Optimal Dynamic Income Taxation under Quasi-Hyperbolic Discounting and Idiosyncratic Productivity Shocks *

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Abstract

In the context of a dynamic (three-period) general equilibrium model, this paper examines the optimal tax rates on capital savings and labor income under quasi-hyperbolic discounting and idiosyncratic productivity shocks. In the absence of skill-type uncertainty, we analytically show that the marginal capital tax wedges on agents' first-period savings are negative for correcting inherent preference internalities, and that these tax rates will be higher when productivity disturbances are incorporated. In the stochastic two-type setting with exogenously-given factor input prices, our calibrated numerical experiments find that the marginal capital wedges for both types on their period-1 savings are positive, indicating the government's motive to relax individuals' incentive-compatibility constraints. We also quantitatively find that the optimal tax rates for both types on their first- and second-period capital savings, as well as the economy's social welfare, are *ceteris paribus* decreasing in the degree of quasi-hyperbolic discounting because of a stronger need to rectify negative utility internalities.

Keywords: Optimal Dynamic Income Taxation; Quasi-Hyperbolic Discounting; Idiosyncratic Productivity Shocks.

JEL Classification: D91, H21, H24.

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1 Introduction

Experimental psychology has extensively delved into the exploration of time preferences, inspecting individuals' inclinations toward rewards acquired at different dates. In addition, behavioral economists have challenged the conventional postulate that an individual's rate of time preference declines exponentially over time. Specifically, the deviation from traditional exponential discounting captures the observed phenomenon with time inconsistency between agents' short-run immediate gratification versus their pursuit of long-run utility maximization. To this end, previous research has adopted the formulations of hyperbolic discounting[1] and quasi-hyperbolic discounting[2] to account for this empirical feature, where economic decisions made by households under the present circumstances may yield negative internalities that in turn exert an enduring impact on their future selves. In such an intertemporal environment, the Pigouvian tax/subsidy mechanism is capable of operating as a corrective measure to address and rectify the underlying preference internality.

On the other hand, each individual's skill type or labor productivity (transforming her effort into output) may evolve stochastically over time, influenced by unpredictable factors such as sudden employment opportunities or unforeseen setbacks like the illness of a skilled worker leading to efficiency loss. In light of the above-mentioned inherent utility internalities and idiosyncratic productivity shocks, this paper examines optimal nonlinear income taxation within a dynamic general equilibrium model. In the absence of a direct linkage between income tax rates and agents' productivity levels, the difficulty faced by a benevolent government is to simultaneously provide insurance and incentives, along with addressing the negative internalities as well as taking into account the long-term preferences of households. Moreover, the government lacks direct observability of individuals' labor productivities (as skill types are private information), thereby limiting its capacity to leverage this informational asymmetry. To tackle these intricate challenges, the Mirrless framework is employed as a theoretical apparatus here to characterize and analyze the socially optimal nonlinear income tax schedule. This is an important research topic not only for its theoretical insights, but also for its broad implications for the design, implementation and evaluation of tax policy rules.

[1]Hyperbolic discounting has emerged as a prevalent assumption for modeling bounded rationality in individual decision-making, particularly with regard to savings. See *e.g.* Frederick et al. (2002), O'Donoghue and Rabin (1999), and Hey and Lotito (2009), among others.

[2]In the macroeconomics literature, quasi-hyperbolic discounting has been frequently employed to analyze the economic outcomes at the aggregate level. Representative examples include Tobacman (2009), Krusell et al. (2010), and Graham and Snower (2013), among others.

In accordance with the New Dynamic Public Finance (NDPF) literature,[3] we consider a three-period stochastic general equilibrium model, under quasi-hyperbolic discounting and idiosyncratic productivity shocks, to qualitatively as well as quantitatively study the optimal nonlinear taxation on capital savings and labor income. At the onset of period 1, households are endowed with different levels of labor productivity. Individuals work in all three periods and save in periods 1 and 2. Agents encounter an idiosyncratic productivity disturbance (or a skill-type shock) at period 2, and are postulated to possess the same level of productivity in the final period of their lifetime. The government imposes nonlinear taxation on capital savings in periods 1 and 2 and on labor income in all three periods, in order to maximize a utilitarian social welfare function based on individuals' true (long-run) preferences. The endogeneity of these tax functions is a key feature for our analysis, representing the delicate balance between providing insurance to households and attaining overall economic efficiency. In addition, the endogeneity of market interest and wage rates, together with exogenous productivity disturbances, capture the intrinsic uncertainties faced by agents in their economic pursuits. As typically assumed in the NDPF literature, the government can commit to its fiscal policy regimes.

To characterize the social planning problem in the macroeconomy, we analytically obtain the law of motion for the cumulative Lagrange multiplier associated with individuals' incentive-compatibility constraints. Under quasi-hyperbolic discounting, the cumulative Lagrange multiplier is found to follow a first-order autoregressive process with a persistence coefficient smaller than one. It follows that the difference between this persistence coefficient and one (under traditional exponential discounting) indicates the extent of immediate temptation for short-run gratification. This cumulative Lagrange multiplier, which ensures agents' truthfully reporting their skill types, is then used to endogenously determine the resultant pseudo Pareto weights that encapsulate both the past and current levels of bindingness/tightness on incentive-compatibility constraints, reflecting the informational rent received by heterogenous households. Next, we derive the model's Inverse Euler equations that will govern how the social planner allocates resources across different time periods at the optimum. The characteristics of optimal marginal tax wedges on capital savings are subsequently deduced from these Inverse Euler equations, shedding light on the nuanced relationship between agents' discounting pattern and the dynamic features of the cumulative Lagrange multiplier.

In the simple two-type (high-skill and low-skill) version of our model economy under quasi-hyperbolic discounting and no productivity shocks, we first prove that the optimal

[3] Seminal works in this literature include Golosov et al. (2006) and Kocherlakota (2010), which trace their root to the static model advanced by Mirrlees (1971) and subsequently generalized by Stiglitz (1982).

marginal tax rates on capital savings at period 1 are negative for both types of individuals, and that skilled workers are subject to a higher capital tax wedge than their unskilled counterparts.[4] When agents gravitate to current rather than future consumption, the social planner will subsidize capital savings to induce households to invest more and correct their preference internalities. Since high-type agents have the incentive to mimic low-type individuals, their respective pseudo Pareto weights will be higher, which in turn raises the social planner's desire to back-load skilled workers' consumption with heavier capital taxation. We also find that adding skill-type disturbances exerts an upward pressure on the first-period capital tax rates, acting as an opposite force to counteract the impact of quasi-hyperbolic discounting that generates a downward pressure. As a result, the signs of these period-1 tax wedges on capital savings will be theoretically ambiguous. At period 2, it is analytically shown that the optimal tax rates on capital savings are equal to a negative constant for all households, regardless of whether productivity uncertainties are present or not.[5] A negative tax wedge on capital income under quasi-hyperbolic discounting provides an incentive for agents' savings, taking into account their preference for immediate consumption and leisure. We finally show that for each period, high-type households face a zero marginal tax rate on their labor income while that for low-type individuals is positive. These are the well-known "no-distortion-at-the-top" and "downward-distortion-at-the-bottom" results that stem from the social planner's intratemporal incentive to prevent skilled workers from mimicking their unskilled counterparts.

To obtain further insights of the preceding theoretical results and their consequent welfare effects, numerical experiments are undertaken within a calibrated version of our model economy under various scenarios, including cases where (i) the interest and wage rates are either endogenously determined or exogenously given, (ii) productivity/skill-type shocks are either present or absent in period 2, and (iii) agents undertake quasi-hyperbolic or traditional exponential discounting. Under quasi-hyperbolic discounting and endogenous interest/wage rates, we find that incorporating idiosyncratic productivity disturbances leads to increases in the optimal (negative) capital tax rates on period-1 savings for both skilled and unskilled workers, while narrowing the quantitative difference between them. When factor prices are exogenous, the marginal tax wedges for both types on their first-period capital savings become positive in the presence of a skill-type shock, indicating the government's motive to distort agents' capital accumulation

[4]We also find that the optimal capital tax rates on period-1 savings will be zero under standard exponential discounting.

[5]The optimal capital tax rates on period-2 savings will be zero under standard exponential discounting.

downwards to relax their incentive-compatibility constraints in a partial-equilibrium environment. Without the presence of such uncertainties, a downward distortion at the intertemporal margin in period 1 will decrease the optimal capital tax rates for both types of agents. Our numerical experiments show that low-skill households will now face negative capital income taxation, whereas a positive capital tax wedge continues to be prescribed for high-skill individuals .

We also find that given quasi-hyperbolic discounting and the presence of productivity shocks, the optimal (negative and constant) tax rate on capital savings for all agents at period 2 is lower under exogenous interest/wage rates than that with endogenous input prices. In either setting, there exists an inclination to encourage greater savings from each household, exerting a downward pressure on the marginal tax wedges for its period-2 investment spending, and this effect will be strengthened when the capital rental rate remains fixed over time. In terms of social welfare, the government will face additional binding constraints under exogenously-given factor prices, which in turn *ceteris paribus* yields notable utility loss. While keeping all other aspects of the model unchanged, the existence of a period-2 skill-type disturbance is shown to raise the economy's aggregate utility level because of enhanced social mobility, caused by more unskilled individuals moving upwards than the opposite movement of skilled workers.

Under traditional exponential discounting, along with endogenous factor prices and the presence of skill-type disturbances as the baseline formulation, the optimal tax rates on first-period capital savings for both types of workers are positive in order to relax their incentive-compatibility constraints, and that the quantitative difference between them is close to zero. Without the presence of productivity shocks, the social planner has no desire to distort individuals' intertemporal margin hence the marginal capital tax rates will be zero on each agent's period-1 and period-2 savings. When the interest/wage rates are exogenously given, we find that the optimal capital tax rates on individuals' first-period savings are positive and higher than the corresponding benchmark counterparts, while those for period-2 investment expenditures are negative because of the requirement of a time-invariant capital rental rate. Putting our quantitative results together implies that the optimal marginal tax wedges for both types of households on their first- as well as second-period capital savings are *ceteris paribus* monotonically decreasing in the degree of quasi-hyperbolic discounting, because the need to correct negative preference internalities will be strengthened. Finally, the economy's aggregate welfare with traditional exponential discounting is found to exhibit a qualitatively identical pattern as that under quasi-hyperbolic discounting across all variants of our model under consideration.

Our work is closely related to Guo and Krause (2015) who also investigate optimal nonlinear income taxation in a three-period environment with quasi-hyperbolic discounting. However, there are important differences between the two studies. In particular, the Guo-Krause analysis considers a deterministic, partial equilibrium framework with exogenously-given interest and wage rates, together with the lack of commitment by the government. By contrast, we analyze a stochastic, general equilibrium model in which factor prices are endogenously determined and the government commits to its fiscal policy rules. In a simplified two-type setting, Guo and Krause (2015) quantitatively find that the optimal capital tax rate on skilled workers' first-period savings is positive, and that for unskilled agents is negative; whereas we analytically prove that these tax wedges are negative for both types of individuals in the no-uncertainty version of our macroeconomy. As a result, this paper provides theoretical as well as quantitative insights on how the presence/absence of idiosyncratic productivity shocks within a general- versus partial-equilibrium model affects the economy's optimal income tax rates and social welfare under quasi-hyperbolic or traditional exponential discounting.

The remainder of our paper is organized as follows. Section 2 describes the analytical framework that we consider. Section 3 examines the social planner's problem and derives the resulting constrained-efficient allocations. Section 4 analytically studies the structure of optimal dynamic nonlinear taxation on capital savings and labor income. Section 5 discusses numerical simulation results to shed light on the effects of quasi-hyperbolic discounting, idiosyncratic productivity shocks, and general- versus partial-equilibrium modelling setups. Section 6 concludes, while proofs are relegated to an appendix..

2 Environment

The economy lasts for three periods, indexed by $t = 1, 2, 3$; and is inhabited by a continuum of heterogeneous agents of unit mass.[6] At the onset of period 1, individuals exhibit different levels of labor productivity. In particular, they are partitioned by θ_1 that takes a positive value in a finite set Θ. We let θ_1 and $\pi(\theta_1)$ denote the the level of productivity and the fraction of population who represents the type θ_1, respectively. Individuals are exposed to an idiosyncratic productivity shock, denoted as θ_2, at period 2. We let θ^2 denote the history of events up to $t = 2$. It follows that $\pi_2(\theta^2) = \pi(\theta_2|\theta_1)\pi(\theta_1)$, where $\pi(\theta_2|\theta_1)$ stands for the conditional probability. By the law of large numbers, $\pi_2(\theta^2)$ also represents the proportion of households who experience a history of event θ^2.

[6]Three periods represent the minimum time horizon needed to analyze the effects of quasi-hyperbolic discounting. Extending the model to an n-period setting with $n > 3$ will not qualitatively change our results.

Individuals can work in all three periods and save in periods 1 and 2. In period 2, agents (in their middle age) encounter an idiosyncratic productivity disturbance θ_2. In the final period of their lifetime, households are postulated to possess the same level of productivity as they experienced in period 2. Individual θ_1's true (long-run) utility function is given by:

$$
\begin{aligned}
& u\left(c_1\left(\theta_1\right)\right)-v\left(\frac{l_1\left(\theta_1\right)}{\theta_1}\right)+\delta \sum_{\theta_1} \sum_{\theta_2}\left[u\left(c_2\left(\theta^2\right)\right)-v\left(\frac{l_2\left(\theta^2\right)}{\theta_2}\right)\right] \pi\left(\theta_2 \mid \theta_1\right) \\
& +\delta^2 \sum_{\theta^2}\left[u\left(c_3\left(\theta^2\right)\right)-v\left(\frac{l_3\left(\theta^2\right)}{\theta_2}\right)\right] \pi\left(\theta_2 \mid \theta_1\right),
\end{aligned} \tag{1}
$$

where $\delta \in (0,1)$ denotes the discount factor, $c_t(\theta^j)$ denotes the consumption of household θ^j at time t, and $l_t(\theta^j)$ is the effective labor supply of agent θ^j at period t. The ratio $\frac{l_t(\theta^j)}{\theta_j}$ represents hours worked or effort of individual θ^j at time t. It is assumed that the utility and disutility functions, given by $u(\cdot)$ and $v(\cdot)$, satisfy the following standard properties: $u' > 0$, $u'' < 0$, $v' > 0$, and $v'' > 0$.

