A Unified Approach to Revealed Preference Theory: The Case of Rational Choice *

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Abstract

The theoretical literature on (non-random) choice largely follows the route of Richter (1966) by working in abstract environments and by stipulating that we see all choices of an agent from a given feasible set. On the other hand, empirical work on consumption choice using revealed preference analysis follows the approach of Afriat (1967), which assumes that we observe only one (and not necessarily all) of the potential choices of an agent. These two approaches are structurally different and are treated in the literature in isolation from each other. This paper introduces a framework in which both approaches can be formulated in tandem. We prove a rationalizability theorem in this framework that simultaneously generalizes the results of Afriat and Richter. This approach also gives a new, 'tight' version of Afriat's Theorem and a continuous version of Richter's Theorem, and leads to a number of novel observations for the theory of consumer demand.

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1 INTRODUCTION

As pioneered by Hendrik Houthakker and Paul Samuelson, the classical theory of revealed preference was conducted for consumption choice problems within the class of *all* budget sets in a given consumption space. In time, this work has been extended, and refined, in mainly two ways. The seminal contributions of Arrow (1959) and Richter (1966, 1971) have shifted the focus of decision theorists to studying the consequences of rational decision-making in richer, and more abstract, settings. The vast majority of modern choice theory, be it modeling rational or boundedly rational decision making, is now couched within this framework. On the other hand, another seminal approach was pursued by Afriat (1967) in the context of standard consumption problems, but under the unexceptionably reasonable premise that a researcher may record one's choice behavior only for *finitely* many budget sets in a given consumption space. This approach has proved useful for econometric tests of rationality, and for the construction of utility and demand functions from consumption choice data.[1]

1.1 Richter and Afriat's Theorems compared. It is striking that even though virtually the entire literature on (non-random) choice can be viewed either as following the abstract route of Richter (1966) or the empirically-oriented route of Afriat (1967), there is little contact between them. This is mainly because these two approaches are structurally different. To be precise about this, let us have a look at the fundamental rationalizability theorems of these papers:

Richter's Theorem. *Let X be a nonempty set and $\mathcal{A}$ a nonempty collection of nonempty subsets of X. A map (choice correspondence) c from $\mathcal{A}$ into $2^X \backslash \{\emptyset\}$ with $c(A) \subseteq A$ for each $A \in \mathcal{A}$ satisfies the congruence axiom if, and only if, there is a complete preference relation (preorder) $\succsim$ on X such that $c(A) = \{x \in A : x \succsim y$ for each $y \in A\}$.*[2]

Afriat's Theorem. *Let k and n be positive integers, and take any $(\mathbf{p}^1, \mathbf{x}^1), ..., (\mathbf{p}^k, \mathbf{x}^k)$ in $\mathbb{R}^n_{++} \times \mathbb{R}^n_+$. Then there is a continuous and strictly increasing (utility) function $u : \mathbb{R}^n_+ \to \mathbb{R}$ such that $u(\mathbf{x}^i) \geq u(\mathbf{x})$ for each $\mathbf{x} \in \mathbb{R}^n_+$ and $i = 1, ..., k$ with $\mathbf{p}^i\mathbf{x} \leq \mathbf{p}^i\mathbf{x}^i$ if, and only if, $(\mathbf{p}^1, \mathbf{x}^1), ..., (\mathbf{p}^k, \mathbf{x}^k)$ obeys cyclical consistency, which means that, for any*

[1] See Vermeulen (2012) for a nice survey on the theory of revealed preference.

[2] Here we wish to examine only the "structure" of this theorem, so for the present discussion, it is not important what the congruence axiom is. This axiom is defined formally in Section 4.1.

$\{t_1, t_2, ..., t_l\} \subseteq \{1, 2, ..., k\}$,

$$\mathbf{p}^{t_2}\mathbf{x}^{t_1} \leq \mathbf{p}^{t_2}\mathbf{x}^{t_2}, ..., \mathbf{p}^{t_l}\mathbf{x}^{t_{l-1}} \leq \mathbf{p}^{t_l}\mathbf{x}^{t_l} \textit{ and } \mathbf{p}^{t_1}\mathbf{x}^{t_l} \leq \mathbf{p}^{t_1}\mathbf{x}^{t_1}$$

imply

$$\mathbf{p}^{t_2}\mathbf{x}^{t_1} = \mathbf{p}^{t_2}\mathbf{x}^{t_2}, ..., \mathbf{p}^{t_l}\mathbf{x}^{t_{l-1}} = \mathbf{p}^{t_l}\mathbf{x}^{t_l} \textit{ and } \mathbf{p}^{t_1}\mathbf{x}^{t_l} = \mathbf{p}^{t_1}\mathbf{x}^{t_1}.$$

Even a casual look at these results witnesses a number of important differences. Richter's Theorem is very abstract. It has the advantage of allowing for *any* kind of choice domain. It presumes that *all* choices of the agent are observed in the case of any choice problem – that is, the entirety of the set $c(A)$ is known for any A in $\mathcal{A}$ – and on the basis of a single axiom on c, delivers a complete preference relation the maximization of which yields all choices of the agent in every choice problem. By contrast, Afriat's Theorem is fairly concrete. It works only with k many budget problems for consumption of bundles of n goods. It presumes that only *one* choice of the agent is observed in the case of any budget set,[3] – we see the bundle $\mathbf{x}^i$ being chosen in the budget set with prices $\mathbf{p}^i$ and income $\mathbf{p}^i\mathbf{x}^i$ – and on the basis of a single axiom on the choices of the agent, delivers a utility function with respect to which the (observed) choices are best within their respective budget sets. Furthermore, this utility function is continuous and strictly increasing, concepts which are not even meaningful in the general alternative space considered in Richter's Theorem. Comparing the central assumptions, we see that the special structure of $\mathbb{R}^n$ is used in an essential manner in the definition of cyclical consistency and it is not possible to state a generalization of this property in an environment where the alternative space lacks an inherent order and/or algebraic structure. By contrast, Richter's Theorem does not need a special mathematical structure on the alternative space X and the congruence axiom is a purely set-theoretic property.

All in all, the theorems above appear to have fairly different characters, even though, conceptually, they are after the same thing, namely, identifying conditions on one's choice behavior that would allow us to view this individual "as if" she is maximizing a preference relation (or a utility function).[4] It thus seems desirable to develop a framework

[3] This is not entirely correct. While this is how the theorem is utilized in practice, Afriat's Theorem allows for $\mathbf{p}^i\mathbf{x}^i = \mathbf{p}^j\mathbf{x}^j$ for distinct i and j, so the data may in principle yield two (or more) choices from a given budget set. It is, however, in the very nature of this theorem that only *some* choices of the agent is observed in the case of any budget set.

[4] While we single out Richter's and Afriat's Theorems in this discussion, we note that these theorems serve as prototypes here. In particular, everything we said so far about the distinction between these theorems remain valid in the case of any of the extensions of Afriat's Theorem provided in the literature.

in which the approaches of Afriat and Richter toward revealed preference theory can be formulated simultaneously. On the one hand, such a framework would allow for a unified approach to (non-random) revealed preference theory that admits the previous approaches as special cases. On the other hand, it would provide an avenue for bringing together the most desirable parts of these approaches together, thereby paving the way toward more powerful revealed preference theories. In particular, such a framework would let us work with choice environments in which one recognizes the fact that often we observe only one (or a few) choice(s) of an agent in a given choice situation (a major advantage of Afriat's theory) without limiting attention only to consumption choice problems (a major advantage of Richter's theory). The primary objective of the present paper is to provide such a framework.

1.2 The framework. We depart from the previous literature on abstract revealed preference theory in two ways. *First*, we model the choice behavior of an agent by a *set* of choice correspondences, instead of a single one. The idea is quite intuitive. Suppose we observe the choice behavior of an agent across a collection, say, $\mathcal{A}$, of feasible sets. For simplicity, suppose we see exactly one choice of the agent, say x_A, from each feasible set A in $\mathcal{A}$. Our model identifies this behavior with the set of all choice correspondences (on $\mathcal{A}$) that declares x_A as a potential choice from A, that is, it says that the agent's "true" choice correspondence c is one with $x_A \in c(A)$ for each A in $\mathcal{A}$. Notice that this is precisely Afriat's approach generalized to an arbitrary choice domain. By contrast, Richter's approach presumes that we are privy to "all" choices of the agent in the case of any feasible set. For instance, it may be the case that we are somehow certain that x_A is the *only* choice of the agent from A for each A in $\mathcal{A}$. In that case, the set of all choice correspondences that is consistent with the data becomes a singleton that contains the "true" choice correspondence c of the agent, where $\{x_A\} = c(A)$ for each A in $\mathcal{A}$. As this example easily generalizes to the case where the agent may have been observed to make multiple choices from a given feasible set, we thus see that modeling an agent as a *set* of choice correspondences captures both approaches as special cases. In Richter's case this set is necessarily a singleton, and in Afriat's case it is not (except in trivial instances).[5]

This framework is, however, not yet enough to formulate Afriat's Theorem within, because that theorem relies crucially on an exogenously given order structure. Indeed,

[5] While this is a side point for the current paper, it is worth noting that this framework allows for modeling certain types of interesting choice situations that cannot be captured by either the Afriat or the Richter approaches. An example to this effect is provided in Section 3.4 (see Example 4).

without some monotonicity requirement on u, the notion of consistency in Afriat's Theorem will impose no restrictions on observations since we can allow the agent to be indifferent across all alternatives. This leads to the *second* novel aspect of the framework we propose here: We assume that the alternative space is a partially ordered set, as opposed to an arbitrary set. This way we keep the standard environment of Richter as a special case (where the partial order is the equality ordering) and include domains, such as $\mathbb{R}^n$, which have intrinsic order structures.

After going through a few mathematical preliminaries in Section 2, we introduce our framework formally in Section 3. In that section we also show by examples how this abstract framework admits numerous choice environments that are studied in the previous literature, ranging from Richter-type choice frameworks to Nash bargaining problems and Afriat-type environments, as well as new ones. It is important to note that our framework is an abstract setup that is primed to capture any sort of choice data that one can encounter in both theory and practice. As such, it is not geared necessarily toward rational choice theory and it can be used to study any type of boundedly rational choice theory as well. In this paper, however, we focus on rational choice theory in the context of this framework with the aim of demonstrating that the approaches of Richter and Afriat are in fact branches of the same tree.

1.3 Summary of the main results. We show that there is a natural way of extending Richter's congruence axiom to our framework such that the axiom recognizes the inherent order on the alternative space. We call this extension the *monotone congruence axiom* (Section 4.3). Similarly, the cyclical consistency condition is extended to our setting in a natural manner; we refer to this extension as the *generalized cyclical consistency* (Section 4.4). Our main theorem shows that these (extended) properties have a close formal connection. Furthermore, the monotone congruence axiom yields a representation very much in the spirit of Richter (but now with monotone preference relations) while generalized cyclical consistency yields precisely an Afriat-type representation (but now over an arbitrary choice domain). Therefore, our main theorem (in Section 4.5) generalizes Richter's Theorem and the choice-theoretic content of Afriat's Theorem simultaneously (see Sections 4.6 and 4.8). However, the structure of rationalization we obtain in the latter case is different than that of Afriat (1967) in an important way. Unlike Afriat's construction, our rationalizing preference is typically not convex; instead the preference realtion we derive provides a *tight* rationalization in the sense that, at any observed budget set, it identifies as optimal *only* those bundles that have been revealed as optimal by the data (see Section 4.7).

Another difference with Afriat's Theorem is that our main result is of "rationalization by a preference relation" form, and not of "rationalization by a utility function" form. The latter form obtains in Afriat's Theorem due to the special structure of the alternative space $\mathbb{R}^n$ and the assumption that the collection of feasible sets under consideration is finite. In Section 5, we show that if we make this finiteness assumption in our framework and posit that the alternative space satisfies fairly general (topological) conditions, then our main theorem can be stated in terms of a continuous and monotonic utility function.[6] Not only this result admits Afriat's Theorem (as stated above) as a special case (Section 6.1), but it also yields a new version of Richter's Theorem which is of "rationalization by a continuous utility function" form (Section 5.3).

Closely related to the rationalizability of choice data is the issue of *extrapolation* introduced by Varian (1982). Loosely speaking, this issue is about predicting the agent's choice behavior in the context of choice situations that are not part of the actual data. Despite its obvious importance, extrapolation has so far been investigated only for consumption choice problems. In Section 5.4, however, we use the new (continuous) version of Richter's Theorem to characterize exactly what can be said about this matter within general choice theory.

