le T} \left(\sum_{i \le n} (\gamma (c_t - \bar{c})(u_{i,t} - \bar{u}_{i,\cdot}) + \varepsilon_{i,t}) \right)^2$$

$$\le 2 (nT)^{-1} \sum_{t \le T} \left(\sum_{i \le n} (\gamma (c_t - \bar{c}) u_{i,t} + \varepsilon_{i,t}) \right)^2 + 2 \gamma^2 (nT)^{-1} \sum_{t \le T} (c_t - \bar{c})^2 \left(\sum_{i \le n} \bar{u}_{i,\cdot} \right)^2$$

$$\le K (nT)^{-1} \sum_{t \le T} (c_t - \bar{c})^2 \left(\left(\sum_{i \le n} \bar{u}_{i,\cdot} \right)^2 + \left(\sum_{i \le n} u_{i,t} \right)^2 \right) + O_p(1). \tag{SA.74}$$

By Assumptions 1(i, ii, iii), we have

$$\mathbb{E} \left[(nT)^{-1} \sum_{t \le T} (c_t - \bar{c})^2 \left(\sum_{i \le n} \bar{u}_{i,\cdot} \right)^2 \right] = (nT)^{-1} \sum_{t \le T} \mathbb{E} \left[(c_t - \bar{c})^2 \right] \sum_{i \le n} \mathbb{E} \left[\bar{u}_{i,\cdot}^2 \right] \le K T^{-1}$$

and

$$\mathbb{E}\left[\left|(nT)^{-1}\sum_{t\leq T}(c_t-\bar{c})^2\left(\sum_{i\leq n}u_{i,t}\right)^2\right|\right]=(nT)^{-1}\sum_{t\leq T}\mathbb{E}\left[(c_t-\bar{c})^2\right]\sum_{i\leq n}\sigma_u^2\leq K,$$

which together with (SA.74) and Markov inequality implies that

$$(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(\gamma(c_t-\bar{c})(u_{i,t}-\bar{u}_{i,\cdot})+\varepsilon_{i,t})\right)^2=O_p(1). \tag{SA.75}$$

By Assumptions 1(ii, iii) and Theorem 1,

$$(nT)^{-1}\sum_{t\leq T}\left((\hat{\beta}_{iv}-\beta)\gamma^2 n(c_t-\bar{c})^2\right)^2=(\hat{\beta}_{iv}-\beta)^2\gamma^4 nT^{-1}\sum_{t\leq T}(c_t-\bar{c})^4=O_p(T^{-1}). \tag{SA.76}$$

Collecting the results in (SA.75) and (SA.76) leads to

$$(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(\gamma(c_t-\bar{c})(u_{i,t}-\bar{u}_{i,\cdot})+\varepsilon_{i,t})-(\hat{\beta}_{iv}-\beta)\gamma^2 n(c_t-\bar{c})^2\right)^2=O_p(1). \tag{SA.77}$$

In view of (SA.73) and (SA.77), the claim of the lemma follows if

$$\sum_{t\leq T}I_{1,t}^2=O_p(1)\qquad\text{and}\qquad(\hat{\beta}_{iv}-\beta)^2\sum_{t\leq T}I_{2,t}^2=O_p(1). \tag{SA.78}$$

To verify the first result in (SA.78), it is sufficient to show that

$$(n-1)^{-2}\sum_{t\leq T}\left(\left(\sum_{i\leq n}\bar{u}_{i,\cdot}\sum_{i'\neq i}\bar{v}_{i',\cdot}\right)^2+\left(\sum_{i\leq n}u_{i,t}\sum_{i'\neq i}\bar{v}_{i',\cdot}\right)^2+\left(\sum_{i\leq n}\bar{u}_{i,\cdot}\sum_{i'\neq i}v_{i',t}\right)^2\right)=O_p(1). \tag{SA.79}$$

By Assumptions 1(i, iii),

$$\begin{aligned}\frac{\mathbb{E}\left[\sum_{t\leq T}\left(\sum_{i\leq n}\bar{u}_{i,\cdot}\sum_{i'\neq i}\bar{v}_{i',\cdot}\right)^2\right]}{(n-1)^2}&=\frac{\sum_{t\leq T}\mathbb{E}\left[\left(\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(\bar{u}_{i,\cdot}\bar{v}_{i',\cdot}+\bar{u}_{i',\cdot}\bar{v}_{i,\cdot})\right)^2\right]}{(n-1)^2}\\&\leq\frac{K\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}\left(\mathbb{E}\left[\bar{u}_{i,\cdot}^2\bar{v}_{i',\cdot}^2\right]+\mathbb{E}\left[\bar{u}_{i',\cdot}^2\bar{v}_{i,\cdot}^2\right]\right)}{(n-1)^2}\leq\frac{Kn}{(n-1)T},\end{aligned}$$

which together with Markov's inequality implies that

$$(n-1)^{-2}\sum_{t\leq T}\left(\sum_{i\leq n}\bar{u}_{i,\cdot}\sum_{i'\neq i}\bar{v}_{i',\cdot}\right)^2=O_p(T^{-1}). \tag{SA.80}$$

Similarly, we can show that

$$(n-1)^{-2}\sum_{t\leq T}\mathbb{E}\left[\left(\sum_{i\leq n}u_{i,t}\sum_{i'\neq i}\bar{v}_{i',\cdot}\right)^2+\left(\sum_{i\leq n}\bar{u}_{i,\cdot}\sum_{i'\neq i}v_{i',t}\right)^2\right]\leq K,$$

and hence

$$(n-1)^{-2}\sum_{t\leq T}\left(\left(\sum_{i\leq n}u_{i,t}\sum_{i'\neq i}\bar{v}_{i',\cdot}\right)^2+\left(\sum_{i\leq n}\bar{u}_{i,\cdot}\sum_{i'\neq i}v_{i',t}\right)^2\right)=O_p(1) \tag{SA.81}$$

by Markov's inequality. The desired result in (SA.79) follows from (SA.80) and (SA.81). In view of Theorem 1, to verify the second result in (SA.78) it is sufficient to show that

$$(nT)^{-1}\sum_{t\leq T}(c_t-\bar{c})^2\left(\sum_{i\leq n}v_{i,t}-n\bar{v}\right)^2=O_p(1), \tag{SA.82}$$

$$(nT)^{-1}\sum_{t\leq T}\left((n-1)^{-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(v_{i,t}-\bar{v}_{i,\cdot})(v_{i',t}-\bar{v}_{i',\cdot})\right)^2=O_p(1). \tag{SA.83}$$

By Assumptions 1(i, iii), we obtain

$$\begin{aligned}\mathbb{E}\left[(nT)^{-1}\sum_{t\leq T}(c_t-\bar{c})^2\left(\sum_{i\leq n}v_{i,t}-n\bar{v}\right)^2\right]&\leq K(nT)^{-1}\sum_{t\leq T}\mathbb{E}\left[\left(\sum_{i\leq n}v_{i,t}-n\bar{v}\right)^2\right]\\&\leq K(nT)^{-1}\sum_{t\leq T}\mathbb{E}\left[\left(\sum_{i\leq n}v_{i,t}\right)^2+n^2\bar{v}^2\right]\\&\leq K(nT)^{-1}\sum_{t\leq T}(n+nT^{-1})\leq K,\end{aligned} \tag{SA.84}$$

which together with Markov's inequality shows (SA.82). Similarly,

$$\begin{aligned}&\mathbb{E}\left[(nT)^{-1}\sum_{t\leq T}\left((n-1)^{-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(v_{i,t}-\bar{v}_{i,\cdot})(v_{i',t}-\bar{v}_{i',\cdot})\right)^2\right]\\&=((n-1)^2nT)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}\mathbb{E}\left[((v_{i,t}-\bar{v}_{i,\cdot})(v_{i',t}-\bar{v}_{i',\cdot}))^2\right]\leq Kn^{-1},\end{aligned} \tag{SA.85}$$

which together with Markov's inequality shows (SA.83). ■

Lemma SA.9 *Under Assumption 1, we have*

$$\begin{aligned}&(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(\gamma(c_t-\bar{c})(u_{i,t}-\bar{u}_{i,\cdot})+\varepsilon_{i,t})-\gamma^2 n(c_t-\bar{c})^2(\hat{\beta}_{iv}-\beta)\right)^2\\&=(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}\gamma(c_t-\bar{c})u_{i,t}-\xi_t\right)^2+\frac{\sigma_u^2\sigma_v^2+\sigma_{u,v}^2}{n-1}+O_p((nT)^{-1/2}).\end{aligned}$$

Proof. In light of the definition of ξ_t in Lemma 2, some elementary algebra leads to:

$$(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(\gamma(c_t-\bar{c})(u_{i,t}-\bar{u}_{i,\cdot})+\varepsilon_{i,t})-\gamma^2 n(c_t-\bar{c})^2(\hat{\beta}_{iv}-\beta)\right)^2$$
$$=(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}\gamma(c_t-\bar{c})u_{i,t}-\xi_t\right)^2$$
$$+2\gamma(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\sum_{i'\leq n}(c_t-\bar{c})u_{i,t}\varepsilon_{i',t}-2\gamma\bar{u}T^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})\varepsilon_{i,t}$$
$$-2\gamma^2(\hat{\beta}_{iv}-\beta)T^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})^2\varepsilon_{i,t}+(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}\varepsilon_{i,t}\right)^2.$$

Therefore, in view of Lemma SA.7, the claim of the lemma follows if

$$\bar{u}T^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})\varepsilon_{i,t}=O_p((nT^2)^{-1/2})\text{ and }(\hat{\beta}_{iv}-\beta)T^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})^2\varepsilon_{i,t}=O_p((nT^2)^{-1/2}).\quad\text{(SA.86)}$$

By Assumption 1(i, ii, iii) and (SA.50),

$$\mathbb{E}\left[\bar{u}^2\right]\leq K(nT)^{-1}\quad\text{and}\quad\mathbb{E}\left[\left(T^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})\varepsilon_{i,t}\right)^2\right]\leq KT^{-1},\qquad\text{(SA.87)}$$

which together with Markov's inequality shows the first result in (SA.86). Similarly,

$$\mathbb{E}\left[\left(T^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})^2\varepsilon_{i,t}\right)^2\right]\leq T^{-2}\sum_{t\leq T}\sum_{i\leq n}\mathbb{E}\left[(c_t-\bar{c})^4\right]\mathbb{E}\left[\varepsilon_{i,t}^2\right]\leq KT^{-1},$$

which, along with Markov's inequality and Theorem 1, shows the second result in (SA.86). ■

Lemma SA.10 *Let $\mu_{i,t}\equiv u_{i,t}\sum_{i'=1}^{i-1}u_{i',t}$. Under Assumption 1, we have*

$$(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(c_t-\bar{c})u_{i,t}\right)^2=\sigma_u^2\hat{\sigma}_c^2+2(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})^2\mu_{i,t}+O_p((nT)^{-1/2})=O_p(1).$$

Proof. By Assumptions 1(i, ii, iii),

$$\mathbb{E}\left[(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(c_t-\bar{c})u_{i,t}\right)^2\right]\leq(nT)^{-1}\sum_{t\leq T}\mathbb{E}[(c_t-\bar{c})^2]\sum_{i\leq n}\mathbb{E}[u_{i,t}^2]\leq K,$$

which together with Markov's inequality shows that

$$(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(c_t-\bar{c})u_{i,t}\right)^2=O_p(1).\qquad\text{(SA.88)}$$

Since

$$\left((c_t-\bar{c})\sum_{i\leq n}u_{i,t}\right)^2=(c_t-\bar{c})^2\left(n\sigma_u^2+\sum_{i\leq n}(u_{i,t}^2-\sigma_u^2)+2\sum_{i=2}^{n}\mu_{i,t}\right),$$

we can write

$$\sum_{t\leq T}\left(\sum_{i\leq n}(c_t-\bar{c})u_{i,t}\right)^2=(nT)\sigma_u^2\hat{\sigma}_c^2+\sum_{t\leq T}(c_t-\bar{c})^2\sum_{i\leq n}(u_{i,t}^2-\sigma_u^2)+2\sum_{t\leq T}(c_t-\bar{c})^2\sum_{i=2}^{n}\mu_{i,t}. \tag{SA.89}$$

By Assumptions 1(i, ii, iii),

$$\mathbb{E}\left[\left((nT)^{-1}\sum_{t\leq T}(c_t-\bar{c})^2\sum_{i\leq n}(u_{i,t}^2-\sigma_u^2)\right)^2\right]\leq(nT)^{-2}\sum_{t\leq T}\mathbb{E}[(c_t-\bar{c})^4]\sum_{i\leq n}\mathbb{E}[u_{i,t}^4]\leq K(nT)^{-1},$$

which together with (SA.88), (SA.89) and Markov's inequality shows the claim of the lemma. ■

