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PATIENT PREFERENCES, INTERGENERATIONAL EQUITY AND THE PRECAUTIONARY PRINCIPLE

URMEE KHAN, MAXWELL B. STINCHCOMBE

Abstract. Patient preferences have a social welfare interpretation consonant with a belief that the society affected by present decisions will last for a very long time. In stochastic settings, these preferences lead to justifications for variants of the precautionary principle.

1. Introduction

An early expression of the desirability of long term social planning can be found in the late 1600's "Traité de la culture des forêts" [13], by Louis XIV's defense minister, Sébastien Le Prestre de Vauban. He noted that forests were systematically over-exploited in France, that after replanting, they start being productive in slightly less than 100 years but don't become fully productive for 200 years, and that no private enterprise could conceivably have so long a time-horizon, essentially for discounting reasons. From these observations, he concluded that the only institutions that could, and should, undertake such projects were the government and the church. His calculations involved summming the un-discounted benefits, delayed and large, and costs, early and small, on the assumption that society would be around for at least the next 200 years to enjoy the net benefits.

This paper systematically explores stochastic dynamic planning problems with patient preferences over long sequences of costs and rewards. These preferences are the limits of various generalizations of "discount factor close to 1" preferences and are meant to represent preferences in optimization problems where the time interval over which payoffs are being optimized is much longer than the time scale over which the decisions are made. For deterministic problems, the preferences value investments in the expansion of long-term productive capacity, for stochatic problems, they value precautionary research into potential benefits and costs and highlight the option value of reversibility.

The next section works through a deterministic and a stochastic version of a natural resource management problem to demonstrate what is involved. In our applications, there will be a result true for the patient preferences and a corresponding limit interpretation of the optimal policy. The subsequent two sections give a formal development of patient preferences and a number of applications.

2. Long Run Resource Management

We work with a deterministic and then a stochastic version of the classical fishery model, often interpreted as a simple one-good macro growth model. At the first time point in $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, $t = 0$, one starts with a stock x_0, makes a consumption choice $c_0 \in [0, x_0]$, the remaining stock (called 'escapement' in fishery contexts, 'investment' in macro contexts) grows to $x_1 = f(x_0 - c_0)$ at $t = 1$, at which point the the decision/growth process repeats. The growth function, $f(\cdot)$, is concave, satisfies $f(0) = 0$, $f(x) > x$ for $x \in (0, x_c)$ for some carrying capacity $x_c > 0$, and $f(x) < x$ for $x \in (x_c, \infty)$. Stochastic versions of the model include measurement error for x_t, errors in the implementation of the consumption/harvest decisions, c_t, and shocks to the growth function. A consumption path $c = (c_0, c_1, \ldots)$ gives rise to a sequence of period utilities $u = (u_0, u_1, \ldots)$ with $u_t = u(c_t)$, $u(\cdot)$ a bounded, increasing, concave function with the normalization $u(0) = 0$.

2.1. **Overview of Preferences.**

Prominent in the class of study preferences on bounded sequences of utilities that we study are those having *tangents* of the form

$$u \mapsto \lim \sum_{t=0}^{T} u_t \eta_t \qquad (1)$$

where the limit is taken along a sequence of weights η^α that: are normalized to sum to 1, $\sum_t \eta_t^\alpha = 1$; are patient in that $\sum_{t\leq\tau} \eta_t^\alpha \to 0$ for all τ; and are intergenerationally equitable in that $\sum_t |\eta_{t+1}^\alpha - \eta_t^\alpha| \to 0$.

Often such preferences take the form

$$u \mapsto \lim \tfrac{1}{\mathbb{E}\,T}\mathbb{E} \textstyle\sum_{t=0}^{T} u_t \tag{2}$$

where the limit is taken along a sequence of random variables T^α with $P(T^\alpha \leq \tau) \to 0$ for any finite τ. For example, in the fishery/growth model: $u = (u_0, u_1, \ldots)$ has $u_t = u(c_t)$; T is a random end of the planning horizon; $\mathbb{E}$ is expectation; and we normalize by $\frac{1}{\mathbb{E}\,T}$.

If we interpret the utility of people born at time t as u_t, and interpret the random variable T as the time at which society will end, then the utility function in (2) represents the expected summation of society's utility. To study preferences appropriate for a society that believes it will last for a long time relative to the time frame for making decisions, we study the limits of such preferences as the expected termination date, $\mathbb{E}\,T$ increases to ∞, that is, as the end of the relevant period becomes more and more remote.

To see the connection between utilities of the form $\sum_{t=0}^{T} u_t\eta_t$ and $\frac{1}{\mathbb{E}\,T}\mathbb{E} \sum_{t=0}^{T} u_t$ for a random T, note that one can calculate $\mathbb{E} \sum_{t=0}^{T} u_t$ in two ways, "horizontally,"

$$\begin{array}{llllll}
P(T=0)\cdot(u_0)+ \\
P(T=1)\cdot(u_0 & + u_1)+ \\
P(T=2)\cdot(u_0 & + u_1 & + u_2)+ \\
P(T=3)\cdot(u_0 & + u_1 & + u_2 & + u_3)+ \\
\quad\vdots \qquad\ \vdots & \quad\vdots & \quad\vdots & \quad\vdots & \ddots \ ,
\end{array} \tag{3}$$

or "vertically" by noting that the u_0 term appears in all of the vertical lines, that u_1 appears in all vertical lines with $T \geq 1$, that that u_2 appears in all vertical lines with

$T \geq 2$, and so on. This rearrangement delivers

$$\mathbb{E} \sum_{t=0}^{T} u_t = \sum_{t \geq 0} u_t \cdot P(T \geq t). \tag{4}$$

If one normalizes, dividing the previous sum by $\mathbb{E}\, T = \sum_{t\geq 0} P(T \geq 0)$, the result is

$$\tfrac{1}{\mathbb{E}\, T}\mathbb{E} \sum_{t=0}^{T} u_t = \sum_{t \geq 0} \eta_t \cdot u_t \tag{5}$$

where $\eta_t = \frac{1}{\mathbb{E}\, T} P(T \geq t)$ yields a non-negative sequence summing to 1. If e.g. T has a geometric distribution on $\mathbb{N}_0 = \{0, 1, \ldots\}$ with $\mathbb{E}\, T = 1/(1-\delta)$, then the term in (5) is the discounted sum $(1-\delta) \sum_{t=0}^{T} \delta^t u_t$.