Under quasi-hyperbolic discounting, we follow Laibson (1997) and postulate that agents maximize the following lifetime utility:

$$
\begin{aligned}
& u\left(c_1\left(\theta_1\right)\right)-v\left(\frac{l_1\left(\theta_1\right)}{\theta_1}\right)+\beta\delta \sum_{\theta^2}\left[u\left(c_2\left(\theta^2\right)\right)-v\left(\frac{l_2\left(\theta^2\right)}{\theta_2}\right)\right] \pi\left(\theta_2 \mid \theta_1\right) \\
& +\beta\delta^2 \sum_{\theta^2}\left[u\left(c_3\left(\theta^2\right)\right)-v\left(\frac{l_3\left(\theta^2\right)}{\theta_2}\right)\right] \pi\left(\theta_2 \mid \theta_1\right),
\end{aligned} \tag{2}
$$

where $\beta \in (0,1)$ denotes the degree of quasi-hyperbolic discounting. When viewed from period 1, it can be seen that an individual's discount factor between periods 1 and 2 is $\beta\delta$, whereas it is δ between periods 2 and 3. But when viewed from period 2, the discount factor between periods 2 and 3 is $\beta\delta$. Thus (2) captures a preference for gravitation toward immediate consumption and leisure, which are not optimal from the viewpoint of long-run utility maximization. As a reference formulation, we will also examine the economy under traditional exponential discounting with $\beta = 1$, therefore the discount factor between two consecutive periods is δ.

2.1 Household Problem Without Taxation

This subsection characterizes individuals' decision-making problem each period in the absence of income taxation. Following Guo and Krause (2015), we assume that despite being aware of the craving for short-run gratification, agents hold the naive belief that their future economic decisions will align with their true/long-run preferences. Such a setup captures the notion that those who succumb to immediate satisfaction often console themselves by promising that they will act more rationally in the future.[7] This modelling assumption is made not only for the sake of analytical simplicity, but also because there is empirical evidence on households' lacking sophistication in this context.

At the beginning of period 1, all agents are endowed with $k_1 > 0$ units of physical capital that earns income $(1 + r_1)k_1$. Each household maximizes (2) by choosing $\{c_1(\theta_1), c_2(\theta^2), l_1(\theta_1), l_2(\theta^2), s_1(\theta_1), s_2(\theta^2)\}$ and faces the following period budget constraints:

$$c_1 + s_1 = (1 + r_1) k_1 + w_1 l_1, \tag{3}$$

$$c_2 + s_2 = w_2 l_2 + (1 + r_2) s_1, \tag{4}$$

$$c_3 = w_3 l_3 + (1 + r_3) s_2, \tag{5}$$

where w_t denotes the wage rate, r_t denotes the interest rate, and s_t denotes the period-t savings that are held in the form of additions to next period's capital stock. Without loss of generality, the capital depreciation rate is set to be 100% after one period. It follows that $1 + r_t$ can also be interpreted as the (gross) rental rate that the individuals receive from providing capital services to firms. The first-order conditions for this dynamic optimization problem are

$$w_t = \frac{v'\left(\frac{l_t(\theta^t)}{\theta_t}\right)}{\theta_t u'(c_t(\theta^t))}, t = 1, 2, \tag{6}$$

$$w_3 = \frac{v'\left(\frac{l_3(\theta^2)}{\theta_2}\right)}{\theta_2 u'(c_3(\theta^2))}, \tag{7}$$

$$1 = \frac{u'(c_1(\theta_1))}{\beta\delta(1 + r_2)\sum_{\theta_2} u'(c_2(\theta^2))\pi(\theta_2|\theta_1)}, \text{ and} \tag{8}$$

[7] By contrast, sophisticated individuals acknowlege their impatience and recognize that such desire will resurface in the future. Consequently, they also factor this awareness into their decision-making process.

$$1 = \frac{u'\left(c_2\left(\theta^2\right)\right)}{\delta\left(1+r_3\right)u'\left(c_3\left(\theta^2\right)\right)}. \tag{9}$$

However, individuals cannot commit themselves to their original period-2 and period-3 consumption and labor supply plans in that they succumb to short-run gratification in period 2. Instead, they will choose $\left\{c_2\left(\theta^2\right), s_2\left(\theta^2\right), l_2\left(\theta^2\right), c_3\left(\theta^2\right), l_3\left(\theta^2\right)\right\}$ to maximize

$$\max u\left(c_2\left(\theta^2\right)\right) - v\left(\frac{l_2\left(\theta^2\right)}{\theta_2}\right) + \beta\delta\left[u\left(c_3\left(\theta^2\right)\right) - v\left(\frac{l_3\left(\theta^2\right)}{\theta_3}\right)\right], \tag{10}$$

subject to (4) and (5). Solving this problem yields that

$$w_2 = \frac{v'\left(\frac{l_2\left(\theta^2\right)}{\theta_2}\right)}{\theta_2 u'\left(c_2\left(\theta^2\right)\right)},\ w_3 = \frac{v'\left(\frac{l_3\left(\theta^2\right)}{\theta_2}\right)}{\theta_2 u'\left(c_3\left(\theta^2\right)\right)}, \text{ and} \tag{11}$$

$$1 = \frac{u'\left(c_2\left(\theta^2\right)\right)}{\beta\delta\left(1+r_3\right)u'\left(c_3\left(\theta^2\right)\right)}. \tag{12}$$

2.2 Firms

There is a standard neoclassical production function, denoted by $F\left(K_t, L_t\right)$, which exhibits constant returns-to-scale with aggregate capital K_t and aggregate labor L_t as inputs. This production technology is operated by a representative firm that employs labor and rents capital from households to maximize their profits. Under the assumption that factor markets are perfectly competitive, the associated first–order conditions are

$$w_t = F_{L,t}(K_t, L_t), \tag{13}$$

$$1 + r_t = F_{K,t}(K_t, L_t), \tag{14}$$

where $F_{L,t}$ and $F_{K,t}$ denote the marginal products of labor and capital in period t, respectively.

2.3 Government

Under the postulated nonlinear income taxation scheme, households with different levels of labor income and capital savings will face separate marginal tax rates. In particular, the government may find it optimal to violate individuals' first-order conditions derived above in the absence of taxation. To implement the government's desired allocations in the market economy, non-zero marginal tax rates on labor income and capital savings may be imposed. As is common in the relevant literature, the implicit marginal tax rates or tax wedges to reflect the resource distortions at the social optimum are defined as[8]

$$\tau_{l,t} := 1 - \frac{v'\left(\frac{l_t(\theta^t)}{\theta_t}\right)}{w_t\theta_t u'\left(c_t\left(\theta^t\right)\right)}, t = 1,2, \tag{15}$$

$$\tau_{l,3} := 1 - \frac{v'\left(\frac{l_3(\theta^2)}{\theta_2}\right)}{w_3\theta_2 u'\left(c_3\left(\theta^2\right)\right)}, \tag{16}$$

$$\tau_{k,1} := 1 - \frac{u'\left(c_1\left(\theta_1\right)\right)}{\beta\delta\left(1+r_2\right)\sum_{\theta_2} u'\left(c_2\left(\theta^2\right)\right)\pi\left(\theta_2|\theta_1\right)}, \text{ and} \tag{17}$$

$$\tau_{k,2} := 1 - \frac{u'\left(c_2\left(\theta^2\right)\right)}{\beta\delta\left(1+r_3\right)u'\left(c_3\left(\theta^2\right)\right)}. \tag{18}$$

We also list households' budget constraints in the presence of income taxation at each period as follows:

$$c_1 + s_1 + T_1\left(w_1 l_1\right) = \left(1+r_1\right)k_1 + w_1 l_1, \tag{19}$$

$$c_2 + s_2 + T_2\left(w_1 l_1, w_2 l_2, \left(1+r_2\right)s_1\right) = w_2 l_2 + \left(1+r_2\right)s_1, \tag{20}$$

$$c_3 + T_3\left(w_1 l_1, w_2 l_2, w_3 l_3, \left(1+r_2\right)s_1, \left(1+r_3\right)s_2\right) = w_3 l_3 + \left(1+r_3\right)s_2, \tag{21}$$

where T_1, T_2, and T_3 are the tax payments in period $t = 1,2,3$. Notice that T_2 and T_3 are contingent on labor as well as capital income, while T_1 is a function of labor income only because the period-1 capital income $\left(1+r_1\right)k_1$ is each agent's non-taxable endowment.[9]

Since there are no public expenditures on goods and services, the government collects taxes T_t in period t for the sole purpose of redistribution. We assume that the government issues bonds denoted as B_t, $t = 1,2,3$.[10] The government needs to pay off the principal of its one-period debt and accrued interest at a rate q_t. Hence, the government's period budget constraints are given by

[8] The tax wedges are defined under the assumption that individuals exhibit naive quasi-linear hyperbolic discounting. The marginal conditions for sophisticated agents, in the absence of taxation, will deviate from (15)-(18). This discrepancy arises because sophisticated agents recognize that their future selves are inclined towards immediate consumption and leisure.

[9] The capital tax wedges are applicale only to period-1 and period-2 savings since individuals have a lifespan of three periods. As a result, they will not engage in savings at the last period.

[10] Assuming the government cannot borrow or save does not qualitatively affect our results. See footnote 11 of Guo and Krause (2015) for the same point.

$$\sum_{\theta_1} T_1 \left(w_1 l_1 \left(\theta_1 \right) \right) \pi \left(\theta_1 \right) + B_1 = 0, \tag{22}$$

$$\sum_{\theta^2} T_2 \left(w_1 l_1, w_2 l_2, \left(1 + r_2\right) s_1 \right) \pi_2 \left(\theta^2 \right) + B_2 = \left(1 + q_2\right) B_1, \text{ and} \tag{23}$$

$$\sum_{\theta^2} T_3 \left(w_1 l_1, w_2 l_2, w_3 l_3, \left(1 + r_2\right) s_1, \left(1 + r_3\right) s_2 \right) \pi_2 \left(\theta^2 \right) + B_3 = \left(1 + q_3\right) B_2. \tag{24}$$

As the economy only lasts for three periods, the government will not issue any bond in the terminal period thus $B_3 = 0$.

2.4 Competitive Equilibrium

Given the initial aggregate capital $K_1 > 0$, a competitive equilibrium for our model is defined as sequences of prices $\{w_t, r_t, q_t\}_{t=1,2,3}$, economy-wide allocations $\{C_t, L_t\}_{t=1,2,3}$ and $\{K_{t+1}\}_{t=1,2}$, together with individuals' choices $\{c_t, l_t\}_{t=1,2,3}$ and $\{s_t\}_{t=1,2}$ such that

1. Given $\{w_t, r_t\}$, $\{c_t, l_t\}_{t=1,2,3}$ and $\{s_t\}_{t=1,2}$ solve the household's dynamic optimization problem.

2. Given $\{w_t, r_t\}$, $\{L_t, K_t\}$ solve the representative firm's problem, for all t.

3. Non-arbitrage condition holds: $q_t = r_t$, for $t = 2, 3$.

4. Government budget constraint holds for all t.

5. Labor market clears for all t:

$$L_1 = \sum_{\theta_1} l_1 \left(\theta_1 \right) \pi \left(\theta_1 \right), \; L_2 = \sum_{\theta^2} l_2 \left(\theta^2 \right) \pi_2 \left(\theta^2 \right), \text{ and } L_3 = \sum_{\theta^2} l_3 \left(\theta^2 \right) \pi_2 \left(\theta^2 \right).$$

6. Asset market clears for $t = 2, 3$:

$$K_2 + B_1 = \sum_{\theta_1} s_1 \left(\theta_1 \right) \pi \left(\theta_1 \right) \text{ and } K_3 + B_2 = \sum_{\theta^2} s_2 \left(\theta^2 \right) \pi_2 \left(\theta^2 \right).$$

Notice that by Walras' Law, combining individuals' period budget constraint and the government budget constraint, along with the corresponding asset-market clearing condition, leads to the economy's resource constraints for each period:

$$\sum_{\theta_1} c_1 \pi \left(\theta_1 \right) + K_2 = F \left(K_1, L_1 \right), \tag{25}$$

$$\sum_{\theta^2} c_2\left(\theta^2\right)\pi\left(\theta_2\right)+K_3=F\left(K_2,L_2\right), \text{ and} \tag{26}$$

$$\sum_{\theta^2} c_3\left(\theta^2\right)\pi_2\left(\theta^2\right)=F\left(K_3,L_3\right). \tag{27}$$

3 Planning Problem

We assume that income taxation cannot be conditioned on individuals' productivity or skill-type heterogeneities in periods 1 and 2, *i.e.* θ_1's and θ_2's are unobservable to the government. Rather, the government can only observe the pre-tax labor income, given by $w_1 l_1$ and $w_2 l_2$. We further assume that the effort levels ($\frac{l_1}{\theta_1}$ and $\frac{l_2}{\theta_2}$) are unobservable to the government as well, hence θ_1's and θ_2's cannot be deduced either. As a result, the government must take into account agents' incentive-compatibility constraints to induce each household to choose its designated tax treatment.