Finally, in Section 6, we turn to the classical theory of consumer demand, and provide two applications of our general results in this context. First, we look into the issue of *recoverability* (Varian (1982)), that is, the extent to which an observer is able to recover information about the agent's preferences from her observed choices, without subscribing to a particular utility function that happens to rationalize the choice data at hand. In Section 6.2, we show that, insofar as we impose only the properties of continuity and monotonicity of the rationalizing utility functions, what we can recover of the agent's preferences is exactly her (indirect) revealed preference relation. Second, we provide an extension of Afriat's Theorem to the case where the data set consists of a finite set of Engel curves (Section 6.3). As even a single Engel curve corresponds to an infinite set of choice observations, this sort of a rationalizability result escapes the coverage of Afriat's Theorem, but falls comfortably within our general rationalizability theorems.

The numerous corollaries we deduce from our main results throughout this paper attest to the unifying structure of the revealed preference framework proposed here. We hope that this framework will also facilitate the development of the recent literature

[6] The crucial technical tool in this result is Levin's (1983) Theorem, which is a continuous version of Szpilrajn's Theorem.

on boundedly rational choice theory, especially in extensions that explicitly account for issues of data availability.

2 PRELIMINARIES

The primary tool of analysis in this paper is order theory. The present section catalogues the definitions of all the order-theoretic notions that we utilize throughout the present work. As these notions are largely standard, this section is mainly for the reader who may need a clarification about them in the main body of the paper.

2.1 Order-Theoretic Nomenclature. Let X be a nonempty set, and denote the diagonal of $X \times X$ by $\triangle_X$, that is, $\triangle_X := \{(x,x) : x \in X\}$. By a **binary relation** on X, we mean any nonempty subset of $X \times X$. For any binary relation R on X, we adopt the usual convention of writing $x\ R\ y$ instead of $(x,y) \in R$. (Thus, $x \triangle_X y$ iff $x = y$ for any $x, y \in X$.) Similarly, for any $x \in X$ and subset A of X, by $x\ R\ A$ we mean $x\ R\ y$ for every $y \in A$, and interpret the expression $A\ R\ x$ analogously. Moreover, for any binary relations R and S on X, we simply write $x\ R\ y\ S\ z$ to mean $x\ R\ y$ and $y\ S\ z$, and so on. For any subset A of X, the **decreasing closure** of A with respect to R is defined as

$$A^{\downarrow,R} := \{x \in X : y\ R\ x \text{ for some } y \in A\},$$

but when R is apparent from the context, we may denote this set simply as $A^{\downarrow}$. The **increasing closure** *of* A is defined dually. By convention, $x^{\downarrow} := \{x\}^{\downarrow}$ and $x^{\uparrow} := \{x\}^{\uparrow}$ for any x in X.

The **inverse** of a binary relation R on X is itself a binary relation, defined as $R^{-1} := \{(y,x) : x\ R\ y\}$. The **composition** of two binary relations R and R' on X is defined as $R \circ R' := \{(x,y) \in X \times X : x\ R\ z\ R'\ y \text{ for some } z \in X\}$. In turn, we let $R^1 := R$ and $R^n := R \circ R^{n-1}$ for any integer $n > 1$; here R^n is said to be the n**th iterate** of R.

The **asymmetric part** of a binary relation R on X is defined as $P_R := R \backslash R^{-1}$ and the **symmetric part** of R is $I_R := R \cap R^{-1}$. We say that a binary relation R on X **extends** another such binary relation S if $S \subseteq R$ and $P_S \subseteq P_R$. For any nonempty subset A of X, the set of all maximal elements with respect to R is denoted as $\text{MAX}(A,R)$, that is, $\text{MAX}(A,R) := \{x \in A : y\ P_R\ x \text{ for no } y \in A\}$. Similarly, the set of all maximum elements with respect to R is denoted as $\max(A,R)$, that is, $\max(A,R) := \{x \in A : x\ R\ y \text{ for all } y \in A\}$. We also define $\text{MIN}(A,R) := \text{MAX}(A,R^{-1})$ and $\min(A,R) := \max(A,R^{-1})$.

A binary relation R on X is said to be **reflexive** if $\triangle_X \subseteq R$, **antisymmetric** if $R \cap R^{-1} \subseteq \triangle_X$, **transitive** if $R \circ R \subseteq R$, and **complete** if $R \cup R^{-1} = X \times X$. The **transitive closure** of R, denoted by $\text{tran}(R)$, is the smallest transitive relation on X that contains R, and is given by $\text{tran}(R) := R \cup R^2 \cup \cdots$. In other words, $x \text{ tran}(R)\, y$ iff we can find a positive integer k and $x_0, ..., x_k \in X$ such that $x = x_0 \; R \; x_1 \; R \cdots R \; x_k = y$.

If R is reflexive and transitive, we refer to it as a **preorder** on X. (In particular, $\text{tran}(R)$ is a preorder on X for any reflexive binary relation R on X.) If R is an antisymmetric preorder, we call it a **partial order** on X. The ordered pair (X, R) is a **preordered set** if R is a preorder on X, and a **poset** if R is a partial order on X. (Throughout the paper, a generic preorder is denoted as $\succsim$, with $\succ$ acting as the asymmetric part of $\succsim$.) Finally, we say that R is **acyclic** if $\triangle_X \cap P_R^n = \emptyset$ for every positive integer n. It is readily verified that transitivity of a binary relation implies its acyclicity, but not conversely.

For any preorder $\succsim$ on X, a complete preorder on X that extends $\succsim$ is said to be a **completion** of $\succsim$. It is a set-theoretical fact that every preorder on a nonempty set admits a completion. This result, which is based on the Axiom of Choice, is known as *Szpilrajn's Theorem.*[7]

Given any preordered set $(X, \succsim)$, a function $f : X \to \mathbb{R}$ is said to be **increasing with respect to** $\succsim$, or simply $\succsim$**-increasing**, if $f(x) \geq f(y)$ holds for every $x, y \in X$ with $x \succsim y$. If, in addition, $f(x) > f(y)$ holds for every $x, y \in X$ with $x \succ y$, we say that f is **strictly increasing with respect to** $\succsim$, or that it is **strictly** $\succsim$**-increasing**.

Finally, given a poset $(X, \succcurlyeq)$ and a subset A of X, we denote by $\bigvee A$ the unique element of $\min(\{x \in X : x \succcurlyeq A\}, \succcurlyeq)$, provided that this set is nonempty (and hence a singleton). Analogously, $\bigwedge A$ is the unique element of $\max(\{x \in X : A \succcurlyeq x\}, \succcurlyeq)$, provided that this set is nonempty. If $\bigvee A$ exists for every nonempty finite $A \subseteq X$, then $(X, \succcurlyeq)$ is said to be a $\vee$**-semilattice**, and if $\bigvee A$ exists for every $A \subseteq X$, then $(X, \succcurlyeq)$ is said to be a **complete** $\vee$**-semilattice**. If $(X, \succcurlyeq^{-1})$ is a $\vee$-semilattice, we say that $(X, \succcurlyeq)$ is a $\wedge$**-semilattice**.

2.2 Topological Nomenclature. Let $(X, \succsim)$ be a preordered set such that X is a topological space. We say that $\succsim$ is a **continuous** preorder on X if it is a closed subset of $X \times X$ (relative to the product topology).[8] We note that the closure of a preorder on X (in $X \times X$) need not be transitive, nor is the transitive closure of a closed binary

[7] Szpilrajn (1938) has proved this result for partial orders, but the result easily generalizes to the case of preorders; see Corollary 1 in Chapter 1 of Ok (2007).

[8] While there are other notions of continuity for a preorder (for instance, openness of its strict part

relation on X in general continuous. One needs additional conditions to ensure such inheritance properties to hold (Ok and Riella (2014)).

Given a continuous preorder $\succsim$ on X, the topological conditions that would ensure the existence of a continuous real map on X that is strictly increasing with respect to $\succsim$ are well-studied in the mathematical literature. It is known that such a function exists if X is a locally compact and separable metric space. This is *Levin's Theorem*.[9]

Notational Convention. Throughout this paper, we write $[k]$ to denote the set $\{1, ..., k\}$ for any positive integer k.

3 CHOICE ENVIRONMENTS AND CHOICE DATA

3.1 Choice Environments. By a **choice environment**, we mean an ordered pair $((X, \geq), \mathcal{A})$, where $(X, \geq)$ is a poset and $\mathcal{A}$ is a nonempty collection of nonempty subsets of X. Here we interpret X as the consumption space, that is, the grand set of all mutually exclusive choice alternatives, that is, the consumption set. We think of $\geq$ as an exogeneously given domination relation on X, and view the statement $x \geq y$ as saying that x is an unambiguously better alternative than y for any individual. (If the environment one wishes to study lacks such a dominance relation, we may set $\geq$ as $\triangle_X$ so that $x \geq y$ holds iff $x = y$.[10]) Finally, $\mathcal{A}$ is interpreted as the set of all feasible sets from which a decision maker is observed to make a choice. For instance, if the data at hand is so limited that we have recorded the choice(s) of an agent in the context of a single feasible set $A \subseteq X$, we would set $\mathcal{A} = \{A\}$. At the other extreme, if we have somehow managed to keep track of the choices of the agent from every possible feasible set $A \subseteq X$ (as sometimes is possible in the controlled environments of laboratory experiments), we would set $\mathcal{A} = 2^X \backslash \{\emptyset\}$.

3.2 Choice Correspondences and Choice Data. Given a nonempty set X and a nonempty subset $\mathcal{A}$ of $2^X \backslash \{\emptyset\}$, by a **choice correspondence** on $\mathcal{A}$, we mean a map

on X), this terminology is adopted quite widely in the literature. See, for instance, Evren and Ok (2011) and references cited therein.

[9] See Levin (1983), where this result is proved in the more general case where X is a locally compact, σ-compact and second countable Hausdorff space.

[10] Every result we report in this paper remains valid, if $\geq$ is an arbitrary preorder (with no substantial change in the proofs). We take $(X, \geq)$ as a poset, instead of a preordered set, only to simplify the exposition, and because the natural dominance relations that arise in the applications we consider here are all partial orders.

$c : \mathcal{A} \to 2^X$ such that $c(A)$ is a nonempty subset of A for each $A \in \mathcal{A}$. We denote the family of all choice correspondences on $\mathcal{A}$ by $\mathfrak{C}_{(X,\mathcal{A})}$.

There is a natural way of ordering the choice correspondences on $\mathcal{A}$. Consider the binary relation $\sqsupseteq$ on $\mathfrak{C}_{(X,\mathcal{A})}$ defined as

$$c \sqsupseteq d \quad \text{iff} \quad c(A) \supseteq d(A) \text{ for every } A \in \mathcal{A}.$$

Clearly, $(\mathfrak{C}_{(X,\mathcal{A})}, \sqsupseteq)$ is a poset. It is also plain that this poset is a complete $\vee$-semilattice, but it is not an $\wedge$-semilattice unless all members of $\mathcal{A}$ are singleton subsets of X.

In the present paper, by a **choice correspondence** on a choice environment $((X, \geq), \mathcal{A})$, we simply mean an element of $\mathfrak{C}_{(X,\mathcal{A})}$. (Notice that this notion does not depend on the preorder $\geq$.) In turn, we refer to any nonempty collection $\mathbb{C}$ of choice correspondences on $((X, \geq), \mathcal{A})$ as a **choice data** on $((X, \geq), \mathcal{A})$. As we have discussed in Section 1.2, we may think of $\mathbb{C}$ as a means of summarizing the choices of a given decision maker across all feasible sets in $\mathcal{A}$ in the sense that $\mathbb{C}$ is precisely the set of *all* choice correspondences on $((X, \geq), \mathcal{A})$ that are compatible with the (observed) choices of that agent.

3.3 Revealed Preference Frameworks. By a **revealed preference (RP) framework**, we mean an ordered triplet

$$((X, \geq), \mathcal{A}, \mathbb{C}),$$

where $((X, \geq), \mathcal{A})$ is a choice environment and $\mathbb{C}$ is a choice data on $((X, \geq), \mathcal{A})$. We note that this model is quite general, and it departs from how revealed preference theory is usually formulated in the literature mainly in two ways. First, it features the notion of an unambiguous ordering of the alternatives (in terms of some form of a domination relation). Second, and more important, this model takes as a primitive not one choice correspondence, but potentially a multiplicity of them. The following subsection aims to demonstrate the advantages of this modeling strategy by means of several examples.