Lemma SA.11 *Under Assumption 1, we have*

$$(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(c_t-\bar{c})\bar{u}_{i,\cdot}+\gamma n(c_t-\bar{c})^2(\hat{\beta}_{iv}-\beta)\right)^2=O_p(T^{-1}).$$

Proof. It is evident that the claim of the lemma follows if

$$n(\hat{\beta}_{iv}-\beta)^2T^{-1}\sum_{t\leq T}(c_t-\bar{c})^4=O_p(T^{-1}),\quad\text{and} \tag{SA.90}$$

$$(nT)^{-1}\sum_{t\leq T}(c_t-\bar{c})^2\left(\sum_{i\leq n}\bar{u}_{i,\cdot}\right)^2=O_p(T^{-1}). \tag{SA.91}$$

By Assumption (iii) and Markov's inequality,

$$T^{-1}\sum_{t\leq T}(c_t-\bar{c})^4=O_p(1), \tag{SA.92}$$

which together with Theorem 1 shows (SA.90). By Assumptions (i, ii, iii),

$$\mathbb{E}\left[(nT)^{-1}\sum_{t\leq T}(c_t-\bar{c})^2\left(\sum_{i\leq n}\bar{u}_{i,\cdot}\right)^2\right]=(nT)^{-1}\sum_{t\leq T}\mathbb{E}[(c_t-\bar{c})^2]\sum_{i\leq n}\mathbb{E}[\bar{u}_{i,\cdot}^2]\leq KT^{-1},$$

which along with Markov's inequality shows (SA.91). ■

SB Auxiliary Lemmas for Results in Section 4

Lemma SB.12 *Under Assumptions 1 and 2, we have:*

$$\hat{\lambda} = \gamma \hat{\Sigma}_w^{-1} \hat{\Gamma}_{w,c} + O_p((nT)^{-1/2}) = O_p(1). \tag{SB.93}$$

Proof. Using the expression for $x_{i,t}$ in (2) and the definition of $\hat{\lambda}$, we can write

$$\hat{\lambda} = \gamma \hat{\Sigma}_w^{-1} \hat{\Gamma}_{w,c} + \hat{\Sigma}_w^{-1} (nT)^{-1} \sum_{t \leq T} \sum_{i \leq n} (w_{i,t} - \bar{w}_{i,\cdot}) v_{i,t}. \tag{SB.94}$$

By Assumptions 1(i, iii) and 2(i, iv), we have

$$\mathbb{E}\left[\left| (nT)^{-1} \sum_{t \leq T} \sum_{i \leq n} (w_{i,t} - \bar{w}_{i,\cdot}) v_{i,t} \right|^2 \right] = (nT)^{-2} \sum_{t \leq T} \sum_{i \leq n} \mathbb{E}\left[(w_{i,t} - \bar{w}_{i,\cdot})^2 \right] \mathbb{E}\left[v_{i,t}^2 \right] \leq K(nT)^{-1},$$

which, together with Markov's inequality, implies that

$$(nT)^{-1} \sum_{t \leq T} \sum_{i \leq n} (w_{i,t} - \bar{w}_{i,\cdot}) v_{i,t} = O_p((nT)^{-1/2}). \tag{SB.95}$$

The first equality in (SB.93) follows from (SB.94) and (SB.95). To show that $\hat{\lambda}$ is stochastically bounded, we begin by applying the Cauchy-Schwarz inequality to obtain

$$\left\| \hat{\Gamma}_{w,c} \right\|^2 \leq (nT)^{-1} \sum_{t \leq T} \sum_{i \leq n} \| w_{i,t} \|^2 \times T^{-1} \sum_{t \leq T} (c_t - \bar{c})^2. \tag{SB.96}$$

Since $(nT)^{-1} \sum_{t \leq T} \sum_{i \leq n} \| w_{i,t} \|^2 = O_p(1)$ by Assumption 2(iv) and Markov's inequality, we can use (SB.96) and Assumption 1(iii) to show that

$$\hat{\Gamma}_{w,c} = O_p(1). \tag{SB.97}$$

By Assumptions 2(ii, iv), we have

$$K^{-1} \leq \rho_{\min}(\hat{\Sigma}_w) \leq \rho_{\max}(\hat{\Sigma}_w) \leq K \tag{SB.98}$$

with probability approaching 1, where $\rho_{\min}(\hat{\Sigma}_w)$ and $\rho_{\max}(\hat{\Sigma}_w)$ denote the smallest and the largest eigenvalues of $\hat{\Sigma}_w$, respectively. Combining (SB.97) and (SB.98), we obtain

$$\hat{\Sigma}_w^{-1} \hat{\Gamma}_{w,c} = O_p(1). \tag{SB.99}$$

The second equality in (SB.93) follows from (SB.94), (SB.95) and (SB.99). ■

Lemma SB.13 *Under Assumptions 1 and 2, we have:*

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\hat{x}_{i,t}z_{i,t}=\gamma^2(\hat{\sigma}_c^2-\hat{\Gamma}_{w,c}^\top\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c})+O_p((nT)^{-1/2}).$$

Proof. By the definitions of $\hat{x}_{i,t}$ and $z_{i,t}$, we can write

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\hat{x}_{i,t}z_{i,t}=(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(x_{i,t}-\bar{x}_{i,\cdot})z_{i,t}-\hat{\lambda}^\top\hat{\Gamma}_{w,z}. \tag{SB.100}$$

Given the expression for $x_{i,t}$ in (2) and the expression for $z_{i,t}$ in (3), we can decompose $\hat{\Gamma}_{w,z}$ as

$$\begin{aligned}\hat{\Gamma}_{w,z}&=(n(n-1)T)^{-1}\sum_{t\leq T}\sum_{i\leq n}\sum_{i'\neq i}(w_{i,t}-\bar{w}_{i,\cdot})x_{i',t}\\&=\gamma\hat{\Gamma}_{w,c}+(n(n-1)T)^{-1}\sum_{t\leq T}\sum_{i\leq n}\sum_{i'\neq i}(w_{i,t}-\bar{w}_{i,\cdot})v_{i',t}.\end{aligned} \tag{SB.101}$$

Therefore, by Lemmas SA.2 and SB.12, and using (SB.97), (SB.100) and (SB.101), the claim of the lemma follows if

$$(n(n-1)T)^{-1}\sum_{t\leq T}\sum_{i\leq n}\sum_{i'\neq i}(w_{i,t}-\bar{w}_{i,\cdot})v_{i',t}=O_p((nT)^{-1/2}). \tag{SB.102}$$

To show (SB.102), we first write

$$\sum_{t\leq T}\sum_{i\leq n}\sum_{i'\neq i}(w_{i,t}-\bar{w}_{i,\cdot})v_{i',t}=\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(w_{i,t}v_{i',t}+w_{i',t}v_{i,t})-T\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(\bar{w}_{i\cdot}\bar{v}_{i'\cdot}+\bar{w}_{i'\cdot}\bar{v}_{i\cdot}). \tag{SB.103}$$

By Assumptions 1(i, iii) and 2(i, iv),

$$\mathbb{E}\left[\left|\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}w_{i,t}v_{i',t}\right|^2\right]=\sum_{t\leq T}\mathbb{E}\left[\left|\sum_{i=1}^{n-1}v_{i,t}\sum_{i'=i+1}^{n}w_{i',t}\right|^2\right]=\sigma_v^2\sum_{t\leq T}\sum_{i=1}^{n-1}\mathbb{E}\left[\left|\sum_{i'=i+1}^{n}w_{i',t}\right|^2\right]\leq Kn^3T,$$

which, together with Markov's inequality, implies that

$$(n(n-1)T)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}w_{i,t}v_{i',t}=O_p((nT)^{-1/2}). \tag{SB.104}$$

Similarly, we can show that

$$(n(n-1)T)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}w_{i',t}v_{i,t}=O_p((nT)^{-1/2}). \tag{SB.105}$$

Next, note that by Assumptions 1(i, iii) and 2(i, iv), and Cauchy-Schwarz inequality,

$$\mathbb{E}\left[\left|\sum_{i=2}^{n}\sum_{i'=1}^{i-1}\bar{w}_{i\cdot}\bar{v}_{i'\cdot}\right|^2\right]=\mathbb{E}\left[\left|\sum_{i=1}^{n-1}\bar{v}_{i\cdot}\sum_{i'=i+1}^{n}\bar{w}_{i'\cdot}\right|^2\right]=\sum_{i=1}^{n-1}\mathbb{E}\left[\bar{v}_{i\cdot}^2\right]\mathbb{E}\left[\left|\sum_{i'=i+1}^{n}\bar{w}_{i'\cdot}\right|^2\right]$$

$$\leq KT^{-1}\sum_{i=1}^{n-1}(n-i)\sum_{i'=i+1}^{n}\mathbb{E}\left[\bar{w}_{i'\cdot}^2\right]\leq Kn^3T^{-1},$$

which, together with Markov's inequality, implies that

$$(n(n-1))^{-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}\bar{w}_{i\cdot}\bar{v}_{i'\cdot}=O_p((nT)^{-1/2}). \tag{SB.106}$$

Similarly, we can show that

$$(n(n-1))^{-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}\bar{w}_{i'\cdot}\bar{v}_{i\cdot}=O_p((nT)^{-1/2}),$$

which, along with (SB.103)-(SB.106), proves (SB.102). ■

Lemma SB.14 *Under Assumptions 1 and 2, we have:*

$$\hat{\beta}_{e,iv}-\beta=\frac{(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(\gamma u_{i,t}(c_t-\bar{c}-\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-1}(w_{i,t}-\bar{w}_{i,\cdot}))+\varepsilon_{i,t})+O_p((nT)^{-1})}{\gamma^2(\hat{\sigma}_c^2-\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c})+O_p((nT)^{-1/2})}.$$

Proof. Using the expression for $y_{i,t}$ in (14) and the definition of $\hat{\pi}$, we can write

$$\hat{\pi}=\hat{\lambda}\beta+\theta+\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,u}, \tag{SB.107}$$

where $\hat{\Gamma}_{w,u}\equiv(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(w_{i,t}-\bar{w}_{i,\cdot})u_{i,t}$. Therefore,

$$\begin{aligned}\hat{y}_{i,t}&=(x_{i,t}-\bar{x}_{i,\cdot})\beta+(u_{i,t}-\bar{u}_{i,\cdot})-(w_{i,t}-\bar{w}_{i,\cdot})^{\top}(\hat{\pi}-\theta)\\&=\hat{x}_{i,t}\beta+(u_{i,t}-\bar{u}_{i,\cdot})-(w_{i,t}-\bar{w}_{i,\cdot})^{\top}\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,u}.\end{aligned} \tag{SB.108}$$

Substituting the expression for $\hat{y}_{i,t}$ in (SB.108) into the definition of $\hat{\beta}_{e,iv}$, we obtain

$$\hat{\beta}_{e,iv}-\beta=\frac{(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(u_{i,t}-\bar{u}_{i,\cdot})z_{i,t}-\hat{\Gamma}_{w,z}^{\top}\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,u}}{(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\hat{x}_{i,t}z_{i,t}}.$$

Therefore, by Assumptions 1(iv) and 2(ii, iii, v), and Lemmas SA.3 and SB.13,

$$\hat{\beta}_{e,iv}-\beta=\frac{(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(\gamma u_{i,t}(c_t-\bar{c})+\varepsilon_{i,t})-\hat{\Gamma}_{w,z}^{\top}\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,u}+O_p((nT)^{-1})}{\gamma^2(\hat{\sigma}_c^2-\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c})+O_p((nT)^{-1/2})}. \tag{SB.109}$$

By (SB.101) and (SB.102),

$$\hat{\Gamma}_{w,z}=\gamma\hat{\Gamma}_{w,c}+O_p((nT)^{-1/2}). \tag{SB.110}$$

Applying similar arguments for showing (SB.95) yields

$$\hat{\Gamma}_{w,u}=(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}w_{i,t}(u_{i,t}-\bar{u}_{i,\cdot})=O_p((nT)^{-1/2}). \tag{SB.111}$$

Combining the results from (SB.97), (SB.98), (SB.110) and (SB.111), we have

$$\hat{\Gamma}_{w,z}^{\top}\hat{\Sigma}_{w}^{-1}\hat{\Gamma}_{w,u} = \gamma\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_{w}^{-1}\hat{\Gamma}_{w,u} + O_p((nT)^{-1}),$$

which together with (SB.109) proves the claim of the lemma. ■

Lemma SB.15 *Under Assumptions 1 and 2, we have:*

$$(nT)^{-1/2}\sum_{t\leq T}\sum_{i\leq n}(\gamma(c_t - \bar{c} - \hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_{w}^{-1}(w_{i,t} - \bar{w}_{i,\cdot}))u_{i,t} + \varepsilon_{i,t}) \rightarrow_d \tilde{\omega}_{e,\infty}Z \qquad (\mathcal{F}_{e,0}\text{-stably}),$$

where $\tilde{\omega}_{e,\infty}^2 \equiv \gamma^2\sigma_u^2\sigma_{e,c}^2 + (n_\infty - 1)^{-1}(\sigma_u^2\sigma_v^2 + \sigma_{u,v}^2)$ *is independent of* $Z \sim N(0,1)$.