The objective of the social planner is to maximize a utilitarian welfare function given by

$$\begin{aligned}
&\sum_{\theta_1}\left[u\left(c_1\left(\theta_1\right)\right)-v\left(\frac{l_1\left(\theta_1\right)}{\theta_1}\right)\right]\pi\left(\theta_1\right)+\delta\sum_{\theta_1}\sum_{\theta_2}\left[u\left(c_2\left(\theta^2\right)\right)-v\left(\frac{l_2\left(\theta^2\right)}{\theta_2}\right)\right]\pi\left(\theta_2|\theta_1\right)\pi\left(\theta_1\right)\\
&+\delta^2\sum_{\theta^2}\left[u\left(c_3\left(\theta^2\right)\right)-v\left(\frac{l_3\left(\theta^2\right)}{\theta_2}\right)\right]\pi_2\left(\theta^2\right),
\end{aligned} \tag{28}$$

subject to the economy's resource constraints (25)-(27) and individuals' incentive-compatibility constraints. For completeness, we will list all incentive-compatibility constraints below, even though some of them may not bind:

$$\begin{aligned}
&u\left(c_1\left(\theta_1\right)\right)-v\left(\frac{l_1\left(\theta_1\right)}{\theta_1}\right)+\beta\delta\sum_{\theta_2}\left[u\left(c_2\left(\theta^2\right)\right)-v\left(\frac{l_2\left(\theta^2\right)}{\theta_2}\right)\right]\pi\left(\theta_2|\theta_1\right)\\
&+\beta\delta^2\sum_{\theta_2}\left[u\left(c_3\left(\theta^2\right)\right)-v\left(\frac{l_3\left(\theta^2\right)}{\theta_2}\right)\right]\pi\left(\theta_2|\theta_1\right)\\
\geq\ &u\left(c_1\left(\tilde{\theta}_1;\theta_1\right)\right)-v\left(\frac{l_1\left(\tilde{\theta}_1;\theta_1\right)}{\theta_1}\right)+\beta\delta\sum_{\theta_2}\left[u\left(c_2\left(\tilde{\theta}^2;\theta^2\right)\right)-v\left(\frac{l_2\left(\tilde{\theta}^2;\theta^2\right)}{\theta_2}\right)\right]\pi\left(\theta_2|\theta_1\right)\\
&+\beta\delta^2\sum_{\theta_2}\left[u\left(c_3\left(\tilde{\theta}^2;\theta^2\right)\right)-v\left(\frac{l_3\left(\tilde{\theta}^2;\theta^2\right)}{\theta_2}\right)\right]\pi\left(\theta_2|\theta_1\right),\ \forall\tilde{\theta}_1,\tilde{\theta}^2,
\end{aligned} \tag{29}$$

where $c\left(\tilde{\theta};\theta\right)$ and $l\left(\tilde{\theta};\theta\right)$ are consumption and labor supply assigned to individuals whose true type is θ but mimics type $\tilde{\theta}$.

Following Fernandes and Phelan (2000), we rewrite (29) as a series of temporary incentive-compatibility constraints, each of which is intended to rule out one-time deviations:

$$\begin{aligned}
& u\left(c_1\left(\theta_1\right)\right)-v\left(\frac{l_1\left(\theta_1\right)}{\theta_1}\right)+\beta\delta\sum_{\theta_2}\left[u\left(c_2\left(\theta^2\right)\right)-v\left(\frac{l_2\left(\theta^2\right)}{\theta_2}\right)\right]\pi\left(\theta_2|\theta_1\right) \\
& +\beta\delta^2\sum_{\theta_2}\left[u\left(c_3\left(\theta^2\right)\right)-v\left(\frac{l_3\left(\theta^2\right)}{\theta_2}\right)\right]\pi\left(\theta_2|\theta_1\right) \\
\geq\ & u\left(c_1\left(\tilde{\theta}_1;\theta_1\right)\right)-v\left(\frac{l_1\left(\tilde{\theta}_1;\theta_1\right)}{\theta_1}\right)+\beta\delta\sum_{\theta_2}\left[u\left(c_2\left(\tilde{\theta}_1,\theta_2;\theta^2\right)\right)-v\left(\frac{l_2\left(\tilde{\theta}_1,\theta_2;\theta^2\right)}{\theta_2}\right)\right]\pi\left(\theta_2|\theta_1\right) \\
& +\beta\delta^2\sum_{\theta_2}\left[u\left(c_3\left(\tilde{\theta}_1,\theta_2;\theta^2\right)\right)-v\left(\frac{l_3\left(\tilde{\theta}_1,\theta_2;\theta^2\right)}{\theta_2}\right)\right]\pi\left(\theta_2|\theta_1\right),\ \forall\tilde{\theta}_1; \qquad (30)
\end{aligned}$$

and

$$\begin{aligned}
& u\left(c_2\left(\theta^2\right)\right)-v\left(\frac{l_2\left(\theta^2\right)}{\theta_2}\right)+\delta\left[u\left(c_3\left(\theta^2\right)\right)-v\left(\frac{l_3\left(\theta^2\right)}{\theta_2}\right)\right] \\
\geq\ & u\left(c_2\left(\theta_1,\tilde{\theta}_2;\theta^2\right)\right)-v\left(\frac{l_2\left(\theta_1,\tilde{\theta}_2;\theta^2\right)}{\theta_2}\right)+\delta\left[u\left(c_3\left(\theta_1,\tilde{\theta}_2;\theta^2\right)\right)-v\left(\frac{l_3\left(\theta_1,\tilde{\theta}_2;\theta^2\right)}{\theta_2}\right)\right],\ \forall\tilde{\theta}_2. \qquad (31)
\end{aligned}$$

Both temporary incentive-compatibility constraints, as in (30)-(31), are exploited to induce individuals' truth-telling for one time. Specifically, eq. (30) is to impel type-θ_1 agents not to mimic type $\tilde{\theta}_1 \neq \theta_1$, given that they will reveal their true types in the second period. Similarly, eq. (31) is to impel the type-(θ_1,θ_2) households not to mimic type $(\theta_1,\tilde{\theta}_2)$ with $\tilde{\theta}_2 \neq \theta_2$, given that both types (θ_1,θ_2) and $(\theta_1,\tilde{\theta}_2)$ have the identical skill θ_1. Note that the discount factor in (31) is δ (instead of $\beta\delta$) because from the perspective of individuals in period 1, they think their future decisions will be consistent with long-term preferences.

It is worth noting that only incentive-compatibility constraints and resource constraints are incorporated into the planning problem.[11] The government will implement the req-

[11] Our setting adheres to the standard framework in the New Dynamic Public Finance literature. See Golosov et al. (2006) and Kocherlakota (2010) for a survey of this strand of research.

uisite conditions of a competitive equilibrium through utilizing the endogenously determined income tax schedule.[12] In particular, the implicit marginal tax rates/wedges (15)-(18) are derived from imposing the first-order conditions of households' as well as firms' optimization problems. In addition, the personalized tax payments $\left\{T_1\left(\theta_1\right), T_2\left(\theta^2\right), T_3\left(\theta^2\right)\right\}$ are determined by agents' budget constraints, once we substitute constrained-efficient allocations, denoted by $\{c_t^*, l_t^*\}_{t=1,2,3}$ and $\{K_t^*\}_{t=1,2,3}$, into (19)-(21) and asset-market clearing conditions. Alternatively, these tax instruments $\left\{T_1\left(\theta_1\right), T_2\left(\theta^2\right), T_3\left(\theta^2\right)\right\}$ and $\{B_1, B_2\}$ serve as residuals to satisfy the household's budget constraints. While constrained by the government's budget constraints (22)-(24), the values of $\left\{T_1\left(\theta_1\right), T_2\left(\theta^2\right), T_3\left(\theta^2\right)\right\}$ and $\{B_1, B_2\}$ exhibit indeterminacy and thus cannot be uniquely pinned down.

3.1 Solving Constrained-Efficient Allocations

Given the planning problem laid out before, we set up the associated Lagrangian, as shown in Appendix 7.1, for the constrained-efficient allocations $\{c_t^*, l_t^*\}_{t=1,2,3}$ and $\{K_t^*\}_{t=2,3}$. In particular, allocations $\{c_t^*\}_{t=1,2,3}$ will satisfy

$$c_1\left(\theta_1\right) : \phi_1\left(\theta_1\right) u'\left(c_1\left(\theta_1\right)\right) = \mu_1, \tag{32}$$

$$c_2\left(\theta^2\right) : \phi_2\left(\theta^2\right) u'\left(c_2\left(\theta^2\right)\right) = \mu_2, \tag{33}$$

$$c_3\left(\theta^2\right) : \phi_2\left(\theta^2\right) u'\left(c_3\left(\theta^2\right)\right) = \mu_3, \tag{34}$$

where μ_t is the Lagrange multiplier attached to the resource constraint in period $t = 1, 2, 3$, and $\left\{\phi_1\left(\theta_1\right), \phi_2\left(\theta^2\right)\right\}$ are the pseudo Pareto weights given by

$$\phi_1\left(\theta_1\right) \equiv 1 + \sum_{\tilde{\theta}_1 \neq \theta_1} \left\{\lambda_1\left(\theta_1, \tilde{\theta}_1\right) - \lambda_1\left(\tilde{\theta}_1, \theta_1\right) \frac{\pi\left(\tilde{\theta}_1\right)}{\pi\left(\theta_1\right)}\right\}, \tag{35}$$

$$\begin{aligned} \phi_2\left(\theta^2\right) \quad \equiv \quad & 1 + \beta \sum_{\tilde{\theta}_1 \neq \theta_1} \left\{\lambda_1\left(\theta_1, \tilde{\theta}_1\right) - \lambda_1\left(\tilde{\theta}_1, \theta_1\right) \frac{\pi\left(\theta_2|\tilde{\theta}_1\right)\pi\left(\tilde{\theta}_1\right)}{\pi\left(\theta_2|\theta_1\right)\pi\left(\theta_1\right)}\right\} \\ & + \sum_{\tilde{\theta}_2 \neq \theta_2} \left\{\lambda_2\left(\theta^2, \tilde{\theta}_2\right) - \lambda_2\left(\left\{\theta_{1,}, \tilde{\theta}_2\right\}; \theta_2\right) \frac{\pi\left(\tilde{\theta}_2|\theta_1\right)}{\pi\left(\theta_2|\theta_1\right)}\right\}, \end{aligned} \tag{36}$$

[12] The assumption of either naive or sophisticated individuals does not alter the constrained-efficient allocations derived from our planning problem.This is because solving the constrained-efficient allocation is unrelated to each individual's marginal conditions in the competitive equilibrium. However, it does exert an impact on the marginal tax rates. Recall that the definitions of tax wedges for sophisticated agents do not align with (15)-(18).

where $\lambda_1(\theta_1, \tilde{\theta}_1)$ and $\lambda_2(\theta^2, \tilde{\theta}_2)$ are the Lagrange multipliers attached to incentive-compatibility constraints that will induce type θ_1 and type $\theta^2 = (\theta_1, \theta_2)$ to truth-tell their skills and not to mimic type $\tilde{\theta}_1$ and type $(\theta_1, \tilde{\theta}_2)$, respectively. In addition, $\lambda_1(\tilde{\theta}_1, \theta_1)$ is the the Lagrange multiplier on the incentive-compatibility constraint for agent-$\tilde{\theta}_1$ who mimics the type-θ_1 household; and $\lambda_2(\{\theta_1, \tilde{\theta}_2\}; \theta_2)$ is the Lagrange multiplier on the incentive-compatibility constraint for individual-$\{\theta_1, \tilde{\theta}_2\}$ who mimics the type-$\{\theta_1, \theta_2\}$ individual.

It is evident from (32)-(34) that $\{\phi_1(\theta_1), \phi_2(\theta^2)\}$ play a role reminiscent of Pareto weights, which are associated with period-1 utility and period-2 utility, respectively. However, $\phi_1(\theta_1)$ and $\phi_2(\theta^2)$ here represent the cumulative Lagrange multipliers that summarize the tightness of temporary incentive-compatibility constraints over the course of history. As a result, $\{\phi_1(\theta_1), \phi_2(\theta^2)\}$ are endogenously determined by incentive-compatibility constraints, rather than being Pareto weights that are exogenously assigned to different skill types. In this sense, we refer to $\{\phi_t(\theta^t)\}$ as pseudo Pareto weights. Their magnitudes stem from the informational rent that type-θ^t individuals enjoy, given the nature of these cumulative Lagrange multipliers. In addition, the pseudo Pareto weight associated with period-3 utility will be $\phi_2(\theta^2)$ because there is no asymmetric information in period 3.

3.2 Characterization of Pseudo Pareto Weights

In this subsection, we delve into investigating the properties of pseudo Pareto weights.

Lemma. *The pseudo Pareto weights* $\{\phi_1(\theta_1), \phi_2(\theta^2)\}$ *satisfy*

$$\sum_{\theta_1} \phi_1(\theta_1)\, \pi(\theta_1) = 1, \tag{37}$$

$$\phi_2\left(\theta^2\right) = 1 - \beta + \beta\phi_1(\theta_1) + \varepsilon\left(\theta^2\right), \text{ where } \sum_{\theta_2} \varepsilon\left(\theta^2\right) \pi(\theta_2|\theta_1) = 0. \tag{38}$$

Proof. The proof is relegated to Appendix 7.2, with the definition of $\varepsilon(\theta^2)$ given by (A.4). ■

Summing $\phi_2(\theta^2)$ in (38) up over θ_2, together with $\sum_{\theta_2} \varepsilon(\theta^2)\, \pi(\theta_2|\theta_1) = 0$, gives

$$\sum_{\theta_2} \phi_2\left(\theta^2\right) \pi(\theta_2|\theta_1) = 1 - \beta + \beta\phi_1(\theta_1), \tag{39}$$

which indicates that $\phi_2(\theta^2)$ and $\phi_1(\theta_1)$ are governed by a stationary AR(1) process with the degree of persistence β and a drift term $1-\beta$.[13]

Clearly, the degree of quasi-hyperbolic discounting β affects the the evolution of pseudo Pareto weights. When $\beta = 1$, (38) becomes a random walk process, given by

$$\phi_2\left(\theta^2\right) = \phi_1\left(\theta_1\right) + \varepsilon\left(\theta^2\right). \tag{40}$$

After comparing (38) and (40), we find that holding the variance of $\varepsilon\left(\theta^2\right)$ fixed, the conditional variance of $\phi_2\left(\theta^2\right)$ in (38) is smaller than that in (40). In addition, the dependence of $\phi_2\left(\theta^2\right)$ on $\phi_1\left(\theta_1\right)$ is discounted by β. This feature plays an essential role in the determination of the marginal tax rate/wedge on capital savings in that the pseudo Pareto weights $\phi_1\left(\theta_1\right)$ and $\phi_2\left(\theta^2\right)$ are derived endogenously by incentive-compatibility constraints. It can be seen from (28) and (30) that compared to the social planner, individuals at period 1 exhibit less concern about their period-2 allocations. It follows that from the government's viewpoint, the presence of time inconsistency alleviates the adverse effect of mimicking with respect to the feasibility of choosing $\{c_t^*, l_t^*\}_{t=1,2,3}$ and $\{K_t^*\}_{t=2,3}$.