3.4 Examples. In many studies of revealed preference, one takes as primitives a finite alternative set X and a choice correspondence c on $2^X \backslash \{\emptyset\}$. (This is, for instance, precisely the model studied by Arrow (1959), and is one of the most commonly adopted choice frameworks in the recent literature on boundedly rational choice.) Especially when X is not finite, however, it is commonplace to posit that we can observe one's choice behavior only in the context of certain types of feasible sets.

Example 1. (*Richter-Type RP Frameworks*) Let X be any nonempty set, $\mathcal{A}$ any nonempty collection of nonempty subsets of X, and c a choice correspondence on $\mathcal{A}$.

Then, $((X, \triangle_X), \mathcal{A}, \{c\})$ is an RP framework that corresponds to the choice model of Richter (1966). The interpretation of the model is that one is able to observe *all* elements that are deemed choosable by the decision maker from any given element of $\mathcal{A}$. No exogeneous order (or otherwise) structure on the consumption set X is postulated. Most of the revealed preference analyses conducted in the literature on choice theory work with instances of this model.

Example 2. (*Ordered Richter-Type RP Frameworks*) A slight modification of the previous model obtains if we endow X with a nontrivial partial order $\geq$, leading us to the RP framework $((X, \geq), \mathcal{A}, \{c\})$. Many classical choice models are obtained as special cases of this framework. We give two illustrations:

a. (*Classical Consumption Choice Problems*) Let n be a positive integer. Take X as $\mathbb{R}^n_+$, $\geq$ as the standard (coordinatewise) ordering of n-vectors, and suppose that $\mathcal{A}$ is a nonempty subset of $\{B(\mathbf{p}, I) : (\mathbf{p}, I) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++}\}$, where $B(\mathbf{p}, I)$ is the budget set at prices $\mathbf{p}$ and income I, that is,

$$B(\mathbf{p}, I) := \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{px} \leq I\}$$

for every positive $(n+1)$-vector $(\mathbf{p}, I)$. The RP framework $((X, \geq), \mathcal{A}, \{c\})$ then corresponds to a classical consumption choice model.

b. (*Nash Bargaining Problems*) Where n is a positive integer, take X as $\mathbb{R}^n_+$, $\geq$ as the standard (coordinatewise) ordering of n-vectors, and put $\mathcal{A}$ as the set of all compact and convex subsets of X that contain the origin $\mathbf{0}$ in their interior. The RP framework $((X, \geq), \mathcal{A}, \{c\})$ then corresponds to the classical n-person cooperative bargaining model. (In this model, elements of X are interpreted as the utility profiles of the involved individuals, while $\mathbf{0}$ is the (normalized) utility profile that corresponds to the disagreement outcome.) When c is single-valued, for instance, this model reduces to the one considered by Nash (1950) and a large fraction of the literature on axiomatic bargaining theory. If we relax the convexity requirement, we obtain the model of non-convex collective choice problems (cf. Ok and Zhou (1999)).

The choice models considered in the previous two examples presume that we can observe *all* choices of an individual in the case of every one of the feasible sets. (Put differently, these models posit that they are given the "true" choice correspondence of the decision-maker in its entirety.) This comprehensiveness assumption is, however, often not met in empirical revealed preference studies in which the researcher has one data point (per individual) for each feasible set. This has led many authors to consider models in which one is privy to only one choice of an individual in a given feasible set.

Example 3. (*Afriat-Type RP Frameworks*) Consider a choice environment of the form $((X, \geq), \mathcal{A})$, where $\mathcal{A}$ is a nonempty *finite* subset of $2^X \backslash \{\emptyset\}$, and c is a choice correspondence on $\mathcal{A}$ such that $c(A)$ is finite for all $A \in \mathcal{A}$.[11] A particularly interesting RP framework is then obtained as $((X, \geq), \mathcal{A}, \mathbb{C})$, where

$$\mathbb{C} := \{C \in \mathfrak{C}_{(X,\mathcal{A})} : C \sqsupseteq c\},$$

that is, $\mathbb{C}$ equals $c^{\uparrow}$, the increasing closure of $\{c\}$ with respect to $\sqsupseteq$. The interpretation is that (i) we observe the choice behavior of the agent for only finitely many choice problems; and (ii) we see only some of the choices of the agent in each problem that she faces. Part (i) is captured by the model through the finiteness of $\mathcal{A}$. In turn, part (ii) is captured by setting c to correspond to the observed choices of the agent (that is, $c(A)$ is what we see the decision maker choose from A for each $A \in \mathcal{A}$). In particular, due to the limited nature of our observations, we do not know if the agent was perhaps indifferent, or indecisive, between her choice from A and some other alternatives in $A \in \mathcal{A}$. Consequently, the framework uses the choice data $\mathbb{C}$ to model the choice behavior of the agent in a coarser way. It presumes that the "true" choice correspondence of the agent may be any one choice correspondence C on $\mathcal{A}$ which is consistent with c in the sense that the elements of $c(A)$ are contained in $C(A)$ for each $A \in \mathcal{A}$. Again, many classical choice models are obtained as special cases of this framework.

a. (*Afriat's Model of Consumption Choice Problems*) Let n be a positive integer. In the classical framework of Afriat (1967), the consumption set is modeled as $\mathbb{R}^n_+$ and viewed as partially ordered by the coordinatewise ordering $\geq$. The primitive of the model is a *finite* collection of price vectors and the choice(s) of the agent at those prices. Formally, we are given a nonempty finite subset P of $\mathbb{R}^n_{++}$, and a map d that assigns to every $\mathbf{p} \in P$ a nonempty finite subset $d(\mathbf{p})$ of $\mathbb{R}^n_+$ such that $\mathbf{py} = \mathbf{pz}$ for every $\mathbf{y}$ and $\mathbf{z}$ in $d(\mathbf{p})$. We interpret P as a set of price profiles, and for each $\mathbf{p}$ in P, think of d as the (observed) **demand correspondence** on P, that is, $d(\mathbf{p})$ is interpreted as the set of the bundles that the individual was observed to choose from the budget set $B(\mathbf{p}, \mathbf{p}d(\mathbf{p}))$. (Here, by a slight abuse of notation, by $\mathbf{p}d(\mathbf{p})$ we mean $\mathbf{py}$ for any $\mathbf{y} \in d(\mathbf{p})$.[12]) This model is captured by the RP framework above by setting $(X, \geq)$ as $\mathbb{R}^n_+$ (with the usual

[11] Afriat's Theorem requires c to be finite-valued, but our analysis remain intact if this assumption is dropped. Indeed, each of our characterization theorems allows for this.

[12] The restriction that bundles in $d(\mathbf{p})$ incur the same expenditure is without loss of generality.

ordering), $\mathcal{A}$ as $\{B(\mathbf{p}, \mathbf{p}d(\mathbf{p})) : \mathbf{p} \in P\}$, and c as mapping each $B(\mathbf{p}, \mathbf{p}d(\mathbf{p}))$ to $d(\mathbf{p})$. The choice data of the model is thus

$$\mathbb{C} := \{C \in \mathfrak{C}_{(X,\mathcal{A})} : d(\mathbf{p}) \subseteq C(B(\mathbf{p}, \mathbf{px}(\mathbf{p}))) \text{ for each } \mathbf{p} \in P\},$$

that is, the collection of all choice correspondences on $\mathcal{A}$ which is consistent with the (observed) demand correspondence being a part of the choice correspondence of the individual.

b. (*The Forges-Minelli Model of Consumption Choice Problems*) The applicability of the Afriat model is strained by the fact that it is concerned only with linear budget sets. To deal with nonlinearities that may arise from price floors/ceilings, price differentiation that may depend on quantity thresholds, and other considerations, many authors have considered Afriat type models with nonlinear budget sets (cf. Matzkin (1991) and Chavas and Cox (1993)). Such cases too are readily modeled by means of the revealed preference framework of the present example. For instance, given any two positive integers n and k, Forges and Minelli (2009) take as a primitive a finite collection of ordered pairs, say, $(g^1, \mathbf{x}^1), ..., (g^k, \mathbf{x}^k)$, where g^i is a strictly increasing and continuous real map on $\mathbb{R}^n_+$ with $g^i(\mathbf{x}^i) = 0$ for each $i \in [k]$. They interpret this data as the situation in which we observe a given decision maker choosing the bundle $\mathbf{x}^i$ from the generalized budget set $B^i(g^i) := \{\mathbf{x} \in \mathbb{R}^n_+ : g^i(\mathbf{x}) \leq 0\}$ for each $i \in [k]$. This setup is then captured by the RP framework $((\mathbb{R}^n_+, \geq), \mathcal{A}, \mathbb{C})$ where $\mathcal{A} := \{B^i(g^i) : i \in [k]\}$ and $\mathbb{C}$ is the set of all choice correspondences C on $\mathcal{A}$ such that $\mathbf{x}^i \in C(B(g^i))$ for each $i \in [k]$.[13]

The examples above accord with viewing the choice data of an agent as the collection of all choice correspondences that are compatible with her observed choices. There are, however, instances where we may get partial information about the potential choices of an agent even though we do not observe them exactly. This situation too can be modeled by using our RP framework technology. We illustrate this in our next example, even though such models will not be investigated in this paper.

Example 4. Consider a choice environment of the form $((X, \geq), \mathcal{A})$, and let us suppose that the agent is supposed to finalize her choice in a second period, but today she is able

Indeed, by modifying the domain P if necessary, we can assume that $\mathbf{p}d(\mathbf{p}) = 1$ for all $\mathbf{p} \in P$. This is because $B(\mathbf{p}, \mathbf{py}) = B(\lambda\mathbf{p}, \lambda\mathbf{py})$ for any $\lambda > 0$ and $\mathbf{y} \in \mathbb{R}^n_+$, so requiring income to equal 1 imposes no restrictions on a budget set provided the price can be scaled up or down.

[13] There are Afriat-type models where the grand set of alternatives X is a discrete subset of $\mathbb{R}^n_+$, endowed with the coordinatewise ordering (cf. Polisson and Quah (2013) and Cosaert and Demuynck (2014)). Such models too can be formulated as revealed preference frameworks wherein our results are applicable.

to commit to choosing something (tomorrow) from a subset of a given feasible set. To formalize this scenario, let us fix a correspondence $t : \mathcal{A} \to 2^X \backslash \{\emptyset\}$ such that $t(A) \subseteq A$ for each $A \in \mathcal{A}$. For each A, the interpretation of $t(A)$ is that the agent commits, today, to not choosing anything (tomorrow) from $A \backslash t(A)$. Given that we observe the commitment decisions of the agent, that is, t, it is natural to model the final choices of the agent (which we do not observe) by means of the choice data

$$\mathbb{C} := \{C \in \mathfrak{C}_{(X,\mathcal{A})} : t \sqsupseteq C\}.$$

(Notice that, mathematically, this choice data is the dual opposite of the one we have considered in Example 3; it is the decreasing closure of $\{t\}$ with respect to $\sqsupseteq$.) For, in the present scenario, all we know about the choice of the agent from a feasible set A is that this choice is contained within $t(A)$.

4 RATIONALIZABILITY OF CHOICE DATA (with arbitrary data sets)

4.1 Rationalizability of Choice Correspondences. Let X be a nonempty set and $\mathcal{A}$ a nonempty subset of $2^X \backslash \{\emptyset\}$. A choice correspondence c on $\mathcal{A}$ is said to be **rationalizable** if there is a complete preorder $\succsim$ on X such that

$$c(A) = \max(A, \succsim) \quad \text{for every } A \in \mathcal{A}. \tag{1}$$

In his seminal paper, Richter (1966) has provided a characterization of such choice correspondences by means of what he dubbed the "congruence axiom." To state this property, let us define the binary relation $R(c)$ on X by

$$x \; R(c) \; y \quad \text{if and only if} \quad (x, y) \in c(A) \times A \text{ for some } A \in \mathcal{A}.$$

This relation, introduced first by Samuelson (1938) in the special case of consumption problems, is often called the **direct revealed preference relation** induced by c in the literature, while the transitive closure of $R(c)$ is referred to as the **revealed preference relation** induced by c. Then, given X and $\mathcal{A}$, a choice correspondence c on $\mathcal{A}$ is said to satisfy the **congruence axiom** if

$$x \; \text{tran}(R(c)) \; y \quad \text{and} \quad y \in c(A) \quad \text{imply} \quad x \in c(A)$$

for every $A \in \mathcal{A}$ that contains x. As we have noted in Section 1.1, Richter's Theorem says that a choice correspondence c on $\mathcal{A}$ is rationalizable iff it satisfies the congruence axiom.

