

# Identification and Estimation of Nonparametric Hedonic Equilibrium Model with Unobserved Quality\*

Ruoyao Shi<sup>†</sup>

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## Abstract

This paper studies a nonparametric hedonic equilibrium model in which certain product characteristics are unobserved. Unlike most previously studied hedonic models, both the observed and unobserved agent heterogeneities enter the structural functions nonparametrically. Prices are endogenously determined in equilibrium. Using both within- and cross-market price variation, I show that all the structural functions of the model are nonparametrically identified up to normalization. In particular, the unobserved product quality function is identified if the relative prices of the agent characteristics differ in at least two markets. Following the constructive identification strategy, I provide easy-to-implement series minimum distance estimators of the structural functions and derive their consistency and uniform rates of convergence. To illustrate the estimation procedure, I estimate the unobserved efficiency of American full-time workers as a function of age and unobserved ability.

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<sup>†</sup>Department of Economics, University of California Riverside. Email: ruoyao.shi@ucr.edu.

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# 1 Introduction

Counterfactual distributions are indispensable components for the evaluation of distributional effects of large-scale policy interventions or social changes; they can also be used to measure the values of public good or natural resources. For example, labor economists might be interested in constructing the counterfactual wage distribution in 1988 had there been no de-unionization or decline in real minimum wage during the 1979-1988 period to evaluate the effect of labor market institutions on inequality (see DiNardo, Fortin, and Lemieux, 1996 for details). Another application of interest would be to measure heterogeneous willingness to pay for clean air as exhibited in housing prices (e.g., Sieg, Smith, Banzhaf, and Walsh, 2004, and Chay and Greenstone, 2005).

Three features should be acknowledged in the counterfactual distributional analysis. First, large-scale interventions usually affect a substantial proportion of the agents (e.g., DiNardo, Fortin, and Lemieux, 1996, and Chernozhukov, Fernández-Val, and Melly, 2013), hence the importance of accounting for the equilibrium effects is of first order (e.g., Sieg, Smith, Banzhaf, and Walsh, 2004). Second, some product characteristics might not be observed by researchers and their importance in price determination is widely recognized (e.g., Berry, Levinsohn, and Pakes, 1995), and workers' efficiency in labor markets is an important example. Third, there is considerable observed and unobserved heterogeneity among the agents. Ignoring any of them (e.g., ignoring changes in return to college education as more college graduates entered the labor force and other factors remained constant) is likely to result in biased counterfactual predictions.

This paper is the first to provide an economic model and an econometric method that admit all these features in a nonparametric setting. In this paper, I study a hedonic equilibrium model with unobserved product quality. I show that the quality function, together with all the other structural functions of the model, can be nonparametrically identified. I also provide easy-to-implement estimators for the structural functions and an algorithm to solve the counterfactual equilibrium. In contrast to widely used distributional decomposition methods, the counterfactuals

thus constructed shall account for equilibrium effects of large-scale interventions.

I incorporate unobserved product quality captured by a *structural quality function*  $e(x, a)$  into standard hedonic equilibrium models (e.g., Heckman, Matzkin, and Nesheim, 2010), which have been used to analyze the market equilibria of differentiated products with heterogeneous agents. Let  $z_i^m$  denote the *effective amount* of the product traded between seller-buyer pair  $i$  in market  $m$  upon which the payment is determined, and assume that

$$z_i^m \equiv h_i^m \cdot e(x_i^m, a_i^m), \quad (1.1)$$

where  $h_i^m$  represents observed quantity, vector  $x_i^m$  and scalar  $a_i^m$  represent the seller's observed and unobserved heterogeneity, respectively. I relax the restriction in standard hedonic equilibrium models that  $e(x, a) \equiv 1$  by allowing the functional form of the quality function (and hence, the values of  $e(x_i^m, a_i^m)$  and  $z_i^m$ ) to be unknown to researchers.

The identification strategy consists of three steps.<sup>1</sup> First, I show that the *reduced form* (equilibrium outcome) payment function  $I^m(x, a)$  and quantity function  $h^m(x, a)$  are nonparametrically identified within each market  $m$ . Second, I exploit within- and cross-market variation in the reduced form functions to identify the unobserved quality function up to normalization. Specifically, equation (1.1) indicates that quantity and quality are substitutes in determining the payment. As a result, variation in quality is manifested inversely in the variation in quantity among sellers who receive the same payment within the same market. Moreover, since quantity is optimally chosen by sellers, it suffers from an endogeneity problem. The different distributions of observed agent characteristics across markets serve as aggregate supply or demand shifters that induce cross-market variation in the payment functions, which facilitates the full identification of the quality function. The identification requirement boils down to a rank condition on the payment functions, which requires that relative prices of the agent characteristics vary across markets.<sup>2</sup> Finally, the third step utilizes the agents' first-order conditions to identify the marginal utility

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<sup>1</sup>This paper focuses on the supply side, since the identification and estimation of the demand side structural functions is completely symmetric.

<sup>2</sup>To focus on the key identification problem that arises because of unobserved quality, I concentrate on the scalar-valued quality function  $e(x, a)$  in the main text. It is, however, easy to extend the argument to a vector-valued quality function  $e(x, a)$  captured by a single-index structure as in Epple and Sieg (1999) and Sieg, Smith, Banzhaf, and Walsh (2004).

functions, in the spirit of the second step of Rosen (1974)’s method.<sup>3</sup>

The constructive identification strategy suggests an easy-to-implement series estimation procedure. I derive consistency and uniform rates of convergence of the estimators and demonstrate the procedure by estimating the unobserved efficiency of American full-time workers using data from the 2015 American Time Use Survey (ATUS).<sup>4</sup>

The literature on the identification and estimation of hedonic equilibrium models is vast. In his seminal work, Rosen (1974) originated a two-step method, of which the first step obtains the hedonic price function and its derivatives by fitting a parametric regression of prices on observed product characteristics, and the second step combines the hedonic price function and agents’ first-order conditions to recover the preference and production parameters. Ekeland, Heckman, and Nesheim (2004) considered the identification of a nonparametric hedonic equilibrium model with additive marginal utility and marginal production functions using single market data. Heckman, Matzkin, and Nesheim (2010) formalized the argument in Brown and Rosen (1982), Epple (1987) and Kahn and Lang (1988) that, in general, cross-market variation in price functions is necessary to nonparametrically identify the structural functions. They then focused on the sufficient restrictions for the identification using single market data, and generalized Rosen (1974)’s two-step method to a nonparametric setting. This paper advances the literature on hedonic equilibrium models in two ways. First, it allows product quality to be unobserved by researchers (but still observed by both seller and buyer), which captures a crucial feature of many applications. Second, this paper is the first to present a nonparametric estimation procedure and to provide convergence rates for the structural function estimators in hedonic equilibrium models using multiple market data.

The counterfactual analysis enabled by this model is closely related to an extensive literature on distributional decomposition methods (elegantly reviewed in Fortin, Lemieux, and Firpo (2011)), which aims to evaluate the distributional effects of policy interventions or historical changes. Several methods have been proposed, in-

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<sup>3</sup>Unlike Rosen (1974), the estimation procedure introduced in Section 4.1 does not require explicitly estimating the price schedule functions.

<sup>4</sup>I also propose an algorithm to solve for the counterfactual equilibrium of the model in Supplemental Appendix A. It is based on the equilibrium condition and Chiappori, McCann, and Nesheim (2010)’s insight that hedonic equilibrium models are mathematically equivalent to an optimal transportation problem. A simple simulation experiment indicates that the numerical equilibrium solution is stable with regard to the estimation errors in the structural functions.

cluding the imputation method (Juhn, Murphy, and Pierce, 1993), the reweighting method (DiNardo, Fortin, and Lemieux, 1996), the quantile regression-based method (Machado and Mata, 2005), the re-centered influence function method (Firpo, Fortin, and Lemieux, 2009), among many others (e.g., Fessler, Kasy, and Lindner, 2013, and Fessler and Kasy, 2016). Moreover, Rothe (2010) and Chernozhukov, Fernández-Val, and Melly (2013) considered inference in the context of distributional decomposition. This literature is based on the “selection on observables” assumption, which excludes general equilibrium effects. On the contrary, this paper establishes an equilibrium model, which allows the prices (e.g., the returns to college education) to change in response to changes in the distribution of the characteristics in the population (e.g., as more college graduates enter the labor force).

Characteristic-based demand models in industrial organization and marketing permit unobserved product characteristic as well. This immense literature dates back at least to Berry (1994) and Berry, Levinsohn, and Pakes (1995) and includes Rossi, McCulloch, and Allenby (1996), Nevo (2001), Petrin (2002), Berry, Levinsohn, and Pakes (2004), Bajari and Benkard (2005), Berry and Pakes (2007), and many others.<sup>5</sup> The econometric methods used to analyze characteristic-based demand models are reviewed by Akerberg, Benkard, Berry, and Pakes (2007).<sup>6</sup> Characteristic-based demand models focus on the demand side and often assume additively separable utility functions and parametric distributions for the random error terms, which facilitates the identification and estimation. In this paper, however, the utility functions are nonparametrically identified and estimated, and the estimators are of least-square type (and hence easy to implement).

The rest of this paper is organized as follows. Section 2.1 sets up the model and describes some important properties of the equilibrium; Section 2.2 discusses that this model can be applied to labor markets to conduct counterfactual policy analysis. Section 3 explains the nonparametric identification of the structural functions of the model. The key step is the identification of the unobserved quality function; the intuition and formal results of this step are given in Section 3.2. Section 4 describes

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<sup>5</sup>The utility functions in Bajari and Benkard (2005) and Berry and Pakes (2007) are similar to that in this paper, where only the characteristics of the products bear utility, but not the products *per se*. The consumers’ utility functions in models of Berry, Levinsohn and Pakes (1995) type have independently and identically distributed random error terms, which represent taste for products for reasons besides product characteristics.

<sup>6</sup>The estimation of production functions, dynamic models and other issues are also reviewed.

the series estimators and derives their uniform rates of convergence. An illustration of the estimation procedure using the 2015 ATUS data is given in Section 5. Section 6 concludes the paper. The algorithm to solve for the counterfactual equilibrium and the proofs are collected in the Supplemental Appendices.

## 2 Model

The hedonic equilibrium model with unobserved quality studied in this paper allows some product characteristics to be unobserved by researchers. Section 2.1 introduces the model and discusses its properties that facilitate identifying the structural functions and solving for the counterfactual equilibrium of the model.<sup>7</sup> Section 2.2 describes labor markets in which the model and the econometric method provided in this paper could be applied to analyze the distributional effects of counterfactual interventions.

### 2.1 Model Setup and Properties of Equilibrium

The model analyzed in this paper pertains to competitive markets (indexed by  $m \in \mathcal{M}$ ) of a product (good or service), of which the quantity is observed by researchers but quality is not. Each seller and buyer only trades once, and chooses the *effective amount*  $z$ , where  $z \in \mathcal{Z}$ . I assume that  $\mathcal{Z} \subset \mathbb{R}$  is compact. Let  $P^m(z)$  be a twice continuously differentiable price schedule function defined on  $\mathcal{Z}$ . Then the value of  $P^m(z)$  is the payment for an effective amount  $z$  of the product in market  $m$ .

The following is the key assumption of this model, and distinguishes it from other hedonic equilibrium models (e.g., Heckman, Matzkin, and Nesheim, 2010).

**Assumption 1.** *Suppose that the unobserved effective amount  $z$  of the product is determined by the unobserved quality  $e$  and observed quantity  $h$  in a multiplicative way, i.e.,  $z = h \cdot e$ .*

Assumption 1 implies that quantity  $h$  and quality  $e$  are substitutes in production. Existing hedonic equilibrium models (e.g., Heckman, Matzkin, and Nesheim, 2010) assume that  $e \equiv 1$ , hence  $z$  is observed. But this paper allows quality  $e$  and  $z$  to be

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<sup>7</sup>Parallel discussion for hedonic equilibrium models without unobserved quality can be found in Heckman, Matzkin, and Nesheim (2010), Ekeland, Heckman, and Nesheim (2004) and Ekeland (2010).

unknown to researchers. Sellers and buyers both observe quality. As a result, there is no principal-agent problem in this model.

Each seller's quality  $e$  is exogenously determined by a *quality function*  $e(x, a)$ , where the  $d_x \times 1$  vector  $x$  is the seller's observed characteristics, and the scalar  $a$  is the seller's unobserved characteristic. Sellers have quasilinear utility  $P^m(z) - U(h, x, a)$ , where  $U(h, x, a)$  is the disutility that a seller with characteristics  $(x, a)$  endures by producing the product of quantity  $h \in \mathcal{H}$  (the set  $\mathcal{H} \subset \mathbb{R}$  is compact).<sup>8</sup> The population of sellers in market  $m$  is described by the density  $f_{x,a}^m$ , which is assumed to be differentiable and strictly positive on the compact sets  $\mathcal{X} \times \mathcal{A} \subset \mathbb{R}^{d_x+1}$ . Sellers may choose not to trade, then they obtain reservation utility  $V_0$ .

Each buyer has a utility function  $R(z, y, b)$ , where the  $d_y \times 1$  vector  $y$  is the buyer's observed characteristics and the scalar  $b$  is the buyer's unobserved characteristic. The population of buyers in market  $m$  is described by the density  $f_{y,b}^m$ , which is assumed to be differentiable and strictly positive on the compact set  $\mathcal{Y} \times \mathcal{B} \subset \mathbb{R}^{d_y+1}$ . If a buyer chooses not to participate, she gets reservation utility  $S_0$ .

For the structural functions  $e(x, a)$ ,  $U(h, x, a)$  and  $R(z, y, b)$ , assume the following assumptions hold.

**Assumption 2.** *Suppose that buyers' utility function  $R(z, y, b)$ , sellers' disutility function  $U(h, x, a)$  and quality function  $e(x, a)$  are all twice continuously differentiable with respect to all arguments on their respective supports. Also suppose that  $e(x, a)$  is bounded below away from zero.*

**Assumption 3.** *Suppose that  $U_h > 0$ ,  $U_a < 0$ ,  $U_{ha} < 0$  and  $U_{hh} > 0$  for all  $(h, x, a) \in \mathcal{H} \times \mathcal{X} \times \mathcal{A}$ , and suppose that  $R_z > 0$ ,  $R_b > 0$ ,  $R_{zb} > 0$  and  $R_{zz} < 0$  for all  $(z, y, b) \in \mathcal{Z} \times \mathcal{Y} \times \mathcal{B}$ .*

**Assumption 4.** *Suppose  $e_a > 0$ , that is, the quality function is strictly increasing in the unobserved characteristic of the seller, for all  $(x, a) \in \mathcal{X} \times \mathcal{A}$ .*

If reservation utilities  $V_0$  and  $S_0$  are sufficiently small, then sellers and buyers always participate.<sup>9</sup> In addition, similar to the discussion in Heckman, Matzkin,

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<sup>8</sup>I concentrate on scalar-valued quantity  $h$  in the main text. But it is easy to extend the argument to a vector-valued  $h$  captured by a single-index structure as those in Epplé and Sieg (1999) and Sieg, Smith, Banzhaf, and Walsh (2004).