Next, we use (37) and (39) to obtain

$$\sum_{\theta^2} \phi_2\left(\theta^2\right) \pi\left(\theta^2\right) = 1. \tag{41}$$

Eqs. (37) and (41) state that the average value across pseudo Pareto weights in both periods 1 and 2 is equal to one, whereas the bindingness/tightness of incentive-compatibility constraints will drive each agent's pseudo Pareto weight to deviate from one. Notice that the average pseudo Pareto weight is the same $(= 1)$ as the original Pareto weight, since the social welfare function is defined in a utilitarian manner. At the aggregate level, the characteristics of the pseudo Pareto weights are identical to those of the original Pareto weights. At the individual level, the pseudo Pareto weights differ among different types of households based on their productivities and, notably, their informational rent. In addition, the expected pseudo Pareto weights in the next period are contingent on the current skill type. In light of (39), all agents' expected pseudo Pareto weights in period 2 share a common component of $1-\beta$, yet they will differ due to the value of $\beta\phi_1\left(\theta_1\right)$. When β is set to 1, not only does the common component decrease to zero, but it also reinforces the linkage between $\sum_{\theta_2} \phi_2\left(\theta^2\right) \pi\left(\theta_2|\theta_1\right)$ and $\phi_1\left(\theta_1\right)$. Therefore, the incorporation of quasi-hyperbolic discounting weakens the connection between the expected future pseudo Pareto weight and its current-period counterpart.

[13] Notice that the lemma arises from the structure of agents' incentive compatibility constraints, and is not dependent on which specific incentive compatibility constraint is binding.

3.3 Deriving the Inverse Euler Equations

By exploiting the properties of pseudo Pareto weights, we derive the Inverse Euler equation that governs how the social planner allocates resources across periods at the optimum. After solving the planning problem, described in Appendix 7.1, we obtain the first-order conditions with respect to capital, given by

$$K_2 : \mu_1 = \delta \mu_2 F_{K,2}, \tag{42}$$

$$K_3 : \mu_2 = \delta \mu_3 F_{K,3}. \tag{43}$$

Combining (32)-(34) and (42)-(43) leads to

$$\frac{u'\left(c_1\left(\theta_1\right)\right)}{u'\left(c_2\left(\theta^2\right)\right)} = \delta F_{K,2} \frac{\phi_2\left(\theta^2\right)}{\phi_1\left(\theta_1\right)}, \text{ and} \tag{44}$$

$$\frac{u'\left(c_2\left(\theta^2\right)\right)}{u'\left(c_3\left(\theta^2\right)\right)} = \delta F_{K,3}. \tag{45}$$

Using (39), eq. (44) can be re-written as

$$\sum_{\theta_2} \frac{1}{u'\left(c_2\left(\theta^2\right)\right)} \pi\left(\theta_2|\theta_1\right) = \frac{\delta F_{K,2}}{u'\left(c_1\left(\theta_1\right)\right)} \underbrace{\left(\frac{1-\beta}{\phi_1\left(\theta_1\right)} + \beta\right)}_{=\sum_{\theta_2} \frac{\phi_2\left(\theta^2\right)}{\phi_1\left(\theta_1\right)} \pi\left(\theta_2|\theta_1\right)}, \tag{46}$$

where (45) and (46) are the Inverse Euler equations that characterize how resources are allocated intertemporally at the social optimum. When $\beta = 1$, (46) reduces to the standard formulation:

$$\sum_{\theta_2} \frac{1}{u'\left(c_2\left(\theta^2\right)\right)} \pi\left(\theta_2|\theta_1\right) = \frac{\delta F_{K,2}}{u'\left(c_1\left(\theta_1\right)\right)}, \tag{47}$$

where the social planner seeks to set the expected $\frac{u'(c_1(\theta_1))}{u'(c_2(\theta^2))}$ equal to $\delta F_{K,2}$ for all θ_1's.

Under quasi-hyperbolic discounting with $\beta \in (0,1)$, the social planner modifies the optimal intertemporal condition (46) to

$$\sum_{\theta_2} \frac{u'\left(c_1\left(\theta_1\right)\right)}{u'\left(c_2\left(\theta^2\right)\right)} = \beta \delta F_{K,2} + \frac{(1-\beta)\, \delta F_{K,2}}{\phi_1\left(\theta_1\right)}. \tag{48}$$

The term $\beta\delta F_{K,2}$ on the RHS of (48) serves to encapsulate the adjustment in an individual's preference for future utility. It is noteworthy that when considering the social planner's perspective, the emphasis placed on period-2 utility is represented by δ alone, not by the composite factor $\beta\delta$. The term $\frac{(1-\beta)\delta F_{K,2}}{\phi_1(\theta_1)}$ provides insight about the contrast between the social planner's perspective on resources at period 2 in comparison to that from period 1, $\delta F_{K,2}$, and that from the individual's viewpoint, $\beta\delta F_{K,2}$. Consequently, the expression $(1-\beta)\,\delta F_{K,2}$ signifies the disparity in the intertemporal resource perceptions between the planner and individual agents. To gauge this distinction in terms of marginal utility, it is divided by the period-1 pseudo Pareto weight, $\phi_1(\theta_1)$. For households assigned with a lower $\phi_1(\theta_1)$, this difference translates into a more pronounced impact on marginal utility in contrast to that for agents with a higher $\phi_1(\theta_1)$. Hence, the second term on the RHS of (48) acts as a corrective factor that will amplify the expected $\frac{u'(c_1(\theta_1))}{u'(c_2(\theta^2))}$. We will illustrate its influence on the implicit marginal capital tax rates in Section 4.

3.4 Solving Constrained-Efficient Labor Supply

We complete the description of constrained-efficient allocations in this subsection. Turning to the first-order conditions with respect to labor supply, solved from the planning problem per Appendix 7.1, we derive that

$$\begin{aligned} l_1(\theta_1) \quad : \quad \mu_1 F_{L,1} = & \left(1+\sum_{\tilde{\theta}_1}\lambda_1\left(\theta_1,\tilde{\theta}_1\right)\right) v'\left(\frac{l_1(\theta_1)}{\theta_1}\right)\frac{1}{\theta_1} \\ & -\left(\sum_{\tilde{\theta}_1}\lambda_1\left(\tilde{\theta}_1,\theta_1\right)\frac{\pi\left(\tilde{\theta}_1\right)}{\pi\left(\theta_1\right)}\right) v'\left(\frac{l_1(\theta_1)}{\tilde{\theta}_1}\right)\frac{1}{\tilde{\theta}_1}; \end{aligned} \tag{49}$$

$$\begin{aligned} l_2\left(\theta^2\right) \quad : \quad \mu_2 F_{L,2} = & \left(1+\beta\sum_{\tilde{\theta}_1}\lambda_1\left(\theta_1,\tilde{\theta}_1\right)+\sum_{\tilde{\theta}_2}\lambda_2\left(\theta^2,\tilde{\theta}_2\right)\right) v'\left(\frac{l_2(\theta^2)}{\theta_2}\right)\frac{1}{\theta_2} \\ & -\beta\left(\sum_{\tilde{\theta}_1}\lambda_1\left(\tilde{\theta}_1,\theta_1\right)\frac{\pi\left(\theta_2|\tilde{\theta}_1\right)}{\pi\left(\theta_2|\theta_1\right)}\frac{\pi\left(\tilde{\theta}_1\right)}{\pi\left(\theta_1\right)}\right) v'\left(\frac{l_2(\theta^2)}{\theta_2}\right)\frac{1}{\theta_2} \\ & -\left(\sum_{\tilde{\theta}_2}\lambda_2\left(\left\{\theta_1,\tilde{\theta}_2\right\},\theta_2\right)\frac{\pi\left(\tilde{\theta}_2|\theta_1\right)}{\pi\left(\theta_2|\theta_1\right)}\right) v'\left(\frac{l_2(\theta^2)}{\tilde{\theta}_2}\right)\frac{1}{\tilde{\theta}_2}; \text{ and} \end{aligned} \tag{50}$$

$$l_3\left(\theta^2\right) \quad : \quad \mu_3 F_{L,3} = \left(1+\beta\sum_{\tilde{\theta}_1}\lambda_1\left(\theta_1,\tilde{\theta}_1\right)+\sum_{\tilde{\theta}_2}\lambda_2\left(\theta^2,\tilde{\theta}_2\right)\right)v'\left(\frac{l_3\left(\theta^2\right)}{\theta_2}\right)\frac{1}{\theta_2}$$

$$-\beta\left(\sum_{\tilde{\theta}_1}\lambda_1\left(\tilde{\theta}_1,\theta_1\right)\frac{\pi\left(\theta_2|\tilde{\theta}_1\right)}{\pi_2\left(\theta_2|\theta_1\right)}\frac{\pi\left(\tilde{\theta}_1\right)}{\pi\left(\theta_1\right)}\right)v'\left(\frac{l_3\left(\theta_1,\theta_2\right)}{\theta_2}\right)\frac{1}{\theta_2}$$

$$-\left(\sum_{\tilde{\theta}_2}\lambda_2\left(\left\{\theta_1,\tilde{\theta}_2\right\},\theta_2\right)\frac{\pi_2\left(\tilde{\theta}_2|\theta_1\right)}{\pi_2\left(\theta_2|\theta_1\right)}\right)v'\left(\frac{l_3\left(\theta_1,\theta_2\right)}{\tilde{\theta}_2}\right)\frac{1}{\tilde{\theta}_2}. \tag{51}$$

In the following discussion, we contrast these socially optimal conditions, (32)-(34), (42)-(43), and (49)-(51), versus (15)-(18) to back out the implicit marginal tax rates that will be implemented to achieve constrained-efficient allocations. Specifically, we utilize (37) and (39) to shed light on how the degree of time inconsistency affects the capital wedges.

4 Implicit Marginal Tax Rates

This section examines the qualitative properties of implicit marginal tax rates at the social optimum. We begin with the tax wedge on capital savings. Using eqs. (14), (17) and (48), it can be derived that

$$1-\tau_{k,1}\left(\theta_1\right) = \left(\frac{1-\beta}{\beta\phi_1\left(\theta_1\right)}+1\right)\underbrace{\left(\sum_{\theta_2}u'\left(c_2\left(\theta^2\right)\right)\pi\left(\theta_2|\theta_1\right)\sum_{\theta_2}\frac{1}{u'\left(c_2\left(\theta^2\right)\right)}\pi\left(\theta_2|\theta_1\right)\right)^{-1}}_{<1}, \tag{52}$$

where $\left(\sum_{\theta_2}u'\left(c_2\left(\theta^2\right)\right)\pi\left(\theta_2|\theta_1\right)\sum_{\theta_2}\frac{1}{u'\left(c_2\left(\theta^2\right)\right)}\pi\left(\theta_2|\theta_1\right)\right)^{-1} < 1$ is due to Jensen's inequality.[14]

It is worth pointing out that when $\beta = 1$, the first term on the right-hand-side of (52) will drop out, leading to $\tau_{k,1}\left(\theta_1\right) > 0, \forall\theta_1$. This is a well-known result in the NDPF literature with traditional exponential discounting and the presence of exogenous shocks. Intuitively, since individuals who save more in period 1 have stronger incentives to misreport and tend to reduce hours worked in period 2, the imposition of a positive implicit

[14]Jensen's inequality states that $\sum_{i=1}^{n}\pi_i\frac{1}{X_i} \geq \frac{1}{\sum_{i=1}^{n}\pi_i X_i}$, where $\sum_{i=1}^{n}\pi_i = 1$; and the equality holds if and only if $X_1 = X_2 = ... = X_n$.

marginal tax wedge on capital investment is able to offset this potential adverse impact. Nevertheless, it remains theoretically ambiguous whether the high- or low-type agents will be subject to a higher positive rate of $\tau_{k,1}$.

To understand how time inconsistency affects the optimal implicit marginal tax rates on capital savings, we combine conditions (14) and (44) to find that without exogenous disturbances in θ_2,

$$\frac{u'\left(c_1\left(\theta_1\right)\right)}{u'\left(c_2\left(\theta_1\right)\right)} = \delta\left(1+r_2\right)\frac{\phi_2\left(\theta_1\right)}{\phi_1\left(\theta_1\right)}, \tag{53}$$

where the pseudo Pareto weight on period-2 utility is now expressed as $\phi_2\left(\theta_1\right)$. When the intertemporal relative pseudo Pareto weight $\frac{\phi_2(\theta_1)}{\phi_1(\theta_1)}$ rises, the social planner's inclination to back-load consumption becomes more pronounced. We then use the no-uncertainty version of (52) and eq. (53) to obtain

$$1-\tau_{k,1}\left(\theta_1\right) = \frac{1}{\beta}\frac{\phi_2\left(\theta_1\right)}{\phi_1\left(\theta_1\right)} = 1+\frac{1-\beta}{\beta\phi_1\left(\theta_1\right)}, \tag{54}$$

where the second equality follows from (38) that abstracts from skill-type shocks at $t=2$. As a result, the social planner will choose to provide subsidies on first-period capital savings:

$$\tau_{k,1}\left(\theta_1\right) < 0, \forall\theta_1 \tag{55}$$

because of $\phi_1\left(\theta_1\right) > 0$ for all θ_1 per condition (32) and $\beta \in (0,1)$ under quasi-hyperbolic discounting.