4.2 Monotonic Rationalizability of Choice Data. The notion of rationalizability readily extends to the more general context of RP frameworks. Where $((X, \geq), \mathcal{A}, \mathbb{C})$ is an RP framework, we say that the choice data $\mathbb{C}$ is **rationalizable** if at least one c in $\mathbb{C}$ is a rationalizable choice correspondence on $\mathcal{A}$. However, this concept does not at all depend on the partial order $\geq$. Given the interpretation of $\geq$ as a dominance relation, it is natural to require the "rationalizability" take place by means of preference relations that are consistent with $\geq$. (For instance, in the context of consumption choice problems where $(X, \geq)$ is $\mathbb{R}^n_+$, it is natural to require preferences derived from choice data be consistent with the usual ordering $\geq$ of $\mathbb{R}^n_+$, thereby reflecting the sentiment that "more is better.") This leads us to the notion of monotonic rationalizability: The choice data $\mathbb{C}$ is said to be **monotonically rationalizable** if there is a $c \in \mathbb{C}$ and a complete preorder $\succsim$ on X such that (1) holds and $\succsim$ extends $\geq$.[14] Obviously, in the context of any Richter-type RP framework (Example 1), the notions of rationalizability and monotonic rationalizability coincide.

4.3 The Monotone Congruence Axiom. Richter's congruence axiom is readily translated into the context of RP frameworks, but this axiom needs to be strengthened to deliver a characterization of monotonic rationalizability. Where $((X, \geq), \mathcal{A}, \mathbb{C})$ is an RP framework, we say that the choice data $\mathbb{C}$ satisfies the **monotone congruence axiom** if there is a $c \in \mathbb{C}$ such that

$$x \text{ tran}(R(c) \cup \geq)\, y \quad \text{and} \quad y \in c(A) \quad \text{imply} \quad x \in c(A) \tag{2}$$

for every $A \in \mathcal{A}$ that contains x, and

$$x \text{ tran}(R(c) \cup \geq)\, y \quad \text{implies} \quad \text{not } y > x. \tag{3}$$

Clearly, in the context of any Richter-type RP framework, this axiom reduces to the congruence axiom. Furthermore, given Richter's theorem, a natural conjecture is that a choice data $\mathbb{C}$ on $\mathcal{A}$ is monotonically rationalizable iff it satisfies the monotone congruence axiom. That this conjecture is true will be proved in Section 4.6 as an immediate consequence of the main theorem of this paper.

4.4 Generalized Cyclical Consistency. Consider the RP framework we have formalized in Example 3.a, where we are given a nonempty finite subset P of $\mathbb{R}^n_{++}$, and

[14] *Reminder.* $\succsim$ extends $\geq$ iff $> \subseteq \succ$, where $>$ and $\succ$ are asymmetric parts of $\geq$ and $\succsim$, respectively.

a demand correspondence d on P such that $d(\mathbf{p})$ is finite for each $\mathbf{p} \in P$. In this context, Afriat (1967) characterizes rational decision making by means of his famous *cyclical consistency axiom*, which may be stated as follows: For every positive integer k, $\mathbf{p}^1, ..., \mathbf{p}^k \in P$ and $\mathbf{x}^1 \in d(\mathbf{p}^1), ..., \mathbf{x}^k \in d(\mathbf{p}^k)$,

$$\mathbf{p}^2\mathbf{x}^1 \leq \mathbf{p}^2\mathbf{x}^2, ..., \mathbf{p}^k\mathbf{x}^{k-1} \leq \mathbf{p}^k\mathbf{x}^k \text{ and } \mathbf{p}^1\mathbf{x}^k \leq \mathbf{p}^1\mathbf{x}^1$$

imply

$$\mathbf{p}^2\mathbf{x}^1 = \mathbf{p}^2\mathbf{x}^2, ..., \mathbf{p}^k\mathbf{x}^{k-1} = \mathbf{p}^k\mathbf{x}^k \text{ and } \mathbf{p}^1\mathbf{x}^k = \mathbf{p}^1\mathbf{x}^1.$$

This axiom is also commonly known in its equivalent formulation, due to Varian (1982), as the *generalized axiom of revealed preference* (GARP).

Let $((X, \geq), \mathcal{A})$ be a choice environment. We can easily extend the cyclical consistency axiom to the context of a choice correspondence c on $((X, \geq), \mathcal{A})$. We say that c satisfies **generalized cyclical consistency** if $c(A) \subseteq \text{MAX}(A, \geq)$ for each $A \in \mathcal{A}$, and for every $k \in \mathbb{N}$, $A_1, ..., A_k \in \mathcal{A}$, and $x_1 \in c(A_1), ..., x_k \in c(A_k)$,

$$x_1 \in A_2^{\downarrow}, ..., x_{k-1} \in A_k^{\downarrow} \text{ and } x_k \in A_1^{\downarrow}$$

imply

$$x_1 \in \text{MAX}(A_2^{\downarrow}, \geq), ..., x_{k-1} \in \text{MAX}(A_k^{\downarrow}, \geq) \text{ and } x_k \in \text{MAX}(A_1^{\downarrow}, \geq).$$ [15]

The first requirement of this property, that is, $c(A) \subseteq \text{MAX}(A, \geq)$ for each $A \in \mathcal{A}$, is implicit in Afriat's modeling where every choice problem is of the form $B(\mathbf{p}, I)$ where $\mathbf{p}$ is a price vector and $I = \mathbf{py}$ with $\mathbf{y}$ being the consumption bundle that corresponds to the choice of the agent at prices $\mathbf{p}$. On the other hand, the second requirement is a straightforward reflection of Afriat's cyclical consistency axiom.

Recall that Afriat's analysis makes substantial use of the linear structure of $\mathbb{R}^n$, and this makes it inapplicable in our general context. Consequently, it is not obvious if the generalized cyclical consistency property can yield a general rationalizability theorem along the lines of Afriat's Theorem. It is also not clear how, if at all, this property relates to the monotone congruence axiom. These issues will be clarified next.

4.5 Characterizations of Rationalizability. The structures of the general cyclical consistency property and the monotone congruence axiom are different at a basic level.

[15] All decreasing closures are taken here with respect to the partial order $\geq$. That is, $A_i^{\downarrow} := \{x \in X : y \geq x \text{ for some } y \in A_i\}$ for each $i \in [k]$.

In particular, the first one applies to a single choice correspondence on $\mathcal{A}$, while the second to a collection $\mathbb{C}$ of choice correspondences on $\mathcal{A}$. (The properties do not become identical when the latter collection is singleton.) However, there is in fact a close connection between these properties: In any RP framework, a choice correspondence satisfies the general cyclical consistency iff the increasing closure of c with respect to $\sqsupseteq$, that is, $c^{\uparrow}$, satisfies the monotone congruence axiom. As the former property yields a rational representation in the sense of Afriat and the latter in the sense of Richter, this fact yields, in turn, a connection that ties these two notions of rationalizability together. The following is, then, the main theorem of this paper. (The proof is in the Appendix.)

The Rationalizability Theorem I. *Let $((X, \geq), \mathcal{A})$ be a choice environment and c a choice correspondence on $\mathcal{A}$. Then, the following are equivalent:*

a. *$c^{\uparrow}$ satisfies the monotone congruence axiom;*

b. *$c^{\uparrow}$ is monotonically rationalizable;*

c. *c satisfies generalized cyclical consistency;*

d. *There is a complete preorder $\succsim$ on X that extends both $\mathrm{tran}(R(c) \cup \geq)$ and $\geq$, and that satisfies*

$$c(A) \subseteq \max(A, \succsim) \quad \textit{for every } A \in \mathcal{A}; \tag{4}$$

e. *There is a complete preorder $\succsim$ on X that extends $\geq$ and that satisfies* (4).

As we will make it precise in the following two sections, this theorem generalizes the Richter- and Afriat-type approaches to revealed preference theory simultaneously. As such, it unifies these two approaches, and demonstrates that, unlike their initial appearance, and how they are treated in the literature, each of these approaches are in fact special cases of a more general viewpoint.

4.6 The Monotone Version of Richter's Theorem. As a corollary of the Rationalizability Theorem I, we obtain a fairly substantial generalization of Richter's Theorem.

Proposition 1. *Let $((X, \geq), \mathcal{A}, \mathbb{C})$ be an RP framework. Then, $\mathbb{C}$ is monotonically rationalizable if, and only if, it satisfies the monotone congruence axiom.*

Proof. We omit the straightforward proof of the "only if" part of this assertion. To prove its "if" part, take any element c of $\mathbb{C}$ that satisfies (2) and (3). Then, $c^{\uparrow}$ satisfies the monotone congruence axiom, so, by the Rationalizability Theorem I, there is a complete preorder $\succsim$ on X that extends both $\succsim' := \mathrm{tran}(R(c) \cup \geq)$ and $\geq$, and that satisfies (4). Fix an arbitrary $A \in \mathcal{A}$, and take any y in $\max(A, \succsim)$. Then, for an arbitrarily picked $x \in c(A)$, we have $x \succsim' y$. As $y \succsim x$ and $\succsim$ extends $\succsim'$, however, we cannot have $x \succ' y$. It follows that we also have $y \succsim' x$, and using (2) yields $y \in c(A)$. Conclusion: $c(A) = \max(A, \succsim)$.

In the context of the ordered Richter-type RP framework $((X, \geq), \mathcal{A}, \{c\})$ that we introduced in Example 2, Proposition 1 says that $\{c\}$ obeys the monotone congruence axiom iff it is monotonically rationalizable. In particular, we recover Richter's Theorem as a special case by setting $\geq\, = \triangle_X$. Note also that if $\{c\}$ satisfies the monotone congruence axiom then c must obey generalized cyclical consistency (but the converse is not true in general). Indeed, if $\{c\}$ satisfies the monotone congruence axiom, then it is monotonically rationalizable. This, in turn, implies that (4) holds, and Rationalizability Theorem I guarantees that c obeys generalized cyclical consistency.

Example 2.a. [**Continued**] Consider the RP-framework $((X, \geq), \mathcal{A}, \{c\})$ we introduced in Example 2.a, which corresponds to the classical consumption choice model. In this framework, c is said to satisfy the **budget identity** if $\mathbf{x} \in c(B(\mathbf{p}, I))$ implies $\mathbf{px} = I$ for every $B(\mathbf{p}, I) \in \mathcal{A}$. Now, note that $\mathbf{x}\, R(c)\, \mathbf{y}$ means here that there is a budget set $B(\mathbf{p}, I)$ in $\mathcal{A}$ such that $\mathbf{x} \in c(B(\mathbf{p}, I))$ and $\mathbf{py} \leq I$. Consequently, if $\mathbf{x} \in c(B(\mathbf{p}, I))$ for some $B(\mathbf{p}, I) \in \mathcal{A}$, then $\mathbf{x}\, R(c)\, \mathbf{y}$ for all $\mathbf{y} \in \mathbb{R}^n_+$ such that $\mathbf{x} \geq \mathbf{y}$. Given this observation, it is easy to check that a choice correspondence c obeys the monotone congruence axiom iff it obeys the congruence axiom and the budget identity. In view of Proposition 1, therefore, we reach the following conclusion in the context of Example 2.a: *A demand correspondence c on $\mathcal{A}$ is monotonically rationalizable iff it satisfies the congruence axiom and the budget identity.* By contrast, Richter's Theorem says that c is rationalizable iff it satisfies the congruence axiom.

4.7 On the Structure of Rationalizability. With the exception of some trivial situations, there are a multitude of complete preference relations that (weakly) rationalize a given choice correspondence as in (4). Part (d) of the Rationalizability Theorem I points to a particular type of rationalization which, as we shall now demonstrate, is linked to the revealed preference relation induced by the choice correspondence in the tightest possible way.

Let $((X, \geq), \mathcal{A})$ be a choice environment and c a choice correspondence on $\mathcal{A}$. We say that a complete preorder $\succsim$ on X is **$\geq$-monotonic** if $x \geq y$ implies $x \succsim y$ for every $x, y \in X$. (Notice that this property is weaker than $\succsim$ being an extension of $\geq$.) In turn, we say that $\succsim$ is a **rationalization for** c if it is $\geq$-monotonic and (4) holds. Clearly, a rationalization for c may be too coarse to be useful. For instance, if $\succsim$ declares every alternative in X as indifferent (that is, $\succsim$ equals $X \times X$), then it is, trivially, a rationalization for c. Indeed, Afriat type theorems look for particular types of rationalizations. In particular, we wish to choose a rationalization for c in a way that is tightly linked to the dominance relation of the environment, as well as the observed choices of the agent. Then, it seems desirable that when $x > y$, or when x is revealed to be strictly preferred to y by c, the rationalization for c should declare x strictly better than y. Part (d) of the Rationalizability Theorem I says that this can be done, provided that c satisfies generalized cyclical consistency. Our next result, whose proof is relegated to the Appendix, demonstrates the precise way in which one can view the preference relation found in that part of the theorem as "minimal" among all possible rationalizations for c.