We will take limits as $\sum_{t \leq \tau} \eta_t \to 0$ for all finite τ. This implies that the accumulation points of any sequence of η's is a purely finitely additive (a.k.a. weightless or remote) probability. The sense of 'accumulation point' is given by the weak* topology, $\eta = \lim_\alpha \eta^\alpha$ iff for all $E \subset \mathbb{N}_0$, $|\eta^\alpha(E) - \eta(E)| \to 0$, and the set of finitely additive probabilities is compact in this topology (by Alaoglu's theorem).

Definition 2.1. *A net/sequence $(\eta^\alpha)_{\alpha \in A}$ is* ***limit patient*** *if $\lim_\alpha \sum_{t \leq \tau} \eta_t^\alpha \to 0$ for any finite τ, and η is* ***patient*** *if for all $E \subset \mathbb{N}_0$, $\eta(E) = \lim_\alpha \eta^\alpha(E)$ for a limit patient net. A net/sequence η^α* ***satisfies approximate intergenerational equity*** *if $\lim_\alpha \sum_t |\eta_{t+1}^\alpha - \eta_t^\alpha| \to 0$, and η* ***satisfies intergenerational equity*** *if $\eta(E) = \lim_\alpha \eta^\alpha(E)$ for all $E \subset \mathbb{N}_0$ and some equitable net.*

Limits of uniform distributions and limits of discounted sums are leading examples.

Example 2.1. *If $\eta_t^\alpha = \frac{1}{T^\alpha + 1}$ for $t = j^\alpha, j^\alpha + 1, \ldots, j^\alpha + T^\alpha$, $T^\alpha \to \infty$, and η is an accumulation point of the η^α, then η is patient and satisfies intergenerational equity. In much the same way, if $\eta_t^\delta = (1-\delta)\delta^t$ and η is an accumulation point of the set $\{\eta^\delta : \delta \uparrow 1\}$, then η is patient and satisfies intergenerational equity.*

The two essential results flow from the observation that a concave $V : \ell_\infty \to \mathbb{R}$ can be expressed as the minimum of the affine functions that majorize it. First, maximizing a concave functional over a convex set is equivalent to maximizing any one of the tangent functionals at the optimum, so that anything true about the optimum for all of the tangent functionals is true for the function V. Second, when the set of tangents, S, are integrals against probabilities, $V(u) = \min_{\eta \in S} \int_{\mathbb{N}_0} u_t \, d\eta(t)$, a form of preferences extensively studied in the literature on choice under ambiguity.

2.2. **Patient Management of a Deterministic Fishery.** A feasible steady state consumption for the fishery is a stock level x° and a consumption level $c^\circ \in [0, x^\circ]$ satisfying $f(x^\circ - c^\circ) = x^\circ$ — if one starts with stock x° and consumes c°, then in the next period the stock will again be x°. With steady state escapement equal to $e^\circ := x^\circ - c^\circ$, steady state consumption is $f(e^\circ) - e^\circ$. The steady state escapement (or investment), e^*, delivering maximal steady state consumption solves

$$\max_{e \in [0, x_c]} [f(e) - e]. \tag{6}$$

It has corresponding consumption $c^* = f(e^*) - e^*$ and utility $u(c^*)$, both called **golden rule levels**. Patient preferences lead to golden rule levels, and this result has a limit interpretation.

Lemma 1 (Patience in a deterministic fishery)**.** *If η is patient and satisfies intergenerational equity, then for any strictly positive starting stock x_0, any optimal plan leads to a sequence of utilities $u \in \ell_\infty$ satisfying*

$$\langle u, \eta \rangle = u(c^*). \tag{7}$$

Further, for any $\eta^\alpha \to \eta$, for every $\epsilon > 0$, there exists an α' such that for all $\alpha \succ \alpha'$, $\max_{u \in H_u(x_0)} \langle u, \eta^\alpha \rangle \in ((1-\epsilon)u(c^*), (1+\epsilon)u(c^*))$ where the maximum is taken over $H_u(x_0)$, the set of feasible utility paths starting from x_0.

We will give a direct proof of this result when the machinery is in place for an easy proof. At present, we give a direct proof of the limit interpretation.

Proof. Suppose that $\eta^\alpha \to \eta$ is a limit patient net satisfying approximate intergenerational equity. From any $x_0 > 0$, there exists a finite τ at which the stock e^* can be achieved if consumption is $c_0 = c_1 = \cdots c_\tau = 0$. Consuming c^* from τ onward delivers utility $V(\eta^\alpha) \sum_{t \leq \tau} u(0)\eta_t^\alpha + \sum_{t > \tau} u(c^*)\eta_t^\alpha$. Since $\sum_{t \leq \tau} \eta_t \to 0$, $V(\eta^\alpha) \to u(c^*)$ showing that some feasible plan gives $V(\eta^\alpha) > (1-\epsilon)u^*$ for sufficiently large α. The argument that $V(\eta^\alpha) < (1+\epsilon)u^*$ proceeds from the observation that consuming more than c^* out of a stock x^* consumes fish that, if left in the escapement, would more than reproduce themselves on average. Combined with $\lim_\alpha \sum_t |\eta_{t+1}^\alpha - \eta_t^\alpha| \to 0$ and the concavity of $u(\cdot)$, this means that for any starting x_0, $V(\eta^\alpha) < (1+\epsilon)u^*$ for large α. □

2.3. **Patient Management of a Stochastic Fishery.** Following [10], there are three main sources of stochasticity in fishery models: measurement errors in the stock, $m_t = z_t^m x_t$ is observed rather than x_t; stochastic growth, next period's stock is $x_{t+1} = z_t^g f(x_t - c_t)$ rather than $f(x_t - c_t)$; errors in the harvest/catch, $c_t = \min(x_t, a_t)$ where $a_t = z_t^h c_t$ rather than c_t where z_t^m and z_t^g are positive random variables. Measurement error and harvest errors can combine to drive the stock to extinction, $x_{t+1} = 0$.

A policy p, is a sequence of catch targets $(c_t)_{t=0}^\infty$ with each c_t being a function of the history of stock measurements, $m_0, m_1, \ldots, m_t$, and previous catches, $c_0, c_1, \ldots, c_{t-1}$. For a policy p, let τ_p denote the random time until the stock is extinct. The following is the limit version of the patient management of a stochastic fishery result that if extinction can be avoided, then it cannot be optimal for extinction to happen in finite time.