It can be seen from eq. (54) that the capital wedge at the intertemporal margin is determined by the ratio of across-period pseudo Pareto weights ($= \frac{\phi_2(\theta_1)}{\phi_1(\theta_1)}$) multiplied by the factor $\frac{1}{\beta}$, illustrating the existence of an interaction between quasi-hyperbolic discounting and the incentive provision. Specifically, when individuals are more likely to gravitate to immediate consumption with a smaller β, the social planner has to subsidize capital savings at a higher rate to induce agents to invest more, *i.e.* $\frac{\partial\tau_{k,1}(\theta_1)}{\partial\beta} > 0$. We also note that the ratio $\frac{\phi_2(\theta_1)}{\phi_1(\theta_1)}$ governs how the social planner views agents' current versus future consumption. Eq. (54) shows that with an increase in this ratio, the social planner will impose a lower level of $\tau_{k,1}\left(\theta_1\right)$ in order to implement a back-loading consumption plan, given that the pseudo Pareto weight on future consumption exceeds that on current consumption.

In terms of how $\tau_{k,1}\left(\theta_1\right)$ varies among households subject to different levels of productivity, we consider a two-type environment with $\theta_s > \theta_u > 0$ in period 1. Since high type-θ_s agents have the incentive to mimic low type-θ_u individuals, their respective pseudo Pareto weights must exhibit $\phi_1(\theta_s) > \phi_1(\theta_u)$. which in turn implies that $\tau_{k,1}(\theta_s) > \tau_{k,1}(\theta_u)$. For households with higher $\phi_1(\theta_1)$, or greater informational rent as shown in (35), the social planner's desire to back-load their consumption will be stronger, thus a lower retention rate $1 - \tau_{k,1}(\theta_1)$ is prescribed.

On the other hand, it is immediately clear from (54) that $\tau_{k,1}(\theta_1) = 0$ under traditional exponential discounting $\beta = 1$. In the NDPF literature, Golosov et al. (2006) also find that a zero capital wedge is socially optimal when each individual's labor productivity or skill type does not change over time. By contrast, condition (55) illustrates that the effects of asymmetric information on $\tau_{k,1}(\theta_1)$ only emerges when quasi-hyperbolic discounting is considered. In a three-period framework with time-invariant input prices, $\beta \in (0,1)$, and no exogenous shocks in period 2, Guo and Krause (2015) quantitatively show that $\tau_{k,1}(\theta_s) > 0$ and $\tau_{k,1}(\theta_u) < 0$. However, we analytically prove that both $\tau_{k,1}(\theta_s)$ and $\tau_{k,1}(\theta_u)$ are negative in the current setting. As mentioned earlier, the key differences between Guo and Krause (2015) and our study are that the former adopts a deterministic, partial equilibrium setting with exogenously-given interest and wage rates, together with no commitment by the government; whereas we analyze the same problem through the lens of a stochastic, general equilibrium model in which factor prices are endogenously determined and the government commits to its fiscal policy rules.

We summarize the above results with the following Proposition:

Proposition 1. *Consider a two-type model with $\theta_s > \theta_u > 0$ in period 1 and assume that there is no productivity shock in period 2. The optimal implicit marginal tax rates on first-period capital savings are prescribed by $\tau_{k,1}(\theta_u) < \tau_{k,1}(\theta_s) < 0$.*

Next, we note that when productivity disturbances in period 2 are incorporated, the stochastic term $\left(\sum_{\theta_2} u'\left(c_2\left(\theta^2\right)\right)\pi\left(\theta_2|\theta_1\right)\sum_{\theta_2}\frac{1}{u'(c_2(\theta^2))}\pi\left(\theta_2|\theta_1\right)\right)^{-1} < 1$ will be added to multiply the no-uncertainty capital wedge, as shown in (52). This term thus exerts an upward pressure on $\tau_{k,1}(\theta_1)$, acting as an opposite force to counteract the impact of quasi-hyperbolic discounting that generates a downward pressure on $\tau_{k,1}(\theta_1)$. Moreover, if the size of the latter effect for high type-θ_s agents is lower than that for low type-θ_u individuals, a positive $\tau_{k,1}(\theta_s)$ now becomes a possibility while simultaneously allowing for a negative $\tau_{k,1}(\theta_u)$.

Turning to examining the properties of $\tau_{k,2}\left(\theta^2\right)$, we combine eqs. (14), (18) and (45) to obtain

$$1 - \tau_{k,2}\left(\theta^2\right) = \frac{u'\left(c_2\left(\theta^2\right)\right)}{\beta\delta\left(1 + r_3\right)u'\left(c_3\left(\theta^2\right)\right)} = \frac{1}{\beta}, \tag{56}$$

which indicates that the optimal constant capital tax rate on period-2 savings is intended to correct negative utility internalities. When $\beta = 1$, $\tau_{k,2}\left(\theta^2\right) = 0$; whereas $\tau_{k,2}\left(\theta^2\right) = \frac{\beta-1}{\beta} < 0$ when $\beta \in (0,1)$. A negative tax wedge on capital investment under quasi-hyperbolic discounting provides an incentive for agents' savings, taking into account their preference for immediate consumption and leisure. It is worth noting that the result $\tau_{k,2}\left(\theta^2\right) = 1 - \frac{1}{\beta}$ holds true, regardless of the realization of θ^2 or whether skill-type shocks are present.[15] To characterize this finding, we present the following proposition:

Proposition 2. *The optimal implicit marginal tax rate on period-2 capital savings is prescribed by* $\tau_{k,2}\left(\theta^2\right) = 1 - \frac{1}{\beta}, \forall \theta^2$*, no matter whether productivity disturbances are present or not.*

Finally, the Proposition below characterizes the implicit marginal tax rates on labor income at the social optimum.

Proposition 3. *Consider a two-type model with* $\theta_s > \theta_u > 0$ *in period 1. The optimal implicit marginal tax rates on labor income are prescribed by*

$$\tau_{l,1}\left(\theta_s\right) = \tau_{l,2}\left(\theta_s, \theta_s\right) = \tau_{l,2}\left(\theta_u, \theta_s\right) = 0, \tag{57}$$

$$\tau_{l,1}\left(\theta_u\right) > 0, \tau_{l,2}\left(\theta_s, \theta_u\right) > 0, \tau_{l,2}\left(\theta_u, \theta_u\right) > 0, \text{ and} \tag{58}$$

$$\tau_{l,3}\left(\theta_s, \theta_s\right) = \tau_{l,3}\left(\theta_u, \theta_s\right) = 0, \tau_{l,3}\left(\theta_s, \theta_u\right) > 0, \tau_{l,3}\left(\theta_u, \theta_u\right) > 0. \tag{59}$$

Proof. The proof is relegated to Appendix 7.3. ■

In accordance with the existing literature, Proposition 3 shows since high type-θ_s households have the incentive to mimic low type-θ_u individuals, skilled workers will encounter no distortion at the intratemporal margin, whereas unskilled workers are subject to a positive marginal labor tax wedge.[16] This result further highlights that the marginal tax rate on capital savings is the pertinent fiscal tool for addressing the time inconsistency problem caused by agents' quasi-hyperbolic discounting. In addition, it is straightforward to show that the optimal marginal labor tax wedges under $\beta = 1$ are qualitatively identical to those in (57)-(59), as the signs of these tax rates are determined without intertemporal considerations.

[15] Guo and Krause (2015) obtain a simiar result on $\tau_{k,2}$ within a partial-equilibrium model where the government lacks commitment to its fiscal policy rules and does not issue bonds.

[16] We exclude the incentive-compatibility constraint for the low-skill type, as it is not expected to be binding due to the government's redistributive objective.

5 Quantitative Results

This section quantitatively examines the effects of quasi-hyperbolic discounting on the optimal implicit marginal tax rates of capital savings and labor income as well as the economy's aggregate welfare. Per the theoretical results of section 4, the inclusion of idiosyncratic productivity disturbances within a general equilibrium model is analytically shown to affect the socially optimal tax schedule. Our objective here is to investigate the quantitative impacts of this discounting variation on the optimal tax wedges and social welfare. To this end, we will conduct numerical experiments under various scenarios, including cases where (i) the interest rates and the wage rates are either endogenous or exogenous, (ii) productivity shocks are either present or absent in period 2, and (iii) agents undertake quasi-hyperbolic or traditional discounting.

Each individual's period preferences $u(\cdot)$ and $v(\cdot)$ are postulated as isoelastic functions, given by

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma} \text{ and } v\left(\frac{l}{\theta}\right) = \frac{\left(\frac{l}{\theta}\right)^{\gamma}}{\gamma}, \tag{60}$$

where $\sigma > 0$, $\sigma \neq 1$ and $\gamma > 1$.[17] The representative firm's production function takes a Cobb-Douglas formulation that exhibits constant returns-to-scale:

$$F(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}, \;\; 0 < \alpha < 1. \tag{61}$$

Table 1 presents the description of all model parameters and their benchmark values that are empirically plausible. We assume that the cardinality of Θ is 2, *i.e.* $\Theta = \{\theta_s, \theta_u\}$ with $\theta_s > \theta_u > 0$. Since the college wage premium is approximately 60% *à la* Fang (2006) and Goldin and Katz (2007), we follow Guo and Krause (2015) to calibrate $\theta_u = 1$ and $\theta_s = 1.6$. In addition, $\pi(\theta_s)$ is set to be 0.45 in order to match with the college share of hours worked in 2000, as reported by Autor (2014). Each period of an agent's lifetime is taken to be 20 calendar years. This feature, together with an annual real interest rate of 3%, yields the discount factor $\delta = 0.97^{20}$. Given the production function (61), the labor share of national income $1-\alpha$ is selected to be 0.7. The household's labor supply elasticity is chosen to be one, implying that $\gamma = 2$. We also consider the risk-aversion parameter $\sigma = 2$, which lies within the empirically realistic range of one to three; and the transition probability $\pi(\theta_s|\theta_s) = \pi(\theta_u|\theta_u) = 0.9$, hence a high(low)-type worker at period 1 faces the probability of 10% to become a low(high)-type worker in the next period. Finally, the

[17]Notice that the social welfare function (28) is formulated in an utilitarian manner. In order to encompass the planner's redistributive motive, we cannot consider quasi-linear preferences.

initial capital stock K_1 is normalized to be one and the baseline degree of quasi-hyperbolic discounting is calibrated to be $\beta = 0.95$.[18]

Table 1: Baseline Parameterization

Parameter	Description	Value
σ	risk aversion coefficient	2
$\frac{1}{\gamma-1}$	Frisch labor supply elasticity	1
δ	discount factor	0.54
θ_s	productivity of high-type agents	1.6
θ_u	productivity of low-type agents	1
α	capital share of national income	0.3
$\pi(\theta_s)$	fraction of high-type agents	0.45
β	degree of quasi-hyperbolic discounting	0.95
$\pi(\theta'\|\theta)$	transitional probability	0.9

5.1 Quasi-Hyperbolic Discounting with $\beta = 0.95$

Tables 2 and 3 report the optimal tax and welfare outcomes under quasi-hyperbolic discounting and other baseline parameter values of Table 1. The first row of these tables provides the description of four different economies. In particular, the first case corresponds to our benchmark model (analyzed in sections 2-4) with endogenously determined interest/wage rates and the presence of a period-2 skill-type shock. Cases A and C are postulated to be deterministic without productivity disturbances. In Cases B and C, factor input prices are exogenously given by $r_t = \frac{1}{\delta} - 1$ and $w_t = (1-\alpha)(\alpha\delta)^{\frac{\alpha}{1-\alpha}}$ each period.[19] We let $\tau_{k,2}(\theta_i, \theta_j)$ and $\tau_{l,2}(\theta_i, \theta_j)$ denote the marginal tax rate on capital savings and labor income, respectively, for individuals who are endowed with θ_i at period 1 and experience θ_j at period 2. Within each model economy, high-skill households face a zero marginal tax rate on their labor income at $t = 1$, while that for low-skill individuals is positive: $\tau_{l,1}(\theta_s) = 0$ and $\tau_{l,1}(\theta_u) > 0$, as stated in Proposition 3. These are the well-known "no-distortion-at-the-top" and "downward-distortion-at-the-bottom" results that typify second-best nonlinear income taxation. Since Cases A and C do not involve productivity shocks at $t = 2$, each agent's skill type will not change across periods and thus

[18]As illustrated by the numerical results below, a slight deviation of β from one suffices to generate notable quantitative differences.

[19]The constant level of $r_t = \frac{1}{\delta} - 1$ for all t is derived from setting the long-run discount factor δ to be $\frac{1}{1+r}$. Using condition (14), the equilibrium capital-to-labor ratio is pinned down by $\frac{K_t}{L_t} = (\alpha\delta)^{\frac{1}{1-\alpha}}, \forall t$. Plugging this expression into (13) yields that the wage rate is given by $w_t = (1-\alpha)(\alpha\delta)^{\frac{\alpha}{1-\alpha}}$ each period.

the capital/labor tax rates associated with (θ_s, θ_u)- or (θ_u, θ_s)-individuals are not available therein.