Proof. For any $k = 1, \ldots, nT$, we define $t_k = \lceil k/n \rceil$ and $i_k = k - n(t_k - 1)$. Let $\mathcal{F}_{e,0,m}$ denote the sigma-field generated by $\{\{c_t\}_{t\leq T_m}, \{w_{i,t}\}_{i\leq n_m, t\leq T_m}\}$. For $k = 1, \ldots, nT$, let $\mathcal{F}_{e,k,m}$ denote the sigma-field generated by $\{\{c_t\}_{t\leq T_m}, \{w_{i,t}\}_{i\leq n_m, t\leq T_m}, \{u_{i_l,t_l}\}_{l\leq k}, \{v_{i_l,t_l}\}_{l\leq k}\}$. Using such notation, we can write

$$\begin{aligned}&(nT)^{-1/2}\sum_{t\leq T}\sum_{i\leq n}(\gamma(c_t - \bar{c} - \hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_{w}^{-1}(w_{i,t} - \bar{w}_{i,\cdot}))u_{i,t} + \varepsilon_{i,t}) \\ &= \sum_{k=1}^{nT}\overbrace{\frac{\gamma u_{i_k,t_k}(c_{t_k} - \bar{c} - \hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_{w}^{-1}(w_{i_k,t_k} - \bar{w}_{i_k,\cdot})) + \varepsilon_{i_k,t_k}}{(nT)^{1/2}}}^{\equiv\tilde{\eta}_{e,k}}. \qquad \text{(SB.112)}\end{aligned}$$

By similar arguments as those used to derive (SA.24) in the proof of Lemma SA.4, we can show that $\{\tilde{\eta}_{e,k}\}_{k\leq nT}$ is an MDA adapted to $\mathcal{F}_{e,k,m}$. We next show that

$$\sum_{k=1}^{nT}\mathbb{E}\left[\tilde{\eta}_{e,k}^2\middle|\mathcal{F}_{e,k-1,m}\right] \rightarrow_p \tilde{\omega}_{e,\infty}^2, \qquad \text{(SB.113)}$$

and for any $\varepsilon > 0$,

$$\sum_{k=1}^{nT}\mathbb{E}[\tilde{\eta}_{e,k}^2 I\{|\tilde{\eta}_{e,k}| > \varepsilon\}|\mathcal{F}_{e,k-1,m}] \rightarrow_p 0. \qquad \text{(SB.114)}$$

Under conditions (SB.113) and (SB.114), the claim of the lemma follows by similar arguments used in proving Lemma SA.4. By the definitions of $\tilde{\eta}_{e,k}$ and $\tilde{\eta}_k$, it follows that

$$\tilde{\eta}_{e,k} = \tilde{\eta}_k - (nT)^{-1/2}\gamma\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_{w}^{-1}(w_{i_k,t_k} - \bar{w}_{i_k,\cdot})u_{i_k,t_k}. \qquad \text{(SB.115)}$$

Therefore,

$$\sum_{k=1}^{nT}\mathbb{E}\left[\tilde{\eta}_{e,k}^2\middle|\mathcal{F}_{e,k-1,m}\right] = \sum_{k=1}^{nT}\mathbb{E}\left[\tilde{\eta}_{k}^2\middle|\mathcal{F}_{e,k-1,m}\right] + \gamma^2\sigma_u^2\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_{w}^{-1}\hat{\Gamma}_{w,c}$$

$$- 2\gamma\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-1}(nT)^{-1/2}\sum_{k=1}^{nT}(w_{i_k,t_k} - \bar{w}_{i_k,\cdot})\mathbb{E}\left[\tilde{\eta}_k u_{i_k,t_k}\middle|\mathcal{F}_{e,k-1,m}\right]. \tag{SB.116}$$

By similar arguments as those used to derive (SA.35) in the proof of Lemma SA.4, Assumptions 1(i, ii, iii) and 2(i), and the definition of $\mathcal{F}_{e,k-1,m}$, we can show that

$$\sum_{k=1}^{nT}\mathbb{E}\left[\tilde{\eta}_k^2\middle|\mathcal{F}_{e,k-1,m}\right] = \omega_{nT}^2 + O_p((nT)^{-1/2}). \tag{SB.117}$$

Since $\tilde{\eta}_k u_{i_k,t_k} = (nT)^{-1/2}(\gamma u_{i_k,t_k}^2(c_{t_k} - \bar{c}) + u_{i_k,t_k}\varepsilon_{i_k,t_k})$, by Assumptions 1(i, ii, iii) and 2(i), we have

$$\begin{aligned}\mathbb{E}\left[\tilde{\eta}_k u_{i_k,t_k}\middle|\mathcal{F}_{e,k-1,m}\right] &= (nT)^{-1/2}\gamma\sigma_u^2(c_{t_k} - \bar{c}) + (nT)^{-1/2}\mathbb{E}\left[u_{i_k,t_k}\varepsilon_{i_k,t_k}\middle|\mathcal{F}_{e,k-1,m}\right] \\ &= (nT)^{-1/2}\gamma\sigma_u^2(c_{t_k} - \bar{c}) + (n(n-1)^2T)^{-1/2}\sum_{i'=1}^{i_k-1}(v_{i',t_k}\sigma_u^2 + u_{i',t_k}\sigma_{u,v}),\end{aligned}$$

which implies

$$\begin{aligned}&(nT)^{-1/2}\sum_{k=1}^{nT}(w_{i_k,t_k} - \bar{w}_{i_k,\cdot})\mathbb{E}\left[\tilde{\eta}_k u_{i_k,t_k}\middle|\mathcal{F}_{e,k-1,m}\right] \\ &= \gamma\sigma_u^2\hat{\Gamma}_{w,c} + (n(n-1)T)^{-1}\sum_{k=1}^{nT}\sum_{i'=1}^{i_k-1}(w_{i_k,t_k} - \bar{w}_{i_k,\cdot})(v_{i',t_k}\sigma_u^2 + u_{i',t_k}\sigma_{u,v}) \\ &= \gamma\sigma_u^2\hat{\Gamma}_{w,c} + (n(n-1)T)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(w_{i,t} - \bar{w}_{i,\cdot})(\sigma_u^2 v_{i',t} + \sigma_{u,v}u_{i',t}).\end{aligned} \tag{SB.118}$$

By similar arguments to those used in proving (SB.102), we can show that

$$(n(n-1)T)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(w_{i,t} - \bar{w}_{i,\cdot})(\sigma_u^2 v_{i',t} + \sigma_{u,v}u_{i',t}) = O_p((nT)^{-1/2}). \tag{SB.119}$$

Combining the results from (SB.116), (SB.117), (SB.118) and (SB.119), we obtain

$$\sum_{k=1}^{nT}\mathbb{E}\left[\tilde{\eta}_{e,k}^2\middle|\mathcal{F}_{e,k-1,m}\right] = \gamma^2\sigma_u^2(\hat{\sigma}_c^2 - \hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c}) + (n-1)^{-1}(\sigma_u^2\sigma_v^2 + \sigma_{u,v}^2) + O_p((nT)^{-1/2}). \tag{SB.120}$$

The desired result in (SB.113) follows from (SB.120) and Assumptions 1(iv) and 2(ii, iii). We now verify (SB.114). By Assumption 1(iii), (SB.115), and the Cauchy-Schwarz inequality

$$\begin{aligned}\sum_{k=1}^{nT}\mathbb{E}[\tilde{\eta}_{e,k}^2 I\{|\tilde{\eta}_{e,k}| > \varepsilon\}|\mathcal{F}_{e,k-1,m}] &\leq \varepsilon^{-2}\sum_{k=1}^{nT}\mathbb{E}[\tilde{\eta}_{e,k}^4|\mathcal{F}_{e,k-1,m}] \\ &\leq K\varepsilon^{-2}\left(\sum_{k=1}^{nT}\mathbb{E}[\tilde{\eta}_k^4|\mathcal{F}_{e,k-1,m}] + (nT)^{-2}\sum_{k=1}^{nT}(\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-1}(w_{i_k,t_k} - \bar{w}_{i_k,\cdot}))^4\right)\end{aligned}$$

$$\leq K\varepsilon^{-2}\left(\sum_{k=1}^{nT}\mathbb{E}[\tilde{\eta}_k^4|\mathcal{F}\mathcal{F}_{e,k-1,m}] + ||\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-2}\hat{\Gamma}_{w,c}||^2(nT)^{-2}\sum_{k=1}^{nT}||w_{i_k,t_k}-\bar{w}_{i_k,\cdot}||^4\right). \tag{SB.121}$$

By similar arguments as those used to derive (SA.36), (SA.37) and (SA.41) in the proof of Lemma SA.4, we have

$$\sum_{k=1}^{nT}\mathbb{E}[\tilde{\eta}_k^4|\mathcal{F}_{e,k-1,m}] = O_p((nT)^{-1}). \tag{SB.122}$$

By Assumption 2(iv) and Markov's inequality,

$$(nT)^{-2}\sum_{k=1}^{nT}||w_{i_k,t_k}-\bar{w}_{i_k,\cdot}||^4 = O_p((nT)^{-1}),$$

which, together with (SB.97) and (SB.98), implies that

$$||\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-2}\hat{\Gamma}_{w,c}||^2(nT)^{-2}\sum_{k=1}^{nT}||w_{i_k,t_k}-\bar{w}_{i_k,\cdot}||^4 = O_p((nT)^{-1}). \tag{SB.123}$$

Combining the results from (SB.121), (SB.122) and (SB.123), we conclude that (SB.114) holds. ■

Lemma SB.16 *Under Assumptions 1 and 2, we have:*

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\hat{u}_{e,i,t}^2 = \sigma_u^2(1-T^{-1}) + O_p((nT)^{-1/2}).$$

Proof. By the definition of $\hat{u}_{e,i,t}$ and the expression for $\hat{y}_{i,t}$ in (SB.108), we can write

$$\hat{u}_{e,i,t} = (u_{i,t}-\bar{u}_{i,\cdot}) - \hat{x}_{i,t}(\hat{\beta}_{e,iv}-\beta) - (w_{i,t}-\bar{w}_{i,\cdot})^{\top}\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,u} = \tilde{u}_{e,i,t} - (w_{i,t}-\bar{w}_{i,\cdot})^{\top}\hat{\phi}_w, \tag{SB.124}$$

where $\hat{\phi}_w \equiv \hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,u} - (\hat{\beta}_{e,iv}-\beta)\hat{\lambda}$ and $\tilde{u}_{e,i,t} \equiv (u_{i,t}-\bar{u}_{i,\cdot}) - (x_{i,t}-\bar{x}_{i,\cdot})(\hat{\beta}_{e,iv}-\beta)$. This implies

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\hat{u}_{e,i,t}^2 = (nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\tilde{u}_{e,i,t}^2 + \hat{\phi}_w^{\top}\hat{\Sigma}_w\hat{\phi}_w - 2(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\tilde{u}_{e,i,t}w_{i,t}^{\top}\hat{\phi}_w. \tag{SB.125}$$

Applying similar arguments to those used in the proof of Lemma SA.5 (replacing $\hat{\beta}_{iv}$ with $\hat{\beta}_{e,iv}$ and Theorem 1 with Theorem 3), we can show that

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\tilde{u}_{e,i,t}^2 = \sigma_u^2(1-T^{-1}) + O_p((nT)^{-1/2}). \tag{SB.126}$$

By Theorem 3, Lemma SB.12, (SB.98) and (SB.111), we can deduce that

$$||\hat{\phi}_w|| = O_p((nT)^{-1/2}) \quad \text{and} \quad \hat{\phi}_w^{\top}\hat{\Sigma}_w\hat{\phi}_w = O_p((nT)^{-1}). \tag{SB.127}$$

Similarly, we can show that

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\tilde{u}_{e,i,t}w_{i,t}^{\top}\hat{\phi}_w=\hat{\Gamma}_{w,u}^{\top}\hat{\phi}_w-(\hat{\beta}_{e,iv}-\beta)\hat{\Gamma}_{w,x}^{\top}\hat{\phi}_w=O_p((nT)^{-1}). \tag{SB.128}$$

The claim of the lemma now follows from (SB.125) to (SB.128). ■

Lemma SB.17 *Under Assumptions 1 and 2, we have:*

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\hat{z}_{i,t}^2=\gamma^2\hat{\sigma}_{e,c}^2+\sigma_v^2(n-1)^{-1}+O_p((nT)^{-1/2}) \tag{SB.129}$$

where $\hat{\sigma}_{e,c}^2\equiv\hat{\sigma}_c^2-\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c}$.