Lemma 2 (Limit patience in a stochastic fishery)**.** *If η^α is a limit patient sequence satisfying approximate intergenerational equity and T^α is the corresponding sequence of random planning horizons, then if there is a sequence of policies p^α with no extinction in finite time,* $\lim_\alpha P(\tau_{p^\alpha} \leq N) \to 0$*, and positive average utility,* $\lim_\alpha \frac{1}{\mathbb{E}\, T^\alpha} \mathbb{E} \sum_{t \leq \tau_{p^\alpha}} u_t^\alpha > 0$*, then no policy with finite time extinction can be optimal.*

We will give stronger results below when we examine the behavior of $\mathbb{E}\,\tau_p/\mathbb{E}\,T$ in more detail.

3. Representations of Patient Preferences

We study monotonic preferences on $\ell_\infty := \{(u_0, u_1, u_2, \ldots) : (\exists B)(\forall t)[\,|u_t| \leq B\,]\}$ that can be represented by continuous concave functions. Continuous concave functions are the lower envelope of the continuous affine functions that majorize them. This makes knowing how to represent continuous linear functionals, i.e. the dual space, a crucial first step.

3.1. The Dual Space.

The dual space of ℓ_∞ is is denoted $\ell_\infty^\dagger$ and defined as the set of continuous linear functionals on ℓ_∞. An element $f \in \ell_\infty^\dagger$ has norm $\|f\| := \sup\{|f(u)| : \|u\|_\infty \leq 1\}$. Well-known elements of $\ell_\infty^\dagger$ include $\ell_1 := \{x \in \mathbb{R}^{\mathbb{N}} : \sum_t |x_t| < \infty\}$, where each $x \in \ell_1$ defines/is identified with an element of $\ell_\infty^\dagger$ by $f_x(u) = \langle u, x\rangle := \sum_t u_t x_t$. To see that ℓ_1 is a strict subset of $\ell_\infty^\dagger$, define $f(u) = \lim_T \frac{1}{T+1}\sum_{t=0}^{T} u_t$ on the linear subspace of ℓ_∞ where the limit exists and use the Hahn-Banach theorem to extend this to a continuous linear functional on all of ℓ_∞.

With $\mathbb{Z} = \{\ldots, -2, -1, 0, +1, +2, \ldots\}$, a **bounded time change** is a one-to-one mapping $\pi : \mathbb{N}_0 \to \mathbb{Z}$ that is onto $\mathbb{N}_0$ and for which there exists $M \in \mathbb{N}$ with $|\pi(t) - t| \leq M$ for all $t \in \mathbb{N}_0$. For $u \in \ell_\infty$, the time changed version of u is denoted u^π and defined by $u^\pi_t = u_{\pi^{-1}(t)}$ for $t \in \mathbb{N}_0$. Every finite time-shift, $\pi(t) = t - M$, is a bounded time change, as is every permutation of $\mathbb{N}_0$ that moves no time point by more than M.

Definition 3.1. *An $f \in \ell_\infty^\dagger$ is*

(1) ***monotonic or positive*** *if $[u \geq v] \Rightarrow [f(u) \geq f(v)]$,*

(2) ***invariant*** *if $f(u) = f(u^\pi)$ for all bounded time changes π, and*

(3) ***normalized*** *if $\|f\| = 1$.*

A monotonic, invariant, normalized $f \in \ell_\infty^\dagger$ is called a ***Banach-Mazur*** *limit.*

The most tractable representations of $\ell_\infty^\dagger$ use constructions from nonstandard analysis (references here). We work in a polysaturated extension of a superstructure $V(S)$ where S contains $\mathbb{R}$. We say that an element of $r \in {}^*\mathbb{R}$ is **limited** if for some $B \in \mathbb{R}$, $|r| \leq B$, and for any limited r, ${}^\circ r$ denotes its standard part. If $\eta \in {}^*\ell_1$ and $\sum_t |\eta_t|$ is limited, then $u \mapsto {}^\circ\langle u, \eta\rangle$ is continuous and linear, and the mapping $E \mapsto {}^\circ \sum_{t \in {}^*E} \eta_t$ is a finitely additive signed measure on the class of subsets of $\mathbb{N}_0$. A great deal more is known.

Theorem A. *$\ell_\infty^\dagger$, its monotonic invariant elements, and the extreme points of its normalized monotonic invariant elements have the following nonstandard representations.*

(1) [*Robinson*] *$f \in \ell_\infty^\dagger$ if and only if there exists a star-finite set $\{0, \ldots, T\} \subset {}^*\mathbb{N}_0$ and $\eta_0, \ldots, \eta_T$ with $\sum_{t=1}^{T} |\eta_t| \simeq \|f\|$ such that $f(u) = {}^\circ\langle {}^*u, \eta\rangle$ for all $u \in \ell_\infty$.*

(2) [*Luxemberg*] *a monotonic $f \in \ell_\infty^\dagger$ is invariant if and only if $\sum_t |\eta_{t+1} - \eta_t| \simeq 0$ where $f(u) = {}^\circ\langle {}^*u, \eta\rangle$.*

(3) [*Keller and Moore*] *f is an extreme point of the class of monotonic, normalized, invariant $f \in \ell_\infty^\dagger$ if and only if $f(u) = {}^{\circ *}u, \eta\rangle$ where $\eta_t = \frac{1}{T+1}$ for all $t \in \{j, j+1, \ldots, j+T\}$, $\eta_t = 0$ otherwise, for some unlimited T.*

Proof. The Luxemberg representation result is [8, Theorem 5.9], the Keller and Moore representation result is [4, Theorem 3.1]. The Robinson representation result has a short and informative proof, which we give here.

If $f \in \ell_\infty^\dagger$, then it is Lipschitz with Lipschitz constant $\|f\|$, hence determined by its values on any dense subset, hence determined by its values on the class of simple functions. For any finite set of simple functions, $\mathcal{U} := \{U_i : i \in I\}$, let $\mathcal{P}_{\mathcal{U}}$ denote the partition of $\mathbb{N}_0$ generated by the non-empty sets of the form $U_i^{-1}(r)$, $r \in \mathbb{R}$, $i \in I$. By basic linear algebra, there exists a finitely supported measure, $\eta_f = \eta_f(\mathcal{U})$, on $\mathbb{N}_0$ with $f(U_a) = \sum_{t \in \mathbb{N}_0} U_a(t)\eta_f(t)$ and $\sum_t |\eta_f(t)| = \|f\|^\dagger$. Let $\mathcal{U}'$ be an exhaustive, *-finite set of *-simple functions (saturation), and let $\eta_f = \eta_f(\mathcal{U}')$. □

Also useful is the representation of $\ell_\infty^\dagger$ as integrals against finitely additive signed measures on the class of all subsets of $\mathbb{N}_0$. For $\eta \in {}^*\ell_1$ with $\sum_t |\eta_t|$ limited, we denote by $\gamma = \mathrm{st}\,(\eta)$ the finitely additive signed measure defined by $\gamma(E) = {}^\circ \sum_{t \in {}^*E} \eta_t$. Thus, γ is the standard part of η in the weak* topology, and Alaoglu's theorem tells us that $\sum_t |\eta_t|$ limited is sufficient for η to be nearstandard.