Table 2: Marginal Capital Tax Rates and Social Welfare under $\beta = 0.95$

	Benchmark Case: Endo. $\{w, r\}$ with shocks	Case A: Endo. $\{w, r\}$ without shocks	Case B: Exog. $\{w, r\}$ with shocks	Case C: Exog. $\{w, r\}$ without shocks
$\tau_{k,1}(\theta_s)$	-0.0108	-0.0401	0.0304	0.0024
$\tau_{k,1}(\theta_u)$	-0.0392	-0.0708	0.0035	-0.0271
$\tau_{k,2}(\theta_s, \theta_s)$	-0.0526	-0.0526	-0.7512	-0.7514
$\tau_{k,2}(\theta_s, \theta_u)$	-0.0526	N.A.	-0.7512	N.A.
$\tau_{k,2}(\theta_u, \theta_s)$	-0.0526	N.A.	-0.7512	N.A.
$\tau_{k,2}(\theta_u, \theta_u)$	-0.0526	-0.0526	-0.7512	-0.7514
SW	-1.4668	-1.4731	-1.4937	-1.5001

Table 3: Marginal Labor Tax Rates under $\beta = 0.95$

	Benchmark Case: Endo. $\{w, r\}$ with shocks	Case A: Endo. $\{w, r\}$ without shocks	Case B: Exog. $\{w, r\}$ with shocks	Case C: Exog. $\{w, r\}$ without shocks
$\tau_{l,1}(\theta_s)$	0	0	0	0
$\tau_{l,1}(\theta_u)$	0.1519	0.1741	0.1520	0.1739
$\tau_{l,2}(\theta_s, \theta_s)$	0	0	-0.0178	-0.0175
$\tau_{l,2}(\theta_s, \theta_u)$	0.2966	N.A.	0.2804	N.A.
$\tau_{l,2}(\theta_u, \theta_s)$	0	N.A.	-0.0178	N.A.
$\tau_{l,2}(\theta_u, \theta_u)$	0.1698	0.1645	0.1547	0.1496
$\tau_{l,3}(\theta_s, \theta_s)$	0	0	0.2844	0.2845
$\tau_{l,3}(\theta_s, \theta_u)$	0.2966	N.A.	0.4941	N.A.
$\tau_{l,3}(\theta_u, \theta_s)$	0	N.A.	0.2844	N.A.
$\tau_{l,3}(\theta_u, \theta_u)$	0.1698	0.1645	0.4057	0.4020

In the Benchmark Case, we note that $\tau_{k,2}(\cdot)$ equals $1 - \frac{1}{\beta} = -0.0526$ for all households, which follows directly from the derivation of eq. (56). It can also be seen that the optimal tax wedges on period-1 capital savings exhibit progressivity $\tau_{k,1}(\theta_s) > \tau_{k,1}(\theta_u)$ due to $\phi_1(\theta_s) > 1 > \phi_1(\theta_u)$. It follows that these marginal capital tax rates are intended to correct the negative internality caused by agents' quasi-hyperbolic discounting. By contrast, Table 3 shows that the optimal labor tax rates do not play a significant role in addressing this intertemporal informational issue, because labor income taxation stems from the planner's within-period incentive to prevent skilled workers from mimicking their unskilled counterparts. Consequently, the levels of the labor tax wedges are quite close to those under traditional exponential discounting with $\beta = 1$, as shown in Table 5.

In Case A where the interest and wage rates are endogenously pinned down in the absence of period-2 productivity shocks, Table 2 shows that $\tau_{k,1}(\theta_u) < \tau_{k,1}(\theta_s) < 0$, as stated in Proposition 1. In comparison with our benchmark formulation, the introduction of idiosyncratic productivity disturbances leads to an increase in both $\tau_{k,1}(\theta_s)$ and $\tau_{k,1}(\theta_u)$ according to eq. (52), while narrowing the quantitative difference between them. We also find that $\tau_{k,2}(\theta_s, \theta_s)$ and $\tau_{k,2}(\theta_u, \theta_u)$ remain unchanged at the same constant level $(= -0.0526)$ as that in our baseline general-equilibrium model such that negative utility internalities associated with quasi-hyperbolic discounting can be rectified. This finding aligns with Proposition 2. As in the benchmark setting, skilled workers are not subject to distortionary labor income taxation in all three periods per Proposition 3; whereas the feasible positive optimal labor tax wedges, given by $\tau_{l,1}(\theta_u)$, $\tau_{l,2}(\theta_u, \theta_u)$ and $\tau_{l,3}(\theta_u, \theta_u)$, turn out to be numerically similar to those in the Benchmark Case. As a result, intratemporal distortions are solely reflected by positive marginal labor tax rates faced by low-skill households.

Next, we turn to Case B where factor prices $\{w_t, r_t\}$ are exogenously given and a skill-type shock is present at period 2. The marginal tax wedges for both types on their first-period capital savings are now positive with $\tau_{k,1}(\theta_s) > \tau_{k,1}(\theta_u) > 0$. Moreover, these tax rates are higher than those in the Benchmark Case, indicating the government's motive to distort agents' period-1 capital accumulation downwards to relax their incentive-compatibility constraints within our partial-equilibrium setup with productivity disturbances. We also note that $\tau_{k,2}(\cdot) = -0.7512$ for both types of agents. As in the benchmark setting, there exists an inclination to encourage greater savings from each individual, exerting a downward pressure on $\tau_{k,2}$. Unlike the benchmark model, the necessity to augment capital accumulation will take place because of the requirement of a fixed $r_3 = \frac{1}{\delta} - 1$, which further intensifies the downward pressure on $\tau_{k,2}$.[20] It follows that the optimal tax wedge on period-2 capital savings $\tau_{k,2}$ in Case B is lower than that in the Benchmark Case. In terms of the optimal labor tax rates, $\tau_{l,2}(\cdot)/\tau_{l,3}(\cdot)$ for all households in Case B are lower/higher than those within the benchmark economy. This non-smooth qualitative pattern underscores the social planner's motive to fulfill the constant-$\{w, r\}$ condition.

Our Case C is similar to Guo and Krause's 2015 model with exogenous interest/wage rates with no productivity shocks, but our setup also postulates that the government can commit to its tax policy rules. In comparison to Case B, there exits a downward distortion at the intertemporal margin in period 1 that will decrease the optimal capital tax rates for both types within this configuration, leading to $\tau_{k,1}(\theta_s) > 0 > \tau_{k,1}(\theta_u)$. Since $r_t = \frac{1}{\delta} - 1$ for Cases B and C, their capital-labor ratios will be equal to the same constant each period.

[20] In the absence of a constant $r_3 = \frac{1}{\delta} - 1$, the period-3 capital level will decrease since $t = 3$ is the terminal period. The inclusion of this condition introduces an additional factor resulting in an increase of K_3.

However, the corresponding levels of aggregate capital and aggregate labor are different because of the presence or absence of skill-level disturbances. Table 2 shows that their resulting capital tax rates on period-2 capital investment $\tau_{k,2}(\cdot)$ are quite close to each other. In a manner akin to Case B, skilled workers are subject to a negative marginal labor tax wedge at $t = 2$ and positive labor income taxation at $t = 3$ within Case C. We also observe that $\tau_{l,2}(\cdot)/\tau_{l,3}(\cdot)$ for all households in Case C are lower/higher than those in Case A.

The last row of Table 2 presents the utility level of social welfare for each case under $\beta = 0.95$. When the interest and wage rates are exogenously given, the social planner faces additional binding constraints that will *ceteris paribus* yield notable welfare loss. While keeping all other aspects of the model unchanged, we find that the presence of a productivity shock at $t = 2$ may raise the economy's aggregate welfare by providing an opportunity for low-type individuals to climb the skill ladder, thus enhancing social mobility. However, high-type workers encounter the possibility of becoming unskilled at the same time period. For our baseline parameterization with $\pi(\theta_u) = 0.55$ and $\pi(\theta_s|\theta_s) = \pi(\theta_u|\theta_u) = 0.9$, the fraction of the population transitioning from low-skill to high-skill is 0.055, while the fraction for the reversed transition is 0.045. Consequently, the welfare benefits of higher social mobility outweigh the corresponding welfare costs.

5.2 Traditional Exponential Discounting with $\beta = 1$

This subsection presents our numerical results of the optimal implicit marginal tax rates and social welfare under traditional exponential discounting with $\beta = 1$. As shown in the Benchmark Case of Table 4, both $\tau_{k,1}(\theta_s)$ and $\tau_{k,1}(\theta_u)$ are positive such that all agents' period-1 capital savings are distorted downwards to relax their incentive-compatibility constraints. Moreover, the difference between these tax wedges is close to zero. In the corresponding setup under quasi-hyperbolic discounting with $\beta = 0.95$, recall that both capital tax rates are negative and that $\tau_{k,1}(\theta_s)$ exceeds $\tau_{k,1}(\theta_u)$ by around 3%. Putting these results together shows that the optimal tax wedges for both types on their first-period capital investment are *ceteris paribus* monotonically decreasing in the degree of quasi-hyperbolic discounting (or increasing in β), because the need to correct negative preference internalities will be strengthened (attenuated).

Next, the optimal tax rate applicable to the high-skill type's labor income is equal to zero for all t and β, as stated in Proposition 3. On the other hand, the intratemporal distortion for low-skill workers reflects the social planner's need to satisfy agents' binding temporary incentive-compatible constraints, preventing skilled agents from imitating unskilled households. Comparing Table 3 versus Table 5 shows that at periods 2 and 3, the optimal intratemporal wedges for type-(s, u) individuals decrease in β, whereas the marginal labor tax rates for type-(u, u) workers increase in β. This quantitative outcome depends on the relative strengths of two contrasting effects. First, stronger quasi-hyperbolic discounting with a lower $\beta \in (0, 1)$ will prompt the social planner to reduce more current-period (relative to previous-period) consumption for the low-skilled type. Second, since an increase in β tightens the skilled worker's incentive-compatibility constraint, low-skill individuals will face a higher marginal tax rate on their labor income. In order to induce truth-telling from type-(s, s) households who garner the highest informational rent, the social planner needs to reduce type-(s, u) agents' period-2 consumption to a greater extent. Under our baseline parameterization, the first effect is numerically stronger to yield $\tau_{l,2}(\theta_s, \theta_u)$ and $\tau_{l,3}(\theta_s, \theta_u)$ higher with $\beta < 1$ than those when $\beta = 1$. Conversely, a slightly reduced consumption assigned to type-(u, u) individuals will provide sufficient incentive for the truth-revealing behavior of type-(u, s) agents. As a consequence, the second effect dominates and thus $\tau_{l,2}(\theta_u, \theta_u)$ and $\tau_{l,3}(\theta_u, \theta_u)$ are lower when $\beta < 1$ than those with $\beta = 1$.

In Case A where factor input prices $\{w_t, r_t\}$ are endogenously determined without the presence of skill-type shocks at period 2, the social planner has no desire to distort individuals' intertemporal margin under traditional exponential discounting. Following directly from eq. (54), the optimal capital tax rates will be zero for all agents on their period-1 and period-2 investment expenditures. In terms of labor income taxation, skilled workers still face no intratemporal wedge; whereas unskilled-(u, u) individuals are subject to higher labor tax rates in periods 2 and 3 when $\beta = 1$, as compared to those with $\beta = 0.95$. According to the above discussion, this is cased by a quantitatively stronger effect from tightening the high-type's incentive-compatibility constraint as β rises.

In Case B where the interest/wage rates are exogenous with the presence of productivity shocks, the optimal tax rates on period-1 capital savings exhibit that $\tau_{k,1}(\theta_s) > \tau_{k,1}(\theta_u) > 0$. Since households lack the temptation to consume immediately under traditional exponential discounting, there is no need to subsidize capital investment made at $t = 1$ for both types of workers. In addition, these tax wedges are higher than those in the Benchmark Case, because the government needs to discourage agents' first-period savings through relaxing their incentive-compatibility constraints. In order to satisfy the requirement of a fixed $r_3 = \frac{1}{\delta} - 1$ that exerts a downward distortion on the marginal tax rates for period-2 capital savings, $\tau_{k,2}(\cdot)$ will be a negative constant $(= -0.6632)$ for all individuals.

When skill-type disturbances are abstracted away in Case C with a downward pressure at the intertemporal margin on period-1 capital investment, we find that $\tau_{k,1}(\theta_s)$ and $\tau_{k,1}(\theta_u)$ remain positive and lower than those in Case B. Per the preceding discussion on our model under $\beta = 0.95$, Table 4 shows that the optimal capital tax rates on period-2 savings $\tau_{k,2}(\cdot)$ are very close to each in Cases B and C under $\beta = 1$. As in the benchmark model and Case A with endogenous factor input prices, the optimal capital tax wedges within our partial-equilibrium Cases B and C, given by $\tau_{k,1}(\cdot)$ and $\tau_{k,2}(\cdot)$, are *ceteris paribus* monotonically increasing with respect to β as well. We also note that there are no noticeable numerical differences between Case B and Case C with regard to the labor tax rates. In particular, the intratemporal distortions for type-(s,u) agents are decreasing in β, whereas the labor tax wedges for type-(u,u) individuals are increasing in β. The intuitive explanations for these results are provided earlier on our Benchmark Case.