<sup>9</sup>Allowing for binding reservation utilities serves as an important topic for future research.

and Nesheim (2010) and Chiappori, McCann, and Nesheim (2010), Assumptions 2-4 (Spence-Mirrlees type single-crossing condition) are sufficient for each seller and buyer who participates to have a unique interior optimum.

A seller with characteristics  $(x, a)$  in market  $m$  chooses  $h \in \mathcal{H}$ , a quantity supplied, to maximize

$$\max_{h \in \mathcal{H}} P^m(h \cdot e(x, a)) - U(h, x, a).$$

Since quality  $e(x, a)$  is fixed for seller  $(x, a)$ , choosing  $h \in \mathcal{H}$  is equivalent to choosing  $z \in \mathcal{Z}$ . Under Assumptions 2-4, there exists an effective amount supply function  $z^s \equiv s^m(x, a)$  (hence a quantity supply function  $h^m(x, a) \equiv s^m(x, a)/e(x, a)$ ) that satisfies the seller's first-order condition (FOC)

$$P_z^m(s^m(x, a)) \cdot e(x, a) - U_h\left(\frac{s^m(x, a)}{e(x, a)}, x, a\right) = 0. \quad (2.1)$$

Applying the Implicit Function Theorem (Hildebrandt and Graves, 1927) to equation (2.1) gives rise to

$$\frac{\partial z^s}{\partial a} = \frac{\partial s^m(x, a)}{\partial a} = \frac{eU_{ha} - P_z^m e e_a - U_{hh} h^m e_a}{P_{zz}^m e^2 - U_{hh}}, \quad (2.2)$$

where the arguments of the functions on the right-hand side of equation (2.2) are suppressed for simplicity. By Assumptions 2 and 3 and the FOC in equation (2.1),  $P_z^m > 0$ . Then Assumptions 2-4 imply that  $s^m(x, a)$  is strictly increasing in  $a$ .<sup>10</sup> Then the inverse effective amount supply function  $a = (s^m)^{-1}(x, z)$  exists and satisfies

$$\frac{\partial (s^m)^{-1}(x, z)}{\partial z^s} = \frac{P_{zz}^m e^2 - U_{hh}}{eU_{ha} - P_z^m e e_a - U_{hh} h^m e_a}.$$

The payment received by seller  $(x, a)$  in market  $m$  is then determined by

$$I^m(x, a) = P^m(s^m(x, a)) = P^m(h^m(x, a) \cdot e(x, a)). \quad (2.3)$$

Note that the payment function  $I^m(x, a)$  is also strictly increasing in  $a$ . But since  $h^m(x, a) = s^m(x, a)/e(x, a)$ , the quantity function  $h^m(x, a)$  is not necessarily monotonic in  $a$ .

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<sup>10</sup>As discussed later, equations (2.6) and (2.7) imply that  $P_{zz}^m e^2 - U_{hh} < 0$ .



Similar argument applies to the buyers. Each buyer chooses  $z \in \mathcal{Z}$  to maximize

$$\max_{z \in \mathcal{Z}} R(z, y, b) - P^m(z).$$

There exists an effective amount demand function  $z^d \equiv d^m(y, b)$  that satisfies the buyers' FOC

$$R_z(d^m(y, b), y, b) - P^m_z(d^m(y, b)) = 0, \quad (2.4)$$

and an inverse effective amount demand function  $b = (d^m)^{-1}(y, z)$  that satisfies

$$\frac{\partial (d^m)^{-1}(y, z)}{\partial z^d} = \frac{R_{zb}}{P^m_{zz} - R_{zz}}.$$

Define the range of equilibrium effective amount supplied

$$\begin{aligned} \mathcal{Z}_s = \{ & z \in \mathcal{Z}: \text{there exists a market } m \in \mathcal{M} \text{ and some} \\ & (x, a) \in \mathcal{X} \times \mathcal{A} \text{ such that in equilibrium } z = h^m(x, a) \cdot e(x, a) \}, \end{aligned}$$

and the range of equilibrium effective amount demanded

$$\begin{aligned} \mathcal{Z}_d = \{ & z \in \mathcal{Z}: \text{there exists a market } m \in \mathcal{M} \text{ and some} \\ & (y, b) \in \mathcal{Y} \times \mathcal{B} \text{ such that in equilibrium } z = d^m(y, b) \}. \end{aligned}$$

In a unique interior equilibrium, the density of effective amount supplied  $z^s$  equals that of effective amount demanded  $z^d$  for all  $z \in \mathcal{Z}$ . Using standard change-of-variables formula, this requires  $\mathcal{Z}_s = \mathcal{Z}_d$  and

$$\begin{aligned} & \int_{\mathcal{X}} f^m_{x,a}(x, (s^m)^{-1}(x, z)) \frac{\partial (s^m)^{-1}(x, z)}{\partial z^s} dx \\ &= \int_{\mathcal{Y}} f^m_{y,b}(y, (d^m)^{-1}(y, z)) \frac{\partial (d^m)^{-1}(y, z)}{\partial z^d} dy, \end{aligned} \quad (2.5)$$

for  $\forall z \in \mathcal{Z}_s \cap \mathcal{Z}_d$ .

Figure 2.1 illustrates the market equilibrium. Under the price schedule function  $P^m$ , each seller  $(x, a)$  (drawn from distribution  $f^m_{x,a}$ ) chooses her optimal effective amount supplied  $z^s$ . The distribution of  $z^s$  is represented by the red line in the figure. Similarly, each buyer  $(y, b)$  (drawn from distribution  $f^m_{y,b}$ ) chooses her optimal

effective amount demanded  $z^d$ . The distribution of  $z^d$  is represented by the blue line in the figure. If the red density equals the blue density for  $\forall z \in \mathcal{Z}$ , then the market is in equilibrium.

On the contrary, Figure 2.2 illustrates a case where the market is not in equilibrium. For example, sellers who are willing to supply the effective amount  $z_1$  outnumber the buyers who demand  $z_1$ , and more buyers than sellers are willing to trade effective amount  $z_2$ . This mismatch between supply and demand will drive the price schedule function  $P^m$  to adjust to clear the market.

Following Chiappori, McCann, and Nesheim (2010), the equilibrium of this model is defined as follows.

**Definition 1.** (Equilibrium) Let  $\mu^m$  be a joint density on the space of effective amount  $z$ , characteristics  $(x, a)$  of sellers and  $(y, b)$  of buyers. A pair  $(\mu^m, P^m)$  is an *equilibrium* if:

- (i) the marginal of  $\mu^m$  with respect to  $(x, a)$  equals  $f_{x,a}^m$ , and that with respect to  $(y, b)$  equals  $f_{y,b}^m$  (market clears); and
- (ii) if  $(z, x, a, y, b)$  is in the support of  $\mu^m$ , then  $z = s^m(x, a) = d^m(y, b)$  (agents optimize).

By the argument provided in Chiappori, McCann, and Nesheim (2010) (also in Ekeland (2010) and Heckman, Matzkin, and Nesheim (2010)), Assumptions 2-4 are sufficient for the equilibrium to exist and to be unique and pure. A pure equilibrium means that each seller matches to a single buyer, and each pair chooses a single effective amount  $z$ .

Note that the effective amount supply function  $s^m(x, a)$  and demand function  $d^m(y, b)$  have a superscript  $m$ , since they both depend on the market-specific price schedule function  $P^m$ . And price schedule function  $P^m$  is itself an equilibrium outcome, which in turn depends on the market primitives  $(f_{x,a}^m, f_{y,b}^m, U, e, R)$ . To see this more clearly, substitute  $\frac{\partial(s^m)^{-1}(x,z)}{\partial z^s}$  and  $\frac{\partial(d^m)^{-1}(y,z)}{\partial z^d}$ , rearrange equation (2.5) and suppress the arguments of the functions, one gets

$$P_{zz}^m(z) = \frac{\int_{\mathcal{Y}} \frac{f_{y,b}^m}{R_{zb}} R_{zz} dy + \int_{\mathcal{X}} \frac{f_{x,a}^m}{-(U_{ha}e - P_z^m e e_a - U_{hh} h^m e_a)} U_{hh} dx}{\int_{\mathcal{Y}} \frac{f_{y,b}^m}{R_{zb}} dy + \int_{\mathcal{X}} \frac{e^2 f_{x,a}^m}{-(U_{ha}e - P_z^m e e_a - U_{hh} h^m e_a)} dx}. \quad (2.6)$$

Equation (2.6) implies that the curvature of the price schedule function  $P^m$  can be regarded as a weighted average of the curvatures of the sellers' disutility and the

buyers' utility functions. Assumptions 2 and 3 imply that the second-order condition (SOC)<sup>11</sup>

$$R_{zz} \cdot e^2 - U_{hh} < 0 \quad (2.7)$$

holds for all  $(h, x, a) \in \mathcal{H} \times \mathcal{X} \times \mathcal{A}$  and all  $(z, y, b) \in \mathcal{Z} \times \mathcal{Y} \times \mathcal{B}$ . Together, equation (2.6) and equation (2.7) imply  $P_{zz}^m e^2 - U_{hh} < 0$ .

Since the structural functions  $(U, e, R)$  remain invariant across markets, equation (2.6) implies that cross-market variation in the price schedule functions  $P^m$  is driven by that in the distributions  $f_{x,a}^m$  and  $f_{y,b}^m$ . As a result, cross-market variation in other reduced form (equilibrium outcome) functions, such as  $s^m$ ,  $d^m$ ,  $h^m$  and  $I^m$ , also depends on that in  $f_{x,a}^m$  and  $f_{y,b}^m$ . Throughout this paper, I summarize this dependence using the superscript  $m$ .

In the same market, all sellers and buyers face the same price schedule function  $P^m$ , so sellers with the same characteristics  $(x, a)$  always choose the same quantity  $h^m(x, a)$  to supply. Without restrictions on sellers' marginal disutility function  $U_h(h, x, a)$ , its identification using single market data is obstructed by this endogeneity problem. With multiple market data, however, the distributions  $f_{x,a}^m$  and  $f_{y,b}^m$  serve as aggregate supply or demand shifters (i.e., instruments) that induce variation in  $P^m$  (and hence  $h^m(x, a)$ ) while maintaining individual values of  $(x, a)$ . In practice, multiple markets could be geographically isolated locations, or repeated observations of the same market over time.

Chiappori, McCann, and Nesheim (2010) showed that the classic hedonic equilibrium model is mathematically equivalent to a stable matching problem and to an optimal transportation problem. The same argument applies to the model in this paper as well, since quality is observable to both sellers and buyers. This insight suggests an algorithm for solving for the counterfactual equilibria, which is provided in Supplemental Appendix A.

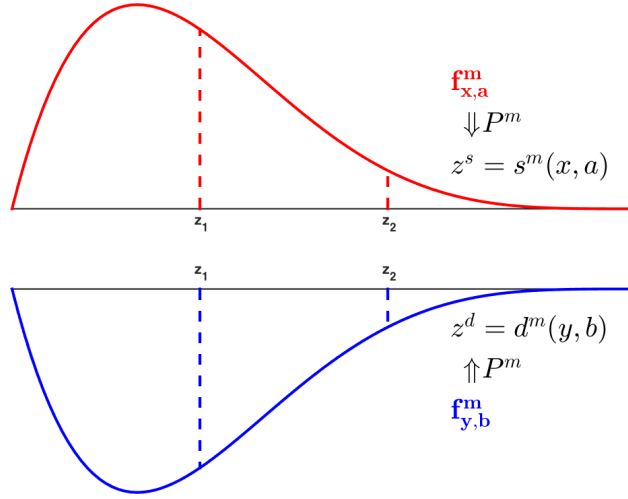
## 2.2 Application to Labor Markets

In labor markets, the product that is traded between sellers (workers) and buyers (employers or single-employed firms) is labor.<sup>12</sup> The quantity of labor, i.e. how much time workers work may be observed by researchers. For example, common

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<sup>11</sup>SOC of a pair-wise surplus maximization problem.

<sup>12</sup>It is also helpful to think of the buyers as job positions.

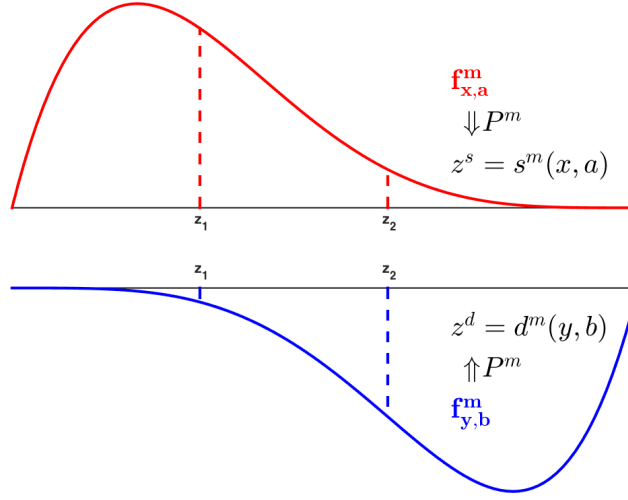


The red line illustrates the distribution of the optimal effective labor supply  $z^s$  under the price schedule function  $P^m$  in market  $m$ , as a function of sellers' observed characteristics  $x$  and unobserved characteristic  $a$ , which follow the distribution  $f_{x,a}^m$ . Similarly, the blue line illustrates the distribution of the optimal effective labor demand  $z^d$  under the same price schedule function  $P^m$  in market  $m$ , as a function of buyers' observed characteristics  $y$  and unobserved characteristic  $b$ , which follow the distribution  $f_{y,b}^m$ . As is shown in this figure, when the distributions of  $z^s$  and  $z^d$  are the same, the market clears.

Figure 2.1: Equilibrium

surveys such as Current Population Survey ask about the respondents' working time; punch card records may also be available in many proprietary data sets. But the workers' efficiency, which is how much work the workers get done within one unit of time, is rarely available to researchers. On the other hand, employers often can obtain some idea about workers' efficiency, at least in the long run, by observing the amount of work that the workers accomplish. In equilibrium, workers' efficiency plays an important role in determining their earnings. For workers on a piece rate job, this is apparent. Even for hourly paid or salaried workers, more efficient workers make higher earnings in the long run by being promoted to higher positions or gaining access to more opportunities.