Corollary A.1. *γ is a finitely additive measure on the class of subsets of $\mathbb{N}_0$ if and only if there exists a star-finite set $\{0, \ldots, T\} \subset {}^*\mathbb{N}_0$ and $\eta_0, \ldots, \eta_T$ with $\sum_{t=1}^T |\eta_t| = \|\gamma\|$ such that $\gamma(E) =^\circ \sum_{t \in {}^*E} \eta_t$. Further, for every $u \in \ell_\infty$, $\int u_t \, d\gamma(t) = {}^\circ\langle {}^*u, \eta\rangle$.*

Proof. Immediate from Robinson's representation result. qed

3.2. **Weight on the Remote Future.** Preferences $\succsim$ on ℓ_∞ are **monotonic** if $[u \geq v] \Rightarrow [u \succsim v]$, they are **patient** if $[\{t : u_t \neq v_t\}$ finite$] \Rightarrow [u \sim v]$, and respect **intergenerational equity** if for any bounded time change π, $u^\pi \sim u$.

Definition 3.2. *An $\eta \in {}^*\ell_1$ with $\sum_t |\eta_t|$ limited* ***puts weight only on the remote future****, or is* ***remote****, if $\sum_{t \leq \tau} |\eta_t| \simeq 0$ for all limited τ, it is* ***monotonic*** *if ${}^\circ \sum_{t \in {}^*E} \eta_t \geq 0$ for all $E \subset \mathbb{N}_0$, and it* ***respects intergenerational equity*** *or is* ***invariant*** *if $\sum_t |\eta_{t+1} - \eta_t| \simeq 0$.*

Theorem B. *A continuous concave $V : \ell_\infty \to \mathbb{R}$ is represents patient monotonic preferences iff every tangent is non-negative and remote, and they respect intergenerational equity iff their tangents are invariant.*

Proof. Suppose that V is monotonic, patient, and respects intergenerational equity. Pick $u \in \ell_\infty$ and suppose that a tangent, η, to V at u fails to be non-negative. By definition, there exits $E \subset \mathbb{N}_0$ such that ${}^\circ \sum_{t \in E} \eta_t < 0$. This means that ${}^\circ\langle u + 1_E, \eta\rangle < {}^\circ\langle u, \eta\rangle$ so that V fails to be monotonic. Suppose now that the tangent, η, fails to be remote. This means that there exists a finite set E such that ${}^\circ\langle -1_E, \eta\rangle < 0$ so that $V(u - 1_E) < V(u)$, so that V fails to be patient. Suppose now that the tangent, η, fails to be invariant. Pick a finite permutation π such that ${}^\circ\langle u - u^\pi, \eta\rangle > 0$ and note that $V(u^\pi) < V(u)$ so that V fails to respect integenerational equity. The arguments for the reverse implications are similar. □

3.3. **Examples of Monotonic, Concave, Patient Preferences.** Let $Unif(j,T) \in \ell_1$ denote the uniform distribution on the set $\{j, j+1, \ldots, j+T\}$, and $Geom(j,\delta)$ denote the geometric distribution on $\{j, j+1, \ldots\}$ with expectation $j + \frac{1}{(1-\delta)}$. Several monotonic, concave, patient utility functions have tangents belong to the remote accumulation points of the following classes.

Definition 3.3. *The set of* ***uniform distributions*** *is* $\mathcal{U} := \{Unif(0,T) : T \in \mathbb{N}_0\}$, *the set of* ***shifted uniform distributions*** *is* $\mathcal{U}_{shift} := \{Unif(j,T) : j, T \in \mathbb{N}_0\}$, *the set of* ***tail-shifted uniform distributions*** *is* $\mathcal{U}_{tail} := \{Unif((1-\epsilon)T, T) : \epsilon \in [0,1]\}$, *the set of* ***discounted distributions*** *is* $\mathcal{U}_{disct} := \{Geom(0,\delta) : \delta \in (0,1)\}$, *and the set of* ***shifted discounted distributions*** *is* $\mathcal{U}_{shift\ disct} := \{Geom(j,\delta) : j \in \mathbb{N}_0, \delta \in (0,1)\}$.

The first four parts of the following are closely related to the representation of patient, monotonic, concave and continuous utility functions studied in [9].

Theorem C (Representation). *The following preferences can be represented by concave utility functionals with patient tangents:*

(1) *worst long-run — for all* $u \in \ell_\infty$, $\liminf_t u_t = \min\{{}^\circ\langle {}^*u, \eta\rangle : \eta \text{ is remote}\}$;
(2) *worst long-run average —* $\eta \in {}^*\ell_1$ *represents a remote accumulation point of* $\mathcal{U}$ *iff* $\eta = Unif(0,T)$ *for some unlimited* T *and for all* $u \in \ell_\infty$, $\liminf_T \frac{1}{T+1}\sum_{t=0}^{T} u_t = \min\{{}^\circ\langle {}^*u, Unif(0,T)\rangle : T \text{ unlimited}\}$;
(3) *worst shifted long-run average —* $\eta \in {}^*\ell_1$ *represents a remote accumulation point of* $\cap_{J,\tau\in\mathbb{N}_0}\{Unif(j,T) : j \geq J, T \geq \tau\}$ *iff* $\eta = Unif(j,T)$ *for some unlimited pair* j, T, *and for all* $u \in \ell_\infty$, $\inf_j \liminf_T \frac{1}{T+1}\sum_{t=j}^{j+T} u_t = \min\{{}^\circ\langle {}^*u, Unif(j,T)\rangle : j, T \text{ unlimited}\}$;
(4) *worst tail-shifted long-run average —* $\eta \in {}^*\ell_1$ *represents a remote accumulation point of* $\cap_{\epsilon\in(0,1),\ n\in\mathbb{N}}\{Unif((1-\epsilon)T,T) : \epsilon T \geq n\}$ *iff* $\eta = Unif((1-\epsilon)T, T)$ *where*