Proposition 2. *Let $((X, \geq), \mathcal{A})$ be a choice environment and c a choice correspondence on $\mathcal{A}$. Let $\succsim$ be a complete preorder satisfying the properties in part* (d) *of the Rationalizability Theorem I. Then,*

$$\max(A, \succsim) = A \cap c(A)^{\uparrow,\mathrm{tran}(R(c)\cup\geq)} \subseteq \max(A, \trianglerighteq) \tag{5}$$

for every $A \in \mathcal{A}$ and every rationalization $\trianglerighteq$ for c.

Given any feasible set A in the choice environment, the second part of (5) says that any element in $c(A)$, or any element in A that is revealed preferred to at least one chosen alternative in A, has to be declared optimal with respect to *every* rationalization of c. Furthermore, the first part of (5) says that it is *precisely* the set of all such elements that the rationalization identified in part (d) of the Rationalizability Theorem I declares optimal. (In particular, an optimal point with respect to this rationalization must be optimal for *any* rationalization of c.) It is in this sense that the rationalization is "tight" among all possible rationalizations for c. The elements that are declared optimal by this preference are the only ones that an observer can robustly conclude to be optimal by the "true" preference relation of the decision maker (which can, in general, only be partially identified). We show in the next section that the classical construction of the preference relations in Afriat's Theorem are not tight in this sense.

4.8 A Non-Finite Version of Afriat's Theorem. Let $((X, \geq), \mathcal{A})$ be a choice environment, and c a choice correspondence on $\mathcal{A}$. Let us consider the RP framework

$((X, \geq), \mathcal{A}, \mathbb{C})$, where

$$\mathbb{C} := \{C \in \mathfrak{C}_{(X,\mathcal{A})} : C \sqsupseteq c\}. \tag{6}$$

This framework generalizes the Afriat-type RP frameworks as we have introduced them in Example 3 by allowing $\mathcal{A}$ to be infinite and also by allowing $c(A)$ to be an infinite set for any $A \in \mathcal{A}$. The equivalence of the statements (c) and (e) of the Rationalizability Theorem I says that c satisfies generalized cyclical consistency iff there is a complete preorder $\succsim$ on X that extends $\geq$ and that satisfies

$$c(A) \subseteq \max(A, \succsim) \quad \text{for every } A \in \mathcal{A}.$$

To demonstrate the power of this observation, let us specialize it to the context of Afriat (1967), but note that precisely the same argument can be made in, say, the context of Forges and Minelli (2009).

Example 3.a. **[Continued]** Consider the RP-framework $((X, \geq), \mathcal{A}, \mathbb{C})$ we introduced in Example 3.a, with P and d defined as in that example, but allowing now both P and any $d(\mathbf{p})$ to be infinite sets. In this case, $\mathcal{A} = \{B(\mathbf{p}, \mathbf{p}d(\mathbf{p})) : \mathbf{p} \in P\}$, $c(B(\mathbf{p}, \mathbf{p}d(\mathbf{p}))) = d(\mathbf{p})$, and $\mathbb{C}$ is given by (6). It is straightforward to check that the demand correspondence d satisfies cyclical consistency (in Afriat's sense, as defined in Section 4.4) iff c obeys generalized cyclical consistency. In turn, by the Rationalizability Theorem I, c has this property iff there is a strictly monotonic preference relation $\succsim$ on $\mathbb{R}^n_+$, that is, a complete preorder on $\mathbb{R}^n_+$ that extends $\geq$, such that $d(\mathbf{p}) \subseteq \max(B(\mathbf{p}, \mathbf{p}d(\mathbf{p})), \succsim)$ for each $\mathbf{p} \in P$. This is very much the choice-theoretic gist of Afriat's Theorem.

It is important to note, however, that the nature of rationalization obtained here is markedly different from that obtained in Afriat's analysis and from the recent extension of that analysis to the case of infinite choice data provided by Rany (2014). Indeed, those procedures guarantee rationalization by a quasi-concave utility function (and hence a convex preference) in $\mathbb{R}^n_+$, so the set of optimal bundles in a given budget set can be strictly larger than that according to the rationalization found in the Rationalization Theorem I. (Recall (5).) To wit, suppose we have choice data about an individual at the same prices $\mathbf{p}$ at two different times, say, $\mathbf{x}^1$ and $\mathbf{x}^2$, with $\mathbf{p}\mathbf{x}^1 = \mathbf{p}\mathbf{x}^2$. Suppose also that $\mathbf{x}^1 \neq \mathbf{x}^2$, so, what we observe is precisely two distinct elements in $d(\mathbf{p})$. Rationalization by a convex preference would then entail that every bundle on the line segment between $\mathbf{x}^1$ and $\mathbf{x}^2$ is also optimal for the individual at prices $\mathbf{p}$, even though there is no choice data to support this. Such a rationalization may seem unduly coarse, especially in situations where there is no strong reason to suppose that convexity holds. By contrast, the rationalization found in the Rationalization Theorem I would declare *only* $\mathbf{x}^1$ and $\mathbf{x}^2$ as optimal at prices $\mathbf{p}$.

5 RATIONALIZABILITY OF CHOICE DATA (with continuous utility)

The classical statement of Afriat's Theorem seems to deliver more information about the structure of rationalization. Indeed, in that theorem, one not only finds a monotonic preference relation that rationalizes the choice data, but also the fact that this relation can be chosen to have a continuous utility representation. This fact owes, obviously, to the particular choice domain that is adopted by Afriat which possesses a well-behaved topological structure. However, even if we impose such a structure on X in the context of Richter's Theorem, we would not be able to guarantee the continuity of the rationalizing preference relation. The difficulty, however, disappears if, as in Afriat's Theorem, we restrict our attention to choice environment with only *finitely* many choice problems. This is actually quite pleasant because the "finiteness" hypothesis is unexceptionable from an empirical point of view. The objective of this section is to prove that, given this hypothesis, the additional structure that Afriat's Theorem delivers would also obtain in the context of any well-behaved RP framework. It turns out that the very special structure of Afriat's Theorem is not at all needed for this fact.

5.1 Rationalizability by a (Continuous) Utility Function. Where $((X, \geq), \mathcal{A}, \mathbb{C})$ is an RP framework, we say that the choice data $\mathbb{C}$ is **rationalizable by a utility function** if there is at least one c in $\mathbb{C}$ and a (utility) function $u : X \to \mathbb{R}$ such that

$$c(A) = \arg\max_{x \in A} u(x) \quad \text{for each } A \in \mathcal{A}.$$

It is natural to ask for u to be strictly increasing with respect to $\geq$ and, when X is a topological space, we would also like u to be continuous.

5.2 Characterizations of Continuous Rationalizability. We now show that, in finite environments, that is, when the set $\mathcal{A}$ of all choice problems to be observed is finite, rather basic assumptions allows restating the Rationalizability Theorem I in terms of continuous utility functions.

The Rationalizability Theorem II. *Let $((X, \geq), \mathcal{A})$ be a choice environment such that X is a locally compact and separable metric space, $\geq$ a continuous partial order on*

X, and $\mathcal{A}$ a finite collection of nonempty compact subsets of X. Let c be a closed-valued choice correspondence on $\mathcal{A}$. Then, the following are equivalent:

a. *$c^{\uparrow}$ satisfies the monotone congruence axiom;*

b. *$c^{\uparrow}$ is rationalizable by a continuous and strictly $\geq$-increasing function $u : X \to \mathbb{R}$;*

c. *c is satisfies generalized cyclical consistency;*

d. *There is a continuous function $u : X \to \mathbb{R}$ which is strictly increasing with respect to both $\mathrm{tran}(R(c) \cup \geq)$ and $\geq$, and which satisfies*

$$c(A) \subseteq \underset{x \in A}{\arg\max}\; u(x) \quad \text{for each } A \in \mathcal{A}; \tag{7}$$

e. *There is a continuous and strictly $\geq$-increasing function $u : X \to \mathbb{R}$ that satisfies (7).*

An examination of the proof of Rationalizability Theorem II in the Appendix shows that it consists of two parts. The first part shows that, under the assumptions of the theorem and, in particular, the finiteness of $\mathcal{A}$, the relation $\mathrm{tran}(R(c) \cup \geq)$ is a closed preorder on X. The second part shows that the closedness of $\mathrm{tran}(R(c) \cup \geq)$ is sufficient to guarantee the equivalence of the statements (a) to (e). This is worth bearing in mind because there are interesting cases where $\mathrm{tran}(R(c) \cup \geq)$ is closed, and thus the conclusion of this theorem holds, even when $\mathcal{A}$ in not finite. Section 5.7 focuses on one such case.

5.3 A Continuous Version of Richter's Theorem. It is not a priori obvious how one may obtain a utility representation in the context of Richter's theorem, for the arbitrariness of $\mathcal{A}$ makes it difficult to ensure the continuity of the rationalizing preference relations. However, at least when $\mathcal{A}$ is finite, this sort of a difficulty does not arise. Just as Proposition 1 follows from Rationalizability Theorem I, so by an analogous argument we know that the following characterization follows from Rationalizability Theorem II: Let $((X, \geq), \mathcal{A})$ be a choice environment obeying the conditions in Rationalizability Theorem II and suppose that $\mathbb{C}$ is a collection of closed-valued choice correspondences on $\mathcal{A}$. Then $\mathbb{C}$ is monotonically rationalizable by a continuous and strictly $\geq$-increasing utility function iff it satisfies the monotone congruence axiom. When $\mathbb{C}$ consists of just a single correspondence, we obtain the following result, which provides a continuous, and continuous and monotonic, version of Richter's Theorem.

Proposition 3. *Let $((X, \geq), \mathcal{A}, \{c\})$ be an RP framework such that X is a locally compact and separable metric space, $\geq$ a continuous partial order on X, $\mathcal{A}$ a nonempty* *finite collection of nonempty compact subsets of X, and c a closed-valued choice correspondence. Then, c satisfies the (monotone) congruence axiom if, and only if, it is rationalizable by a continuous (and strictly $\geq$-increasing) utility function on X.*

5.4 Extrapolation to choice behavior outside $\mathcal{A}$. The key feature of Rationalizability Theorem II is that it delivers a *continuous* utility function. This property is important when we are interested in understanding the agent's behavior, not in $\mathcal{A}$ as such, but in a larger (and possibly infinite) collection $\mathcal{B}$ of nonempty compact subsets of X. In that case, $\mathcal{A}$ should be interpreted as a random sample of constraints sets drawn from $\mathcal{B}$ and the observer is interested in testing the hypothesis that the agent chooses by maximizing a utility function for every set in $\mathcal{B}$. The existence of a continuous utility function that rationalizes the choice data in $\mathcal{A}$ then furnishes us with a utility function that could potentially be the one used by the agent when making her choices more generally.[16]

In fact, by applying our continuous and monotone version of Richter's Theorem, we could characterize an agent's choice behavior in out-of-sample constraint sets. To see this, let c be a choice correspondence on $\mathcal{A}$, and C a choice correspondence on $\mathcal{B}$. We say that C is a **rational prediction on $\mathcal{B}$ induced by** c if there is a continuous and strictly $\geq$-increasing utility function $u : X \to \mathbb{R}$ such that

$$c(A) \subseteq \operatorname*{arg\,max}_{x \in A} u(x) \quad \text{for each } A \in \mathcal{A}$$

and

$$C(B) = \operatorname*{arg\,max}_{x \in B} u(x) \quad \text{for each } B \in \mathcal{B}.$$

The first question concerns the existence of rational predictions.

Proposition 4. (Existence of Rational Predictions) *Let $((X, \geq), \mathcal{A}, \{c\})$ be an RP framework as in Proposition 3, but assume that $\mathcal{A}$ is finite. Let $\mathcal{B}$ be any collection of nonempty compact subsets of X with $\mathcal{A} \subseteq \mathcal{B}$. Then, there exists a rational prediction on $\mathcal{B}$ induced by c if, and only if, c satisfies generalized cyclical consistency.*

Proof. The "only if" part of this assertion is straightforward. On the other hand, its

[16] If a rationalizing utility function is not continuous, it would be difficult to view that function as one's "true" utility function, as there is then no guarantee that an optimum exists for every compact subset of X. As a matter of fact, if the agent has a utility function but it is not continuous, then it is not even clear why, in the first place, we should hypothesize that the constraint sets in the random sample $\mathcal{A}$ should each contain an optimal choice.