Both workers and firms exhibit considerable heterogeneity. Workers differ in observed characteristics  $x$  (e.g., age, education and skills) and unobserved characteristic



The red line illustrates the distribution of the optimal effective labor supply  $z^s$  under the price schedule function  $P^m$  in market  $m$ , as a function of sellers' observed characteristics  $x$  and unobserved characteristic  $a$ , which follow the distribution  $f_{x,a}^m$ . Similarly, the blue line illustrates the distribution of the optimal effective labor demand  $z^d$  under the same price schedule function  $P^m$  in market  $m$ , as a function of buyers' observed characteristics  $y$  and unobserved characteristic  $b$ , which follow the distribution  $f_{y,b}^m$ . As is shown in this figure, when the distributions of  $z^s$  and  $z^d$  are different (for example, density of the effective labor supply is larger than that of the demand at  $z_1$ , and is the opposite at  $z_2$ ), the market is off equilibrium and the price schedule function  $P^m$  will adjust.

Figure 2.2: Off Equilibrium

$a$  (e.g., ability). Likewise, employers differ in observed characteristics  $y$  (e.g., capital stock) and unobserved characteristic  $b$  (e.g., firm's productivity). For a worker with characteristics  $(x, a)$ , her efficiency is given by the function  $e(x, a)$ , which is the amount of work she gets done within one unit of time. For the same characteristics  $(x, a)$ , we assume that the worker's efficiency is the same across markets and is unknown to researchers.<sup>13</sup> On the other hand, distributions of agent heterogeneity ( $f_{x,a}^m$  and  $f_{y,b}^m$ ) could vary among markets, which induce market-specific earnings schedule functions  $P^m(z)$ . As a result, workers with the same characteristics may choose to work different amount of time  $h^m(x, a)$  and make different earnings  $I^m(x, a)$  in different markets. Workers' working time and efficiency are substitutes in production, and firms care about how much work is done, but not the working time in itself.<sup>14</sup> There-

<sup>13</sup>Firms know  $z$  and  $e(x, a)$  by looking at how much work the worker gets done.

<sup>14</sup>Ideally, researchers would want to measure the actual time workers spend in working. The

fore, earnings depend on the effective amount of labor  $z$  via the earnings schedule functions  $P^m(z)$ , but not on working time  $h^m(x, a)$  or efficiency  $e(x, a)$  *per se*.

The model in this paper could be used to answer various counterfactual questions that labor economists are interested in. For example, to understand the distributional effects of the changes in labor market institutions during 1979-1988, one may want to construct counterfactual earnings distribution in 1988 had there been no de-unionization since 1979 (e.g., DiNardo, Fortin, and Lemieux, 1996). This corresponds to the counterfactual equilibrium earnings of a market in which workers' union status (one element of  $x$ ) had remained what it was in 1979 and other agent characteristics (other variables in  $(x, a, y, b)$ ) had shifted to their 1988 values. The example of college expansion requires the counterfactual that many people attained higher education status (also one element of  $x$ ) such that the return to college education might change as a result. Technological advances, on the other hand, entail the counterfactual where one element of  $y$  increased for many employers.

### 3 Identification

This section explains identification of the reduced form (equilibrium outcome) functions and the structural functions of the model. The analysis in this section assumes that seller characteristics  $x$ , buyer characteristics  $y$ , equilibrium payment  $I$  and equilibrium quantity  $h$  in all markets are observed. The effective amount  $z$ , however, is unknown to researchers.

The identification consists of three steps. First, identify the reduced form payment functions  $I^m(x, a)$  and the quantity functions  $h^m(x, a)$  using single market data. This step employs an existing method (Matzkin, 2003) and facilitates the identification of structural functions. Second, exploit the variation of the payment and quantity functions within and across markets to identify the quality function  $e(x, a)$ . This is the key step, and I will provide both graphical illustration of intuition and general results. The key identification condition requires that the relative returns to sellers' characteristics differ across markets. Finally, combine the functions identified from the first two steps and sellers' FOC to recover sellers' marginal disutility function  $U_h(h, x, a)$ . To overcome the endogeneity problem of  $h$ , this final step requires multiple

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required working time written on the contract deviates from the actual time, since workers could shirk or work over-time.

market data as well. Section 3.1, 3.2 and 3.3 elaborate these steps, respectively.

This section focuses on the quality function  $e(x, a)$  and sellers' marginal disutility function  $U_h(h, x, a)$ . The identification of buyers' marginal utility function  $R_z(z, y, b)$  can be achieved via the same method as that used for  $U_h(h, x, a)$ , and is briefly discussed in Section 3.4. Although  $f_{x,a}^m$  and  $f_{y,b}^m$  are also primitives of the model and serve as aggregate supply or demand shifters that generate cross-market variation in equilibria, their identification is straightforward. The convergence rate results in Section 4.2 account for the fact that they need to be estimated.

### 3.1 Identification of Payment Functions $I^m(x, a)$ and Quantity Functions $h^m(x, a)$ Using Single Market Data

In each market  $m$ , there is a payment function  $I^m(x, a)$  and a quantity function  $h^m(x, a)$  in equilibrium. This section uses the method developed by Matzkin (2003) to identify these reduced form functions using data from their own markets.

**Assumption 5.** *Suppose that  $x \perp a$  and  $y \perp b$  within each market  $m \in \mathcal{M}$ .<sup>15</sup>*

**Assumption 6.** *Suppose that the sellers' unobserved characteristic  $a$  follows the uniform distribution  $U[0, 1]$  in all markets.*

Assumption 6 may seem restrictive at first glance. But an equivalent interpretation is that  $a$  is the quantile of the seller's unobserved characteristic. Based on this interpretation, Assumption 6 requires that the sellers' unobserved characteristic has the same distribution (probably unknown) across all markets.<sup>16</sup> Assumption 6 is also a normalization that facilitates identification of nonseparable functions like  $I^m(x, a)$  (see Matzkin, 2003 for details).<sup>17</sup> But this normalization does not affect counterfactuals.

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<sup>15</sup>Like Heckman, Matzkin, and Nesheim (2010), because  $a$  enters the quality function and sellers' marginal disutility function nonparametrically, this independence assumption is much weaker than it would be if  $a$  entered additively.

<sup>16</sup>To see this clearly, suppose that  $F_a$  is the distribution function of  $a$ , and suppose  $\tilde{U}(h, x, a)$  and  $\tilde{e}(x, a)$  are the "real" supply side structural functions. Then, based on the quantile interpretation, the supply side structural functions identified in this paper are compounds of  $F_a$  and the "real" structural functions. That is,  $U(h, x, a) = \tilde{U}(h, x, F_a^{-1}(a))$  and  $e(x, a) = \tilde{e}(x, F_a^{-1}(a))$ . Therefore, Assumption 7 implicitly requires that  $F_a$  is invariant across markets.

<sup>17</sup>One could normalize the distribution of  $a$  to any other distributions.

**Lemma 1.** *Under Assumptions 1-6, the payment function  $I^m(x, a)$  is strictly increasing in the seller's unobserved characteristic  $a$ , and  $I^m(x, a)$  is nonparametrically identified within each market  $m$ .*

**Corollary 1.** *Under the conditions for Lemma 1, the partial derivatives of the payment function  $I_{x_j}^m(x, a)$  ( $j = 1, \dots, d_x$ ) and  $I_a^m(x, a)$  are nonparametrically identified within each market  $m$ .*

Once one identifies the payment function  $I^m$ , she can invert it with respect to  $a$  to obtain  $a = (I^m)^{-1}(x, I)$ . Now that  $a$  is known, it is easy to identify the quantity function  $h^m(x, a)$ . Unlike  $I^m(x, a)$ , monotonicity is not necessary for identification of  $h^m(x, a)$ .

**Lemma 2.** *Under the conditions for Lemma 1, the quantity function  $h^m(x, a)$  is nonparametrically identified within each market  $m$ . Moreover, its partial derivatives  $h_{x_j}^m(x, a)$  ( $j = 1, \dots, d_x$ ) and  $h_a^m(x, a)$  are nonparametrically identified within each market  $m$  as well.*

Note that the functional forms of  $I^m(x, a)$  and  $h^m(x, a)$  vary from market to market due to the cross-market variation in  $f_x^m$  and  $f_y^m$ , and they are identified within each market. Their variation within and across markets reveals enough information to identify the quality function  $e(x, a)$ .

### 3.2 Identification of Quality Function $e(x, a)$ Using Multiple Market Data

This section explains how to use within- and cross-market variation in the reduced form functions to identify the structural quality function  $e(x, a)$ . Section 3.2.1 illustrates the intuition for scalar-valued  $x$ . The intuition applies to vector-valued  $x$  as well. Section 3.2.2 gives general results.

Since quality  $e$  and effective amount  $z$  are both unobserved, one can always re-scale the price schedule function to make two quality functions observationally equivalent. So we need the following normalization.

**Assumption 7.** *Suppose that for a known fixed vector  $(\bar{x}, \bar{a}) \in \mathcal{X} \times \mathcal{A}$ , we have  $e(\bar{x}, \bar{a}) = 1$ .*

The vector  $(\bar{x}, \bar{a})$  corresponds to a normalization seller, and the quality of other sellers will be expressed as ratio relative to her.



### 3.2.1 Intuition

This section illustrates the intuition for identifying the unobserved quality function  $e(x, a)$  for scalar-valued  $x$ . The interpretation of the key identification condition is that relative returns to sellers' characteristics differ across markets.

Recall the payment equation (2.3),

$$I^m(x, a) = P^m(s^m(x, a)) = P^m(h^m(x, a) \cdot e(x, a)).$$

Since all sellers in the same market face the same price schedule function  $P^m(z)$ , those sellers who receive the same payment must have sold the same effective amount  $z$  of the product. In other words, if  $I_i^m = I_j^m$  for two sellers  $i$  and  $j$  in the same market  $m$ , then

$$h^m(x_i^m, a_i^m) \cdot e(x_i^m, a_i^m) = h^m(x_j^m, a_j^m) \cdot e(x_j^m, a_j^m),$$

which implies

$$\frac{e(x_i^m, a_i^m)}{e(x_j^m, a_j^m)} = \frac{h^m(x_j^m, a_j^m)}{h^m(x_i^m, a_i^m)}. \quad (3.1)$$

That is, the quality ratio between sellers who receive the same payment in the same market equals the inverse ratio of their quantities.

This is illustrated by Figure 3.1. The solid green line in Step 1 of Figure 3.1 represents the iso-payment curve in Market 1 that contains the normalization seller  $(\bar{x}, \bar{a})$ . By equation (3.1), the quality of any seller  $(x_1, a_1)$  on the same iso-payment curve can be identified as

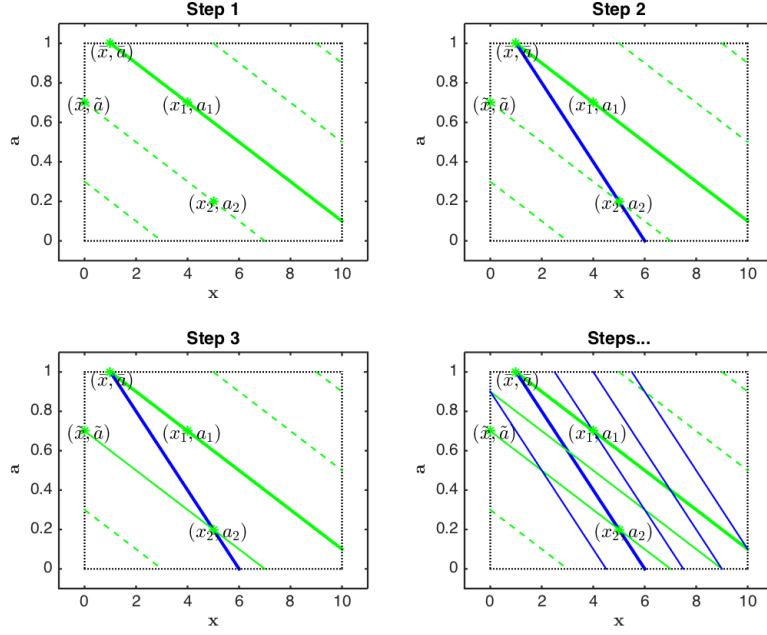
$$e(x_1, a_1) = \frac{h^1(\bar{x}, \bar{a})}{h^1(x_1, a_1)}.$$

The same argument applies to other iso-payment curves in Market 1, which are represented by dashed green lines in Step 1. For example, for sellers  $(\tilde{x}, \tilde{a})$  and  $(x_2, a_2)$  on another iso-payment curve, we get

$$\frac{e(x_2, a_2)}{e(\tilde{x}, \tilde{a})} = \frac{h^1(\tilde{x}, \tilde{a})}{h^1(x_2, a_2)}. \quad (3.2)$$

Since iso-payment curves in the same market are disjoint, neither  $e(x_2, a_2)$  nor  $e(\tilde{x}, \tilde{a})$  could be identified relative to the normalization seller  $(\bar{x}, \bar{a})$ . The dashed green lines in Step 1 indicate that the quality of the sellers on those iso-payment curves are not identified yet. This is the most one can get from variation of reduced form functions

in one market.



Green lines (solid and dashed) illustrate the disjoint iso-payment curves in Market 1 and blue lines (solid and dashed) illustrate the disjoint iso-payment curves in Market 2. The quality  $e(\bar{x}, \bar{a})$  is normalized to be one. In each market, the relative qualities for sellers on the same iso-payment curves can be identified, but not for those on different iso-payment curves. For example,  $e(x_1, a_1)/e(\bar{x}, \bar{a})$  and  $e(x_2, a_2)/e(\bar{x}, \bar{a})$  are identified from Market 1 (illustrated in Step 1), but not  $e(x_2, a_2)/e(\bar{x}, \bar{a})$ . From Market 2, however,  $e(x_2, a_2)/e(\bar{x}, \bar{a})$  can be identified (illustrated in Step 2). As a result,  $e(x_2, a_2)/e(\bar{x}, \bar{a})$  can be identified using the data from both markets (illustrated in Step 3). This idea could be applied repeatedly to identify the quality function  $e(x, a)$  (illustrated in the last panel). The identification requires a rank condition on the derivatives of the payment functions  $I^m(x, a)$  across markets. As is shown in the figure, this condition can be understood as requiring that the slopes of the iso-payment curves across markets are different.