$\epsilon \simeq 0$, $\epsilon{\cdot}T$ *is unlimited, and* $\inf_{\epsilon\in(0,1)} \liminf_T \frac{1}{\epsilon\cdot T}\sum_{t=(1-\epsilon)T}^{T} u_t = \min\{{}^\circ\langle {}^*u, Unif((1-\epsilon)T,T)\rangle : \epsilon \simeq 0,\ \epsilon\cdot T \text{ unlimited}\}$ *for all* $u \in \ell_\infty$*; and*

(5) *worst long-run discounted* — $\eta \in {}^*\ell_1$ *represents a remote accumulation point of* $\lim_{\underline{\delta}\uparrow 1}\{Geom(0,\delta) : \delta \in [\underline{\delta}, 1)\}$ *iff* $\eta_t = (1-\delta)\delta^t$ *for some* $\delta \simeq 1$*, and for all* $u \in \ell_\infty$*,* $\liminf_{\delta\uparrow 1}(1-\delta)\sum_t u_t\delta^t = \min\{{}^\circ{}^*u, Geom(0,\delta) : \delta \simeq 1\}$*; and*

(6) *worst long-run shifted discounted* — $\eta \in {}^*\ell_1$ *represents a remote accumulation point of* $\cap_{J,\underline{\delta}<1}\{Geom(j,\delta) : j \geq J, \delta \in [\underline{\delta}, 1)\}$ *iff* $\eta_{t+j} = (1-\delta)\delta^t$ *for some* $\delta \simeq 1$*, and* $\inf_j \liminf_{\delta\uparrow 1}(1-\delta)\sum_{t\geq j} u_t\delta^t = \min\{{}^\circ{}^*u, Geom(j,\delta) : \delta \simeq 1,\ j \text{ remote}\}$ *for all* $u \in \ell_\infty$.

The usual fashion for describing, say, a remote accumulation point of $\{Unif(0,T) : T \in \mathbb{N}_0\}$, is to let $T^k \uparrow \infty$ be a sequence in $\mathbb{N}_0$, let $\mathbb{L}$ be the linear subspace of ℓ_∞ on which $\frac{1}{T^k+1}\sum_{t=0}^{T^k} u_t$ converges, and use the Hahn-Banach theorem to extend this continuous linear functional from $\mathbb{L}$ to all of ℓ_∞. Theorem C(2) tells us that all of the Hahn-Banach extensions for all of the sequences $T^k \uparrow \infty$ are of the form $u \mapsto {}^\circ\langle {}^*u, Unif(0,T)\rangle$ for some unlimited T.

To compare Theorem C(2) and (5), or (3) and (6), recall that the Hardy and Littlewood Tauberian theorem says that for any $u \in \ell_\infty$,

$$\liminf_T \frac{1}{T+1}\sum_{t=0}^{T} u_t \leq \liminf_{\delta\uparrow 1}\sum_{t=0}^{\infty} u_t\delta^t \leq \limsup_{\delta\uparrow 1}\sum_{t=0}^{\infty} u_t\delta^t \leq \limsup_T \frac{1}{T+1}\sum_{t=0}^{T} u_t. \tag{8}$$

Further, if $u \in \ell_\infty$ is the optimal sequence of utilities for many dynamic programming problems, the inequalities become equalities. (See [12] for a concise proof of the Hardy and Littlewood result and an example with all four inequalities strict, see [3], [7], [5], and [6] for the properties of optimal sequences of utilities.)

Proof. **Worst long-run**. Let $\underline{u} = \liminf_{t\to\infty} u_t$. For any remote η, ${}^\circ\langle {}^*u, \eta\rangle \geq \underline{u}$, so it is sufficient to find a remote sfp with ${}^\circ\langle {}^*u, \eta\rangle = \underline{u}$.

For each $n \in \mathbb{N}$, let $\mathbb{T}_n$ be the internal set $\{t \in {}^*\mathbb{N} : t \geq n, {}^*u_t < \underline{u} + \frac{1}{n}\}$ so that $\mathbb{T}_n \supset \mathbb{T}_{n-1}$ and $\mathbb{T}_n \neq \emptyset$. By the internal extension principle, the mapping $n \mapsto \mathbb{T}_n$ has an internal extension to a mapping from ${}^*\mathbb{N}$ to ${}^*\mathcal{P}(\mathbb{N})$. For that extension, the internal set N of all n such that $\mathbb{T}_n \supset \mathbb{T}_{n-1}$ and $\mathbb{T}_n \neq \emptyset$ contains all finite n, hence contains an infinite n'. Because any $t \in \mathbb{T}_{n'}$ is greater than or equal to n', any $\eta \in \Delta(\mathbb{T}_{n'})$ is remote, and for any such η, $|\langle {}^*u, \eta\rangle - \underline{u}| \leq \frac{1}{n'} \simeq 0$.

Worst long-run average. The standard part of ${}^*\{Unif(0,T) : T \in \mathbb{N}_0\} = \{Unif(0,T) : T \in {}^*\mathbb{N}_0\}$ is its closure, the remote points in the closure are the ones with T unlimited.

Let $\underline{u}_{Av} = \liminf_T \frac{1}{T+1}\sum_{t=0}^{T} u_t$, and repeat a variant of the last arguments.

The rest are essentially the same. □

3.4. **Connections with Choice Under Ambiguity.** There is a formal similarity to some models of choice in presence of ambiguity. All of the utility functions in the representation theorem are of the form $u \mapsto \min_{\gamma \in S} \int u \, d\gamma$ where $S = \{\mathrm{st}\,(\eta) : \eta \in S'\}$ where the sets S' are described in the representation Theorem. This means that one can understand the caution demonstrated by optimal policies for patient preferences as arising from extreme ambiguity aversion coupled with having a wide range of weights given to generations living before the end of the planning horizon.

4. Dynamic Programming Applications

A **deterministic dynamic programming problem** is given by $((S, \mathcal{S}), \Gamma, v)$ where: $(S, \mathcal{S})$ is a Polish state space; $\Gamma : S \rightrightarrows S$ is a non-empty valued correspondence with $\Gamma(s)$ being the set choices that are feasible at $s \in S$; and $v : \Gamma \to \mathbb{R}$ is a bounded measurable stage utility function. Starting at any $s_0 \in S$, a **strategy** is a sequence of history dependent choices, $t \in \mathbb{N}_0$ with $s_{t+1} \in \Gamma(s_t)$. Every strategy induces an **outcome** $h(s_0) = (s_0, s_1, \ldots)$, the set of possible outcomes starting at s is denote $H(s)$, each outcome h has a corresponding sequence of stage utilities, $(v(s_0, s_1), v(s_1, s_2), \ldots)$, and the set of possible sequences of utilities is denoted $H_u(s)$.