"if" part is an immediate consequence of the Rationalizability Theorem II and the fact that a continuous real map on X attains its maximum on every element of $\mathcal{B}$.

As another application, we show how one may identify whether or not a choice correspondence on $\mathcal{B}$ is a rational prediction induced by a given choice correspondence on $\mathcal{A}$.

Proposition 5. (Characterization of Rational Predictions) *Let $((X, \geq), \mathcal{A}, \{c\})$ be an RP framework as in Proposition 3. Let $\mathcal{B}$ be a finite collection of nonempty compact subsets of X with $\mathcal{A} \subseteq \mathcal{B}$. Then, a choice correspondence C on $\mathcal{B}$ is a rational prediction on $\mathcal{B}$ induced by c if, and only if, $c \sqsubseteq C|_{\mathcal{A}}$ and C satisfies the monotone congruence axiom.*

Proof. The "only if" part of the assertion is again straightforward. On the other hand, its "if" part follows by applying Proposition 3 to C.

6 APPLICATIONS TO THE THEORY OF CONSUMER DEMAND

The remainder of this paper focuses on the theory of consumer demand. Our objective is to show how our general rationalizability theorems can be used in this classical framework to obtain novel results.

6.1 Rationalizability. Let us return to the context described in Example 3.a, where an analyst observes the demand correspondence $d : P \rightrightarrows \mathbb{R}^n_+$ of a consumer, with P being a nonempty finite set of prices in $\mathbb{R}^n_{++}$. We assume here that d takes compact (but not necessarily finite) values. (Recall also that we have $\mathbf{pz} = \mathbf{py}$ for every $\mathbf{z}$ and $\mathbf{y}$ in $d(\mathbf{p})$ and $\mathbf{p} \in P$.) Formally, the choice environment at hand is $((\mathbb{R}^n_+, \geq), \mathcal{A})$, where $\mathcal{A} = \{B(\mathbf{p}, \mathbf{px}(\mathbf{p})) : \mathbf{p} \in P\}$, and the (observed) choice correspondence c on $\mathcal{A}$ is given by $c(B(\mathbf{p}, \mathbf{px}(\mathbf{p}))) = d(\mathbf{p})$. As we have already noted in Section 4.8, c obeys generalized cyclical consistency iff d satisfies cyclical consistency. It is also straightforward to check that c satisfies the congruence axiom iff d has the following property: For every positive integer k, $\mathbf{p}^1, ..., \mathbf{p}^k \in P$ and $\mathbf{x}^1 \in d(\mathbf{p}^1), ..., \mathbf{x}^k \in d(\mathbf{p}^k)$,

$$\mathbf{p}^2\mathbf{x}^1 \leq \mathbf{p}^2\mathbf{x}^2, ..., \mathbf{p}^k\mathbf{x}^{k-1} \leq \mathbf{p}^k\mathbf{x}^k \text{ and } \mathbf{p}^1\mathbf{x}^k \leq \mathbf{p}^1\mathbf{x}^1 \tag{8}$$

imply

$$\mathbf{x}^1 \in d(\mathbf{p}^2), ..., \mathbf{x}^{k-1} \in d(\mathbf{p}^k) \text{ and } \mathbf{x}^k \in d(\mathbf{p}^1).$$

With a minor abuse of terminology, let us agree to say that $\mathbf{x}$ satisfies the congruence axiom if it has this property. It is plain that if d satisfies the congruence axiom, it must satisfy cyclical consistency.

Proposition 6. *Suppose that P and d are as in Example 3.a, but allow now that d be compact-valued. Then,*

(i) *d obeys the congruence axiom if, and only if, there is a continuous and strictly increasing utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ such that*

$$d(\mathbf{p}) = \underset{\mathbf{x} \in B(\mathbf{p}, \mathbf{p}d(\mathbf{p}))}{\arg\max} \; u(\mathbf{x}) \quad \textit{for every } \mathbf{p} \in P; \tag{9}$$

(ii) *d obeys cyclical consistency if, and only if, there is a continuous and strictly increasing utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ such that*

$$d(\mathbf{p}) \subseteq \underset{\mathbf{x} \in B(\mathbf{p}, \mathbf{p}d(\mathbf{p}))}{\arg\max} \; u(\mathbf{x}) \quad \textit{for every } \mathbf{p} \in P. \tag{10}$$

Proof. We have seen in Section 4.6 that c satisfies the monotone congruence axiom iff it obeys the budget identity and the congruence axiom. Here c satisfies the budget identity by construction, while it obeys the congruence axiom iff d obeys the congruence axiom. Consequently, part (i) of Proposition 6 follows readily from Proposition 3. Its part (ii), on the other hand, follows from the equivalence of the statements (c) and (d) in the Rationalizability Theorem II and the fact that c satisfies generalized cyclical consistency iff d satisfies cyclical consistency.

We emphasize that part (ii) of Proposition 6 is none other than a version of Afriat's Theorem, but it is stronger than the standard version of that result since d need not be finite-valued here. Moreover, for the reasons outlined in Section 4.8, the utility function we identify here is not the same as the concave utility function constructed from the classical Afriat inequalities.

Part (i) of this Proposition 6 is simply the continuous and monotone version of Richter's Theorem specialized to the context of consumer demand. The literature on the rationalizability of consumer demand is quite large but, to the best of our knowledge, there is no result characterizing rationalization (of the form (9)) with a continuous utility function.[17] This lacuna may be due to the reliance on convex analysis techniques, which

[17] Chiappori and Rochet (1987) characterize finite data sets that are rationalizable (as in (9)) by a strictly quasi-concave, strictly increasing, and smooth utility function. The choice data in their case excludes distinct bundles chosen at the same budget and the same bundle chosen at distinct budgets.

typically lead to concave utility functions, whereas a rationalization of the form (9) will, for some data sets, *require* rationalization with a non-concave (but still continuous) utility function.

It is worth noting that the gap between the two conditions on the demand correspondence d in (i) and (ii) is negligible from a practical point of view. While empirical work on revealed preference analysis has invariably focused on testing cyclical consistency, data sets that obey cyclical consistency will almost always do so because there are simply no observations obeying (8). In these cases, both cyclical consistency and the congruence axiom are satisfied, and therefore, one can invoke part (ii) of Proposition 6 to obtain a utility function obeying (9) rather than just (10).

6.2 Recoverability. Suppose that the demand correspondence d obeys cyclical consistency. Let $\mathcal{U}(d)$ stand for the collection of all continuous and strictly increasing $u : \mathbb{R}^n_+ \to \mathbb{R}$ such that (10) holds. By part (ii) of Proposition 6, $\mathcal{U}(d) \neq \emptyset$. However, different members of $\mathcal{U}(d)$ would entail different preference rankings (which are required to be in agreement only in the case of the observed choices of the agent). A natural question is to what extent we may identify the actual preference relation of the agent without subscribing to any one utility function that rationalizes d as in (10). Formally, we would like to characterize the subsets $S(d)$ and $S'(d)$ of $\mathbb{R}^n_+ \times \mathbb{R}^n_+$ such that

$$(\mathbf{x}, \mathbf{y}) \in S(d) \quad \text{iff} \quad u(\mathbf{x}) \geq u(\mathbf{y}) \text{ for each } u \in \mathcal{U}(d)$$

and

$$(\mathbf{x}, \mathbf{y}) \in S'(d) \quad \text{iff} \quad u(\mathbf{x}) > u(\mathbf{y}) \text{ for each } u \in \mathcal{U}(d).$$

This formulation of the problem is the same as what Varian (1982) calls the *Recoverability Problem*, except that Varian considers the case where $\mathcal{U}(d)$ consists of strictly increasing, continuous, *and concave* utility functions with (10). Given the weaker assumptions on the utility functions we impose here, the relations $S(d)$ and $S'(d)$ are bound to be smaller than those studied by Varian (1982). Moreover, in certain contexts, it is certainly sensible not to impose the concavity, or even the quasiconcavity, requirement on the utility functions (even when it is possible to rationalize d by such a utility function). To wit, consider the case where the consumer chooses a contingent consumption subject to a linear budget set, where a bundle $\mathbf{x} \in \mathbb{R}^n_+$ specifies the levels of the representative good in different states of the world (of which there are n), and

$\mathbf{p} \in \mathbb{R}^n_{++}$ are the state prices.[18] In such a context, we may well wish to draw inferences of the agent's preferences (based on her observed choices) without assuming that she has a quasiconcave utility function, since that assumption would exclude risk-seeking and/or elation-seeking preferences. (Halevy et al. (2014) make a similar point.)[19]

For bundles $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n_+$, we say that **$\mathbf{x}$ is revealed preferred to $\mathbf{y}$** if $\mathbf{x}\, \text{tran}(R(c) \cup \geq)\, \mathbf{y}$, where, as usual, $c(B(\mathbf{p}, \mathbf{p}d(\mathbf{p}))) = d(\mathbf{p})$ for each $\mathbf{p} \in P$. This is equivalent to saying that there are finitely many $\mathbf{p}^1, ..., \mathbf{p}^k \in P$ and $\mathbf{x}^1 \in d(\mathbf{p}^1), ..., \mathbf{x}^k \in d(\mathbf{p}^k)$ such that

$$\mathbf{x} \geq \mathbf{x}^1,\ \mathbf{p}^1\mathbf{x}^1 \geq \mathbf{p}^1\mathbf{x}^2,\ ..., \mathbf{p}^{k-1}\mathbf{x}^{k-1} \geq \mathbf{p}^{k-1}\mathbf{x}^k \text{ and } \mathbf{p}^k\mathbf{x}^k \geq \mathbf{p}^k\mathbf{y}. \tag{11}$$

We say **$\mathbf{x}$ is revealed strictly preferred to $\mathbf{y}$** if any of the inequalities in (11) is strict. It is clear that $\mathbf{x}$ is revealed preferred (revealed strictly preferred) to $\mathbf{y}$ then $(\mathbf{x}, \mathbf{y})$ is in $S(d)$ (respectively $S'(d)$). The next result characterizes $S(d)$ and $S'(d)$ by showing that the converse is also true. When P is finite and d is finite-valued, it is clear that working out the pairs of bundles that are related by revealed preference (or revealed strict preference) is computationally straightforward. It is then also straightforward to compute $S(d)$ and $S'(d)$.

Proposition 7. *Suppose that P and d are as in Example 3.a, but allow now that d be compact-valued. In addition, suppose that d obeys cyclical consistency. Then, for any $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n_+$,*

(i) *$(\mathbf{x}, \mathbf{y}) \in S(d)$ if, and only if, $\mathbf{x}$ is revealed preferred to $\mathbf{y}$; and*

(ii) *$(\mathbf{x}, \mathbf{y}) \in S'(d)$ if, and only if, $\mathbf{x}$ is revealed strictly preferred to $\mathbf{y}$.*

Proof. (i) Suppose $\mathbf{x}$ is not revealed preferred to $\mathbf{y}$. In that case, put $\mathcal{A}^* := \mathcal{A} \cup \{\mathbf{x}, \mathbf{y}\}$, where $\mathcal{A} = \{B(\mathbf{p}, \mathbf{p}d(\mathbf{p})) : \mathbf{p} \in P\}$, and define the correspondence $c^* : \mathcal{A}^* \rightrightarrows \mathbb{R}^n_+$ by $c^*(B(\mathbf{p}, \mathbf{p}d(\mathbf{p}))) := d(\mathbf{p})$ for each $p \in P$, and $c^*(\{\mathbf{x}, \mathbf{y}\}) := \mathbf{y}$. Since $\mathbf{x}$ is not revealed preferred to $\mathbf{y}$, that is, $(\mathbf{x}, \mathbf{y})$ is not in $\text{tran}(R(c) \cup \geq)$, and c obeys generalized cyclical consistency, c^* also obeys generalized cyclical consistency. By construction, $(\mathbf{y}, \mathbf{x}) \in R(c^*)$ but $(\mathbf{x}, \mathbf{y})$ does not belong to $\text{tran}(R(c^*) \cup \geq)$, and thus, by the equivalence of (c)

[18] Data from laboratory experiments of this type have been collected and tested for cyclical consistency, for instance, by Choi et al. (2007), among others.