Table 4: Marginal Capital Tax Rates and Social Welfare under $\beta = 1$

	Benchmark Case: Endo. $\{w,r\}$ with shocks	Case A: Endo. $\{w,r\}$ without shocks	Case B: Exog. $\{w,r\}$ with shocks	Case C: Exog. $\{w,r\}$ without shocks
$\tau_{k,1}(\theta_s)$	0.0290	0	0.0700	0.0426
$\tau_{k,1}(\theta_u)$	0.0273	0	0.0687	0.0426
$\tau_{k,2}(\theta_s,\theta_s)$	0	0	-0.6632	-0.6633
$\tau_{k,2}(\theta_s,\theta_u)$	0	N.A.	-0.6632	N.A.
$\tau_{k,2}(\theta_u,\theta_s)$	0	N.A.	-0.6632	N.A.
$\tau_{k,2}(\theta_u,\theta_u)$	0	0	-0.6632	-0.6633
SW	-1.4663	-1.473	-1.4932	-1.5000

Table 5: Marginal Labor Tax Rates under $\beta = 1$

	Benchmark Case: Endo. $\{w,r\}$ with shocks	Case A: Endo. $\{w,r\}$ without shocks	Case B: Exog. $\{w,r\}$ with shocks	Case C: Exog. $\{w,r\}$ without shocks
$\tau_{l,1}(\theta_s)$	0	0	0	0
$\tau_{l,1}(\theta_u)$	0.1469	0.1690	0.1469	0.1688
$\tau_{l,2}(\theta_s,\theta_s)$	0	0	-0.0184	-0.0183
$\tau_{l,2}(\theta_s,\theta_u)$	0.2954	N.A.	0.2789	N.A.
$\tau_{l,2}(\theta_u,\theta_s)$	0	N.A.	-0.0184	N.A.
$\tau_{l,2}(\theta_u,\theta_u)$	0.1730	0.1690	0.1574	0.1536
$\tau_{l,3}(\theta_s,\theta_s)$	0	0	0.2842	0.2843
$\tau_{l,3}(\theta_s,\theta_u)$	0.2954	N.A.	0.4932	N.A.
$\tau_{l,3}(\theta_u,\theta_s)$	0	N.A.	0.2842	N.A.
$\tau_{l,3}(\theta_u,\theta_u)$	0.1730	0.1690	0.4078	0.4051

Finally, the last row of Table 4 illustrates that the social welfare under $\beta = 1$ displays a qualitatively identical pattern as that under $\beta = 0.95$ across all four model economies.

In addition, comparing Table 4 and Table 2 illustrates that the economy's aggregate utility increases with respect to β for each formulation under consideration. Our numerical results thus complement Guo and Krause's (2015, section 6) analysis, which shows that the impact of changing β on social welfare is theoretically ambiguous in a deterministic, partial-equilibrium framework where the government can commit to its fiscal policy. In this environment, these authors report that the welfare effects of quasi-hyperbolic discounting depend on the modelling details and the calibrated parameter values. Here, we find that whether the model economy is a general- or partial-equilibrium setting plays a critically important role in determining its social welfare.

6 Conclusion

In this paper we have examined, both theoretically and numerically, the effects of incorporating quasi-hyperbolic discounting and idiosyncratic productivity shocks into a dynamic (three-period) general equilibrium model of optimal nonlinear income taxation with commitment. Using the derived cumulative Lagrange multiplier associated with agents' incentive-compatibility constraints, we obtain the model's Inverse Euler equations that will govern the socially optimal resource allocations across different time periods. In a simplified two-type environment without skill-type uncertainties, it is analytically shown that the optimal tax rates on capital savings at period 1 are negative for both types of individuals, and that unskilled workers are subject to a lower capital tax wedge than their skilled counterparts. We further prove that adding productivity disturbances will raise these capital tax rates on first-period savings. When factor input prices are exogenously given, our calibrated numerical simulations show that the marginal tax wedges for both types on their period-1 capital investment become positive in the presence of a skill-type shock. We also quantitatively find that the optimal tax rates for both types of households on their first- and second-period capital savings, as well as the economy's social welfare, are *ceteris paribus* decreasing with respect to the degree of quasi-hyperbolic discounting.

This paper can be extended in several directions. In particular, it would be worthwhile to investigate an overlapping generations setting that allows for economic interactions between individuals of different stages of their respective lifetime. Moreover, it would be valuable to study an economy inhabited by sophisticated quasi-hyperbolic discounters, who are aware of their gratification toward immediate consumption and leisure. This in turn affects how agents behave in the absence of taxation (section 2.1) and thus the resulting equations that characterize the implicit marginal tax rates at the social optimum (section 4). These possible extensions will enhance our understanding of the qualitative and quantitative interrelations between the optimal tax wedges versus quasi-hyperbolic discounting along with other modelling features. We plan to pursue these research projects in the near future.

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7 Appendix

7.1 Solving the Planning Problem

We formulate the Lagrangian:

$$
\begin{aligned}
\mathcal{L} \;=\; & \sum_{\theta_1}\left[u\left(c_1\left(\theta_1\right)\right)-v\left(\frac{l_1\left(\theta_1\right)}{\theta_1}\right)\right]\pi\left(\theta_1\right)+\delta\sum_{\theta_1}\sum_{\theta_2}\left[u\left(c_2\left(\theta^2\right)\right)-v\left(\frac{l_2\left(\theta^2\right)}{\theta_2}\right)\right]\pi_2\left(\theta^2\right) \\
& +\delta^2\sum_{\theta^2}\left[u\left(c_3\left(\theta^2\right)\right)-v\left(\frac{l_3\left(\theta^2\right)}{\theta_2}\right)\right]\pi_2\left(\theta^2\right) \\
& +\sum_{\theta_1,\tilde{\theta}_1}\lambda_1\left(\theta_1,\tilde{\theta}_1\right)\left[\begin{array}{c}
u\left(c_1\left(\theta_1\right)\right)-v\left(\frac{l_1(\theta_1)}{\theta_1}\right)+\beta\delta\sum_{\theta_2}\left[u\left(c_2\left(\theta^2\right)\right)-v\left(\frac{l_2(\theta^2)}{\theta_2}\right)\right]\pi\left(\theta_2|\theta_1\right) \\
+\beta\delta^2\sum_{\theta_2}\left[u\left(c_3\left(\theta^2\right)\right)-v\left(\frac{l_3(\theta^2)}{\theta_2}\right)\right]\pi\left(\theta_2|\theta_1\right) \\
-u\left(c_1\left(\tilde{\theta}_1;\theta_1\right)\right)+v\left(\frac{l_1(\tilde{\theta}_1;\theta_1)}{\theta_1}\right) \\
-\beta\delta\sum_{\theta_2}\left[u\left(c_2\left(\tilde{\theta}_1,\theta_2;\theta^2\right)\right)-v\left(\frac{l_2(\tilde{\theta}_1,\theta_2;\theta^2)}{\theta_2}\right)\right]\pi\left(\theta_2|\theta_1\right) \\
-\beta\delta^2\sum_{\theta_2}\left[u\left(c_3\left(\tilde{\theta}_1,\theta_2;\theta^2\right)\right)-v\left(\frac{l_3(\tilde{\theta}_1,\theta_2;\theta^2)}{\theta_2}\right)\right]\pi\left(\theta_2|\theta_1\right)
\end{array}\right]\pi\left(\theta_1\right) \\
& +\delta\sum_{\theta^2,\tilde{\theta}_2}\lambda_2\left(\theta^2,\tilde{\theta}_2\right)\left[\begin{array}{c}
u\left(c_2\left(\theta^2\right)\right)-v\left(\frac{l_2(\theta^2)}{\theta_2}\right)+\delta\left[u\left(c_3\left(\theta^2\right)\right)-v\left(\frac{l_3(\theta^2)}{\theta_2}\right)\right] \\
-u\left(c_2\left(\theta_1,\tilde{\theta}_2;\theta^2\right)\right)+v\left(\frac{l_2(\theta_1,\tilde{\theta}_2;\theta^2)}{\theta_2}\right) \\
-\delta\left[u\left(c_3\left(\theta_1,\tilde{\theta}_2;\theta^2\right)\right)-v\left(\frac{l_3(\theta_1,\tilde{\theta}_2;\theta^2)}{\theta_2}\right)\right]
\end{array}\right]\pi_2\left(\theta^2\right) \\
& +\mu_1\left[F\left(K_1,\sum_{\theta_1}l_1\left(\theta_1\right)\pi\left(\theta_1\right)\right)-\sum_{\theta_1}c_1\left(\theta_1\right)\pi\left(\theta_1\right)-K_2\right] \\
& +\delta\mu_2\left[F\left(K_2,\sum_{\theta^2}l_2\left(\theta^2\right)\pi_2\left(\theta^2\right)\right)-\sum_{\theta^2}c_2\left(\theta^2\right)\pi_2\left(\theta_2\right)-K_3\right] \\
& +\delta^2\mu_3\left[F\left(K_3,\sum_{\theta^2}l_3\left(\theta^2\right)\pi_2\left(\theta^3\right)\right)-\sum_{\theta^2}c_3\left(\theta^2\right)\pi_2\left(\theta^3\right)\right], \qquad \text{(A.1)}
\end{aligned}
$$

where the Lagrange multipliers $\left\{\lambda_1\left(\theta_1,\tilde{\theta}_1\right),\lambda_2\left(\theta^2,\tilde{\theta}_2\right)\right\}$ associated with incentive-compatibility constraints will induce type θ_1 and type $\theta^2=(\theta_1,\theta_2)$ to truth-tell their skills and not to mimic type $\tilde{\theta}_1$ and type $(\theta_1,\tilde{\theta}_2)$, respectively.

Next, we simplify the Lagrangian to

$$
\begin{aligned}
\mathcal{L} \;=\; & \sum_{\theta_1}\left[\phi_1(\theta_1)\,u\left(c_1(\theta_1)\right)-v\left(\frac{l_1(\theta_1)}{\theta_1}\right)\right]\pi(\theta_1) \\
& +\delta\sum_{\theta^2}\left[\phi_2\left(\theta^2\right)u\left(c_2\left(\theta^2\right)\right)-v\left(\frac{l_2\left(\theta^2\right)}{\theta_2}\right)\right]\pi_2\left(\theta^2\right) \\
& +\delta^2\sum_{\theta^2}\left[\phi_2\left(\theta^2\right)u\left(c_3\left(\theta^2\right)\right)-v\left(\frac{l_3\left(\theta^2\right)}{\theta_2}\right)\right]\pi_2\left(\theta^2\right) \\
& +\sum_{\theta_1,\tilde{\theta}_1}\lambda_1\left(\theta_1,\tilde{\theta}_1\right)\left[\begin{array}{c}
-v\left(\frac{l_1(\theta_1)}{\theta_1}\right)-\beta\delta\sum_{\theta_2}v\left(\frac{l_2\left(\theta^2\right)}{\theta_2}\right)\pi\left(\theta_2|\theta_1\right) \\
-\beta\delta^2\sum_{\theta_2}v\left(\frac{l_3\left(\theta^2\right)}{\theta_2}\right)\pi\left(\theta_2|\theta_1\right) \\
+v\left(\frac{l_1\left(\tilde{\theta}_1;\theta_1\right)}{\theta_1}\right)+\beta\delta\sum_{\theta_2}v\left(\frac{l_2\left(\tilde{\theta}_1,\theta_2;\theta^2\right)}{\theta_2}\right)\pi\left(\theta_2|\theta_1\right) \\
+\beta\delta^2\sum_{\theta_2}v\left(\frac{l_3\left(\tilde{\theta}_1,\theta_2;\theta^2\right)}{\theta_2}\right)\pi\left(\theta_2|\theta_1\right)
\end{array}\right]\pi(\theta_1) \\
& +\delta\sum_{\theta^2,\tilde{\theta}_2}\lambda_2\left(\theta^2,\tilde{\theta}_2\right)\left[\begin{array}{c}
-v\left(\frac{l_2\left(\theta^2\right)}{\theta_2}\right)-\delta v\left(\frac{l_3\left(\theta^2\right)}{\theta_2}\right) \\
+v\left(\frac{l_2\left(\theta_1,\tilde{\theta}_2;\theta^2\right)}{\theta_2}\right)+\delta v\left(\frac{l_3\left(\theta_1,\tilde{\theta}_2;\theta^2\right)}{\theta_2}\right)
\end{array}\right]\pi\left(\theta^2\right) \\
& +\mu_1\left[F\left(\sum_{\theta_1}l_1(\theta_1)\,\pi(\theta_1)\right)-\sum_{\theta_1}c_1(\theta_1)\,\pi(\theta_1)-K_2\right] \\
& +\delta\mu_2\left[F\left(K_2,\sum_{\theta^2}l_2\left(\theta^2\right)\pi_2\left(\theta^2\right)\right)-\sum_{\theta^2}c_2\left(\theta^2\right)\pi_2(\theta_2)-K_3\right] \\
& +\delta^2\mu_3\left[F\left(K_3,\sum_{\theta^2}l_3\left(\theta^2\right)\pi_2(\theta_2)\right)-\sum_{\theta^2}c_3\left(\theta^2\right)\pi_2(\theta_2)\right], \qquad \text{(A.2)}
\end{aligned}
$$

where the definitions of $\phi_1(\theta_1)$ and $\phi_2\left(\theta^2\right)$ are given by (35) and (36). Solving this planning problem directly yields the optimal conditions (32)-(34), (42)-(43), and (49)-(51).