Proof. By the definition of $\hat{z}_{i,t}$, we begin by writing

$$\begin{aligned}&(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\hat{z}_{i,t}^2\\&=(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\left(z_{i,t}-\bar{z}_{i,\cdot}-(w_{i,t}-\bar{w}_{i,\cdot})^{\top}\hat{\varphi}\right)^2\\&=(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(z_{i,t}-\bar{z}_{i,\cdot})^2+\hat{\varphi}^{\top}\hat{\Sigma}_w\hat{\varphi}-2\hat{\Gamma}_{w,z}^{\top}\hat{\varphi}.\end{aligned} \tag{SB.130}$$

By the definition of $\hat{\varphi}$, (SB.98), (SB.99) and (SB.110), we have

$$\hat{\varphi}\equiv\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,z}=\gamma\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c}+O_p((nT)^{-1/2})=O_p(1), \tag{SB.131}$$

which together with (SB.97) and (SB.98) implies that

$$\hat{\varphi}^{\top}\hat{\Sigma}_w\hat{\varphi}=\gamma^2\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c}+O_p((nT)^{-1/2}). \tag{SB.132}$$

Similarly, by (SB.97), (SB.98), (SB.110) and (SB.131),

$$\hat{\Gamma}_{w,z}^{\top}\hat{\varphi}=\gamma^2\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c}+O_p((nT)^{-1/2}). \tag{SB.133}$$

Combining the results from Lemma SA.6, (SB.130), (SB.132) and (SB.133), we establish the claim of the lemma. ■

Lemma SB.18 *Under Assumptions 1 and 2, we have:*

$$(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(u_{i,t}-\bar{u}_{i,\cdot})(w_{i,t}-\bar{w}_{i,\cdot})^{\top}\hat{\varphi}\right)^2=(1-T^{-1})\sigma_u^2\gamma^2\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c}+O_p(T^{-1/2}). \tag{SB.134}$$

Proof. The term on the left-hand side of (SB.134) can be expressed as:

$$(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(u_{i,t}-\bar{u}_{i,\cdot})(w_{i,t}-\bar{w}_{i,\cdot})^{\top}\hat{\varphi}\right)^2$$
$$=(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(u_{i,t}-\bar{u}_{i,\cdot})^2((w_{i,t}-\bar{w}_{i,\cdot})^{\top}\hat{\varphi})^2$$
$$+2(nT)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(u_{i,t}-\bar{u}_{i,\cdot})(u_{i',t}-\bar{u}_{i',\cdot})(w_{i,t}-\bar{w}_{i,t})^{\top}\hat{\varphi}(w_{i',t}-\bar{w}_{i',\cdot})^{\top}\hat{\varphi}. \quad \text{(SB.135)}$$

We first analyze the double summation term on the right-hand side of (SB.135). Let $w_{i,t}^k$ and $\bar{w}_{i,t}^k$ denote the kth entries of $w_{i,t}$ and $\bar{w}_{i,t}$, respectively. By Assumptions 1(i, iii) and 2(i, iv), we can show that for any $k_1, k_2 \leq d_w$:

$$\mathbb{E}\left[\left|(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(u_{i,t}^2-\sigma_u^2)(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})\right|^2\right]$$
$$\leq (nT)^{-2}\sum_{t\leq T}\sum_{i\leq n}\mathbb{E}\left[u_{i,t}^4\right]\mathbb{E}[(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})^2(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})^2]\leq K(nT)^{-1}. \quad \text{(SB.136)}$$

Using Assumptions 1(i, iii), we have

$$\mathbb{E}\left[(\bar{u}_{i,\cdot}^2-\sigma_u^2T^{-1})^2\right]=\mathbb{E}\left[\left(T^{-2}\sum_{t\leq T}(u_{i,t}^2-\sigma_u^2)+2T^{-2}\sum_{t=2}^{T}u_{i,t}\sum_{t'=1}^{t-1}u_{i,t'}\right)^2\right]$$
$$=T^{-4}\sum_{t\leq T}\mathbb{E}\left[(u_{i,t}^2-\sigma_u^2)^2\right]+4T^{-4}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\mathbb{E}[u_{i,t}^2u_{i,t'}^2]$$
$$\leq T^{-4}\sum_{t\leq T}\mathbb{E}\left[u_{i,t}^4\right]+4T^{-4}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sigma_u^4\leq KT^{-2}.$$

Combining this result with Assumptions 2(i, iv) leads to:

$$\mathbb{E}\left[\left|(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(\bar{u}_{i,\cdot}^2-\sigma_u^2T^{-1})(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})\right|^2\right]$$
$$=(nT)^{-2}\sum_{i\leq n}\mathbb{E}\left[(\bar{u}_{i,\cdot}^2-\sigma_u^2T^{-1})^2\right]\mathbb{E}\left[\left|\sum_{t\leq T}(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})\right|^2\right]$$
$$\leq K(nT)^{-2}\sum_{i\leq n}\mathbb{E}\left[\left|T^{-1}\sum_{t\leq T}(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})\right|^2\right]$$
$$\leq K(n^2T^3)^{-1}\sum_{i\leq n}\sum_{t\leq T}\mathbb{E}\left[\left|(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})\right|^2\right]\leq K(nT^2)^{-1}. \quad \text{(SB.137)}$$

Next, observe that:

$$\begin{aligned}
&\sum_{t\leq T}\left(u_{i,t}\sum_{t'\leq T}u_{i,t'}-\sigma_u^2\right)(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})\\
&=\sum_{t\leq T}(u_{i,t}^2-\sigma_u^2)(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})\\
&+\sum_{t=2}^{T}\sum_{t'=1}^{t-1}u_{i,t}u_{i,t'}\left((w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})+(w_{i,t'}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t'}^{k_2}-\bar{w}_{i,\cdot}^{k_2})\right). \qquad \text{(SB.138)}
\end{aligned}$$

By Assumptions 1(i, iii) and 2(i, iv), we have

$$\begin{aligned}
&\mathbb{E}\left[\left|\sum_{t\leq T}(u_{i,t}^2-\sigma_u^2)(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})\right|^2\right]\\
&=\sum_{t\leq T}\mathbb{E}\left[(u_{i,t}^2-\sigma_u^2)^2\right]\mathbb{E}[(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})^2(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})^2]\leq KT, \qquad \text{(SB.139)}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left[\left|\sum_{t=2}^{T}\sum_{t'=1}^{t-1}u_{i,t}u_{i,t'}\left((w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})+(w_{i,t'}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t'}^{k_2}-\bar{w}_{i,\cdot}^{k_2})\right)\right|^2\right]\\
&=\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\mathbb{E}[u_{i,t}^2u_{i,t'}^2]\mathbb{E}[|(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})+(w_{i,t'}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t'}^{k_2}-\bar{w}_{i,\cdot}^{k_2})|^2]\\
&\leq K\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sigma_u^4\leq KT^2. \qquad \text{(SB.140)}
\end{aligned}$$

Collecting the results from (SB.138), (SB.139) and (SB.140), we conclude

$$\mathbb{E}\left[\left|\sum_{t\leq T}\left(u_{i,t}\sum_{t'\leq T}u_{i,t'}-\sigma_u^2\right)(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})\right|^2\right]\leq KT^2. \qquad \text{(SB.141)}$$

By Assumptions 1(i) and 2(i) and using (SB.141), we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left|(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(u_{i,t}\bar{u}_{i,\cdot}-\sigma_u^2T^{-1})(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})\right|^2\right]\\
&=(nT^2)^{-2}\sum_{i\leq n}\mathbb{E}\left[\left|\sum_{t\leq T}\left(u_{i,t}\sum_{t'\leq T}u_{i,t'}-\sigma_u^2\right)(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})\right|^2\right]\leq K(nT^2)^{-1},
\end{aligned}$$

which, along with (SB.136), (SB.137) and Markov's inequality, implies that for any $k_1,k_2\leq d_w$:

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}((u_{i,t}-\bar{u}_{i,\cdot})^2-(1-T^{-1})\sigma_u^2)(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})=O_p((nT)^{-1/2}). \qquad \text{(SB.142)}$$

Therefore, combining this with (SB.131) and (SB.132), it follows that

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(u_{i,t}-\bar{u}_{i,\cdot})^2((w_{i,t}-\bar{w}_{i,\cdot})^\top\hat{\varphi})^2 = (1-T^{-1})\sigma_u^2\hat{\varphi}^\top\hat{\Sigma}_w\hat{\varphi}+O_p((nT)^{-1/2})$$
$$= (1-T^{-1})\sigma_u^2\gamma^2\hat{\Gamma}_{w,c}^\top\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c}+O_p((nT)^{-1/2}). \quad \text{(SB.143)}$$

Next, we consider the triple summation term on the right-hand side of (SB.135). Some elementary algebra yields:

$$\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(u_{i,t}-\bar{u}_{i,\cdot})(u_{i',t}-\bar{u}_{i',\cdot})(w_{i,t}-\bar{w}_{i,t})^\top\hat{\varphi}(w_{i',t}-\bar{w}_{i',\cdot})^\top\hat{\varphi}$$
$$=\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(u_{i,t}u_{i',t}-u_{i,t}\bar{u}_{i',\cdot}-\bar{u}_{i,\cdot}u_{i',t}+\bar{u}_{i,\cdot}\bar{u}_{i',\cdot})(w_{i,t}-\bar{w}_{i,t})^\top\hat{\varphi}(w_{i',t}-\bar{w}_{i',\cdot})^\top\hat{\varphi}. \quad \text{(SB.144)}$$

By Assumptions 1(i, iii) and 2(i, iv), we have

$$\mathbb{E}\left[\left|\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}u_{i,t}u_{i',t}(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i',t}^{k_2}-\bar{w}_{i',\cdot}^{k_2})\right|^2\right]$$
$$=\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}\mathbb{E}\left[u_{i,t}^2u_{i',t}^2(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})^2(w_{i',t}^{k_2}-\bar{w}_{i',\cdot}^{k_2})^2\right]\leq Kn^2T,$$

and

$$\mathbb{E}\left[\left|\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}u_{i,t}\bar{u}_{i',\cdot}(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i',t}^{k_2}-\bar{w}_{i',\cdot}^{k_2})\right|\right]$$
$$\leq\sum_{t\leq T}\mathbb{E}\left[\left|\sum_{i=2}^{n}\sum_{i'=1}^{i-1}u_{i,t}\bar{u}_{i',\cdot}(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i',t}^{k_2}-\bar{w}_{i',\cdot}^{k_2})\right|\right]$$
$$\leq\sum_{t\leq T}\left(\mathbb{E}\left[\left|\sum_{i=2}^{n}\sum_{i'=1}^{i-1}u_{i,t}\bar{u}_{i',\cdot}(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i',t}^{k_2}-\bar{w}_{i',\cdot}^{k_2})\right|^2\right]\right)^{1/2}$$
$$=\sum_{t\leq T}\left(\sum_{i=2}^{n}\sum_{i'=1}^{i-1}\mathbb{E}\left[u_{i,t}^2\bar{u}_{i',\cdot}^2(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})^2(w_{i',t}^{k_2}-\bar{w}_{i',\cdot}^{k_2})^2\right]\right)^{1/2}\leq KnT^{1/2}.$$

Thus, by Markov's inequality:

$$(nT)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(u_{i,t}u_{i',t}-u_{i,t}\bar{u}_{i',\cdot})(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i',t}^{k_2}-\bar{w}_{i',\cdot}^{k_2})=O_p(T^{-1/2}). \quad \text{(SB.145)}$$

Similarly, we can show that

$$(nT)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(\bar{u}_{i,\cdot}u_{i',t}-\bar{u}_{i,\cdot}\bar{u}_{i',\cdot})(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i',t}^{k_2}-\bar{w}_{i',\cdot}^{k_2})=O_p(T^{-1/2}),$$

which, combined with (SB.131), (SB.144) and (SB.145) implies that

$$(nT)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(u_{i,t}-\bar{u}_{i,\cdot})(u_{i',t}-\bar{u}_{i',\cdot})(w_{i,t}-\bar{w}_{i,t})^{\top}\hat{\varphi}(w_{i',t}-\bar{w}_{i',\cdot})^{\top}\hat{\varphi}=O_p(T^{-1/2}). \tag{SB.146}$$