Figure 3.1: Identification of  $e(x, a)$  in Two Markets

With data from another market, however, it is possible to connect the disjoint iso-payment curves. Suppose that in Market 2, there is an iso-payment curve that contains both  $(\bar{x}, \bar{a})$  and  $(x_2, a_2)$ , then

$$e(x_2, a_2) = \frac{h^2(\bar{x}, \bar{a})}{h^2(x_2, a_2)}. \quad (3.3)$$

Combining equation (3.2) and equation (3.3), we now can identify the quality for

seller  $(\tilde{x}, \tilde{a})$  as

$$e(\tilde{x}, \tilde{a}) = \frac{h^1(x_2, a_2)}{h^1(\tilde{x}, \tilde{a})} \cdot \frac{h^2(\bar{x}, \bar{a})}{h^2(x_2, a_2)}.$$

Once  $e(\tilde{x}, \tilde{a})$  is identified, so is the quality of other sellers on the same iso-payment curve.

In Step 2 of Figure 3.1, the iso-payment curve in Market 2 is represented by the solid blue line. It connects the Market 1 iso-payment curve that contains  $(\bar{x}, \bar{a})$  with the one that contains  $(\tilde{x}, \tilde{a})$ , and thus helps determine the quality level of the latter. In Step 3 of Figure 3.1, the latter becomes solid green as the quality of those sellers are identified. Step 4 shows that by applying this idea recursively to the iso-payment curves from the two markets that cross with each other, one will be able to identify the quality of all sellers with characteristics in the support of their distribution.

As suggested by Figure 3.1, the key identification condition is that for any seller characteristics  $(x, a)$ , one could find two markets that have iso-payment curves with different slopes. Otherwise, all the iso-payment curves are disjoint, and it is impossible to connect a seller  $(x, a)$  with the normalization seller  $(\bar{x}, \bar{a})$  if they do not belong to the same iso-payment curve.

Note that the slope of an iso-payment curve can be expressed in terms of the partial derivatives of the payment function, then the identification condition is

$$\frac{I_x^1(x, a)}{I_a^1(x, a)} \neq \frac{I_x^2(x, a)}{I_a^2(x, a)},$$

for  $\forall (x, a) \in \mathcal{X} \times \mathcal{A}$ , scalar-valued  $x$  and two markets. This condition is also equivalent to that the matrix

$$\begin{pmatrix} I_a^1(x, a) & -I_x^1(x, a) \\ I_a^2(x, a) & -I_x^2(x, a) \end{pmatrix}$$

has full column rank.

This key condition is easy to understand. Partial derivatives of the payment functions represent the equilibrium market returns to respective seller characteristics. For example,  $I_x^m(x, a)$  could represent labor market return to education, and  $I_a^m(x, a)$  to ability. Then the identification condition requires that the relative equilibrium returns to education and to ability differ in at least two markets. This in turn requires that cross-market variation in  $f_x^m$  and  $f_y^m$  is sufficiently rich to induce such cross-market variation in equilibria.

### 3.2.2 General Results

It is not hard to generalize the intuition explained in Section 3.2.1 to vector-valued  $x$ . This section formalizes this intuition and gives general results on the identification of the unobserved quality function  $e(x, a)$ .

When  $x$  is vector-valued ( $d_x > 1$ ), the key identification condition is still that relative market returns to seller characteristics differ in at least two markets. Without loss of generality, one could measure returns as relative to that to the unobserved characteristic  $a$ . Suppose  $I_a^m(x, a) \neq 0$  and  $I_a^{m'}(x, a) \neq 0$  for markets  $m$  and  $m'$ . Then it is required that

$$\left( \frac{I_{x_1}^m(x, a)}{I_a^m(x, a)}, \dots, \frac{I_{x_{d_x}}^m(x, a)}{I_a^m(x, a)}, 1 \right) \neq \left( \frac{I_{x_1}^{m'}(x, a)}{I_a^{m'}(x, a)}, \dots, \frac{I_{x_{d_x}}^{m'}(x, a)}{I_a^{m'}(x, a)}, 1 \right). \quad (3.4)$$

These are just the gradient vectors of the payment functions  $I^m(x, a)$  and  $I^{m'}(x, a)$ .

Cross-market variation in equilibria is crucial for identifying the quality function. The following assumption requires that neither sellers nor buyers move across markets on a large scale. Otherwise, the distributions  $f_x^m$  and  $f_y^m$  will tend to equalize across markets, which diminishes the cross-market variation.

**Assumption 8.** *Suppose that the sellers and buyers do not move across markets.*

In order to state the formal identification condition and the theorem, I need some notation. Let  $\nabla_x I^m(x, a)$  denote the  $d_x \times 1$  vector of the derivatives of  $I^m(x, a)$  with respect to  $(x_1, \dots, x_{d_x})'$ , let  $\nabla_x h^m(x, a)$  denote those of  $h^m(x, a)$  and let  $\nabla_x e(x, a)$  denote those of  $e(x, a)$ . For any integer  $d$ , let  $\mathbb{I}_d$  denote a  $d \times d$  identity matrix.

**Assumption 9.** *Suppose that there exist  $M$  markets such that the  $(Md_x) \times (d_x + 1)$  matrix  $B(x, a)$  defined as*

$$B(x, a) \equiv \begin{pmatrix} \mathbb{I}_{d_x} \otimes I_a^1(x, a) & -\nabla_x I^1(x, a) \\ \vdots & \vdots \\ \mathbb{I}_{d_x} \otimes I_a^M(x, a) & -\nabla_x I^M(x, a) \end{pmatrix}$$

*has full column rank for all  $(x, a) \in \mathcal{X} \times \mathcal{A}$ .*<sup>18</sup>

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<sup>18</sup>Note that a necessary condition for  $B(x, a)$  to have full column rank is that there are  $d_x + 1$  linearly independent rows in  $B(x, a)$ . Therefore we need at least two markets. But when data

It only takes some basic algebra to see that if equation (3.4) holds for all  $(x, a) \in \mathcal{X} \times \mathcal{A}$ , then Assumption 9 is satisfied. Moreover, if Assumption 9 holds, there could be more than two markets satisfying equation (3.4).

Define the  $(Md_x) \times 1$  vector  $A(x, a)$  as

$$A(x, a) \equiv \begin{pmatrix} [h_a^1(x, a) \nabla_x I^1(x, a) - I_a^1(x, a) \nabla_x h^1(x, a)] / h^1(x, a) \\ \vdots \\ [h_a^M(x, a) \nabla_x I^M(x, a) - I_a^M(x, a) \nabla_x h^M(x, a)] / h^M(x, a) \end{pmatrix}.$$

And define  $d_x + 1$  real-valued functions  $g_1(x, a), \dots, g_{d_x+1}(x, a)$  as

$$(g_1(x, a), \dots, g_{d_x+1}(x, a))' \equiv [B(x, a)' B(x, a)]^- [B(x, a)' A(x, a)],$$

where the superscript “ $-$ ” indicates the generalized inverse of a matrix.

**Theorem 1.** *Suppose that Assumptions 7-9 and the conditions for Lemma 1 are satisfied. The quality function is then nonparametrically identified on  $\mathcal{X} \times \mathcal{A}$  as*

$$e(x, a) = \exp \left( \sum_{j=1}^{d_x} \int_{\bar{x}_j}^{x_j} g_j(x_1, \dots, x_{j-1}, s_j, \bar{x}_{j+1}, \dots, \bar{x}_{d_x}, \bar{a}) ds_j + \int_{\bar{a}}^a g_{d_x+1}(x, t) dt \right) \quad (3.5)$$

and  $\bar{x}_j$  ( $j = 1, \dots, d_x$ ) and  $\bar{a}$  are coordinates of the normalization vector  $(\bar{x}, \bar{a})$ .

Define the range of equilibrium effective amount supplied in market  $m$  as

$$\begin{aligned} \mathcal{Z}_s^m &= \{z \in \mathcal{Z}: \text{there exists some } (x, a) \in \mathcal{X} \times \mathcal{A} \text{ in market} \\ &\quad m \in \mathcal{M} \text{ such that in equilibrium } z = h^m(x, a) \cdot e(x, a)\}. \end{aligned}$$

**Corollary 2.** *Under the conditions for Theorem 1, the unobserved effective amount  $z = h^m(x, a) \cdot e(x, a)$  is identified.*

**Corollary 3.** *Under the conditions for Theorem 1, the price schedule function  $P^m(z)$  for market  $m \in \mathcal{M}$  is nonparametrically identified on  $\mathcal{Z}_s^m$ .*

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from more markets is available, and multiple combinations of rows satisfy the requirement, we get over-identification.

### 3.3 Identification of Sellers' Marginal Disutility Function $U_h(h, x, a)$ Using Multiple Market Data

The next important result is the identification of the marginal disutility function  $U_h$ . Before stating the theorem, define the equilibrium support for sellers' marginal disutility function as:

$$\begin{aligned} \mathcal{HXA} = \{ & (h, x, a) \in \mathcal{H} \times \mathcal{X} \times \mathcal{A}: \text{there exists a market } m \in \mathcal{M} \text{ and} \\ & \text{some } (x, a) \in \mathcal{X} \times \mathcal{A} \text{ such that in equilibrium } h = h^m(x, a)\}. \end{aligned}$$

If  $|\mathcal{M}| = 1$ , then  $\mathcal{HXA}$  is degenerate since  $h$  is endogenous. As discussed in Section 2.1, different distributions  $f_x^m$  and  $f_y^m$  serve as aggregate supply or demand shifters (i.e., instruments) that induce variation in  $P^m$  (and hence  $h^m(x, a)$ ) while maintaining individual values of  $(x, a)$ . The richer the variation in  $f_x^m$  and  $f_b^m$ , the larger the set  $\mathcal{HXA}$  will be.

**Theorem 2.** *Under the conditions for Theorem 1, the sellers' marginal disutility function  $U_h(h, x, a)$  is nonparametrically identified on  $\mathcal{HXA}$ .*

### 3.4 Identification of Buyers' Marginal Utility Function $R_z(z, y, b)$ Using Multiple Market Data

Identifying buyers' marginal utility function  $R_z(z, y, b)$  and the effective amount demand function  $d^m(y, b)$  makes little difference from Heckman, Matzkin, and Nesheim (2010)'s method. The only tweak stems from the fact that  $z$  is not directly observed. Once one recovers  $z$  from the supply side, Heckman, Matzkin, and Nesheim (2010)'s method can be applied without modification. The relevant definition, assumption and results are given below.

Define the equilibrium support for buyers' marginal utility function  $R_z(z, y, b)$  as:

$$\begin{aligned} \mathcal{ZXB} = \{ & (z, y, b) \in \mathcal{Z} \times \mathcal{Y} \times \mathcal{B}: \text{there exists a market } m \in \mathcal{M} \\ & \text{and some } (x, a) \in \mathcal{X} \times \mathcal{A} \text{ such that } z = d^m(y, b) \\ & \text{and } z = h^m(x, a) \cdot e(x, a) \text{ in equilibrium}\}. \end{aligned}$$

**Assumption 10.** *Suppose that the buyers' unobserved characteristic  $b$  follows the uniform distribution  $U[0, 1]$  in all markets.*

**Lemma 3.** *(Heckman, Matzkin, and Nesheim 2010 Theorem 4.1) Under Assumption 10 and the conditions for Theorem 1, the buyers' marginal utility function  $R_z(z, y, b)$  is nonparametrically identified on  $\mathcal{ZYB}$ .<sup>19</sup>*

## 4 Estimation

This section provides an estimation procedure for the structural functions. Section 4.1 describes the estimation procedure step by step, and in Section 4.2 I derive the uniform rates of convergence for the estimators.

### 4.1 Series Estimation of Structural Functions

The estimators introduced in this section are premised on the following data structure. Suppose that linked seller-buyer data for  $M$  independent markets are available. Within each market  $m$ , suppose that there are  $N^m$  seller-buyer pairs, and each pair is indexed by  $i$ . Researchers observe which seller is matched with which buyer. For each pair  $i$  ( $i = 1, \dots, N^m$  and  $m = 1, \dots, M$ ), researchers observe  $(I_i^m, x_i^m, h_i^m, y_i^m)$ .<sup>20</sup>

**Assumption 11.** *Suppose  $\{(I_i^m, x_i^m, h_i^m, y_i^m)\}_{i=1}^{N^m}$  are i.i.d. for  $m = 1, \dots, M$ .*

**Assumption 12.** *For notational simplicity, suppose that the sample sizes from all the markets are equal, i.e.,  $N^1 = N^2 = \dots = N^M = N$ .*

In the rest of this paper, I maintain Assumptions 1-12. Assumption 12 is not essential for deriving the convergence rates, but relaxing it will complicate the notation and will not provide any new insights. In principle, even though the sample sizes from all the markets are the same, one still could use market-specific numbers of series basis functions  $k_{Q,N}^m$ ,  $k_{I,N}^m$  and  $k_{h,N}^m$  to estimate  $\hat{a}^m$ ,  $\hat{I}^m(x, a)$  and  $\hat{h}^m(x, a)$  respectively within each market. To keep the notation simple, however, I assume that one uses the same tuning parameters for all markets for the rest of the paper, i.e.,

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<sup>19</sup>In labor markets, if the firms' revenue is observed by researchers, then the function  $R(z, y, b)$  is also nonparametrically identified under the conditions of Lemma 3.

<sup>20</sup>In labor markets, it is possible that the employers' revenue  $R_i^m$  is also observed in the data.

$k_{Q,N}^m = k_{Q,N}$ ,  $k_{I,N}^m = k_{I,N}$  and  $k_{h,N}^m = k_{h,N}$ . All the convergence rate results in Section 4.2 hold if one relaxes this assumption.<sup>21</sup>

For any vector  $v$ , let  $\|v\| \equiv (v'v)^{1/2}$  denote its Euclidean norm; for any matrix  $A$ , let  $\|A\| \equiv [\text{trace}(A'A)]^{1/2}$  denote its Euclidean norm.

The estimation of the structural functions  $(U_h, e, R_z)$  follows the steps suggested by the identification strategy. I start with the within market estimation of two reduced form functions, namely, the payment function  $I^m(x, a)$  and the quantity function  $h^m(x, a)$ , as well as their partial derivative functions for each market. Then the quality function  $e(x, a)$  can be estimated by an estimated version of equation (3.5). Finally, sellers' marginal disutility function  $U_h(h, x, a)$  can be estimated by a series minimum distance (MD) estimator using the sellers' FOCs.