[19] For example, suppose that $n = 2$ and the consumer's true utility function is $u(x_1, x_2) = \pi_1 v(x_1) + \pi_2 v(x_2)$, where $\pi_i > 0$ for $i = 1$, 2, and v is strictly increasing but not concave (so the agent is not everywhere risk averse). Then u will not be quasiconcave and predicting the consumer's preference from d while assuming quasiconcavity can lead to false conclusions. On the other hand, the predictions captured by $S(d)$ and $S'(d)$ will be correct (even though they will be coarser than what would have been if the observer assumed, correctly in this case, that u is additive across states).

and (d) in Rationalizability Theorem II, there exists u^* with (7), and hence $u^* \in \mathcal{U}(d)$ such that $u^*(\mathbf{y}) > u^*(\mathbf{x})$. Thus, $(\mathbf{x}, \mathbf{y})$ does not belong to $S(d)$.

(ii) We define $\mathcal{A}^*$ and c^* as in the proof of (i). If $\mathbf{x}$ is not revealed preferred to $\mathbf{y}$, then we know from the proof of (i) that $(\mathbf{x}, \mathbf{y})$ is not in $S(d)$ and hence not in $S'(d)$. Now suppose $\mathbf{x}$ is revealed preferred to $\mathbf{y}$, but not strictly so. In that case, one could check that c^* still satisfies cyclical consistency and hence Rationalizability Theorem II tells us that there is a strictly increasing and continuous function u^{**} with (7). Since $\mathbf{x}$ $\text{tran}(R(c^*) \cup \geq)\, \mathbf{y}$ and, by construction, $\mathbf{y}\, R(c^*)\, \mathbf{x}$, we obtain $u^{**}(\mathbf{x}) = u^{**}(\mathbf{y})$. Finally, note that $u^{**} \in \mathcal{U}(d)$, and thus $(\mathbf{x}, \mathbf{y})$ does not belong to $S'(d)$.

6.3 Rationalizability of Engel curves. We pointed out in Section 5.2 that continuous rationalizability relies on the closedness of $\text{tran}(R(c) \cup \geq)$ rather than the finiteness of the set of observations. This can be an important distinction, as we shall illustrate with the following application.

Consider a situation in which we happen to know the entirety of finitely many Engle curves of an individual. When is it the case that these curves correspond to those of a continuous and strictly increasing utility maximizing individual? This rationalizability problem cannot be attacked by Afriat's approach, for even a single Engel curve presumes uncountably many choice situations.[20] We can, however, provide an answer fairly easily by using the (proof of) Rationalizability Theorem II.

Put precisely, the choice environment we consider is $((\mathbb{R}^n_+, \geq), \mathcal{A})$, where

$$\mathcal{A} := \{B(\mathbf{p}, I) : \mathbf{p} \in P \text{ and } I > 0\},$$

with P being a nonempty finite set of prices. In this context, by a **demand correspondence on $\mathcal{A}$** we mean a correspondence $d : P \times \mathbb{R}_{++} \rightrightarrows \mathbb{R}^n_+$ such that $\emptyset \neq d(\mathbf{p}, I) \subseteq B(\mathbf{p}, I)$ for any $\mathbf{p} \in P$ and $I > 0$. (Therefore, the map $I \mapsto d(B(\mathbf{p}, I))$ is the **Engel curve** of the agent in consideration.) Abusing the terminology again, we say that d satisfies monotone congruence (generalized cyclical consistency) if the choice correspondence $c : B(\mathbf{p}, I) \mapsto d(\mathbf{p}, I)$ on $\mathcal{A}$ satisfies monotone congruence (generalized cyclical consistency).

This setup has proved useful in empirical demand analysis that use the revealed preference approach (cf. Blundell et al. (2003)). However, to the best of our knowledge,

[20] While the rationalizability theorem of Reny (2014) applies here, it does not deliver a strictly increasing utility function.

rationalizability of d has not been characterized along the lines of either Afriat's or Richter's theorem. The final result of this paper provides two such characterizations.

Proposition 8. *Let* $((\mathbb{R}^n_+, \geq), \mathcal{A})$ *be the choice environment defined above and* d *an upper hemicontinuous demand correspondence on* $\mathcal{A}$*. Then,*

(i) d *obeys the monotone congruence axiom if, and only if, there is a continuous and strictly increasing utility function* $u : \mathbb{R}^n_+ \to \mathbb{R}$ *such that*

$$d(\mathbf{p}, I) = \underset{\mathbf{y} \in B(\mathbf{p},I)}{\arg\max} u(\mathbf{y}) \quad \textit{for every } \mathbf{p} \in P \textit{ and } I > 0;$$

(ii) d *obeys the generalized cyclical consistency if, and only if, there is a continuous and strictly increasing utility function* $u : \mathbb{R}^n_+ \to \mathbb{R}$ *such that*

$$d(\mathbf{p}, I) \subseteq \underset{\mathbf{y} \in B(\mathbf{p},I)}{\arg\max} u(\mathbf{y}) \quad \textit{for every } \mathbf{p} \in P \textit{ and } I > 0.$$

6 CONCLUSION

We have introduced in this paper a framework for revealed preference theory in which the grand alternative space is modeled as a partially ordered set and the traditional role of a "choice correspondence" is replaced with what we call "choice data" which is simply a *set* of choice correspondences. This framework allows us to formulate generalized versions of the fundamental rationality postulates of Richter (1966) and Afriat (1967). While this is not immediately transparent from the original formulations of these axioms, it is shown here that they are in fact closely connected, thereby pointing to a way of seeing the main rationalizability results of these two seminal papers, as well as numerous other "rationalizability by a preference relation" type theorems obtained in the earlier literature, as special cases of a single rationalizability result. Furthermore, introducing some basic topological structure and presuming that we can observe an agent making choice decisions only finitely many times allow us to formulate this result in the "rationalizability by a continuous utility function" form. This extends the work of Richter (1966) to rationalizations with continuous utility functions and that of Afriat (1967) to arbitrary choice domains.

The rationalizability results we have reported in this paper demonstrate the unifying nature of the choice framework we have introduced. This framework also has the important advantage of allowing us to model choice data availability constraints explicitly, regardless of the nature of choice problems. We hope that this framework will prove useful for modeling any type of choice situation, be it rational or boundedly rational.[21]

APPENDIX

Proof of the Rationalizability Theorem I.(a)$\Rightarrow$(b) Assume that (a) is valid. Then, there is a choice correspondence d on $\mathcal{A}$ such that (i) $d \sqsupseteq c$ and (ii) d satisfies the two requirements of the monotone congruence axiom. Put $\mathcal{B} := \mathcal{A} \cup \{\{x, y\} \in 2^X : x \geq y\}$, and define $e : \mathcal{B} \to 2^X$ as:

$$e(B) := \begin{cases} d(B), & \text{if } B \in \mathcal{A}, \\ \max(B, \geq), & \text{if } B \in \mathcal{B} \backslash \mathcal{A}. \end{cases}$$

Obviously, e is a choice correspondence on $\mathcal{B}$. Moreover, e satisfies the congruence axiom. (To see this, take any $x, y \in X$ such that $x \ \mathrm{tran}(R(e)) \ y$ and $y \in e(B)$ for some $B \in \mathcal{B}$ with $x \in B$. But it is readily checked that $R(e) = R(d) \cup \geq$. Consequently, if $B \in \mathcal{A}$, the monotone congruence axiom yields $x \in d(B) = e(B)$, and if $B \in \mathcal{B} \backslash \mathcal{A}$, then $B = \{x, y\}$ and $y \geq x$ (by definition of e), so again by the monotone congruence axiom, we find $x = y \in e(B)$.)

We now use Szpilrajn's Theorem to find a complete preorder $\succsim$ on X that extends $\mathrm{tran}(R(e))$. Given an arbitrarily fixed B in $\mathcal{A}$, notice that if $x \in e(B)$ and $y \in B$, then $x \ R(e) \ y$, and hence $x \succsim y$, which shows that $e(B) \subseteq \max(B, \succsim)$. Conversely, suppose there is an x in $\max(B, \succsim) \backslash e(B)$. Then, pick any $y \in e(B)$ so that $y \ R(e) \ x$, and hence, $y \ \mathrm{tran}(R(e)) \ x$. The reverse of this relation cannot hold, because, otherwise, we would get $x \in e(B)$ by the congruence axiom (on e). Thus, $y \ \mathrm{tran}(R(e)) \ x$ holds strictly, that is, $y \ P_{\mathrm{tran}(R(e))} \ x$. As $\succsim$ extends $\mathrm{tran}(R(e))$, therefore, we find $y \succ x$, contradicting x being a $\succsim$-maximum in B. Conclusion:

$$e(B) = \max(B, \succsim) \quad \text{for every } B \in \mathcal{B}.$$

Obviously, this implies that $d(A) = \max(A, \succsim)$ for each $A \in \mathcal{A}$. It remains to show that $\succsim$ extends $\geq$. To this end, take any $x, y \in X$ with $x > y$. If $\{x, y\} \in \mathcal{A}$, then $y \in d\{x, y\}$ cannot hold due to the monotone congruence axiom, and hence $\{x\} = d\{x, y\}$, while if $\{x, y\} \notin \mathcal{A}$, we trivially have $\{x\} = d\{x, y\}$. Consequently, $\{x\} = \max(\{x, y\}, \succsim)$, that is, $x \succ y$, as we sought.

(b)$\Rightarrow$(c) Assume that (b) is valid. Then, there is a complete preorder $\succsim$ on X and a d in $c^{\uparrow}$ such that $\succsim$ extends $\geq$ and $d(A) = \max(A, \succsim)$ for each $A \in \mathcal{A}$. It follows that

$$c(A) \subseteq \max(A, \succsim) \subseteq \ \mathrm{MAX}(A, \geq) \quad \text{for every } A \in \mathcal{A}. \tag{12}$$

Now take any $k \in \mathbb{N}$, $A_1, ..., A_k \in \mathcal{A}$, and $(x_1, ..., x_k) \in c(A_1) \times \cdots \times c(A_k)$ such that $x_1 \in A_2^{\downarrow}, ..., x_{k-1} \in A_k^{\downarrow}$ and $x_k \in A_1^{\downarrow}$. Then, there exists a $(y_1, ..., y_k) \in A_1 \times \cdots \times A_k$ such that $x_2 \succsim y_2 \geq x_1, ..., x_k \succsim y_k \geq x_{k-1}$ and $x_1 \succsim y_1 \geq x_k$. As $\succsim$ extends $\geq$, therefore, $x_1 \succsim x_2 \succsim \cdots \succsim x_1$, so, by transitivity of $\succsim$, we find $x_{i-1} \in \max(A_i, \succsim)$ for each $i \in [k]$ and $x_k \in \max(A_0, \succsim)$. In view of (12), then, $x_{i-1} \in \mathrm{MAX}(A_i^{\downarrow}, \geq)$ for each $i \in [k]$ and $x_k \in \mathrm{MAX}(A_1^{\downarrow}, \geq)$, as sought.

[21] For boundedly rational choice theories, however, there is the added difficulty of checking whether or not one can extend a representation on a given (observable) collection of feasible sets to a larger (potentially unobservable) collection of feasible sets. This important point, which is readily formalized in terms of RP frameworks, has recently been made forcefully by de Clippel and Rozen (2013).

(c)⇒(d) Assume that (c) is valid. Define

$$\succsim' := \text{ tran}\left(R\left(c\right) \cup \geq\right),$$

where R is the direct revealed preference relation induced by c (Section 4.1). Clearly, $\succsim'$ is a preorder on X. We use Szpilrajn's Theorem to find a complete preorder $\succsim$ on X that extends $\succsim'$. As $R\left(c\right) \subseteq \succsim$, we have $x \succsim y$ if there is an $A \in \mathcal{A}$ with $(x, y) \in c(A) \times A$. It follows that $c(A) \subseteq \max(A, \succsim)$ for every $A \in \mathcal{A}$. It remains to show that $\succsim$ extends $\geq$, and for this, it is enough to show that $> \subseteq \succ'$. To this end, take any two elements x and y of X such that $x > y$. By definition of $\succsim'$, we have $x \succsim' y$. To derive a contradiction, suppose $y \succsim' x$ holds as well. Then there are x_0, $x_1, ..., x_k$ in X such that

$$y = x_0 \ (R(c) \cup \geq) \cdots \ (R(c) \cup \geq) \ x_k = x. \tag{13}$$

Put $I := \{i \in [k] : x_{i-1} \ R(c) \ x_i\}$. If $I = \emptyset$, then transitivity of $\geq$ and (13) yield $y \geq x$, a contradiction. If I is a singleton, say, $I = \{i\}$, then again by transitivity of $\geq$, we get $x_i \geq x > y \geq x_{i-1}$, while x_{i-1} $R(c)$ x_i. But then there is an $A \in \mathcal{A}$ such that $x_{i-1} \in c(A)$ and $x_i \in A$, while $x_A \notin \text{MAX}(A, \geq)$, and this contradicts (c). Finally, suppose $l := |I| \geq 2$, and enumerate I as $\{i_1, ..., i_l\}$, where $i_l > \cdots > i_1$. By definition of I, for each $j \in [l]$ there is an $A_j \in \mathcal{A}$ such that $x_{i_j - 1} \in c(A_j)$ and $x_{i_j} \in A_j$. On the other hand, again by definition of I, we have $x_{i_{j-1}} \geq x_{i_j}$ for each $j = 2, ..., l$, while

$$x_{i_l} \geq x_k = x > y \geq x_{i_1}. \tag{14}$$

Consequently, $x_{i_2} \in A_1^{\downarrow}$, ..., $x_{i_l} \in A_{l-1}^{\downarrow}$ and $x_{i_1} \in A_l^{\downarrow}$. It then follows from (c) that $x_{i_1} \in \text{MAX}(A_l^{\downarrow}, \geq)$, but as $x_{i_l} \in A_l$, this contradicts (14).