The claim of the lemma follows from (SB.135), (SB.143) and (SB.146). ■

Lemma SB.19 *Under Assumptions 1 and 2, we have:*

$$\begin{aligned}(nT)^{-1}&\sum_{t\leq T}\left(\sum_{i\leq n}(c_t-\bar{c})(u_{i,t}-\bar{u}_{i,\cdot})\right)\left(\sum_{i\leq n}(u_{i,t}-\bar{u}_{i,\cdot})(w_{i,t}-\bar{w}_{i,\cdot})^{\top}\hat{\varphi}\right)\\&=(1-T^{-1})\sigma_u^2\gamma\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c}+O_p(T^{-1/2}).\end{aligned} \tag{SB.147}$$

Proof. Some elementary algebra yields:

$$\begin{aligned}(nT)^{-1}&\sum_{t\leq T}\left(\sum_{i\leq n}(c_t-\bar{c})(u_{i,t}-\bar{u}_{i,\cdot})\right)\left(\sum_{i\leq n}(u_{i,t}-\bar{u}_{i,\cdot})(w_{i,t}-\bar{w}_{i,\cdot})\right)\\&=(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})(u_{i,t}-\bar{u}_{i,\cdot})^2(w_{i,t}-\bar{w}_{i,\cdot})\\&+(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\sum_{i'\neq i}(c_t-\bar{c})(u_{i,t}-\bar{u}_{i,\cdot})(u_{i',t}-\bar{u}_{i',\cdot})(w_{i',t}-\bar{w}_{i',\cdot}).\end{aligned} \tag{SB.148}$$

By Assumptions 1(i, ii, iii) and 2(i, iv), we can use similar arguments to those for proving (SB.142) to show that

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}((u_{i,t}-\bar{u}_{i,\cdot})^2-(1-T^{-1})\sigma_u^2)(c_t-\bar{c})(w_{i,t}-\bar{w}_{i,\cdot})=O_p((nT)^{-1/2}).$$

Combining this with (SB.97), (SB.131) and (SB.132) leads to

$$\begin{aligned}(nT)^{-1}&\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})(u_{i,t}-\bar{u}_{i,\cdot})^2(w_{i,t}-\bar{w}_{i,\cdot})^{\top}\hat{\varphi}\\&=(1-T^{-1})\sigma_u^2\hat{\Gamma}_{w,c}^{\top}\hat{\varphi}+O_p((nT)^{-1/2})=(1-T^{-1})\sigma_u^2\gamma\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c}+O_p((nT)^{-1/2}).\end{aligned} \tag{SB.149}$$

The triple summation on the right-hand side of (SB.148) can be written as

$$\begin{aligned}(nT)^{-1}&\sum_{t\leq T}\sum_{i\leq n}\sum_{i'\neq i}(c_t-\bar{c})(u_{i,t}-\bar{u}_{i,\cdot})(u_{i',t}-\bar{u}_{i',\cdot})(w_{i',t}-\bar{w}_{i',\cdot})\\&=(nT)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(c_t-\bar{c})(u_{i,t}-\bar{u}_{i,\cdot})(u_{i',t}-\bar{u}_{i',\cdot})(w_{i',t}-\bar{w}_{i',\cdot})\end{aligned}$$

$$+ (nT)^{-1} \sum_{t \le T} \sum_{i=2}^{n} \sum_{i'=1}^{i-1} (c_t - \bar{c})(u_{i',t} - \bar{u}_{i',\cdot})(u_{i,t} - \bar{u}_{i,\cdot})(w_{i,t} - \bar{w}_{i,\cdot}). \tag{SB.150}$$

Using similar arguments to those used for proving (SB.146), we can show that

$$(nT)^{-1} \sum_{t \le T} \sum_{i=2}^{n} \sum_{i'=1}^{i-1} (c_t - \bar{c})(u_{i,t} - \bar{u}_{i,\cdot})(u_{i',t} - \bar{u}_{i',\cdot})(w_{i',t} - \bar{w}_{i',\cdot})^\top \hat{\varphi} = O_p(T^{-1/2}),$$

and

$$(nT)^{-1} \sum_{t \le T} \sum_{i=2}^{n} \sum_{i'=1}^{i-1} (c_t - \bar{c})(u_{i',t} - \bar{u}_{i',\cdot})(u_{i,t} - \bar{u}_{i,\cdot})(w_{i,t} - \bar{w}_{i,\cdot})^\top \hat{\varphi} = O_p(T^{-1/2}),$$

which, together with (SB.131) and (SB.150), implies that

$$(nT)^{-1} \sum_{t \le T} \sum_{i \le n} \sum_{i' \le n} (c_t - \bar{c})(u_{i,t} - \bar{u}_{i,\cdot})(u_{i',t} - \bar{u}_{i',\cdot})(w_{i',t} - \bar{w}_{i',\cdot})^\top \hat{\varphi} = O_p(T^{-1/2}). \tag{SB.151}$$

The claim of the lemma follows from (SB.148), (SB.149) and (SB.151) shows the claim of the lemma. ■

Lemma SB.20 *Under Assumptions 1 and 2, we have:*

$$(nT)^{-1} \sum_{t \le T} \left(\sum_{i \le n} (u_{i,t} - \bar{u}_{i,\cdot})(w_{i,t} - \bar{w}_{i,\cdot}) \right) \left(\sum_{i \le n} \varepsilon_{i,t} \right) = O_p(T^{-1/2}).$$

Proof. Without loss of generality, we assume that $w_{i,t}$ is a scalar throughout the proof of this lemma. If $w_{i,t}$ is a vector, the proof can be applied componentwise. We begin the proof by writing:

$$\begin{aligned}
&(nT)^{-1} \sum_{t \le T} \left(\sum_{i \le n} (u_{i,t} - \bar{u}_{i,\cdot})(w_{i,t} - \bar{w}_{i,\cdot}) \right) \left(\sum_{i \le n} \varepsilon_{i,t} \right) \\
&= (nT)^{-1} \sum_{t \le T} \sum_{i \le n} \sum_{i' \le n} (w_{i,t} - \bar{w}_{i,\cdot}) u_{i,t} \varepsilon_{i',t} - (nT)^{-1} \sum_{t \le T} \sum_{i \le n} \sum_{i' \le n} (w_{i,t} - \bar{w}_{i,\cdot}) \bar{u}_{i,\cdot} \varepsilon_{i',t}.
\end{aligned}$$

Applying similar arguments to those used for proving Lemma SA.7(ii) with $c_t - \bar{c}$ replaced by $w_{i,t} - \bar{w}_{i,\cdot}$, we can show that under Assumptions 1(i, iii) and 2(i),

$$(nT)^{-1} \sum_{t \le T} \sum_{i \le n} \sum_{i' \le n} (w_{i,t} - \bar{w}_{i,\cdot}) u_{i,t} \varepsilon_{i',t} = O_p((nT)^{-1/2}).$$

Therefore, the claim of the lemma follows if

$$(nT)^{-1} \sum_{t \le T} \sum_{i \le n} \sum_{i' \le n} (w_{i,t} - \bar{w}_{i,\cdot}) \bar{u}_{i,\cdot} \varepsilon_{i',t} = O_p((nT)^{-1/2}). \tag{SB.152}$$

To demonstrate the result in (SB.152), we first decompose the triple summation term on its left-hand side as follows

$$\sum_{t\leq T}\sum_{i\leq n}\sum_{i'\leq n}(w_{i,t}-\bar{w}_{i,\cdot})\bar{u}_{i,\cdot}\varepsilon_{i',t}=\sum_{t\leq T}\sum_{i\leq n}(w_{i,t}-\bar{w}_{i,\cdot})\bar{u}_{i,\cdot}\varepsilon_{i,t}$$
$$+\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(\bar{u}_{i,\cdot}(w_{i,t}-\bar{w}_{i,\cdot})\varepsilon_{i',t}+\bar{u}_{i',\cdot}(w_{i',t}-\bar{w}_{i',\cdot})\varepsilon_{i,t}). \tag{SB.153}$$

The first term on the right-hand side can be further decomposed as:

$$\sum_{t\leq T}\sum_{i\leq n}(w_{i,t}-\bar{w}_{i,\cdot})\bar{u}_{i,\cdot}\varepsilon_{i,t}=T^{-1}\sum_{t\leq T}\sum_{i\leq n}\varepsilon_{i,t}(w_{i,t}-\bar{w}_{i,\cdot})u_{i,t}$$
$$+T^{-1}\sum_{i\leq n}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}(\varepsilon_{i,t'}(w_{i,t'}-\bar{w}_{i,\cdot})u_{i,t}+\varepsilon_{i,t}(w_{i,t}-\bar{w}_{i,\cdot})u_{i,t'}). \tag{SB.154}$$

Using similar arguments as those employed in deriving (SA.61) with $c_t-\bar{c}$ replaced by $w_{i,t}-\bar{w}_{i,\cdot}$, we can show that under Assumptions 1(i, iii) and 2(i),

$$(nT^2)^{-1}\sum_{t\leq T}\sum_{i\leq n}(w_{i,t}-\bar{w}_{i,\cdot})u_{i,t}\varepsilon_{i,t}=O_p((nT^3)^{-1/2}). \tag{SB.155}$$

For any $i_2>i_1$ and any $t_k\neq t'_k$ for $k=1,2$, we have

$$\begin{aligned}&\mathbb{E}[\varepsilon_{i_1,t'_1}(w_{i_1,t'_1}-\bar{w}_{i_1,\cdot})u_{i_1,t_1}\varepsilon_{i_2,t'_2}(w_{i_2,t'_2}-\bar{w}_{i_2,\cdot})u_{i_2,t_2}]\\&=\mathbb{E}[(w_{i_1,t'_1}-\bar{w}_{i_1,\cdot})(w_{i_2,t'_2}-\bar{w}_{i_2,\cdot})]\mathbb{E}[\varepsilon_{i_1,t'_1}u_{i_1,t_1}\varepsilon_{i_2,t'_2}u_{i_2,t_2}]\\&=\mathbb{E}[(w_{i_1,t'_1}-\bar{w}_{i_1,\cdot})(w_{i_2,t'_2}-\bar{w}_{i_2,\cdot})]\mathbb{E}[\varepsilon_{i_1,t'_1}u_{i_1,t_1}\varepsilon_{i_2,t'_2}]\mathbb{E}[u_{i_2,t_2}]=0,\end{aligned}$$

where the first equality is by Assumption 2(i), and the subsequent equalities follow from Assumptions 1(i, iii) and 2(iv). Therefore,

$$\begin{aligned}\mathbb{E}\left[\left|\sum_{i\leq n}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\varepsilon_{i,t'}(w_{i,t'}-\bar{w}_{i,\cdot})u_{i,t}\right|^2\right]&=\sum_{i\leq n}\mathbb{E}\left[\left|\sum_{t=2}^{T}u_{i,t}\sum_{t'=1}^{t-1}\varepsilon_{i,t'}(w_{i,t'}-\bar{w}_{i,\cdot})\right|^2\right]\\&=K\sum_{i\leq n}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\mathbb{E}[(w_{i,t'}-\bar{w}_{i,\cdot})^2]\mathbb{E}[\varepsilon_{i,t'}^2]\mathbb{E}[u_{i,t}^2]\\&\leq K(n-1)^{-2}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sum_{i\leq n}(i-1)\leq KT^2,\end{aligned}$$

which together with Markov's inequality shows that

$$(nT^2)^{-1}\sum_{i\leq n}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\varepsilon_{i,t'}(w_{i,t'}-\bar{w}_{i,\cdot})u_{i,t}=O_p((nT)^{-1}). \tag{SB.156}$$

Similarly,

$$\mathbb{E}\left[\left|\sum_{i\leq n}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\varepsilon_{i,t}(w_{i,t}-\bar{w}_{i,\cdot})u_{i,t'}\right|^2\right] = \sum_{i\leq n}\mathbb{E}\left[\left|\sum_{t=1}^{T-1}\sum_{t'=t+1}^{T}u_{i,t}\varepsilon_{i,t'}(w_{i,t'}-\bar{w}_{i,\cdot})\right|^2\right]$$
$$= \sum_{i\leq n}\sum_{t=1}^{T-1}\sum_{t'=t+1}^{T}\mathbb{E}[(w_{i,t'}-\bar{w}_{i,\cdot})^2]\mathbb{E}[\varepsilon_{i,t'}^2]\mathbb{E}[u_{i,t}^2]$$
$$\leq K(n-1)^{-2}\sum_{t=1}^{T-1}\sum_{t'=t+1}^{T}\sum_{i\leq n}(i-1)\leq KT^2.$$