Following the identification steps in Section 3, this section describes the steps for estimating  $e(x, a)$  and  $U_h(h, x, a)$  in details. The steps for the buyers' marginal utility function  $R_z(z, y, b)$  are similar and will be briefly summarized at the end.

#### 4.1.1 Estimation of Payment Functions $I^m(x, a)$ and Quantity Functions $h^m(x, a)$ Using Single Market Data

Let me first clarify some notation used in this section:  $I^m(x, a)$  and  $h^m(x, a)$  indicate the reduced form functions;  $I^m$  (or  $h^m$ ,  $x^m$ , or  $a^m$ ) is a random variable, denoting the payment received by (or the quantity supplied by, the observed characteristics of, or the unobserved characteristic of) a randomly chosen seller from market  $m$ ; and  $I_i^m$  (or  $h_i^m$ ,  $x_i^m$ , or  $a_i^m$ ) represents the observed payment (or the observed quantity, the observed characteristics, or the unobserved characteristic) value of a specific seller  $i$  in market  $m$ .

In Section 2.1, I showed that the payment function  $I^m(x, a)$  is strictly increasing in  $a$  under Assumptions 1-4. Recall that  $a^m$  is the conditional quantile of the payment  $I^m$  given observed characteristics  $x^m$  of the seller in market  $m$ . That is

$$F_{I^m|x^m=x}(I^m(x, a)) = F_{a^m}(a) = a.$$

Use a series of basis functions  $\Lambda_{k_{Q,N}}(x) \equiv (\lambda_1(x), \dots, \lambda_{k_{Q,N}}(x))'$  to approximate the indicator function  $\mathbb{I}(I^m \leq I_i^m)$ , where  $k_{Q,N}$  is the number of basis functions. Then

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<sup>21</sup>With minor changes in notation to accommodate market-specific tuning parameters.



one can estimate  $a_i^m$ , the conditional quantile of  $I^m$  given  $x^m$  by

$$\begin{aligned}
& \hat{a}_i^m \\
& \equiv \hat{F}_{I^m|x^m=x_i^m}(I_i^m) \\
& \equiv \Lambda_{k_{Q,N}}(x_i^m)' \left( \sum_{j=1}^N \Lambda_{k_{Q,N}}(x_j^m) \Lambda_{k_{Q,N}}(x_j^m)' \right)^{-} \left( \sum_{j=1}^N \Lambda_{k_{Q,N}}(x_j^m) \mathbb{I}(I_j^m \leq I_i^m) \right) \quad (4.1)
\end{aligned}$$

Note that the tuning parameter  $k_{Q,N}$  might depend on the sample size  $N$ . Here,  $\hat{a}_i^m$  serves as a generated regressor when we estimate functions  $I^m(x, a)$  and  $h^m(x, a)$ .

Use a series of basis functions  $\Phi_{k_{I,N}}(x, a) \equiv (\phi_1(x, a), \dots, \phi_{k_{I,N}}(x, a))'$  to approximate the unknown payment function  $I^m(x, a)$ , where  $k_{I,N}$  is the number of basis functions. Then, the estimated series coefficients for the payment function  $I^m(x, a)$  are the solution to the following least square problem

$$\hat{\xi}_{I,k_{I,N}}^m \equiv \arg \min_{\xi \in \mathbb{R}^{k_{I,N}}} \sum_{i=1}^N (I_i^m - \Phi_{k_{I,N}}(x_i^m, \hat{a}_i^m)' \xi)^2.$$

Therefore, the estimated payment function is

$$\hat{I}^m(x, a) \equiv \Phi_{k_{I,N}}(x, a)' \hat{\xi}_{I,k_{I,N}}^m. \quad (4.2)$$

Note that there is an explicit solution for  $\hat{\xi}_{I,k_{I,N}}^m$ ,

$$\hat{\xi}_{I,k_{I,N}}^m = \left( \sum_{i=1}^N \Phi_{k_{I,N}}(x_i^m, \hat{a}_i^m) \Phi_{k_{I,N}}(x_i^m, \hat{a}_i^m)' \right)^{-} \left( \sum_{i=1}^N \Phi_{k_{I,N}}(x_i^m, \hat{a}_i^m) I_i^m \right). \quad (4.3)$$

Because  $\Phi_{k_{I,N}}(x, a)$  is a series of known functions, their first-order derivatives are also known. Therefore, the series estimator of the partial derivatives of  $I^m(x, a)$  can be obtained immediately

$$\hat{I}_{x_j}^m(x, a) \equiv \left( \frac{\partial \phi_1(x, a)}{\partial x_j}, \dots, \frac{\partial \phi_{k_{I,N}}(x, a)}{\partial x_j} \right) \hat{\xi}_{I,k_{I,N}}^m, \quad (4.4)$$

for  $j = 1, \dots, d_x$ , and

$$\hat{I}_a^m(x, a) \equiv \left( \frac{\partial \phi_1(x, a)}{\partial a}, \dots, \frac{\partial \phi_{k_{I,N}}(x, a)}{\partial a} \right) \hat{\xi}_{I,k_{I,N}}^m. \quad (4.5)$$

Similarly, use the series of basis functions  $\Phi_{k_h,N}(x, a) \equiv (\phi_1(x, a), \dots, \phi_{k_h,N}(x, a))'$  to approximate the unknown quantity function  $h^m(x, a)$ .

Then the estimated series coefficients for the quantity function  $h^m(x, a)$  is

$$\hat{\xi}_{h,k_h,N}^m \equiv \left( \sum_{i=1}^N \Phi_{k_h,N}(x_i^m, \hat{a}_i^m) \Phi_{k_h,N}(x_i^m, \hat{a}_i^m)' \right)^{-1} \left( \sum_{i=1}^N \Phi_{k_h,N}(x_i^m, \hat{a}_i^m) h_i^m \right). \quad (4.6)$$

Therefore, the estimated quantity function and its first-order derivatives are

$$\hat{h}^m(x, a) \equiv \Phi_{k_h,N}(x, a)' \hat{\xi}_{h,k_h,N}^m, \quad (4.7)$$

$$\hat{h}_a^m(x, a) \equiv \left( \frac{\partial \phi_1(x, a)}{\partial a}, \dots, \frac{\partial \phi_{k_h,N}(x, a)}{\partial a} \right) \hat{\xi}_{h,k_h,N}^m, \quad (4.8)$$

and

$$\hat{h}_{x_j}^m(x, a) \equiv \left( \frac{\partial \phi_1(x, a)}{\partial x_j}, \dots, \frac{\partial \phi_{k_h,N}(x, a)}{\partial x_j} \right) \hat{\xi}_{h,k_h,N}^m. \quad (4.9)$$

for  $j = 1, \dots, d_x$ .

#### 4.1.2 Estimation of Quality Function $e(x, a)$ Using Multiple Market Data

Just like the identification strategy, estimating the quality function  $e(x, a)$  starts with replacing  $I^m(x, a)$ ,  $h^m(x, a)$  and their derivatives in  $B(x, a)$  and  $A(x, a)$  with their counterparts estimated in Section 4.1.1. Use the series of basis functions  $\Phi_{k_{x_j},MN}(x, a) = (\phi_1(x, a), \dots, \phi_{k_{x_j},MN}(x, a))'$  to approximate  $e_{x_j}(x, a)/e(x, a)$  and  $\Phi_{k_a,MN}(x, a) = (\phi_1(x, a), \dots, \phi_{k_a,MN}(x, a))'$  to approximate  $e_a(x, a)/e(x, a)$ . Let the series coefficients be  $\beta_{x_j,k_{x_j},MN}$  ( $j = 1, \dots, d_x$ ) and  $\beta_{a,k_a,MN}$ , respectively. And let  $\beta_{MN} \equiv (\beta'_{x_1,k_{x_1},MN}, \dots, \beta'_{x_{d_x},k_{x_{d_x}},MN}, \beta'_{a,k_a,MN})'$ . Then, one estimate  $\hat{\beta}_{MN}$  by minimizing the distance between the left and the right sides of the following equation:

$$\hat{B}^m(x_i^m, \hat{a}_i^m) \cdot \begin{pmatrix} \Phi_{k_{x_1},MN}(x_i^m, \hat{a}_i^m)' \hat{\beta}_{x_1,k_{x_1},MN} \\ \vdots \\ \Phi_{k_{x_{d_x}},MN}(x_i^m, \hat{a}_i^m)' \hat{\beta}_{x_{d_x},k_{x_{d_x}},MN} \\ \Phi_{k_a,MN}(x_i^m, \hat{a}_i^m)' \hat{\beta}_{a,k_a,MN} \end{pmatrix} = \hat{A}^m(x_i^m, \hat{a}_i^m),$$

where the  $d_x \times (d_x + 1)$  matrix  $\hat{B}^m(x_i^m, \hat{a}_i^m)$  is

$$\hat{B}^m(x_i^m, \hat{a}_i^m) \equiv \left( \mathbb{I}_{d_x} \otimes \hat{I}_a^m(x_i^m, \hat{a}_i^m), -\nabla_x \hat{I}^m(x_i^m, \hat{a}_i^m) \right),$$

and the  $d_x \times 1$  vector  $\hat{A}^m(x_i^m, \hat{a}_i^m)$  is

$$\hat{A}^m(x_i^m, \hat{a}_i^m) \equiv \left[ \hat{h}_a^m(x_i^m, \hat{a}_i^m) \nabla_x \hat{I}^m(x_i^m, \hat{a}_i^m) - \hat{I}_a^m(x_i^m, \hat{a}_i^m) \nabla_x \hat{h}^m(x_i^m, \hat{a}_i^m) \right] / \hat{h}^m(x_i^m, \hat{a}_i^m).$$

Therefore, the estimated series coefficients are the solutions to the following least square problem

$$\hat{\beta}_{MN} \equiv \arg \min_{\beta} \sum_{m=1}^M \sum_{i=1}^N LS(x_i^m, \hat{a}_i^m; \beta),$$

where

$$LS(x_i^m, \hat{a}_i^m; \beta) \equiv \left\| \hat{B}^m(x_i^m, \hat{a}_i^m) \cdot \begin{pmatrix} \Phi_{k_{x_1}, MN}(x_i^m, \hat{a}_i^m)' \beta_{x_1} \\ \vdots \\ \Phi_{k_{x_{d_x}}, MN}(x_i^m, \hat{a}_i^m)' \beta_{x_{d_x}} \\ \Phi_{k_a, MN}(x_i^m, \hat{a}_i^m)' \beta_a \end{pmatrix} - \hat{A}^m(x_i^m, \hat{a}_i^m) \right\|^2.$$

There is an explicit expression for  $\hat{\beta}_{MN}$  as follows:

$$\hat{\beta}_{MN} = \hat{S}_{\Phi\Phi}^{-1} \hat{S}_{\Phi A},$$

where

$$\hat{S}_{\Phi\Phi} \equiv (MN)^{-1} \sum_{m=1}^M \sum_{i=1}^N \hat{S}_{\Phi}(x_i^m, \hat{a}_i^m)' \hat{S}_{\Phi}(x_i^m, \hat{a}_i^m), \quad (4.10)$$

$$\hat{S}_{\Phi A} \equiv (MN)^{-1} \sum_{m=1}^M \sum_{i=1}^N \hat{S}_{\Phi}(x_i^m, \hat{a}_i^m)' \hat{A}^m(x_i^m, \hat{a}_i^m). \quad (4.11)$$

In equations (4.10) and (4.11),

$$\hat{S}_{\Phi}(x_i^m, \hat{a}_i^m) \equiv \left( \hat{S}_{\Phi,1}(x_i^m, \hat{a}_i^m), \hat{S}_{\Phi,2}(x_i^m, \hat{a}_i^m) \right),$$

where

$$\hat{S}_{\Phi,1}(x_i^m, \hat{a}_i^m) \equiv \begin{pmatrix} \Phi_{k_{x_1},MN}(x_i^m, \hat{a}_i^m)' & & 0 \\ & \ddots & \\ 0 & & \Phi_{k_{x_{d_x}},MN}(x_i^m, \hat{a}_i^m)' \end{pmatrix} \otimes \hat{I}_a^m(x_i^m, \hat{a}_i^m),$$

$$\hat{S}_{\Phi,2}(x_i^m, \hat{a}_i^m) \equiv -\nabla_x \hat{I}^m(x_i^m, \hat{a}_i^m) \otimes \Phi_{k_a,MN}(x_i^m, \hat{a}_i^m)'.$$

Then the estimated ratios of the quality function are

$$\begin{cases} \widehat{\frac{e_{x_1}(x,a)}{e(x,a)}} & \equiv \hat{g}_1(x,a) = \Phi_{k_{x_1},MN}(x,a)' \hat{\beta}_{x_1,k_{x_1},MN}, \\ \vdots & \vdots \\ \widehat{\frac{e_{x_{d_x}}(x,a)}{e(x,a)}} & \equiv \hat{g}_{d_x}(x,a) = \Phi_{k_{x_{d_x}},MN}(x,a)' \hat{\beta}_{x_{d_x},k_{x_{d_x}},MN}, \\ \widehat{\frac{e_a(x,a)}{e(x,a)}} & \equiv \hat{g}_{d_x+1}(x,a) = \Phi_{k_a,MN}(x,a)' \hat{\beta}_{a,k_a,MN}. \end{cases} \quad (4.12)$$

By replacing the relevant ratios of the quality function in equation (3.5) with their estimators given in equation (4.12), one obtains the estimator of the quality function

$$\begin{aligned} \hat{e}(x,a) &= \exp \left( \sum_{j=1}^{d_x} \int_{\bar{x}_j}^{x_j} \hat{g}_j(x_1, \dots, x_{j-1}, s_j, \bar{x}_{j+1}, \dots, \bar{x}_{d_x}, \bar{a}) ds_j + \int_{\bar{a}}^a \hat{g}_{d_x+1}(x, t) dt \right) \\ &= \exp \left( \sum_{j=1}^{d_x} \int_{\bar{x}_j}^{x_j} \left[ \Phi_{k_{x_j},MN}(x_1, \dots, x_{j-1}, s_j, \bar{x}_{j+1}, \dots, \bar{x}_{d_x}, \bar{a})' \hat{\beta}_{x_j,k_{x_j},MN} \right] ds_j \right. \\ &\quad \left. + \int_{\bar{a}}^a \left[ \Phi_{k_a,MN}(x, t)' \hat{\beta}_{a,k_a,MN} \right] dt \right). \end{aligned} \quad (4.13)$$