4.1. **Controllable Problems.** The following restriction asks that it be possible to go from any state s to any other state s' in at most a uniformly bounded number of steps.

Definition 4.1. *The problem* $((S, \mathcal{S}), \Gamma, u)$ *is* ***finitely controllable*** *if there exists an* M *such that for all* $s, s' \in S$*, there is an outcome* $h(s) = (s = s_0, s_1, \ldots)$ *with* $s_m = s'$ *for some* $m \leq M$*.*

Patience will imply golden rule utility levels much more generally than we saw above. Note the use of "lim" rather than "lim sup" in the following. We will see that for finitely controllable problems, these levels exist.

Definition 4.2. *For* $s \in S$*, the* ***golden rule utility level*** *is*

$$U^*(s) := \sup_{u \in H_u(s)} \lim_T \tfrac{1}{T+1} \textstyle\sum_{t=0}^{T} u_t. \tag{9}$$

The deterministic fishery model given above has growth function $f : [\underline{x}, \overline{x}] \to \mathbb{R}_+$ and stage consumption utilities $u(c_t)$. In terms of the formality here: $S = [\underline{x}, \overline{x}]$: $\Gamma(s) = [0, f(s)]$; and $v(s_t, s_{t+1}) = u(s_t - f^{-1}(s_{t+1}))$. The following tells us that for any finitely controllable problem, the golden rule utility level are independent of starting point and for any patient intergenerationally equitable preferences, the golden rule utility level is achieved.

Theorem D. *If* $((S, \mathcal{S}), \Gamma, u)$ *is finitely controllable,* $V : \ell_\infty \to \mathbb{R}$ *is patient, monotonic continuous, concave, and respects intergenerational equity, then*

(1) *for any* $s, s' \in S$*, the golden rule utility levels exist and are equal,* $U^*(s) = U^*(s')$*; and*
(2) *for any* $s \in S$*, any optimal* h^**, and any* η *defining a tangent to* V*,* $\circ u^*, \eta\rangle = \|\eta\| \cdot U^*(s)$ *where* u^* *is the vector of utilities associated with* h^**.*

In particular, for finitely controllable problems, there is a single golden rule utility, and for any of the intergenerationally equitable $V : \ell_\infty \to \mathbb{R}$ in Theorem C, the optimal utility is equal to that golden rule utility.

Proof. Pick $s \in S$. Let T, T' be unlimited. To show that the golden rule utility levels exist and are equal, it is sufficient to show that

$$\sup_{v \in H_u(s)} \tfrac{1}{T+1} \sum_{t=0}^{T'} v_t \simeq \sup_{v' \in H_u(s)} \tfrac{1}{T'+1} \sum_{t=0}^{T'} v'_t \tag{10}$$

and that there exists $u \in H_u(s)$ such that

$$\tfrac{1}{T+1} \sum_{t=0}^{T} u_t \simeq \tfrac{1}{T'+1} \sum_{t=0}^{T'} u_t \simeq \sup_{v \in H_u(s)} \tfrac{1}{T+1} \sum_{t=0}^{T} v_t \tag{11}$$

To show that $\sup_{v \in H_u(s)} \frac{1}{T+1} \simeq \sup_{v' \in H_u(s)} \frac{1}{T'+1} \sum_{t=0}^{T'} v'_t$: Relabeling if necessary, we suppose that $T < T'$. Let $U^*_T(s) = \sup_{u \in H_u(s)} {}^\circ\langle u, Unif(0,T)\rangle$. For each limited $n \in \mathbb{N}$, take $u_{1/n} \in H_u(s)$ such that $\langle u_{1/n}, Unif(0,T)\rangle > U^*_T(s) - 1/n$. The mapping $n \mapsto u_{1/n}$ from $\mathbb{N}$ to ${}^*\ell_\infty$ has $\|u_{1/n}\|$ limited because v is uniformly bounded. An internal extension of this mapping takes ${}^*\mathbb{N}$ to ${}^*\ell_\infty$, and by overspill, there exists an unlimited n such that $u_{1/n} \in {}^*H_u(s)$ and ${}^\circ\langle u_{1/n}, Unif(0,T)\rangle = U^*_T(s)$. If $T/T' \simeq 1$, then $\langle u_{1/n}, Unif(0,T)\rangle \simeq \langle u_{1/n}, Unif(0,T')\rangle$, and we are done, so suppose that $\epsilon := 1 - {}^\circ(T/T') > 0$. By the Lake Woebegone principle (see [7, Proposition 2]), the average payoff of $u_{1/n}$ over some small enough intervals must be at least $U^*_T(s)$. In particular, this must be true for intervals that are infinite in length but still only infinitesimal relative to ϵT. Using finite controllability, we can repeat these intervals with finite resets and lose at most an infinitesimal average amount.
To show that $\frac{1}{T+1} \sum_{t=0}^{T} u_t \simeq \frac{1}{T'+1} \sum_{t=0}^{T'} u_t \simeq \sup_{v \in H(s)} \frac{1}{T+1} \sum_{t=0}^{T} v_t$: This follows from the last line of the previous argument.

We now turn to the tangent arguments. Re-scaling, we can suppose w.l.o.g. that $\|\eta\| = 1$. From the Keller and Moore representation above, any invariant probability γ is in the convex hull of the class of γ's with representations of the form $\int u\, d\gamma = {}^\circ\langle {}^*u, \eta\rangle$

where $\eta = Unif(j,T)$, j,T unlimited. The previous gives the arguments if $j = 0$. By finite controllability, from any state at $t = j - M$ we can restart in any state s at $t = j$ so that the previous arguments apply. □

The following example from [7, §2] shows that without controllability, golden rule utility levels are not necessarily equal, and that the tangent analysis is not productive.

Example 4.1. *Take $S = \mathbb{N} \times \mathbb{N}_0$ so that $s = (n,m)$, $n \geq 1$, $m \geq 0$. Define $\Gamma(n,0) = \{(n{+}1,0),(n,1)\}$ and $\Gamma(n,m) = \{(n,m{+}1)\}$ if $m > 1$. Define $v((n_t,m_t),(n_{t+1},m_{t+1}) = 1$ if $1 \leq m_t \leq n_t$, and $v = 0$ else. At each state $(n,0)$, one can choose to receive 1 for n periods and 0 ever thereafter, or else can choose to go to $(n+1,0)$. Starting at $s = (1,0)$, $U_T^*(1,0) = \max_{u \in H_u(s)} \langle u, Unif(0,T) \rangle = \frac{1}{2}$, but $U_T^*(n,m) = 0$ for $m > n$. Further, $\max_{u \in H_u(s)} \min{}^\circ \langle u, Unif(0,T) \rangle = 0$ where the minimum is over unlimited T, and the same is true for all of the utility functions in Theorem* C.