Thus, by Markov's inequality

$$(nT^2)^{-1}\sum_{i\leq n}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\varepsilon_{i,t}(w_{i,t}-\bar{w}_{i,\cdot})u_{i,t'} = O_p((nT)^{-1}).$$

Combining this with (SB.154), (SB.155) and (SB.156) yields:

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\varepsilon_{i,t}(w_{i,t}-\bar{w}_{i,\cdot})\bar{u}_{i,\cdot} = O_p((nT)^{-1}). \tag{SB.157}$$

Next, we examine the second term after the equality in (SB.153), which can be decomposed as

$$\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(\bar{u}_{i,\cdot}(w_{i,t}-\bar{w}_{i,\cdot})\varepsilon_{i',t}+\bar{u}_{i',\cdot}(w_{i',t}-\bar{w}_{i',\cdot})\varepsilon_{i,t})$$
$$= T^{-1}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}((w_{i,t}-\bar{w}_{i,\cdot})u_{i,t}\varepsilon_{i',t}+(w_{i',t}-\bar{w}_{i',\cdot})u_{i',t}\varepsilon_{i,t})$$
$$+T^{-1}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(u_{i,t}(w_{i,t'}-\bar{w}_{i,\cdot})\varepsilon_{i',t'}+u_{i,t'}(w_{i,t}-\bar{w}_{i,\cdot})\varepsilon_{i',t})$$
$$+T^{-1}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(u_{i',t}(w_{i',t'}-\bar{w}_{i',\cdot})\varepsilon_{i,t'}+u_{i',t'}(w_{i',t}-\bar{w}_{i',\cdot})\varepsilon_{i,t}). \tag{SB.158}$$

Using similar arguments as those used to derive (SA.67) with $c_t-\bar{c}$ replaced by $w_{i,t}-\bar{w}_{i,\cdot}$, we can show that

$$(nT^2)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}((w_{i,t}-\bar{w}_{i,\cdot})u_{i,t}\varepsilon_{i',t}+(w_{i',t}-\bar{w}_{i',\cdot})u_{i',t}\varepsilon_{i,t}) = O_p((nT^3)^{-1/2}). \tag{SB.159}$$

For any $i_2>i_1$, any $i'_k<i_k$, any t_k and t'_k for $k=1,2$, we have

$$\mathbb{E}[u_{i_1,t_1}(w_{i_1,t'_1}-\bar{w}_{i_1,\cdot})\varepsilon_{i'_1,t'_1}u_{i_2,t_2}(w_{i_2,t'_2}-\bar{w}_{i_2,\cdot})\varepsilon_{i'_2,t'_2}]$$
$$= \mathbb{E}[(w_{i_1,t'_1}-\bar{w}_{i_1,\cdot})(w_{i_2,t'_2}-\bar{w}_{i_2,\cdot})]\mathbb{E}[\varepsilon_{i'_1,t'_1}u_{i_1,t_1}\varepsilon_{i'_2,t'_2}u_{i_2,t_2}]$$
$$= \mathbb{E}[(w_{i_1,t'_1}-\bar{w}_{i_1,\cdot})(w_{i_2,t'_2}-\bar{w}_{i_2,\cdot})]\mathbb{E}[\varepsilon_{i'_1,t'_1}u_{i_1,t_1}\varepsilon_{i'_2,t'_2}]\mathbb{E}[u_{i_2,t_2}] = 0, \tag{SB.160}$$

where the first equality is by Assumption 2(i), and the subsequent equalities follow from Assumptions 1(i, iii) and 2(iv). Additionally, for any $t_2 > t_1$, and for any $i'_k < i$ and $t'_k < t_k$ for $k = 1, 2$, we have

$$\begin{aligned}&\mathbb{E}[u_{i,t_1}(w_{i,t'_1} - \bar{w}_{i,\cdot})\varepsilon_{i'_1,t'_1}u_{i,t_2}(w_{i,t'_2} - \bar{w}_{i,\cdot})\varepsilon_{i'_2,t'_2}]\\ &= \mathbb{E}[(w_{i,t'_1} - \bar{w}_{i,\cdot})(w_{i,t'_2} - \bar{w}_{i,\cdot})]\mathbb{E}[\varepsilon_{i'_1,t'_1}u_{i,t_1}\varepsilon_{i'_2,t'_2}]\mathbb{E}[u_{i,t_2}] = 0. \qquad \text{(SB.161)}\end{aligned}$$

By (SB.160) and (SB.161), we have

$$\begin{aligned}\mathbb{E}\left[\left|\sum_{i=2}^{n}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sum_{i'=1}^{i-1}u_{i,t}(w_{i,t'} - \bar{w}_{i,\cdot})\varepsilon_{i',t'}\right|^2\right] &= \sum_{i=2}^{n}\sum_{t=2}^{T}\mathbb{E}\left[u_{i,t}^2\left|\sum_{t'=1}^{t-1}\sum_{i'=1}^{i-1}(w_{i,t'} - \bar{w}_{i,\cdot})\varepsilon_{i',t'}\right|^2\right]\\ &= \sigma_u^2\sum_{i=2}^{n}\sum_{t=2}^{T}\mathbb{E}\left[\left|\sum_{t'=1}^{t-1}(w_{i,t'} - \bar{w}_{i,\cdot})\sum_{i'=1}^{i-1}\varepsilon_{i',t'}\right|^2\right]\\ &= \sigma_u^2\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sum_{i=2}^{n}\mathbb{E}[(w_{i,t'} - \bar{w}_{i,\cdot})^2]\mathbb{E}\left[\left|\sum_{i'=1}^{i-1}\varepsilon_{i',t'}\right|^2\right]\\ &= \sigma_u^2\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}\mathbb{E}[(w_{i,t'} - \bar{w}_{i,\cdot})^2]\mathbb{E}[\varepsilon_{i',t'}^2]\\ &\leq K(n-1)^{-2}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(i'-1) \leq KnT^2, \qquad \text{(SB.162)}\end{aligned}$$

where the second, the third, and the fourth equalities are by Assumptions 1(i) and 2(i), and the first inequality follows from Assumptions 1(i, iii) and 2(iv). Thus, by Markov's inequality

$$(nT^2)^{-1}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}u_{i,t}(w_{i,t'} - \bar{w}_{i,\cdot})\varepsilon_{i',t'} = O_p((nT^2)^{-1/2}). \qquad \text{(SB.163)}$$

For any $t_2 > t_1$, and for any $i'_k < i$ and $t'_k < t_k$ for $k = 1, 2$, we have

$$\begin{aligned}&\mathbb{E}[u_{i,t'_1}(w_{i,t_1} - \bar{w}_{i,\cdot})\varepsilon_{i'_1,t_1}u_{i,t'_2}(w_{i,t_2} - \bar{w}_{i,\cdot})\varepsilon_{i'_2,t_2}]\\ &= \mathbb{E}[(w_{i,t_1} - \bar{w}_{i,\cdot})(w_{i,t_2} - \bar{w}_{i,\cdot})]\mathbb{E}[u_{i,t'_1}\varepsilon_{i'_1,t_1}u_{i,t'_2}\varepsilon_{i'_2,t_2}]\\ &= \mathbb{E}[(w_{i,t_1} - \bar{w}_{i,\cdot})(w_{i,t_2} - \bar{w}_{i,\cdot})]\mathbb{E}[u_{i,t'_1}\varepsilon_{i'_1,t_1}u_{i,t'_2}]\mathbb{E}[\varepsilon_{i'_2,t_2}] = 0, \qquad \text{(SB.164)}\end{aligned}$$

where the first equality is by Assumption 2(i), and the subsequent equalities follow from Assumptions 1(i, iii) and 2(iv). By (SB.160) and (SB.164), and using similar arguments as those for deriving (SB.162), we obtain:

$$\mathbb{E}\left[\left|\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}u_{i,t'}(w_{i,t} - \bar{w}_{i,\cdot})\varepsilon_{i',t}\right|^2\right] = \sum_{i=2}^{n}\sum_{t=2}^{T}\mathbb{E}\left[(w_{i,t} - \bar{w}_{i,\cdot})^2\left|\sum_{i'=1}^{i-1}\sum_{t'=1}^{t-1}u_{i,t'}\varepsilon_{i',t}\right|^2\right]$$

$$= \sum_{i=2}^{n} \sum_{t=2}^{T} \mathbb{E}[(w_{i,t} - \bar{w}_{i,\cdot})^2] \mathbb{E}\left[\left|\sum_{t'=1}^{t-1} u_{i,t'} \sum_{i'=1}^{i-1} \varepsilon_{i',t}\right|^2\right]$$

$$= \sum_{i=2}^{n} \sum_{t=2}^{T} \sum_{i'=1}^{i-1} \sum_{t'=1}^{t-1} \mathbb{E}[(w_{i,t} - \bar{w}_{i,\cdot})^2] \mathbb{E}[u_{i,t'}^2] \mathbb{E}[\varepsilon_{i',t}^2]$$

$$\leq K(n-1)^2 \sum_{i=2}^{n} \sum_{t=2}^{T} \sum_{t'=1}^{t-1} \sum_{i'=1}^{i-1} (i'-1) \leq KnT^2.$$

Thus, by Markov's inequality

$$(nT^2)^{-1} \sum_{t=2}^{T} \sum_{t'=1}^{t-1} \sum_{i=2}^{n} \sum_{i'=1}^{i-1} u_{i,t'}(w_{i,t} - \bar{w}_{i,\cdot})\varepsilon_{i',t} = O_p((nT^2)^{-1/2}),$$

which, along with (SB.163), shows that

$$(nT^2)^{-1} \sum_{t=2}^{T} \sum_{t'=1}^{t-1} \sum_{i=2}^{n} \sum_{i'=1}^{i-1} (u_{i,t}(w_{i,t'} - \bar{w}_{i,\cdot})\varepsilon_{i',t'} + u_{i,t'}(w_{i,t} - \bar{w}_{i,\cdot})\varepsilon_{i',t}) = O_p((nT^2)^{-1/2}). \tag{SB.165}$$

For any $t_2 > t_1$, any $t'_k < t_k$, any i_k and i'_k for $k = 1, 2$, we have

$$\begin{aligned} &\mathbb{E}[u_{i'_1,t_1}(w_{i'_1,t'_1} - \bar{w}_{i'_1,\cdot})\varepsilon_{i_1,t'_1} u_{i'_2,t_2}(w_{i'_2,t'_2} - \bar{w}_{i'_2,\cdot})\varepsilon_{i_2,t'_2}] \\ &= \mathbb{E}[(w_{i'_1,t'_1} - \bar{w}_{i'_1,\cdot})(w_{i'_2,t'_2} - \bar{w}_{i'_2,\cdot})] \mathbb{E}[u_{i'_1,t_1}\varepsilon_{i_1,t'_1}\varepsilon_{i_2,t'_2}] \mathbb{E}[u_{i'_2,t_2}] = 0, \end{aligned} \tag{SB.166}$$

where the first equality follows from Assumptions 1(i, iii) and 2(iv). Additionally, for any $i_2 > i_1$, and for any $t'_k < t$ and $i'_k < i_k$ for $k = 1, 2$, we have

$$\begin{aligned} &\mathbb{E}[u_{i'_1,t}(w_{i'_1,t'_1} - \bar{w}_{i'_1,\cdot})\varepsilon_{i_1,t'_1} u_{i'_2,t}(w_{i'_2,t'_2} - \bar{w}_{i'_2,\cdot})\varepsilon_{i_2,t'_2}] \\ &= \mathbb{E}[(w_{i'_1,t'_1} - \bar{w}_{i'_1,\cdot})(w_{i'_2,t'_2} - \bar{w}_{i'_2,\cdot})] \mathbb{E}[u_{i'_1,t}\varepsilon_{i_1,t'_1} u_{i'_2,t}] \mathbb{E}[\varepsilon_{i_2,t'_2}] = 0. \end{aligned} \tag{SB.167}$$