#### 4.1.3 Estimation of Sellers' Marginal Disutility Function $\hat{U}_h(h, x, a)$ Using Multiple Market Data

Estimation of the sellers' marginal disutility function starts from the partial derivatives of the payment equation (2.3). Combined with the sellers' FOC in equation (2.1), they imply that for  $\forall(x, a) \in \mathcal{X} \times \mathcal{A}$ ,

$$\begin{cases} \nabla_x I^m(x, a) &= \left[ \nabla_x h^m(x, a) + h^m(x, a) \frac{\nabla_x e(x, a)}{e(x, a)} \right] \cdot U_h(h^m(x, a), x, a), \\ I_a^m(x, a) &= \left[ h_a^m(x, a) + h^m(x, a) \frac{e_a(x, a)}{e(x, a)} \right] \cdot U_h(h^m(x, a), x, a). \end{cases} \quad (4.14)$$

Now, use a series of basis functions  $\Psi_{k_{U,MN}}(h, x, a) \equiv (\psi_1(h, x, a), \dots, \psi_{k_{U,MN}}(h, x, a))'$  to approximate the unknown marginal disutility function. Then, one wants to choose the series coefficients  $\hat{\gamma}_{k_{U,MN}}$  to minimize the sum of the squared distances between the left-hand sides and the right-hand sides of the equations (4.14). Specifically, define

$$G_{x,MN}(h_i^m, x_i^m, \hat{a}_i^m; \gamma) \equiv \left[ \nabla_x \hat{h}^m(x_i^m, \hat{a}_i^m) + \hat{h}^m(x_i^m, \hat{a}_i^m) \frac{\nabla_x \widehat{e(x_i^m, \hat{a}_i^m)}}{e(x_i^m, \hat{a}_i^m)} \right] \Psi_{k_{U,MN}}(h_i^m, x_i^m, \hat{a}_i^m)' \gamma - \nabla_x \hat{I}^m(x_i^m, \hat{a}_i^m),$$

and

$$G_{a,MN}(h_i^m, x_i^m, \hat{a}_i^m; \gamma) \equiv \left[ \hat{h}_a^m(x_i^m, \hat{a}_i^m) + \hat{h}^m(x_i^m, \hat{a}_i^m) \frac{e_a(\widehat{e(x_i^m, \hat{a}_i^m)})}{e(x_i^m, \hat{a}_i^m)} \right] \Psi_{k_{U,MN}}(h_i^m, x_i^m, \hat{a}_i^m)' \gamma - \hat{I}_a^m(x_i^m, \hat{a}_i^m).$$

And the minimum distance (MD) estimator of the series coefficients are defined as

$$\hat{\gamma}_{k_{U,MN}} \equiv \arg \min_{\gamma \in \mathbb{R}^{k_{U,MN}}} \sum_{m=1}^M \sum_{i=1}^N \left\| \begin{pmatrix} G_{x,MN}(h_i^m, x_i^m, \hat{a}_i^m; \gamma) \\ G_{a,MN}(h_i^m, x_i^m, \hat{a}_i^m; \gamma) \end{pmatrix} \right\|^2.$$

The estimator  $\hat{\gamma}_{k_{U,MN}}$  has a closed-form expression given by

$$\hat{\gamma}_{k_{U,MN}} = \hat{S}_{\Psi\Psi}^{-1} \hat{S}_{\Psi I}$$

where

$$\hat{S}_{\Psi\Psi} \equiv (MN)^{-1} \sum_{m=1}^M \sum_{i=1}^N \hat{S}_{\Psi}(h_i^m, x_i^m, \hat{a}_i^m)' \hat{S}_{\Psi}(h_i^m, x_i^m, \hat{a}_i^m), \quad (4.15)$$

$$\hat{S}_{\Psi I} \equiv (MN)^{-1} \sum_{m=1}^M \sum_{i=1}^N \hat{S}_{\Psi}(h_i^m, x_i^m, \hat{a}_i^m)' \hat{S}_I(h_i^m, x_i^m, \hat{a}_i^m). \quad (4.16)$$

In equations (4.15) and (4.16), the  $(d_x + 1) \times k_{U,MN}$  matrix  $\hat{S}_\Psi(h_i^m, x_i^m, \hat{a}_i^m)$  is

$$\hat{S}_\Psi(h_i^m, x_i^m, \hat{a}_i^m) \equiv \begin{pmatrix} \left[ \nabla_x \hat{h}^m(x_i^m, \hat{a}_i^m) + \hat{h}^m(x_i^m, \hat{a}_i^m) \frac{\nabla_x \widehat{e(x_i^m, \hat{a}_i^m)}}{\widehat{e(x_i^m, \hat{a}_i^m)}} \right] \otimes \Psi_{k_{U,MN}}(h_i^m, x_i^m, \hat{a}_i^m)' \\ \left[ \hat{h}_a^m(x_i^m, \hat{a}_i^m) + \hat{h}^m(x_i^m, \hat{a}_i^m) \frac{e_a(x_i^m, \hat{a}_i^m)}{\widehat{e(x_i^m, \hat{a}_i^m)}} \right] \Psi_{k_{U,MN}}(h_i^m, x_i^m, \hat{a}_i^m)' \end{pmatrix}$$

and the  $(d_x + 1) \times 1$  vector  $\hat{S}_I(h_i^m, x_i^m, \hat{a}_i^m) \equiv (\nabla_x \hat{I}^m(x_i^m, \hat{a}_i^m)', \hat{I}_a^m(x_i^m, \hat{a}_i^m))'$ . As a result, the estimated sellers' marginal disutility function is

$$\hat{U}_h(h, x, a) \equiv \Psi_{k_{U,MN}}(h, x, a)' \hat{\gamma}_{k_{U,MN}}.$$

The steps described in Sections 4.1.1-4.1.3 complete the estimation of the supply side structural functions  $(e, U_h)$ .

#### 4.1.4 Estimation of Buyers' Marginal Utility Function $\hat{R}_z(z, y, b)$ Using Multiple Market Data

The buyers' marginal utility function  $R_z$  can be estimated by similar steps. First, within each market  $m$ , estimate the conditional quantile  $b_i^m$  of the payment  $I_i^m$  using a formula similar to equation (4.1). The unobserved effective amounts can also be estimated as  $\hat{z}_i^m \equiv h_i^m \cdot \hat{e}(x_i^m, \hat{a}_i^m)$ , since researchers observe which seller is matched with which buyer. Second, estimate the reduced form payment function  $I^m(y, b)$  and effective amount demand function  $d^m(y, b)$  using the generated regressor  $\hat{b}_i^m$  and generated dependent variable  $\hat{z}_i^m$  from the single market  $m$ . Third, taking the partial derivatives of the payment equation for the buyers yields

$$\begin{aligned} \nabla_y I^m(y, b) &= P_z^m(d^m(y, b)) \cdot \nabla_y d^m(y, b), \\ I_a^m(y, b) &= P_z^m(d^m(y, b)) \cdot d_a^m(y, b). \end{aligned}$$

Combine these equations with the buyers' FOC in equation (2.4), and use a series of basis functions  $\Theta_{k_N}(z, y, b) \equiv (\theta_1(z, y, b), \dots, \theta_{k_N}(z, y, b))$  to approximate the unknown buyers' marginal utility function  $R_z(z, y, b)$ . Then, the function can be estimated by an MD estimator similar to that in Section 4.1.3. Moreover, if the buyers' utility values  $R_i^m$  are observed,<sup>22</sup> then the second and third steps are not necessary.

<sup>22</sup>For example, firm revenue in labor markets.

The series estimation of  $R$  and its derivative functions boils down to a linear regression of  $R_i^m$  on  $\Theta_{k_N}(\hat{z}_i^m, y_i^m, \hat{b}_i^m)$  using multiple market data.

## 4.2 Uniform Rates of Convergence of Structural Function Estimators

In this section and proofs,  $C$  denotes a sufficiently large, generic positive constant, and  $c$  denotes a sufficiently small, generic positive constant, both of which may take different values in different uses.

### 4.2.1 Convergence Rate of Unobserved Heterogeneity Estimators $\hat{a}_i^m$

This subsection derives the convergence rates of the within market series estimators of the conditional quantile  $a_i^m$  given in equation (4.1).

**Assumption 13.** *Suppose that  $F_{I^m|x^m}(I|x) \equiv F_{I^m|x^m=x}(I)$  is continuously differentiable of order  $d_1 > d_x$  on the support with derivatives uniformly bounded in  $I$  and  $x$ .*

Define

$$\nu_{a,N} \equiv \left( \frac{k_{Q,N}}{N} + k_{Q,N}^{1-2d_1/d_x} \right)^{1/2}.$$

**Theorem 3.** *Suppose that Assumption 13 is satisfied. Then,*

$$\sum_{i=1}^N |\hat{a}_i^m - a_i^m|^2 / N = \mathcal{O}(\nu_{a,N}^2).$$

*And if  $k_{Q,N}/N \rightarrow 0$  and  $k_{Q,N} \rightarrow \infty$ , then  $\sum_{i=1}^N |\hat{a}_i^m - a_i^m|^2 / N = o_p(1)$ .*

### 4.2.2 Convergence Rates of Payment Function Estimators $\hat{I}^m(x, a)$ and Quantity Function Estimators $\hat{h}^m(x, a)$

This subsection derives the convergence rates of the within market series estimators of the reduced form payment functions  $I^m(x, a)$  and quantity functions  $h^m(x, a)$  and their first-order derivatives.

**Assumption 14.** *Suppose that  $\mathcal{X}$  and  $\mathcal{Y}$  are Cartesian products of closed intervals.*

**Assumption 15.** Suppose that  $\Phi_k(x, a) = \Phi_{1,k_1}(x_1) \otimes \cdots \otimes \Phi_{d_x,k_{d_x}}(x_{d_x}) \otimes \Phi_{a,k_a}(a)$ . This implies that  $k = k_a \cdot \prod_{j=1}^{d_x} k_j$ .

In Assumption 15, if  $k$  denotes the number of series basis functions used to approximate an unknown function of  $(x, a)$  (or of  $(h, x, a)$ ), then let  $k_h$ ,  $k_j$  and  $k_a$  denote the numbers of series basis functions used to approximate the  $h$  component,  $x_j$  component and  $a$  component in the Cartesian space, respectively. For simplicity, we assume that  $k_h$ ,  $k_j$  and  $k_a$  grow at the same rate.

Let  $\zeta_0(k) \equiv k$ ,  $\zeta_a(k) \equiv k_a^2 k$ , and  $\zeta_j(k) \equiv k_j^2 k$ .

**Assumption 16.** Suppose that for all  $m \in \mathcal{M}$ ,  $I^m(x, a)$  and  $h^m(x, a)$  are continuously differentiable of order  $d \geq 2$  on the support.<sup>23</sup>

For a function  $l(x, a) : \mathcal{X} \times \mathcal{A} \rightarrow \mathbb{R}$ , define the norm  $|l|_\delta$  as  $|l|_\delta \equiv \max_{|\mu| \leq \delta} \sup_{(x,a) \in \mathcal{X} \times \mathcal{A}} |\partial^\mu l(x, a) / \partial x_1^{\mu_1} \cdots \partial x_{d_x}^{\mu_{d_x}} \partial a^{\mu_a}|$ , with  $\mu_1 + \cdots + \mu_{d_x} + \mu_a = \mu$  ( $\mu_1, \dots, \mu_{d_x}, \mu_a$  are integers).

One implication of Assumptions 5, 6 and 16 is that there exist some positive constants  $B_I$  and  $B_{hu}$  such that for all  $m \in \mathcal{M}$ ,  $|I^m|_2 \leq B_I$ , and  $|h^m|_2 \leq B_{hu}$ .

Suppose that the following assumption about the approximation error by the basis functions holds.

**Assumption 17.** Suppose that for a positive integer  $\delta_I \geq 1$ , there exist a constant  $\alpha_I > 2$  and pseudo-true series coefficients  $\xi_{0,I,k_I}^m \in \mathbb{R}^{k_I}$  such that  $|I^m - \Phi'_{k_I} \xi_{0,I,k_I}^m|_{\delta_I} \leq C k_I^{-\alpha_I}$  for all positive integers  $k_I$ . Suppose as well that for a positive integer  $\delta_h \geq 1$ , there exist a constant  $\alpha_h > 2$  and pseudo-true series coefficients  $\xi_{0,h,k_h}^m \in \mathbb{R}^{k_h}$  such that  $|h^m - \Phi'_{k_h} \xi_{0,h,k_h}^m|_{\delta_h} \leq C k_h^{-\alpha_h}$  for all positive integers  $k_h$ .<sup>24</sup>

Let  $l^m(x, a)$  denote either the payment function  $I^m(x, a)$  or the quantity function  $h^m(x, a)$ . Let  $\hat{l}^m(x, a)$  denote the series estimator of  $l^m(x, a)$  defined in equation (4.2) or equation (4.7), and let  $\hat{l}_{x_j}^m(x, a)$  ( $j = 1, \dots, d_x$ ) and  $\hat{l}_a^m(x, a)$  denote the series estimators of the first-order derivatives of  $l^m(x, a)$  defined in equation (4.4), equation (4.5), equation (4.8) or equation (4.9).

Define

$$\nu_{l,N} \equiv \zeta_0(k_{l,N}) (\nu_{a,N} + k_{l,N}^{-\alpha_l}),$$

<sup>23</sup>Without loss of generality, here I assume that  $d$  is the same across all markets  $m \in \mathcal{M}$ .