4.2. **Stochasticity and Uncontrollability.** To extend the dynamic programming model to a stochastic context, we add an action space A, let $q : S \times A \to \Delta(S)$ be a transition probability, re-define the bounded utility function to have $S \times A$ as its domain, $v : S \times A \to \mathbb{R}$. A strategy is now a vector of measurable functions σ_t from histories to $\Delta(A)$. Starting at s_0, for $E \in \mathcal{S}$, the $t = 1$ distribution is given by $\nu_1(E) = \int_{\{s_0\} \times A} q(E|s,a)\, \sigma_1(da, s_0)$, the $t+1$ distribution is given by $\nu_{t+1}(E) = \int_{S \times A} q(E|s,a)\, d\sigma_t(da,s)\nu_t(ds)$. Corresponding to a strategy σ is the vector of expected utilities.

With the notion of controllability changed to the expected time to get the system to a better state having a uniform upper bound, the same types of results are available (see e.g. the survey [1] for the long-run average case). However, this does not capture the possibility that species extinction can happen in a fishery, in which case it is impossible to get the system to a better state.

If for any $p \in \mathbb{R}_{++}$, the event that the future extinction probability at t is at least p is recurrent, then extinction will happen with probability 1. We therefore reduce the problem to the tradeoff between sequences that have a constant utility until some random time τ_p,

$$u_{\tau_p} := (\underbrace{u, u, \ldots, u}_{\tau_p}, 0, 0, \ldots). \tag{12}$$

We are interested in the dependence of $\mathbb{E}\, V(u_{\tau_p})$ on the utility level $u \geq 0$ and the distribution of τ_p when V is patient, monotonic, and satisfies intergenerational equity. We say that τ_p has **an incomplete distribution with incompleteness** q if $P(\tau_p \in {}^*\mathbb{N}_0) = 1 - q$.

The main result is that patient preferences of the form $V(u) = \min_{\eta \in S} {}^\circ\langle {}^*u, \eta\rangle$ allow no *tradeoff* between yield and extinction when S belongs to a class of probabilities that include any of the sets given in Theorem C.

Definition 4.3. *A set $S \subset {}^*\ell_1$ is* ***very remote*** *if for any unlimited $\tau \in {}^*\mathbb{N}_0$, there exists an $\eta \in S$ with $\sum_{t \leq \tau} |\eta_t| \simeq 0$.*

All of the sets of η's given in Theorem C are very remote.

Lemma 3. *If $V(u) = \min\{{}^\circ\langle {}^*u, \eta\rangle : \eta \in S\}$ and S is very remote, then $\mathbb{E}\, V(u_{\tau_p}) = q \cdot u$ where q is the incompleteness of τ_p.*

To put it another way, since extinction is forever, being very patient means that all policies that give us extinction possibilities over any time range are all equally bad.

Proof. $\mathbb{E}\,\langle u_{\tau_p}, \eta\rangle = u \cdot \sum_{t \in {}^*\mathbb{N}_0} P(\tau_p = t)\eta_t$. For any unlimited τ such that $\sum_{t \leq \tau} P(\tau_p = t) \simeq (1-q)$ and $\sum_{t \leq \tau} \eta_t \simeq 0$, this expectation is infinitely close to $q \cdot u$. □

5. Research, Reversibility, and the Precautionary Principle

Model: have a utility flow u in each period; at any time τ, can "pull the trigger," adopting an innovation that irreversibly changes the utility flow to $u + X \cdot 1_{\{\tau, \tau+1, \ldots\}}$. Here X is a random variable with prior distribution β_0, and we assume that there are both upsides, $beta_0(X > 0) > 0$, and downsides, $\beta_0(X < 0) > 0$. At a per period cost of c, can gather more information, modeled as β_{s+1} where $\mathbb{E}\left(\beta_{s+1}|\beta_s\right) = \beta_s$ where s is the number of periods in which research has been done.

First question: with patient preferences, what is optimal timing of the decision to adopt or to abandon research? Second question: if the utility change X is reversible, but only at a flow utility cost of c_R, what is the optimal amount of research? Answers to three variants of this second question give three variants of the precautionary principle.

Throughout, we maximize the expected value of $V : \ell_\infty \to \mathbb{R}$ where V has remote tangents that respect intergenerational equity.

5.1. **Research.** We will suppose first that the only way to find out about the value of X is by making the irreversible decision, by pulling the trigger. In terms of modeling beliefs, this is the assumption that $\beta_s \equiv \beta_0$. We then examine what happens when research is imperfect, allowing β_s to converge to random beliefs β_∞ that are not necessarily point masses. The easiest case, examined last, has completely informatve research, that is, the probability that β_s converges to point mass beliefs is 1.

5.1.1. *Uninformative Research.* If $\beta_s \simeq \beta_0$ for all s, no information can ever be had. If $\int x\, d\beta_0(x) > 0$, then an optimal strategy for any patient preference is $\tau^* = 0$, immediate adoption, while if $\int x\, d\beta_0(x) < 0$, the optimal strategy is to never adopt. For any patient η, the value is $u + \max\{0, \int x\, d\beta_0(x)\}$, so the same is true for any preferences having remote tangents (e.g. those in Theorem C).

5.1.2. *Partially Informative Research.* Suppose now that the beliefs martingale, β_s, converges to β_∞, that $q \in \Delta(\Delta(\mathbb{R}))$ satisfies $q(\{\beta_\infty \text{ is a point mass }\} < 1$ where q is the distribution of β_∞. Let B_+ be the set of $\beta \in \Delta(\Delta(\mathbb{R}))$ such that $\int_{\Delta(\mathbb{R})} \left[\int_{\mathbb{R}} x\, d\beta(x)\right]\, dq(\beta) > 0$. For any of the patient preferences under consideration, the utility is

$$u + q(B_+) \cdot \int_{B_+} \left[\int_{\mathbb{R}} x\, d\beta(x)\right]\, dq(\beta). \tag{13}$$

This reduces to the previous case when $q = \delta_{\beta_0}$ so that $q(B_+)$ is either equal to 0 or equal to 1.