By (SB.166) and (SB.167),

$$\mathbb{E}\left[\left|\sum_{t=2}^{T} \sum_{t'=1}^{t-1} \sum_{i=2}^{n} \sum_{i'=1}^{i-1} u_{i',t}(w_{i',t'} - \bar{w}_{i',\cdot})\varepsilon_{i,t'}\right|^2\right] = \sum_{t=2}^{T} \sum_{i=2}^{n} \mathbb{E}\left[\left|\sum_{i'=1}^{i-1} u_{i',t} \sum_{t'=1}^{t-1} (w_{i',t'} - \bar{w}_{i',\cdot})\varepsilon_{i,t'}\right|^2\right]$$

$$= \sum_{i=2}^{n} \sum_{t=2}^{T} \sum_{i'=1}^{i-1} \mathbb{E}[u_{i',t}^2] \mathbb{E}\left[\left|\sum_{t'=1}^{t-1} (w_{i',t'} - \bar{w}_{i',\cdot})\varepsilon_{i,t'}\right|^2\right]$$

$$= \sum_{i=2}^{n} \sum_{t=2}^{T} \sum_{i'=1}^{i-1} \sum_{t'=1}^{t-1} \mathbb{E}[u_{i',t}^2] \mathbb{E}[(w_{i',t'} - \bar{w}_{i',\cdot})^2] \mathbb{E}[\varepsilon_{i,t'}^2]$$

$$\leq K(n-1)^2 \sum_{i=2}^{n} \sum_{t=2}^{T} \sum_{t'=1}^{t-1} \sum_{i'=1}^{i-1} (i'-1) \leq KnT^2,$$

which, together with Markov's inequality, shows that

$$(nT^2)^{-1}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}u_{i',t}(w_{i',t'}-\bar{w}_{i',\cdot})\varepsilon_{i,t'}=O_p((nT^2)^{-1/2}). \tag{SB.168}$$

For any $t_2>t_1$, any i_k and i'_k, any $t'_k<t_k$ for $k=1,2$, we have

$$\begin{aligned}&\mathbb{E}[u_{i'_1,t'_1}(w_{i'_1,t_1}-\bar{w}_{i'_1,\cdot})\varepsilon_{i_1,t_1}u_{i'_2,t'_2}(w_{i'_2,t_2}-\bar{w}_{i'_2,\cdot})\varepsilon_{i_2,t_2}]\\&=\mathbb{E}[(w_{i'_1,t_1}-\bar{w}_{i'_1,\cdot})(w_{i'_2,t_2}-\bar{w}_{i'_2,\cdot})]\mathbb{E}[u_{i'_1,t'_1}\varepsilon_{i_1,t_1}u_{i'_2,t'_2}]\mathbb{E}[\varepsilon_{i_2,t_2}]=0,\end{aligned} \tag{SB.169}$$

where the first equality follows from Assumptions 1(i, iii) and 2(iv). Additionally for any $i_2>i_1$, and for any $t'_k<t$ and $i'_k<i_k$ for $k=1,2$, we have

$$\begin{aligned}&\mathbb{E}[u_{i'_1,t'_1}(w_{i'_1,t}-\bar{w}_{i'_1,\cdot})\varepsilon_{i_1,t}u_{i'_2,t'_2}(w_{i'_2,t}-\bar{w}_{i'_2,\cdot})\varepsilon_{i_2,t}]\\&=\mathbb{E}[(w_{i'_1,t}-\bar{w}_{i'_1,\cdot})(w_{i'_2,t}-\bar{w}_{i'_2,\cdot})]\mathbb{E}[u_{i'_1,t'_1}\varepsilon_{i_1,t}u_{i'_2,t'_2}]\mathbb{E}[\varepsilon_{i_2,t}]=0.\end{aligned} \tag{SB.170}$$

By (SB.169) and (SB.170),

$$\begin{aligned}\mathbb{E}\left[\left|\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}u_{i',t'}(w_{i',t}-\bar{w}_{i',\cdot})\varepsilon_{i,t}\right|^2\right]&=\sum_{t=2}^{T}\sum_{i=2}^{n}\mathbb{E}\left[\varepsilon_{i,t}^2\left|\sum_{i'=1}^{i-1}\sum_{t'=1}^{t-1}u_{i',t'}(w_{i',t}-\bar{w}_{i',\cdot})\right|^2\right]\\&=\sum_{t=2}^{T}\sum_{i=2}^{n}\mathbb{E}[\varepsilon_{i,t}^2]\mathbb{E}\left[\left|\sum_{i'=1}^{i-1}\sum_{t'=1}^{t-1}u_{i',t'}(w_{i',t}-\bar{w}_{i',\cdot})\right|^2\right]\\&=\sum_{i=2}^{n}\sum_{t=2}^{T}\sum_{i'=1}^{i-1}\sum_{t'=1}^{t-1}\mathbb{E}[\varepsilon_{i,t}^2]\mathbb{E}[u_{i',t'}^2]\mathbb{E}[(w_{i',t}-\bar{w}_{i',\cdot})^2]\\&\leq K(n-1)^2\sum_{i=2}^{n}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sum_{i'=1}^{i-1}(i'-1)\leq KnT^2.\end{aligned}$$

Thus, by Markov's inequality

$$(nT^2)^{-1}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}u_{i',t'}(w_{i',t}-\bar{w}_{i',\cdot})\varepsilon_{i,t}=O_p((nT^2)^{-1/2}),$$

which along with (SB.168) shows that

$$(nT^2)^{-1}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(u_{i',t}(w_{i',t'}-\bar{w}_{i',\cdot})\varepsilon_{i,t'}+u_{i',t'}(w_{i',t}-\bar{w}_{i',\cdot})\varepsilon_{i,t})=O_p((nT^2)^{-1/2}). \tag{SB.171}$$

Collecting the results from (SB.158), (SB.159), (SB.165) and (SB.171) leads to

$$(nT)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(\bar{u}_{i,\cdot}(w_{i,t}-\bar{w}_{i,\cdot})\varepsilon_{i',t}+\bar{u}_{i',\cdot}(w_{i',t}-\bar{w}_{i',\cdot})\varepsilon_{i,t})=O_p((nT)^{-1/2}). \tag{SB.172}$$

The desired result in (SB.152) follows from (SB.153), (SB.157) and (SB.172). ■

Lemma SB.21 *Under Assumptions 1 and 2, we have:*

$$(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(\gamma(u_{i,t}-\bar{u}_{i,\cdot})(c_t-\bar{c})+\varepsilon_{i,t})\right)^2=\gamma^2\sigma_u^2\hat{\sigma}_c^2+(n-1)^{-1}(\sigma_u^2\sigma_v^2+\sigma_{u,v}^2)+O_p(T^{-1/2}).$$

Proof. By Assumptions 1(i, ii, iii), we have

$$\mathbb{E}\left[\left|T^{-1}\sum_{t\leq T}(c_t-\bar{c})^2\sum_{i\leq n}u_{i,t}\right|^2\right]\leq T^{-2}\sum_{t\leq T}\mathbb{E}[(c_t-\bar{c})^4]\sum_{i\leq n}\mathbb{E}[u_{i,t}^2]\leq nT^{-1},$$

which, together with (SA.87) in the proof of Lemma SA.9 and Markov's inequality, shows that

$$\bar{u}T^{-1}\sum_{t\leq T}(c_t-\bar{c})^2\sum_{i\leq n}u_{i,t}=O_p(T^{-1}). \tag{SB.173}$$

Combining this with Lemma SA.10, (SA.91) in the proof of Lemma SA.11, and (29) in the proof of Lemma 2, we obtain

$$\begin{aligned}(nT)^{-1}\sum_{t\leq T}\left((c_t-\bar{c})\sum_{i\leq n}(u_{i,t}-\bar{u}_{i,\cdot})\right)^2&=(nT)^{-1}\sum_{t\leq T}\left((c_t-\bar{c})\sum_{i\leq n}u_{i,t}\right)^2-2\bar{u}T^{-1}\sum_{t\leq T}(c_t-\bar{c})^2\sum_{i\leq n}u_{i,t}\\&+(nT)^{-1}\sum_{t\leq T}\left((c_t-\bar{c})\sum_{i\leq n}\bar{u}_{i,\cdot}\right)^2=\sigma_u^2\hat{\sigma}_c^2+O_p(T^{-1/2}).\end{aligned} \tag{SB.174}$$

By Lemma SA.7(ii) and (SA.86) in the proof of Lemma SA.9,

$$\begin{aligned}(nT)^{-1}\sum_{t\leq T}(c_t-\bar{c})\left(\sum_{i\leq n}(u_{i,t}-\bar{u}_{i,\cdot})\right)\left(\sum_{i\leq n}\varepsilon_{i,t}\right)&=(nT)^{-1}\sum_{t\leq T}(c_t-\bar{c})\left(\sum_{i\leq n}u_{i,t}\right)\left(\sum_{i\leq n}\varepsilon_{i,t}\right)\\&-\bar{u}T^{-1}\sum_{t\leq T}(c_t-\bar{c})\sum_{i\leq n}\varepsilon_{i,t}=O_p((nT)^{-1/2}).\end{aligned} \tag{SB.175}$$

Since

$$\begin{aligned}&(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(\gamma(u_{i,t}-\bar{u}_{i,\cdot})(c_t-\bar{c})+\varepsilon_{i,t})\right)^2\\&=\gamma^2(nT)^{-1}\sum_{t\leq T}\left((c_t-\bar{c})\sum_{i\leq n}(u_{i,t}-\bar{u}_{i,\cdot})\right)^2+(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}\varepsilon_{i,t}\right)^2\\&\quad+2\gamma(nT)^{-1}\sum_{t\leq T}(c_t-\bar{c})\left(\sum_{i\leq n}(u_{i,t}-\bar{u}_{i,\cdot})\right)\left(\sum_{i\leq n}\varepsilon_{i,t}\right),\end{aligned}$$

the claim of the lemma follows from Lemma SA.7(i), (SB.174) and (SB.175). ∎

Lemma SB.22 *Under Assumptions 1 and 2, we have:*

$$(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}((u_{i,t}-\bar{u}_{i,\cdot})(\gamma(c_t-\bar{c})-(w_{i,t}-\bar{w}_{i,\cdot})^{\top}\hat{\varphi})+\varepsilon_{i,t})\right)^2$$
$$=\gamma^2\sigma_u^2(\hat{\sigma}_c^2-(1-T^{-1})\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c})+(n-1)^{-1}(\sigma_u^2\sigma_v^2+\sigma_{u,v}^2)+O_p(T^{-1/2}). \tag{SB.176}$$

Proof. The term on the left-hand side of the equality in (SB.176) can be expressed as

$$(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}((u_{i,t}-\bar{u}_{i,\cdot})(\gamma(c_t-\bar{c})-(w_{i,t}-\bar{w}_{i,\cdot})^{\top}\hat{\varphi})+\varepsilon_{i,t})\right)^2$$
$$=(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(\gamma(u_{i,t}-\bar{u}_{i,\cdot})(c_t-\bar{c})+\varepsilon_{i,t})\right)^2+(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(u_{i,t}-\bar{u}_{i,\cdot})(w_{i,t}-\bar{w}_{i,\cdot})^{\top}\hat{\varphi}\right)^2$$
$$-2(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(\gamma(u_{i,t}-\bar{u}_{i,\cdot})(c_t-\bar{c})+\varepsilon_{i,t})\right)\left(\sum_{i\leq n}(u_{i,t}-\bar{u}_{i,\cdot})(w_{i,t}-\bar{w}_{i,\cdot})^{\top}\hat{\varphi}\right).$$

Therefore, by Lemmas SB.18, SB.19 and SB.20, and using (SB.131), we obtain

$$(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}((u_{i,t}-\bar{u}_{i,\cdot})(\gamma(c_t-\bar{c})-(w_{i,t}-\bar{w}_{i,\cdot})^{\top}\hat{\varphi})+\varepsilon_{i,t})\right)^2$$
$$=(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(\gamma(u_{i,t}-\bar{u}_{i,\cdot})(c_t-\bar{c})+\varepsilon_{i,t})\right)^2-\gamma^2\sigma_u^2(1-T^{-1})\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c}+O_p(T^{-1/2}),$$

which, combined with Lemma SB.21, establishes the claim of the lemma. ■

Lemma SB.23 *Under Assumptions 1 and 2, we have*

$$(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}\hat{u}_{e,i,t}\left(z_{i,t}-\bar{z}_{i,\cdot}-(w_{i,t}-\bar{w}_{i,\cdot})^{\top}\hat{\varphi}\right)\right)^2$$
$$=(nT)^{-1}\sum_{t\leq T}\left(\begin{array}{c}\sum_{i\leq n}((u_{i,t}-\bar{u}_{i,\cdot})(\gamma(c_t-\bar{c})-(w_{i,t}-\bar{w}_{i,\cdot})^{\top}\hat{\varphi})+\varepsilon_{i,t})\\ -\gamma(c_t-\bar{c})(\hat{\beta}_{e,iv}-\beta)\sum_{i\leq n}(\gamma(c_t-\bar{c})-(w_{i,t}-\bar{w}_{i,\cdot})^{\top}\hat{\varphi})\\ -\sum_{i\leq n}\hat{\phi}_w^{\top}(w_{i,t}-\bar{w}_{i,\cdot})(\gamma(c_t-\bar{c})-(w_{i,t}-\bar{w}_{i,\cdot})^{\top}\hat{\varphi})\end{array}\right)^2+O_p((nT)^{-1/2}).$$