<sup>24</sup>Without loss of generality, here I assume that  $\alpha_I$  and  $\alpha_h$  are the same across all markets  $m \in \mathcal{M}$ .



$$\begin{aligned}\nu_{l_j,N} &\equiv \zeta_j(k_{l,N}) (\nu_{a,N} + k_{l,N}^{-\alpha_l}), \\ \nu_{l_a,N} &\equiv \zeta_a(k_{l,N}) (\nu_{a,N} + k_{l,N}^{-\alpha_l}).\end{aligned}$$

**Theorem 4.** *Suppose that Assumptions 14-17 and the conditions of Theorem 3 are satisfied. Suppose as well that the numbers of series basis functions used to approximate each component in  $k_{l,N}$  all increase to infinity with  $N$ , and  $\sqrt{k_{l,N}}\nu_{a,N}\zeta_a(k_{l,N}) \rightarrow 0$  as  $N \rightarrow \infty$ . Then*

$$\sup_{(x,a) \in \mathcal{X} \times \mathcal{A}} |\hat{l}^m(x,a) - l^m(x,a)| = \mathcal{O}_p(\nu_{l,N}).$$

And if  $k_{l,N}\nu_{a,N} \rightarrow 0$  and  $k_{l,N} \rightarrow \infty$ , then

$$\sup_{(x,a) \in \mathcal{X} \times \mathcal{A}} |\hat{l}^m(x,a) - l^m(x,a)| = o_p(1).$$

**Theorem 5.** *Suppose that the conditions for Theorem 4 are satisfied. Then*

$$\begin{aligned}\sup_{(x,a) \in \mathcal{X} \times \mathcal{A}} |\hat{l}_{x_j}^m(x,a) - l_{x_j}^m(x,a)| &= \mathcal{O}_p(\nu_{l_j,N}), \\ \sup_{(x,a) \in \mathcal{X} \times \mathcal{A}} |\hat{l}_a^m(x,a) - l_a^m(x,a)| &= \mathcal{O}_p(\nu_{l_a,N}).\end{aligned}$$

And if  $k_{j,l,N}^2 k_{l,N} \nu_{a,N} \rightarrow 0$  and  $k_{j,l,N} \rightarrow \infty$ , then

$$\sup_{(x,a) \in \mathcal{X} \times \mathcal{A}} |\hat{l}_{x_j}^m(x,a) - l_{x_j}^m(x,a)| = o_p(1);$$

if  $k_{a,l,N}^2 k_{l,N} \nu_{a,N} \rightarrow 0$  and  $k_{a,l,N} \rightarrow \infty$ , then

$$\sup_{(x,a) \in \mathcal{X} \times \mathcal{A}} |\hat{l}_a^m(x,a) - l_a^m(x,a)| = o_p(1).$$

Moreover, note that  $\nu_{l,N} = \mathcal{O}(\nu_{l_j,N})$ , and  $\nu_{l,N} = \mathcal{O}(\nu_{l_a,N})$ .

Since  $\hat{a}_i^m$  is used as a generated regressor,<sup>25</sup> the convergence rates of the reduced form functions and their derivatives depend on the estimation errors of  $\hat{a}_i^m$  as well as on the series approximation errors of the functions themselves.

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<sup>25</sup>Recall equations (4.3) and (4.6).

### 4.2.3 Convergence Rate of Quality Function Estimator $\hat{e}(x, a)$

This subsection derives the convergence rates of the cross-market series estimators of the quality function  $e(x, a)$  and its first-order derivative ratios.

**Assumption 18.** Suppose that for a positive integer  $\delta_e \geq 0$ , there exist a constant  $\alpha_e > 1$  and pseudo-true series coefficients  $\beta_{0,x_j,k_{x_j}} \in \mathbb{R}^{k_{x_j}}$  (for  $j = 1, \dots, d_x$ ) and  $\beta_{0,a,k_a} \in \mathbb{R}^{k_a}$  such that  $|e_{x_j}/e - \Phi'_{k_{x_j}} \beta_{0,x_j,k_{x_j}}|_{\delta_e} \leq Ck_{x_j}^{-\alpha_e}$  and  $|e_a/e - \Phi'_{k_a} \beta_{0,a,k_a}|_{\delta_e} \leq Ck_a^{-\alpha_e}$  for all positive integers  $k_{x_j}$  ( $j = 1, \dots, d_x$ ) and  $k_a$ .

Define

$$S_{\Phi\Phi} \equiv (MN)^{-1} \sum_{m=1}^M \sum_{i=1}^N S_{\Phi}(x_i^m, a_i^m)' S_{\Phi}(x_i^m, a_i^m),$$

where

$$S_{\Phi}(x_i^m, a_i^m) \equiv (S_{\Phi,1}(x_i^m, a_i^m), S_{\Phi,2}(x_i^m, a_i^m)),$$

$$S_{\Phi,1}(x_i^m, a_i^m) \equiv \begin{pmatrix} I_a^m(x_i^m, a_i^m) \Phi_{k_{x_1,MN}}(x_i^m, a_i^m)' & & 0 \\ & \ddots & \\ 0 & & I_a^m(x_i^m, a_i^m) \Phi_{k_{x_{d_x},MN}}(x_i^m, a_i^m)' \end{pmatrix},$$

and

$$S_{\Phi,2}(x_i^m, a_i^m) \equiv -\nabla_x I^m(x_i^m, a_i^m) \otimes \Phi_{k_a,MN}(x_i^m, a_i^m)'.$$

**Assumption 19.** Suppose that there exist some positive constants  $B_{eu}$  and  $B_{el}$  such that the quality function  $e(x, a)$  satisfies  $|e|_2 \leq B_{eu}$  and  $|e|_0 \geq B_{el}$ .

**Assumption 20.** Suppose:

- (i)  $\lambda_{\min}(\mathbb{E}(S_{\Phi\Phi})) \geq c > 0$ ;
- (ii) There exists some positive constant  $B_{hl}$  such that for all  $m \in \mathcal{M}$ ,  $|h^m|_0 \geq B_{hl}$ .

For  $j = 1, \dots, d_x$ , define

$$\nu_{e_j,M,N} \equiv \zeta_0(k_{x_j,MN}) \left[ \sum_{j=1}^{d_x} \nu_{h_j,N} + \nu_{h_a,N} + \sum_{j=1}^{d_x} \nu_{I_j,N} + \nu_{I_a,N} + k_{a,MN}^{-\alpha_e} + \sum_{j=1}^{d_x} k_{x_j,MN}^{-\alpha_e} \right],$$

$$\nu_{e_a,M,N} \equiv \zeta_0(k_{a,MN}) \left[ \sum_{j=1}^{d_x} \nu_{h_j,N} + \nu_{h_a,N} + \sum_{j=1}^{d_x} \nu_{I_j,N} + \nu_{I_a,N} + k_{a,MN}^{-\alpha_e} + \sum_{j=1}^{d_x} k_{x_j,MN}^{-\alpha_e} \right].$$

**Lemma 4.** Suppose that Assumptions 18-20 and the conditions of Theorems 3-5 are satisfied. Suppose as well that the numbers of series basis functions  $k_{x_j,MN} \rightarrow \infty$ ,  $\nu_a^2(\sigma_N) (\zeta_a^2(k_{x_j,MN}) + \zeta_a^2(k_{a,MN})) \rightarrow 0$ ,  $\nu_{a,N}(\zeta_a(k_{x_j,MN}) + \zeta_a(k_{a,MN}))(\zeta_0(k_{x_j,MN}) + \zeta_0(k_{a,MN})) \rightarrow 0$ ,  $[\nu_{l_a,N}\zeta_0(k_{x_j,MN}) + \zeta_0(k_{a,MN})\nu_{l_j,N}](\zeta_0(k_{x_j,MN}) + \zeta_0(k_{a,MN})) \rightarrow 0$  for  $j = 1, \dots, d_x$ ,  $k_{a,MN} \rightarrow \infty$ , and  $[\zeta_0^2(\max_{j=1,\dots,d_x} k_{x_j,MN}) + \zeta_0^2(k_{a,MN})](\max_{j=1,\dots,d_x} k_{x_j,MN} + k_{a,MN})/(MN) \rightarrow 0$  as  $N \rightarrow \infty$ . Then

$$\|\hat{\beta}_{MN} - \beta_{0,MN}\| = \mathcal{O}_p \left( \sum_{j=1}^{d_x} \nu_{h_j,N} + \nu_{h_a,N} + \sum_{j=1}^{d_x} \nu_{I_j,N} + \nu_{I_a,N} + k_{a,MN}^{-\alpha_e} + \sum_{j=1}^{d_x} k_{x_j,MN}^{-\alpha_e} \right).$$

**Theorem 6.** Suppose that the conditions for Lemma 4 are satisfied. Then, for  $j = 1, \dots, d_x$

$$\begin{aligned} \sup_{(x,a) \in \mathcal{X} \times \mathcal{A}} \left| \widehat{e_{x_j}(x,a)/e(x,a)} - e_{x_j}(x,a)/e(x,a) \right| &= \mathcal{O}_p(\nu_{e_j,M,N}), \\ \sup_{(x,a) \in \mathcal{X} \times \mathcal{A}} \left| \widehat{e_a(x,a)/e(x,a)} - e_a(x,a)/e(x,a) \right| &= \mathcal{O}_p(\nu_{e_a,M,N}). \end{aligned}$$

And if for  $l = I, h$ ,  $k_{x_j,MN}\nu_{l_a,N} \rightarrow 0$ ,  $k_{x_j,MN}\nu_{l_j,N} \rightarrow 0$  ( $j = 1, \dots, d_x$ ),  $k_{x_j,MN}k_{a,MN}^{-\alpha_e} \rightarrow 0$  and  $k_{x_j,MN} \rightarrow \infty$ , then

$$\sup_{(x,a) \in \mathcal{X} \times \mathcal{A}} \left| \widehat{e_{x_j}(x,a)/e(x,a)} - e_{x_j}(x,a)/e(x,a) \right| = o_p(1).$$

Similarly, if we swap  $k_{x_j,MN}$  and  $k_{a,MN}$  in the above conditions, then

$$\sup_{(x,a) \in \mathcal{X} \times \mathcal{A}} \left| \widehat{e_a(x,a)/e(x,a)} - e_a(x,a)/e(x,a) \right| = o_p(1).$$

**Theorem 7.** Suppose that the conditions for Theorem 6 are satisfied. Then

$$\sup_{(x,a) \in \mathcal{X} \times \mathcal{A}} |\hat{e}(x,a) - e(x,a)| = \mathcal{O}_p \left( \sum_{j=1}^{d_x} \nu_{e_j,M,N} + \nu_{e_a,M,N} \right).$$

And if both of the following hold:

$$\sup_{(x,a) \in \mathcal{X} \times \mathcal{A}} \left| \widehat{e_{x_j}(x,a)/e(x,a)} - e_{x_j}(x,a)/e(x,a) \right| = o_p(1),$$

$$\sup_{(x,a) \in \mathcal{X} \times \mathcal{A}} \left| \widehat{e_a(x,a)/e(x,a)} - e_a(x,a)/e(x,a) \right| = o_p(1),$$

then we have

$$\sup_{(x,a) \in \mathcal{X} \times \mathcal{A}} |\hat{e}(x,a) - e(x,a)| = o_p(1).$$

The convergence rates of the quality function and its derivative ratios depend on the estimation errors of the reduced form functions and the series approximation errors of the quality function itself. Note that the estimation errors in  $\hat{a}_i^m$  affect the estimation errors of  $e_{x_j}(x,a)/e(x,a)$ ,  $e_a(x,a)/e(x,a)$  and  $e(x,a)$  only through  $\hat{I}^m$ ,  $\hat{h}^m$  and their partial derivatives.

#### 4.2.4 Convergence Rate of Sellers' Marginal Disutility Function Estimator $\hat{U}_h(h, x, a)$

This subsection derives the convergence rate of the cross-market series estimator of the sellers' marginal disutility function  $U_h(h, x, a)$ .

For a function  $l(h, x, a): \mathcal{H} \times \mathcal{X} \times \mathcal{A} \rightarrow \mathbb{R}$ , define the norm  $|l|_\delta$  as  $|l|_\delta \equiv \max_{|\mu| \leq \delta} \sup_{(h,x,a) \in \mathcal{H} \times \mathcal{X} \times \mathcal{A}} |\partial^\mu l(x, a) / \partial h^{\mu_h} \partial x_1^{\mu_1} \cdots \partial x_{d_x}^{\mu_{d_x}} \partial a^{\mu_a}|$ , with  $\mu_h + \mu_1 + \cdots + \mu_{d_x} + \mu_a = \mu$  ( $\mu_h, \mu_1, \dots, \mu_{d_x}, \mu_a$  are integers).

**Assumption 21.** Suppose that for a positive integer  $\delta_U \geq 0$ , there exist a constant  $\alpha_U > 1$  and pseudo-true series coefficients  $\gamma_{0,k_U} \in \mathbb{R}^{k_U}$  such that  $|U_h - \Psi'_{k_U} \gamma_{0,k_U}|_{\delta_U} \leq C k_U^{-\alpha_U}$  for all positive integers  $k_U$ .

**Assumption 22.** Suppose that there exists some positive constant  $B_U$  such that  $|U_h|_1 \leq B_U$ .

**Assumption 23.** Suppose that  $\Psi_k(h, x, a) = \Psi_{h,k_h}(h) \otimes \Psi_{1,k_1}(x_1) \otimes \cdots \otimes \Psi_{d_x,k_{d_x}}(x_{d_x}) \otimes \Psi_{a,k_a}(a)$ . This implies that  $k = k_h \cdot k_a \cdot \prod_{j=1}^{d_x} k_j$ .

**Assumption 24.** Suppose that  $\mathcal{H}$  is a compact set and the cross-market variation in  $f_x^m$  and  $f_y^m$  is rich enough that the equilibrium cross-market joint density of  $(h, x, a)$  is bounded away from zero.

Define

$$S_{\Psi\Psi} \equiv (MN)^{-1} \sum_{m=1}^M \sum_{i=1}^N S_{\Psi}(h_i^m, x_i^m, a_i^m)' S_{\Psi}(h_i^m, x_i^m, a_i^m),$$

where

$$S_\Psi(h_i^m, x_i^m, a_i^m) \equiv \begin{pmatrix} \left[ \nabla_x h^m(x_i^m, a_i^m) + h^m(x_i^m, a_i^m) \frac{\nabla_x e(x_i^m, a_i^m)}{e(x_i^m, a_i^m)} \right] \otimes \Psi_{k_{U,MN}}(h_i^m, x_i^m, a_i^m)' \\ \left[ h_a^m(x_i^m, a_i^m) + h^m(x_i^m, a_i^m) \frac{e_a(x_i^m, a_i^m)}{e(x_i^m, a_i^m)} \right] \Psi_{k_{U,MN}}(h_i^m, x_i^m, a_i^m)' \end{pmatrix}.$$

**Assumption 25.** Suppose that  $\lambda_{\min}(\mathbb{E}(S_{\Psi\Psi})) \geq c > 0$ .

**Lemma 5.** Suppose that Assumptions 21-25 and the conditions for Theorem 6 are satisfied. Suppose as well that  $k_{U,MN} \rightarrow \infty$ ,  $\sqrt{k_{U,MN}}\nu_{a,N}\zeta_a(k_{U,MN}) \rightarrow 0$ ,  $k_{U,MN}\nu_{e_j,N} \rightarrow 0$  and  $k_{U,MN}\nu_{e_a,N} \rightarrow 0$  as  $N \rightarrow \infty$ . Then

$$\|\hat{\gamma}_{k_{U,MN}} - \gamma_{0,k_{U,MN}}\| = \mathcal{O}_p \left( \nu_{e_a,M,N} + \sum_{j=1}^{d_x} \nu_{e_j,M,N} + k_{U,MN}^{-\alpha_U} \right).$$

In addition, define

$$\nu_{U_h,M,N} \equiv \zeta_0(k_{U,MN}) \left[ \nu_{e_a,M,N} + \sum_{j=1}^{d_x} \nu_{e_j,M,N} + k_{U,MN}^{-\alpha_U} \right].$$

**Theorem 8.** Suppose that the conditions of Lemma 5 are satisfied. Then

$$\sup_{(h,x,a) \in \mathcal{H} \times \mathcal{X} \times \mathcal{A}} \left| \hat{U}_h(h, x, a) - U_h(h, x, a) \right| = \mathcal{O}_p(\nu_{U_h,M,N}).$$

And if  $k_{U,MN} \rightarrow \infty$ ,  $k_{U,MN}\nu_{e_a,M,N} \rightarrow 0$  and for  $j = 1, \dots, d_x$ ,  $k_{U,MN}\nu_{e_j,M,N} \rightarrow 0$ , then we have

$$\sup_{(h,x,a) \in \mathcal{H} \times \mathcal{X} \times \mathcal{A}} \left| \hat{U}_h(h, x, a) - U_h(h, x, a) \right| = o_p(1).$$

The convergence rate of the sellers' marginal disutility function depends on the estimation errors of the quality function and on the series approximation error of the sellers' marginal disutility function itself. Note that the estimation errors of  $\hat{a}_i^m$  and the reduced form functions and their derivatives directly affect the convergence rate of the sellers' marginal disutility function, but they are dominated by the estimation errors of the quality function and its derivatives.