7.2 Proof of Lemma

Using eqs. (35) and (36), we obtain that

$$
\phi_2\left(\theta^2\right)=1-\beta+\beta\phi_1(\theta_1)+\beta\varepsilon\left(\theta^2\right), \qquad \text{(A.3)}
$$

where

$$\varepsilon\left(\theta^2\right) \equiv \sum_{\tilde{\theta}_2 \neq \theta_2} \left\{ \lambda_2\left(\theta^2, \tilde{\theta}_2\right) - \lambda_2\left(\theta_1, \tilde{\theta}_2, \theta_2\right) \frac{\pi\left(\tilde{\theta}_2|\theta_1\right)}{\pi\left(\theta_2|\theta_1\right)} \right\}$$
$$+\beta \left[\sum_{\tilde{\theta}_1 \neq \theta_1} \left\{ \lambda_1\left(\theta_1, \tilde{\theta}_1\right) - \lambda_1\left(\tilde{\theta}_1, \theta_1\right) \frac{\pi\left(\tilde{\theta}_1\right)}{\pi\left(\theta_1\right)} \left(\frac{\pi\left(\theta_2|\tilde{\theta}_1\right)}{\pi\left(\theta_2|\theta_1\right)} - 1 \right) \right\} \right]. \quad \text{(A.4)}$$

Next, we manipulate eq. (A.4) to find that

$$\sum_{\theta_2} \varepsilon\left(\theta^2\right) \pi\left(\theta_2|\theta_1\right) = \sum_{\theta_2} \left[\sum_{\tilde{\theta}_2 \neq \theta_2} \lambda_2\left(\theta^2, \tilde{\theta}_2\right) - \sum_{\tilde{\theta}_2 \neq \theta_2} \lambda_2\left(\theta_1, \tilde{\theta}_2, \theta_2\right) \frac{\pi\left(\tilde{\theta}_2|\theta_1\right)}{\pi\left(\theta_2|\theta_1\right)} \right] \pi\left(\theta_2|\theta_1\right)$$
$$-\beta \sum_{\theta_2} \left[\sum_{\tilde{\theta}_1 \neq \theta_1} \lambda_1\left(\tilde{\theta}_1, \theta_1\right) \frac{\pi\left(\tilde{\theta}_1\right)}{\pi\left(\theta_1\right)} \left(\frac{\pi\left(\theta_2|\tilde{\theta}_1\right)}{\pi\left(\theta_2|\theta_1\right)} - 1 \right) \right] \pi\left(\theta_2|\theta_1\right)$$
$$= 0 - \beta \sum_{\tilde{\theta}_1 \neq \theta_1} \lambda_1\left(\tilde{\theta}_1, \theta_1\right) \frac{\pi\left(\tilde{\theta}_1\right)}{\pi\left(\theta_1\right)} \sum_{\theta_2} \left[\pi\left(\theta_2|\tilde{\theta}_1\right) - \pi\left(\theta_2|\theta_1\right)\right] = 0, \quad \text{(A.5)}$$

where $\sum_{\theta_2} \sum_{\tilde{\theta}_2 \neq \theta_2} \lambda_2\left(\theta^2, \tilde{\theta}_2\right) \pi\left(\theta_2|\theta_1\right) = \sum_{\theta_2} \sum_{\tilde{\theta}_2 \neq \theta_2} \lambda_2\left(\theta_1, \tilde{\theta}_2, \theta_2\right) \pi\left(\tilde{\theta}_2|\theta_1\right)$ holds through an index exchange. Based on (A.3), we also derive that

$$\phi_2\left(\theta^2\right) = \phi_1\left(\theta_1\right) + (1-\beta)\left(1 - \phi_1\left(\theta_1\right)\right) + \varepsilon\left(\theta^2\right),$$

which can be re-arranged to result in (38). Finally, we manipulate (35) to yield

$$\sum_{\theta_1} \phi_1\left(\theta_1\right) \pi\left(\theta_1\right) = 1. \quad \text{(A.6)}$$

7.3 Proof of Proposition 3

Since type-θ_s agents have the incentive to mimic type-θ_u individuals, it must be the case that $\lambda_1\left(\theta_s, \theta_u\right) > 0$, $\lambda_1\left(\theta_u, \theta_s\right) = 0$, together with $\lambda_2\left(\{\theta_i, \theta_s\}, \theta_u\right) > 0$ and $\lambda_2\left(\{\theta_i, \theta_u\}, \theta_s\right) = 0$, where $i \in \{s, u\}$. As a result, eqs. (49)-(50) can be re-expressed as

$$\left(1 + \lambda_1\left(\theta_s, \theta_u\right)\right) v'\left(\frac{l_1\left(\theta_s\right)}{\theta_s}\right) \frac{1}{\theta_s} = \mu_1 F_{L,1}, \quad \text{(A.7)}$$

$$v'\left(\frac{l_1\left(\theta_u\right)}{\theta_u}\right) \frac{1}{\theta_u} - \lambda_1\left(\theta_s, \theta_u\right) \frac{\pi\left(\theta_s\right)}{\pi\left(\theta_u\right)} v'\left(\frac{l_1\left(\theta_u\right)}{\theta_s}\right) \frac{1}{\theta_s} = \mu_1 F_{L,1}, \quad \text{(A.8)}$$

$$\left(1+\beta\lambda_1\left(\theta_s,\theta_u\right)+\lambda_2\left(\left\{\theta_s,\theta_s\right\},\theta_u\right)\right)v'\left(\frac{l_2\left(\theta_s,\theta_s\right)}{\theta_s}\right)\frac{1}{\theta_s}=\mu_2 F_{L,2}, \tag{A.9}$$

$$\left(1+\beta\lambda_1\left(\theta_s,\theta_u\right)\right)v'\left(\frac{l_2\left(\theta_s,\theta_u\right)}{\theta_u}\right)\frac{1}{\theta_u}-\left(\lambda_2\left(\left\{\theta_s,\theta_s\right\},\theta_u\right)\frac{\pi\left(\theta_s|\theta_s\right)}{\pi\left(\theta_u|\theta_s\right)}\right)v'\left(\frac{l_2\left(\theta_s,\theta_u\right)}{\theta_s}\right)\frac{1}{\theta_s}=\mu_2 F_{L,2}, \tag{A.10}$$

$$\begin{aligned}&\left(1+\lambda_2\left(\left\{\theta_u,\theta_s\right\},\theta_u\right)\right)v'\left(\frac{l_2\left(\theta_u,\theta_s\right)}{\theta_s}\right)\frac{1}{\theta_s}\\ &-\beta\left(\lambda_1\left(\theta_s,\theta_u\right)\frac{\pi\left(\theta_s|\theta_s\right)}{\pi\left(\theta_s|\theta_u\right)}\frac{\pi\left(\theta_s\right)}{\pi\left(\theta_u\right)}\right)v'\left(\frac{l_2\left(\theta_u,\theta_s\right)}{\theta_s}\right)\frac{1}{\theta_s}=\mu_2 F_{L,2}, \text{ and}\end{aligned} \tag{A.11}$$

$$\begin{aligned}&\left(1-\beta\lambda_1\left(\theta_s,\theta_u\right)\frac{\pi\left(\theta_u|\theta_s\right)}{\pi\left(\theta_u|\theta_u\right)}\frac{\pi\left(\theta_s\right)}{\pi\left(\theta_u\right)}\right)v'\left(\frac{l_2\left(\theta_u,\theta_u\right)}{\theta_u}\right)\frac{1}{\theta_u}\\ &-\left(\lambda_2\left(\left\{\theta_u,\theta_s\right\},\theta_u\right)\frac{\pi\left(\theta_s|\theta_u\right)}{\pi\left(\theta_u|\theta_u\right)}\right)v'\left(\frac{l_2\left(\theta_u,\theta_u\right)}{\theta_s}\right)\frac{1}{\theta_s}=\mu_2 F_{L,2}.\end{aligned} \tag{A.12}$$

Notice that the first-order conditions for l_3 are similar with those for l_2. By exploiting $\lambda_1\left(\theta_s,\theta_u\right)>0$ and $\lambda_1\left(\theta_u,\theta_s\right)=0$, we rewrite (35) and (36) as

$$\phi_1\left(\theta_s\right)=1+\lambda_1\left(\theta_s,\theta_u\right), \tag{A.13}$$

$$\phi_1\left(\theta_u\right)=1-\lambda_1\left(\theta_s,\theta_u\right)\frac{\pi\left(\theta_s\right)}{\pi\left(\theta_u\right)}, \tag{A.14}$$

$$\phi_2\left(\theta_s,\theta_s\right)=\phi_1\left(\theta_s\right)+\left(1-\beta\right)\left(1-\phi_1\left(\theta_s\right)\right)+\lambda_2\left(\left\{\theta_s,\theta_s\right\},\theta_u\right), \tag{A.15}$$

$$\phi_2\left(\theta_s,\theta_u\right)=\phi_1\left(\theta_s\right)+\left(1-\beta\right)\left(1-\phi_1\left(\theta_s\right)\right)-\lambda_2\left(\left\{\theta_s,\theta_s\right\},\theta_u\right)\frac{\pi\left(\theta_s|\theta_s\right)}{\pi\left(\theta_u|\theta_s\right)}, \tag{A.16}$$

$$\phi_2\left(\theta_u,\theta_s\right)=1-\beta\lambda_1\left(\theta_s,\theta_u\right)\frac{\pi\left(\theta_s|\theta_s\right)\pi\left(\theta_s\right)}{\pi\left(\theta_s|\theta_u\right)\pi\left(\theta_u\right)}+\lambda_2\left(\left\{\theta_u,\theta_s\right\},\theta_u\right), \text{ and} \tag{A.17}$$

$$\phi_2\left(\theta_u,\theta_u\right)=1-\beta\lambda_1\left(\theta_s,\theta_u\right)\frac{\pi\left(\theta_s|\theta_s\right)\pi\left(\theta_s\right)}{\pi\left(\theta_s|\theta_u\right)\pi\left(\theta_u\right)}-\lambda_2\left(\left\{\theta_u,\theta_s\right\};\theta_u\right)\frac{\pi\left(\theta_s|\theta_u\right)}{\pi\left(\theta_u|\theta_u\right)}. \tag{A.18}$$

Combining (13), (32), (A.7) and (A.8), along with (A.13) and (A.14), yields that

$$\frac{v'\left(\frac{l_1\left(\theta_s\right)}{\theta_s}\right)}{w_1\theta_s u'\left(c_1\left(\theta_s\right)\right)}=1, \tag{A.19}$$

$$\frac{v'\left(\frac{l_1\left(\theta_u\right)}{\theta_u}\right)}{w_1\theta_u u'\left(c_1\left(\theta_u\right)\right)}=\frac{1-\lambda_1\left(\theta_s,\theta_u\right)\frac{\pi\left(\theta_s\right)}{\pi\left(\theta_u\right)}}{1-\lambda_1\left(\theta_s,\theta_u\right)\frac{\pi\left(\theta_s\right)}{\pi\left(\theta_u\right)}\frac{v'\left(\frac{l_1\left(\theta_u\right)}{\theta_s}\right)}{v'\left(\frac{l_1\left(\theta_u\right)}{\theta_u}\right)}\frac{\theta_u}{\theta_s}}, \tag{A.20}$$

$$\frac{v'\left(\frac{l_2(\theta_s,\theta_s)}{\theta_s}\right)}{w_2\theta_s u'\left(c_2\left(\theta_s,\theta_s\right)\right)}=1, \tag{A.21}$$

$$\frac{v'\left(\frac{l_2(\theta_s,\theta_u)}{\theta_u}\right)}{w_2\theta_u u'\left(c_2\left(\theta_s,\theta_u\right)\right)}=\frac{1+\beta\lambda_1\left(\theta_s,\theta_u\right)-\lambda_2\left(\theta_s,\theta_s,\theta_u\right)\frac{\pi(\theta_s|\theta_s)}{\pi(\theta_u|\theta_s)}}{1+\beta\lambda_1\left(\theta_s,\theta_u\right)-\lambda_2\left(\left\{\theta_s,\theta_s\right\},\theta_u\right)\frac{\pi(\theta_s|\theta_s)}{\pi(\theta_u|\theta_s)}\frac{v'\left(\frac{l_2(\theta_s,\theta_u)}{\theta_s}\right)}{v'\left(\frac{l_2(\theta_s,\theta_u)}{\theta_u}\right)}\frac{\theta_u}{\theta_s}}, \tag{A.22}$$

$$\frac{v'\left(\frac{l_2(\theta_u,\theta_s)}{\theta_s}\right)}{w_2\theta_s u'\left(c_2\left(\theta_u,\theta_s\right)\right)}=1, \text{ and} \tag{A.23}$$

$$\frac{v'\left(\frac{l_2(\theta_u,\theta_u)}{\theta_u}\right)}{w_2\theta_u u'\left(c_2\left(\theta^2\right)\right)}=\frac{1-\beta\lambda_1\left(\theta_s,\theta_u\right)\frac{\pi(\theta_s|\theta_s)\pi(\theta_s)}{\pi(\theta_s|\theta_u)\pi(\theta_u)}-\lambda_2\left(\left\{\theta_u,\theta_s\right\};\theta_u\right)\frac{\pi(\theta_s|\theta_u)}{\pi(\theta_u|\theta_u)}}{1-\beta\lambda_1\left(\theta_s,\theta_u\right)\frac{\pi(\theta_u|\theta_s)}{\pi(\theta_u|\theta_u)}\frac{\pi(\theta_s)}{\pi(\theta_u)}-\lambda_2\left(\left\{\theta_u,\theta_s\right\},\theta_u\right)\frac{\pi(\theta_s|\theta_u)}{\pi(\theta_u|\theta_u)}\frac{v'\left(\frac{l_2(\theta_u,\theta_u)}{\theta_s}\right)}{v'\left(\frac{l_2(\theta_u,\theta_u)}{\theta_u}\right)}\frac{\theta_u}{\theta_s}}. \tag{A.24}$$

Since the household utility exhibits $v''>0$, we find that (i) $\frac{v'\left(\frac{l_1(\theta_u)}{\theta_s}\right)}{v'\left(\frac{l_1(\theta_u)}{\theta_u}\right)}\frac{\theta_u}{\theta_s}<1$, (ii) $\frac{v'\left(\frac{l_2(\theta_s,\theta_u)}{\theta_s}\right)}{v'\left(\frac{l_2(\theta_s,\theta_u)}{\theta_u}\right)}\frac{\theta_u}{\theta_s}<1$, and (iii) $\frac{v'\left(\frac{l_2(\theta_u,\theta_u)}{\theta_s}\right)}{v'\left(\frac{l_2(\theta_u,\theta_u)}{\theta_u}\right)}\frac{\theta_u}{\theta_s}<1$. Substituting (i)-(iii) into (15) results in (57) and (58). By the same procedure, we will obtain (59).