Proof. Applying the expression for $z_{i,t}$ in (3), the expression for $x_{i,t}-\bar{x}_{i,\cdot}$ in (SA.1) and the expression for $\hat{u}_{e,i,t}$ in (SB.124), we can express

$$\sum_{i\leq n}\hat{u}_{e,i,t}\left(z_{i,t}-\bar{z}_{i,\cdot}-(w_{i,t}-\bar{w}_{i,\cdot})^{\top}\hat{\varphi}\right)$$

$$
\begin{aligned}
&= \sum_{i\le n} \hat{u}_{e,i,t} \left((n-1)^{-1} \sum_{i'\neq i} \left(x_{i',t} - \bar{x}_{i',\cdot} \right) - (w_{i,t} - \bar{w}_{i,\cdot})^\top \hat{\varphi} \right) \\
&= \sum_{i\le n} ((u_{i,t} - \bar{u}_{i,\cdot})(\gamma(c_t - \bar{c}) - (w_{i,t} - \bar{w}_{i,\cdot})^\top \hat{\varphi}) + \varepsilon_{i,t}) - (\hat{\beta}_{e,iv} - \beta)\gamma^2 n (c_t - \bar{c})^2 \\
&- \gamma(c_t - \bar{c}) \sum_{i\le n} \hat{\phi}_w^\top (w_{i,t} - \bar{w}_{i,\cdot}) + \gamma(c_t - \bar{c})(\hat{\beta}_{e,iv} - \beta) \sum_{i\le n} (w_{i,t} - \bar{w}_{i,\cdot})^\top \hat{\varphi} \\
&+ \hat{\phi}_w^\top \sum_{i\le n} (w_{i,t} - \bar{w}_{i,\cdot})(w_{i,t} - \bar{w}_{i,\cdot})^\top \hat{\varphi} + I_{1,t} + (\hat{\beta}_{e,iv} - \beta)(I_{3,t} - I_{2,t}) - \hat{\phi}_w^\top I_{4,t}, \qquad \text{(SB.177)}
\end{aligned}
$$

where $\hat{\phi}_w \equiv \hat{\Sigma}_w^{-1} \hat{\Gamma}_{w,u} - (\hat{\beta}_{e,iv} - \beta)\hat{\lambda}$, $I_{1,t}$ and $I_{2,t}$ are defined in (SA.71) and (SA.72) respectively. Additionally,

$$
I_{3,t} \equiv \sum_{i\le n} (v_{i,t} - \bar{v}_{i,\cdot})(w_{i,t} - \bar{w}_{i,\cdot})^\top \hat{\varphi} \quad \text{and} \quad I_{4,t} \equiv (n-1)^{-1} \sum_{i\le n} \sum_{i'\neq i} (w_{i,t} - \bar{w}_{i,\cdot}) \left(v_{i',t} - \bar{v}_{i',\cdot} \right).
$$

By Assumption 1(iii) and Theorem 3, we have

$$
(nT)^{-1} \sum_{t\le T} \left((\hat{\beta}_{e,iv} - \beta)\gamma^2 n (c_t - \bar{c})^2 \right)^2 = O_p(T^{-1}). \qquad \text{(SB.178)}
$$

By Assumptions 1(iii) and 2(iv), and applying Markov's inequality,

$$
(nT)^{-1} \sum_{t\le T} \sum_{i\le n} (c_t - \bar{c})^2 ||w_{i,t} - \bar{w}_{i,\cdot}||^2 = O_p(1). \qquad \text{(SB.179)}
$$

Therefore,

$$
(nT)^{-1} \sum_{t\le T} (c_t - \bar{c})^2 \left(\sum_{i\le n} \hat{\phi}_w^\top (w_{i,t} - \bar{w}_{i,\cdot}) \right)^2 \le ||\hat{\phi}_w||^2 T^{-1} \sum_{t\le T} (c_t - \bar{c})^2 \sum_{i\le n} ||w_{i,t} - \bar{w}_{i,\cdot}||^2 = O_p(T^{-1}), \qquad \text{(SB.180)}
$$

where the first inequality follows from the Cauchy-Schwarz inequality, and the equality is due to (SB.127) and (SB.179). Similarly, we can show that due to (SB.131)

$$
(\hat{\beta}_{e,iv} - \beta)^2 (nT)^{-1} \sum_{t\le T} \left((c_t - \bar{c}) \sum_{i\le n} (w_{i,t} - \bar{w}_{i,\cdot})^\top \hat{\varphi} \right)^2 = O_p(T^{-1}). \qquad \text{(SB.181)}
$$

Applying the Cauchy-Schwarz inequality,

$$
\begin{aligned}
(nT)^{-1} \sum_{t\le T} \left(\sum_{i\le n} \hat{\phi}_w^\top (w_{i,t} - \bar{w}_{i,\cdot})(w_{i,t} - \bar{w}_{i,\cdot})^\top \hat{\varphi} \right)^2 &\le T^{-1} \sum_{t\le T} \sum_{i\le n} \left(\hat{\phi}_w^\top (w_{i,t} - \bar{w}_{i,\cdot})(w_{i,t} - \bar{w}_{i,\cdot})^\top \hat{\varphi} \right)^2 \\
&\le T^{-1} ||\hat{\phi}_w||^2 ||\hat{\varphi}||^2 \sum_{t\le T} \sum_{i\le n} \|w_{i,t} - \bar{w}_{i,\cdot}\|^4
\end{aligned}
$$

$$= O_p(T^{-1}), \tag{SB.182}$$

where the last equality is due to Assumption 2(iv), (SB.127) and (SB.131). Therefore, considering Lemma SB.22, (SB.177), (SB.178), (SB.180), (SB.181) and (SB.182), the claim of the lemma follows if

$$\sum_{t\leq T} I_{1,t}^2 + (\hat{\beta}_{e,iv} - \beta)^2 \sum_{t\leq T}(I_{2,t}^2 + I_{3,t}^2) + \sum_{t\leq T}(\hat{\phi}_w^\top I_{4,t})^2 = O_p(1). \tag{SB.183}$$

By similar arguments to those used in proving Lemma SB.18, we can show that

$$\begin{aligned}(nT)^{-1}\sum_{t\leq T} I_{3,t}^2 &= (nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(v_{i,t} - \bar{v}_{i,\cdot})(w_{i,t} - \bar{w}_{i,\cdot})^\top \hat{\varphi}\right)^2 \\ &= (1 - T^{-1})\sigma_v^2\gamma^2\hat{\Gamma}_{w,c}^\top\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c} + O_p(T^{-1/2}) = O_p(1),\end{aligned}$$

where the last equality follows from (SB.97) and (SB.99). Combining this with Theorem 3, we have

$$(\hat{\beta}_{e,iv} - \beta)^2 \sum_{t\leq T} I_{3,t}^2 = O_p(1). \tag{SB.184}$$

By Theorem 3 and the same arguments as those used for deriving (SA.78),

$$\sum_{t\leq T} I_{1,t}^2 + (\hat{\beta}_{e,iv} - \beta)^2 \sum_{t\leq T} I_{2,t}^2 = O_p(1). \tag{SB.185}$$

Considering (SB.184) and (SB.185), the desired result in (SB.183) follows if

$$\sum_{t\leq T}(\hat{\phi}_w^\top I_{4,t})^2 = O_p(1). \tag{SB.186}$$

To show (SB.186), we begin by writing

$$\begin{aligned}\hat{\phi}_w^\top I_{4,t} &= (n-1)^{-1}\sum_{i\leq n}\sum_{i'\neq i}\hat{\phi}_w^\top(w_{i,t} - \bar{w}_{i,\cdot})\left(v_{i',t} - \bar{v}_{i',\cdot}\right) \\ &= (n-1)^{-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(\hat{\phi}_w^\top(w_{i,t} - \bar{w}_{i,\cdot})\left(v_{i',t} - \bar{v}_{i',\cdot}\right) + \hat{\phi}_w^\top(w_{i',t} - \bar{w}_{i',\cdot})\left(v_{i,t} - \bar{v}_{i,\cdot}\right)).\end{aligned} \tag{SB.187}$$

Therefore, by the Cauchy-Schwarz inequality,

$$\begin{aligned}\sum_{t\leq T}(\hat{\phi}_w^\top I_{4,t})^2 \leq{}& 2||\hat{\phi}_w||^2\sum_{t\leq T}\left\|(n-1)^{-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(w_{i,t} - \bar{w}_{i,\cdot})\left(v_{i',t} - \bar{v}_{i',\cdot}\right)\right\|^2 \\ &+ 2||\hat{\phi}_w||^2\sum_{t\leq T}\left\|(n-1)^{-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(w_{i',t} - \bar{w}_{i',\cdot})\left(v_{i,t} - \bar{v}_{i,\cdot}\right)\right\|^2.\end{aligned} \tag{SB.188}$$

For any $k = 1, \dots, d_w$, by the Cauchy-Schwarz inequality and Assumptions 1(i, iii) and 2(i, iv), we have

$$\begin{aligned}
\mathbb{E}\left[\sum_{t\le T}\left|\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(w_{i,t}^k - \bar{w}_{i,\cdot}^k)\left(v_{i',t} - \bar{v}_{i',\cdot}\right)\right|^2\right] &= \sum_{t\le T}\mathbb{E}\left[\left|\sum_{i=1}^{n-1}(v_{i,t} - \bar{v}_{i,\cdot})\sum_{i'=i+1}^{n}(w_{i',t}^k - \bar{w}_{i',\cdot}^k)\right|^2\right] \\
&= \sum_{t\le T}\sum_{i=1}^{n-1}\mathbb{E}\left[(v_{i,t} - \bar{v}_{i,\cdot})^2\left|\sum_{i'=i+1}^{n}(w_{i',t}^k - \bar{w}_{i',\cdot}^k)\right|^2\right] \\
&\le K\sum_{t\le T}\sum_{i=1}^{n-1}\mathbb{E}\left[(n-i)\sum_{i'=i+1}^{n}(w_{i',t}^k - \bar{w}_{i',\cdot}^k)^2\right] \le Kn^3T,
\end{aligned}$$

which, together with Markov's inequality, implies that

$$\sum_{t\le T}\left|(n-1)^{-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(w_{i,t}^k - \bar{w}_{i,\cdot}^k)\left(v_{i',t} - \bar{v}_{i',\cdot}\right)\right|^2 = O_p(nT). \qquad \text{(SB.189)}$$

Similarly, we can show that

$$\sum_{t\le T}\left|(n-1)^{-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(w_{i',t}^k - \bar{w}_{i',\cdot}^k)\left(v_{i,t} - \bar{v}_{i,\cdot}\right)\right|^2 = O_p(nT).$$

Combining this with (SB.127), (SB.188) and (SB.189) establishes (SB.186). ■

Lemma SB.24 *Suppose that Assumptions 1 and 2 hold. If $T \to \infty$ as $m \to \infty$, then*

$$(nT)^{-1}\sum_{t\le T}\left(\sum_{i\le n}\hat{u}_{e,i,t}\left(z_{i,t} - \bar{z}_{i,\cdot} - (w_{i,t} - \bar{w}_{i,\cdot})^\top\hat{\varphi}\right)\right)^2 \to_p \gamma^2\sigma_u^2\sigma_{e,c}^2 + (n_\infty - 1)^{-1}(\sigma_u^2\sigma_v^2 + \sigma_{u,v}^2).$$

Proof. By (SB.178), (SB.180), (SB.181) and (SB.182), and applying Lemmas SB.22 and SB.23, we have

$$\begin{aligned}
&(nT)^{-1}\sum_{t\le T}\left(\sum_{i\le n}\hat{u}_{e,i,t}\left((n-1)^{-1}\sum_{i'\ne i}\left(x_{i',t} - \bar{x}_{i',\cdot}\right) - (w_{i,t} - \bar{w}_{i,\cdot})^\top\hat{\varphi}\right)\right)^2 \\
&= \gamma^2\sigma_u^2(\hat{\sigma}_c^2 - (1 - T^{-1})\hat{\Gamma}_{w,c}^\top\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c}) + (n-1)^{-1}(\sigma_u^2\sigma_v^2 + \sigma_{u,v}^2) + O_p(T^{-1/2}),
\end{aligned}$$

which, along with Assumptions 1(iv) and 2(ii, iii), establishes the claim of the lemma. ■

References

Hall, P., and C. C. Heyde (1980): *Martingale Limit Theory and Its Application.* Academic Press, Inc. New York.

Häusler, E., and H. Luschgy (2015): *Stable Convergence and Stable Limit Theorems*, vol. 74. Springer.

Kuersteiner, G. M., and I. R. Prucha (2013): "Limit theory for panel data models with cross sectional dependence and sequential exogeneity," *Journal of Econometrics*, 174(2), 107–126.