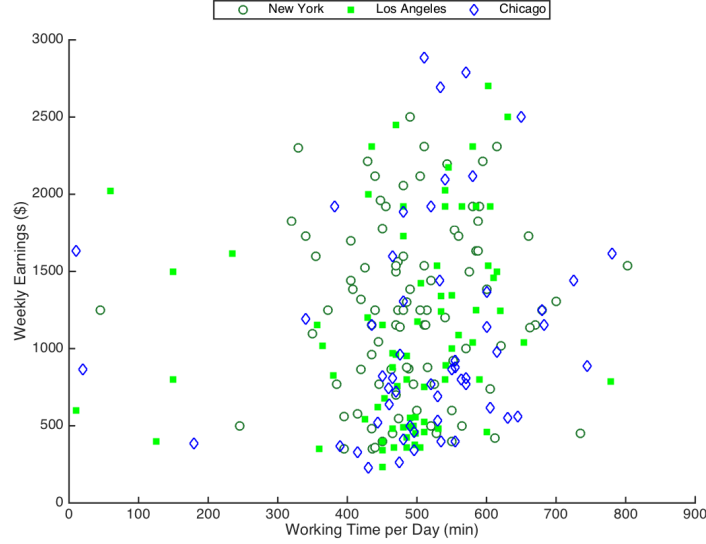


Figure 5.1: Scatter Plots of Weekly Earnings and Working Time in the Three Cities

## 5 Empirical Illustration in Labor Markets

In this section, I apply the estimation procedure provided in Sections 4.1 to estimate the efficiency (quality) function  $e$  in labor markets. Section 5.1 introduces the data set, and Section 5.2 estimates the workers' unobserved efficiency function.

### 5.1 Data: the 2015 American Time Use Survey

The data set I use is the American Time Use Survey (ATUS, see Hofferth, Flood, and Sobek, 2013 for details). The ATUS randomly chooses one individual from a subsample of the households that are completing their participation in the Current Population Survey (CPS) and asks them to recall their time spent, minute by minute, on various activities within a randomly picked 24-hour period in the past. The ATUS classifies activities into 17 major categories and many more sub-categories, and provides a quite precise measure of the time that workers actually spent in working.<sup>26</sup>

I consider the 2015 ATUS respondents,<sup>27</sup> and focus on full-time workers in the

<sup>26</sup>Major categories include working and work-related activities, household activities, education, traveling and others. For working and work-related activities, it further breaks down to working, looking for a job, eating and drinking on the job (e.g., lunch breaks), security procedures, and so on. I use the time spent in the working sub-category as the measure of working time.

<sup>27</sup>The data were obtained via ATUS-X Extract Builder: Sandra L. Hofferth, Sarah M. Flood, and

three largest cities: New York, Los Angeles and Chicago<sup>28</sup>. After dropping observations on Saturdays and Sundays and making some other minor adjustments, I end up with a sample of 92 workers in New York, 74 workers in Los Angeles, and 55 workers in Chicago.

I use the time spent in the “working” sub-category of the ATUS as the measure of working time  $h_i^m$ , the weekly earnings in the CPS as the measure of earnings  $I_i^m$ , and the age reported in the CPS as the observed characteristic  $x_i^m$  of the workers.<sup>29</sup>

Figure 5.1 shows the scatter plots of working time per day and weekly earnings of each worker in the three cities. Within- and cross-market variation appears prominent: (i) both working time and earnings vary substantially within all the markets; (ii) for the same working time, earnings in New York tend to be higher than those in Los Angeles, which in turn, tend to be higher than those in Chicago. In fact, the median of the earnings-to-working-time ratio is 2.47 for the workers in New York, 2.03 in Los Angeles, and 1.62 in Chicago. Such within- and cross-market variation is crucial for the identification of the unobserved efficiency function.

## 5.2 Estimation of Unobserved Efficiency Function

With the observed data  $(I_i^m, h_i^m, x_i^m)$  from the three cities, one is able to estimate the efficiency function  $e(x, a)$ .

As discussed in Sections 2.1 and 3, distributions  $f_x^m$  of workers’ observed characteristic  $x_i^m$  (age) serve as aggregate instruments that induce cross-market variation in the earnings functions. Figure 5.2 plots the kernel estimated densities of the workers’ age distributions in the three cities. It shows that in the 2015 ATUS sample, full-time workers in Chicago are slight younger than in the other two cities. The age distributions in Los Angeles and Chicago are slightly more dispersed than that in New York.

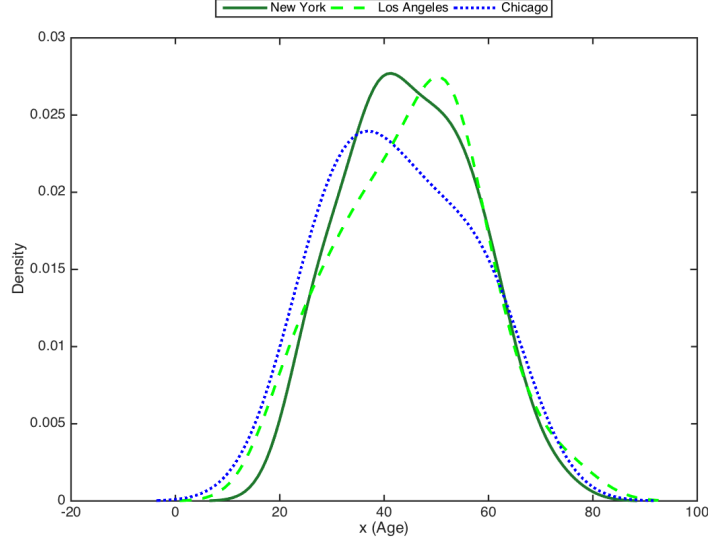
Such variation in the distributions  $f_x^m$  appears to be sufficient to generate ade-

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Matthew Sobek. 2013. American Time Use Survey Data Extract System: Version 2.4 [Machine-readable database]. Maryland Population Research Center, University of Maryland, College Park, Maryland, and Minnesota Population Center, University of Minnesota, Minneapolis, Minnesota.

<sup>28</sup>To be precise, the three largest metro areas: New York-Newark-Bridgeport (NY-NH-CT-PA), Los Angeles-Long Beach-Riverside (CA), and Chicago-Naperville-Michigan City (IL-IN-WI).

<sup>29</sup>Individuals in the ATUS can be linked to their observations in the CPS to obtain rich demographic information. In this illustration, I use age as the only observed characteristic for simplicity. The application to more observed variables poses no theoretical problem, but it may take more computing time.



For ease of illustration, age is used as the single observed characteristic ( $x$ ) of the workers. This figure shows that there is decent cross-market variation in the distributions of  $x$ , which drives (partially) the cross-market variation in the equilibrium payment functions.

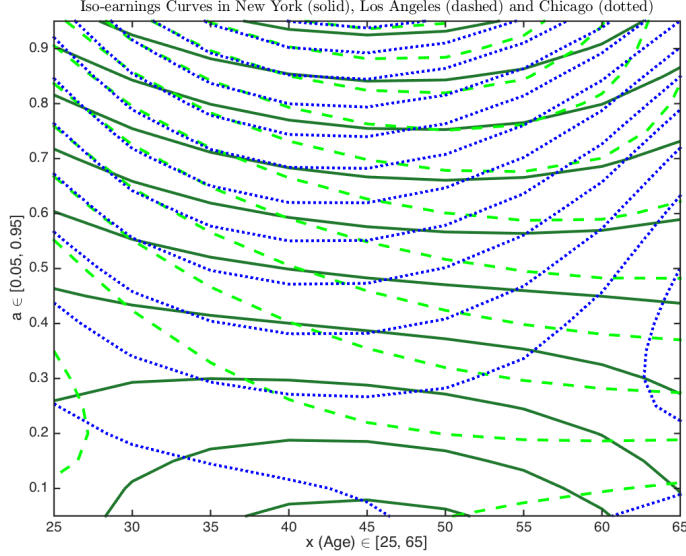
Figure 5.2: Distributions of Age in the Three Cities

quate variation in the earnings functions. Figure 5.3 draws representative iso-earnings curves for the three cities on the support  $\mathcal{X} \times \mathcal{A} = [25, 65] \times [0.05, 0.95]$ . Recall that Assumption 9 for identifying the efficiency function requires that the iso-earnings curves from at least two cities have different slopes. For each value of  $(x, a)$  on the support, this is the case, except in the very small region with  $a > 0.9$  and  $x \in [35, 55]$ . This suggests that Assumption 9 is satisfied. Moreover, using estimated derivatives of the earnings functions  $\hat{I}_x^m(x, a)$  and  $\hat{I}_a^m(x, a)$ ,  $m = 1, \dots, M$ , I compute  $\hat{B}(x, a)$ , the estimate of the coefficient matrix  $B(x, a)$  defined in Assumption 9 for a grid of  $(x, a)$  values on the support  $\mathcal{X} \times \mathcal{A}$ . The determinants of  $\hat{B}(x, a)' \hat{B}(x, a)$  for all these  $(x, a)$  values are bounded well away from zero. This indicates that the matrix  $B(x, a)$  has full column rank. As a result, I am convinced that the key identification condition for the efficiency function  $e(x, a)$  is satisfied.

The normalization worker I choose is  $(\bar{x}, \bar{a}) \equiv (25, 0)$ . I used the tensor product of quadratic polynomials of  $x$  and  $a$  to approximate  $e_x(x, a)/e(x, a)$  and  $e_a(x, a)/e(x, a)$ .<sup>30</sup>

<sup>30</sup>That is, I approximate the two ratio functions using  $\beta_0 + \beta_1 x + \beta_2 x^2 + \beta_4 a + \beta_5 x a + \beta_6 a x^2 + \beta_7 a^2 + \beta_8 a^2 x + \beta_9 a^2 x^2$ . There is no obvious rule for how one should determine the order of the polynomials for the efficiency function or for the other structural functions in this model. This may





This figure shows representative iso-earnings curves for the three cities, on the support of age ( $x$ ) and “ability” ( $a$ ). For majority of the support, the iso-earnings curves from at least two markets cross. This suggests that the identification condition for the efficiency function  $e(x, a)$  is satisfied.

Figure 5.3: Iso-Earnings Curves in the Three Cities

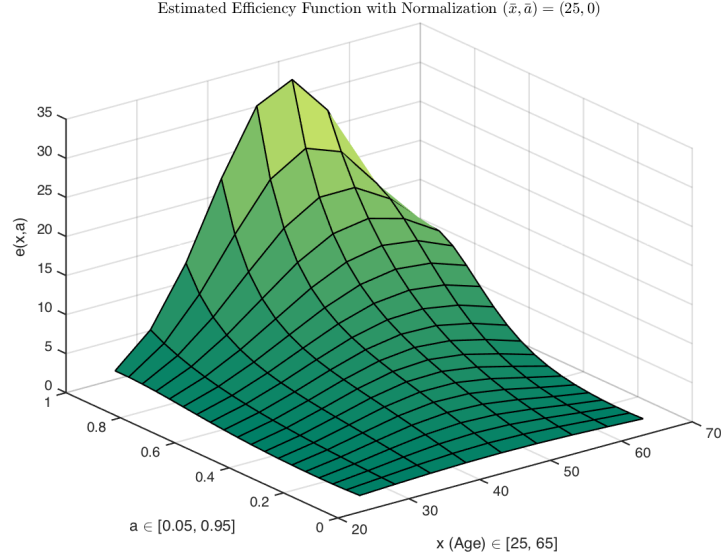
With the two estimated ratio functions, one could obtain the estimates of the efficiency function defined as in equation (4.13). Figure 5.4 plots the estimated efficiency function  $\hat{e}(x, a)$  on the support  $\mathcal{X} \times \mathcal{A}$ .

Figure 5.4 presents a prominent and interesting pattern of the efficiency function. For workers with the same level of unobserved characteristic  $a$  (“ability”), efficiency first increases with age, and then decreases. For workers of the same age, efficiency increases with  $a$ . At age 25, workers with the highest ability do not exhibit much higher efficiency than their lower ability peers. As they mature, however, their efficiency could be much higher than their peers with the lowest ability.<sup>31</sup>

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serve as a topic for further research.

<sup>31</sup>Since I only control for age and neglect the dynamic perspective of the workers, one should be cautious when interpreting this estimate. But this issue will be investigated in future research, and an in-depth empirical analysis is beyond the scope of this section.



Estimated worker efficiency function increases with “ability” ( $a$ ), and is hump-shaped with age ( $x$ ).

Figure 5.4: Estimated Efficiency Function  $e(x, a)$

## 6 Conclusion and Extensions

In this paper, I study the identification and estimation of a nonparametric hedonic equilibrium model with unobserved quality. I explain how to use within- and cross-market variation in equilibrium prices and quantities to identify and estimate the structural functions of the model. Using the estimated structural functions and the equilibrium-solving algorithm suggested in this paper, researchers could solve the counterfactual equilibrium to analyze the distributional effects of policy interventions. In contrast to other widely used methods, the counterfactuals thus constructed account for unobserved quality and equilibrium effects of policy interventions in a nonparametric setting.

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