5.1.3. *Completely Informative Research.* Suppose now that the belief martingale, $\beta_s \to \beta_\infty$, has the property that q puts mass one on the point masses, that is, $\beta_t \to \delta_x$ where $x \sim \beta_0$. In this case, the utility value to any of the patient preferences under consideration is $u + \int_{X>0} x\, d\beta_0(x) = u + \beta_0(X > 0) \cdot \mathbb{E}\,(X|X > 0)$, and this is achieved by waiting until beliefs β_t put all but an infinitesimal mass on $X > 0$. This reduces to the previous case when B_+ is the set of point masses on points in $\mathbb{R}_{++}$.

Since the q in the informative case is a dilation of the q in the partially informative case, and the q in the partially informative case are a dilation of the q in the uninformative case, we have the following ranking,

$$\max\{0, \textstyle\int x\, d\beta_0(x)\} \leq q(B_+) \cdot \int_{B_+} \left[\int_{\mathbb{R}} x\, d\beta(x)\right]\, dq(\beta) \leq \int_{X>0} x\, d\beta_0(x). \tag{14}$$

5.2. **Reversibility.** The words "precautionary principle" have been used to describe three major types of policies toward public choice problems in the presence of uncertainty about irreversible or expensive-to-reverse consequences to actions.

Strong version. Deutsch [2] identifies the principle as a kind of "blind pessimism," as an injunction to take no action if it is not certain that the action is harmless. Sunstein [11] attacks a strong version of the principle as "paralyzing, forbidding every imaginable step" because "risks are on all sides of social situations." By this he means that there are both upsides and downsides: not taking an action risks losing the $X > 0$ part of the realization; while taking an actions risks losing the $X < 0$ part. He then argues that the "extraordinary influence" of this strong version of the principle is best explained by a number of failures of human cognitive processes.

International version. The definition of the principle in the 1992 Rio Declaration is, "Where there are threats of serious or irreversible damage, lack of full scientific certainty shall not be used as a reason for postponing cost-effective measures to prevent environmental degradation."[1]

[1]See also Part VII of the 1987 Ministerial Declaration of the OSPAR Convention, "Accepting that, in order to protect the North Sea from possibly damaging effects of the most dangerous substances, a precautionary approach is necessary which may require action to control inputs of such substances

Allocational version. A third version of the principle is an extension of the 'polluter pays' principle, allocating the costs of research into potential harms and costs of remediation from an action to those taking the action.

To investigate these different versions of the precautionary principle in the presence of patient preferences and the possibility of research into the values of adoption, we introduce the reversal cost as a flow utility cost of c_R. When research is uninformative, this changes the criterion to adopting if $\int \max\{x, -c_R\}\, d\beta_0(x) > 0$; and when research is partly informative, this changes B_+ to $B_+(c_r)$, defined as the set of $\beta \in \Delta(\Delta(\mathbb{R}))$ such that $\int_{\Delta(\mathbb{R})} \left[\int_{\mathbb{R}} \max\{x, -c_R\}\, d\beta(x)\right] dq(\beta) > 0$. Note that costly reversibility is a first order stochastically dominating upward shift in the distribution of long-term benefits, and that as $c_R \downarrow 0$, all decisions X with $\beta_0(X > 0) > 0$ are adopted. When c_R is itself stochastic, the previous analyses of optimal learning can be applied to the random variable $X \vee (-c_R)$.

Strong version. Sunstein's attack on the precautionary principle casts it as saying that no decision can be optimal in situations with both upsides and downsides. In terms of the present analysis, this is the requirement that $c_R = 0$ and the observation that $c_R > 0$. Deutsch's attack on any policy limiting risky undertakings has $c_R < 0$ because new problems, $X < 0$, give rise to new knowledge, and the net benefit of the new knowledge outweighs the costs on average.

International version. This can be understood as requiring enough research that it is reasonably certain either that X is not too negative or that c_R is not too large before adoption.

Allocational version. This can be understood as an attempt to get the incentives for research correct by more closely aligning private and social benefits. In terms of the model of patient preferences used here, Vauban's observation is that no private enterprise will have the patient preferences appropriate for society.

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even before a causal link has been established by absolutely clear scientific evidence." Sunstein's concise summary is "that a lack of decisive evidence of harm should not be a ground for refusing to regulate."

6. Investments that Extend the Long Run

As a first example, in many areas in the arid states, agricultural concerns are pumping so hard that the internal structures of the underlying aquifers collapse. This means that the aquifers will not hold as much water in the future, damaging the long run productive capacity of areas subject to intermittent rainfall. This rational reaction by agricultural concerns with short-run planning horizons is bound to be a bad idea if social preferences are patient, and desertification has leveled previous civilizations.

Following Deutsch, research into expanding control of the physical universe out to the boundaries imposed by physical laws expands the set of things that society can do and enjoy. This is bound to be a good idea if social preferences are patient and if the downside can be limited. A more interesting aspect of Deutcsch's argument is that innovations push back the time that society might end, a plausible argument from a historical perspective. If this is the case, the normalization built into the present approach to patient preferences drastically understates the benefits: if e.g. taking a decision changes the random time to the end of society from T to T' and $\mathbb{E}\,T' > \mathbb{E}\,T$, then comparing

$$\tfrac{1}{\mathbb{E}\,T}\,\mathbb{E}\sum_{t=0}^{T} u_t \text{ to } \tfrac{1}{\mathbb{E}\,T'}\,\mathbb{E}\sum_{t=0}^{T'} u'_t \tag{15}$$

normalizes away the benefits accruing to the generations between T and T'.

When choosing between actions that increase or decrease the distribution of T, essentially by shifting the limit η's to the right or the left, we can consider a class of problems far broader and examine the tradeoff between higher flows for shorter time horizons and lower flows for longer horizons. To compare actions a and b, we should to compare $V_a := \mathbb{E}\,\sum_{t=1}^{T_a} u(X_t^a)$ and $V_b := \mathbb{E}\,\sum_{t=1}^{T_b} u(X_t^b)$. If $V_a, V_b \simeq +\infty$, we ask that the ratio ${}^\circ(V_b/V_a) \in [0, 1)$ for a to be preferred to b. Here, policies that potentially increase the expected termination date of society, say research into non-polluting power sources or healthier lives, become tremendously valuable, while policies that potentially decrease the expected termination date, say agricultural policies leading to desertification or less healthy living conditions, become highly inadvisable.

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