(d)⇒(e) This is obvious.

(e)⇒(a) Assume that (e) is valid. Where $\succsim$ is as given in the statement of (e), define $d : \mathcal{A} \to 2^X$ as $d(A) := \max(A, \succsim)$. As $\max(A, \succsim)$ contains $c(A)$, it is nonempty for any A in $\mathcal{A}$, so d is a choice correspondence on $\mathcal{A}$ such that $c \sqsupseteq d$, that is, $d \in c^{\uparrow}$. Take any x and y in X with x $\text{tran}(R(d) \cup \geq)$ y. Then, there is a positive integer k, elements $A_0, ..., A_k$ of $\mathcal{A}$, and $(x_0, ..., x_k) \in A_0 \times \cdots \times A_k$ such that $x = x_0$, $(x_{i-1}, x_i) \in d(A_i) \times A_i$ for each $i \in [k]$, and $y = x_k$. It follows from the definition of d that $x = x_0 \succsim \cdots \succsim x_k = y$, so, by transitivity of $\succsim$, we find $x \succsim y$. As $\succsim$ extends $\geq$, therefore, we cannot have $y > x$. Furthermore, if $y \in d(A)$ for some $A \in \mathcal{A}$ with $x \in A$, then $x \succsim y \succsim z$ for all $z \in A$, and hence, $x \in d(A)$. Thus: $\mathbb{C}$ satisfies the monotone congruence axiom.

Proof of Proposition 2. Fix an arbitrary rationalization $\trianglerighteq$ for c and put $R := R(c) \cup \geq$. Let us first prove that

$$\text{tran}(R) \subseteq \trianglerighteq \tag{15}$$

(but note that $\trianglerighteq$ need not be an extension of $\text{tran}(R)$). To this end, take any distinct $x, y \in X$ with x $\text{tran}(R)$ y. Then, there is a positive integer k and $x_0, ..., x_k \in X$ such that $x = x_0 \ R \ x_1 \ R \cdots R$ $x_k = y$. If x_{i-1} $R(c)$ x_i for any $i \in [k]$, then $(x_{i-1}, x_i) \in c(A) \times A$ for some $A \in \mathcal{A}$, and hence $x_{i-1} \trianglerighteq x_i$ because $c(A) \subseteq \max(A, \trianglerighteq)$. If, on the other hand, $x_{i-1} \geq x_i$ for any $i \in [k]$, then $x_{i-1} \trianglerighteq x_i$ because $\trianglerighteq$ is $\geq$-monotonic. Therefore, $x = x_0 \trianglerighteq x_1 \trianglerighteq \cdots \trianglerighteq x_k = y$, so, by transitivity of $\trianglerighteq$, we find $x \trianglerighteq y$, as we sought.

We now move to prove (5). Fix an arbitrary A in $\mathcal{A}$, and take any $x \in A$ with x tran(R) y for some $y \in c(A)$. Then, by (15), $x \trianglerighteq y$ while $y \in \max(A, \trianglerighteq)$ because $c(A) \subseteq \max(A, \trianglerighteq)$. It follows that $x \in \max(A, \trianglerighteq)$, establishing the second part of (5). Next, notice that $\succsim$ is obviously a rationalization for c, so the second part of (5) entails the $\supseteq$ part of the asserted equality in (5). To complete our proof, then, take any x in $\max(A, \succsim)$. Now pick any y in $c(A)$ and notice that, by (4), we must have $x \sim y$. On the other hand, as $(y, x) \in c(A) \times A$, we have y $R(c)$ x, and hence, y tran(R) x. As $\succsim$ extends tran(R) by hypothesis, and $x \sim y$, therefore, y tran(R) x cannot hold strictly, that is, we have x tran(R) y, which means $x \in c(A)^{\uparrow, \text{tran}(R)}$, as we sought.

Proof of the Rationalizability Theorem II. It is plain that (b) and (e) are equivalent, and (d) implies (e). From Rationalizability Theorem I we know that (a) implies (c) and that (e) implies (a). We will complete the proof of the theorem by showing that (c) implies (d). Let us denote the direct revealed preference induced by c as R, that is, we put $R := R(c)$. We first show that $\succsim' :=$ tran$(R \cup \geq)$ is a closed preorder on X. We couch the argument in a few easy steps.

[Step 1] If S and T are two compact binary relations on X, then $S \circ T$ is compact as well. As X is a metric space, we may work with sequential compactness instead of compactness. Let (x_m) and (y_m) be two sequences in X with x_m $S \circ T$ y_m for each m. Then, there is a sequence (z_m) in X such that x_m S z_m T y_m for each m. As S is compact, there is a strictly increasing sequence (m_k) of positive integers such that $(x_{m_k}, z_{m_k}) \to (x, z)$ for some $(x, z) \in S$. As z_{m_k} T y_{m_k} for each k, and T is compact, there is a subsequence (m_{k_l}) of (m_k) such that $(z_{m_{k_l}}, y_{m_{k_l}}) \to (z', y)$ for some $(z', y) \in T$. As $(z_{m_{k_l}})$ is a subsequence of (z_{m_k}), we must have $z' = z$, and it follows that x S z T y, that is, x $S \circ T$ y.

[Step 2] R^k is a compact subset of $X \times X$ for each $k = 1, 2, ...$ To prove this, observe first that

$$R = \bigcup \{c(A) \times A : A \in \mathcal{A}\}.$$

As X is compact and c is closed-valued, $c(A)$ is a compact subset of X for any $A \in \mathcal{A}$. Therefore, R is the union of finitely many compact sets in $X \times X$ (relative to the product topology), so it is compact. Applying what we have found in Step 1 inductively, therefore, yields our claim.

[Step 3] tran(R) is a compact subset of $X \times X$. The key observation here is:

$$\text{tran}(R) = R^1 \cup \cdots \cup R^{|\mathcal{A}|+1}. \tag{16}$$

To see this, take any integer $k > |\mathcal{A}|+1$, and any $x, y \in X$ with x R^k y. Then, there exist $x_0, ..., x_{k+1} \in X$ such that $x = x_0$ R x_1 R $\cdots$ R x_k R $x_{k+1} = y$. This means that there exist $A_0, ..., A_k \in \mathcal{A}$ such that $(x_{i-1}, x_i) \in c(A_{i-1}) \times A_{i-1}$ for each $i \in [k]$. As $k > |\mathcal{A}| + 1$, there must be an $i \in [k]$ such that $A_i = A_j$ for some $j \in \{i+1, ..., k\}$ here. Let i be the smallest such index. Then, $x = x_0$ R x_1 R $\cdots$ R x_i R x_{j+1} $R \cdots R$ $x_{k+1} = y$, that is, x $R^{k-(j-i)}$ y. This proves that $R^k \subseteq R^1 \cup \cdots \cup R^{|\mathcal{A}|+1}$ for every $k > |\mathcal{A}| + 1$, and hence follows (16). But then, in view of what we have found in Step 2, we see that tran(R) is the union of finitely many compact subsets of $X \times X$, and hence, it is itself compact in $X \times X$.

Now, for any x and y in X, we have x tran(tran$(R) \cup \geq$) y iff there exist $x_0, ..., x_k \in X$ such that

$$x = x_0 \text{ tran}(R) \cup \geq \cdots \text{ tran}(R) \cup \geq x_k = y.$$

As both tran(R) and $\geq$ are transitive, it is without loss of generality to take $k = 2l+1$ for some positive integer l here to write

$$x = x_0 \geq x_1 \text{ tran}(R) \; x_2 \geq \cdots \text{ tran}(R) \; x_{2k} \geq x_{2k+1} = y. \tag{17}$$

For any positive integer l, we next define the binary relation S^l on X by $x \; S^l \; y$ iff (17) holds. Consequently:

$$\text{tran}(\text{tran}(R) \cup \geq) = S^1 \cup S^2 \cup \cdots. \tag{18}$$

[Step 4] $\text{tran}(R) \cup \geq \, = S^1 \cup \cdots \cup S^{|\mathcal{A}|+1}$. Indeed, we can show that $S^l \subseteq S^l \cup \cdots \cup S^{|\mathcal{A}|+1}$ for every $l > |\mathcal{A}| + 1$, exactly as we have done this for R in Step 3. In view of (18), therefore, we have $\text{tran}(\text{tran}(R) \cup \geq) = S^1 \cup \cdots \cup S^{|\mathcal{A}|+1}$. Our claim thus follows from the obvious observation that tran(R) $\cup \geq \, = \text{tran}(\text{tran}(R) \cup \geq)$.

[Step 5] S^l is a closed subset of $X \times X$ for each $l = 1, 2, ...$ Take any two sequences (x^m) and (y^m) in X such that $x^m \to x$ and $y^m \to y$ for some $(x, y) \in X \times X$. Then, for each m, there exist $z_0^m, ..., z_{2l+1}^m \in X$ such that

$$x^m = z_0^m \geq z_1^m \text{ tran}(R) \; z_2^m \geq \cdots \text{ tran}(R) \; z_{2l}^m \geq z_{2l+1}^m = y. \tag{19}$$

As z_i^m tran(R) z_{i+1}^m for each odd $i \in [2l+1]$, and tran(R) is compact in $X \times X$ (Step 3), there exists a strictly increasing sequence (m_k) of positive integers such that $(z_i^{m_k})$ and $(z_{i+1}^{m_k})$ converge for each $i \in [2l+1]$. Since both tran(R) and $\geq$ are closed in $X \times X$, taking the subsequential limits in (19) yields

$$x \geq \lim z_1^{m_k} \text{ tran}(R) \; \lim z_2^{m_k} \geq \cdots \text{ tran}(R) \; \lim z_{2l}^{m_k} \geq \lim z_{2l+1}^{m_k} = y.$$

Thus $x \; S^l \; y$, as we sought.

We are now ready to complete the proof that (c) implies (d). Combining what is established in Steps 4 and 5, we see that $\succsim' := \text{tran}(R \cup \geq)$ is a continuous preorder on X. We may thus apply Levin's Theorem to find a continuous real map u on X such that u is strictly increasing with respect to $\succsim'$. From the proof of Rationalizability Theorem I, we know that $\succsim'$ is an extension of $\geq$. Therefore, u is also strictly increasing with respect to $\geq$. Lastly, since $R \subseteq \, \succsim'$, for any $A \in \mathcal{A}$ with $x \in c(A)$ we have $x \succsim' y$, and hence, $u(x) \geq u(y)$, for all $y \in A$. Our proof is complete.

Proof of Proposition 8. We only need to show that tran($R(c) \cup \geq$) is closed for the choice correspondence $c : B(\mathbf{p}, I) \mapsto d(\mathbf{p}, I)$. Note that, by generalized cyclical consistency, $\mathbf{x} \in d(\mathbf{p}, I)$ implies $\mathbf{px} = I$ for any $\mathbf{p} \in P$ and $I > 0$. Also, note that if $\mathbf{y}$ tran($R(c) \cup \geq$) $\mathbf{z}$, then there are $(\mathbf{p}^1, I^1), \ldots, (\mathbf{p}^k, I^k) \in P \times I$ and $\mathbf{x}^1 \in d(\mathbf{p}^1, I^1), \ldots, \mathbf{x}^k \in d(\mathbf{p}^k, I^k)$ such that

$$\mathbf{y} \geq \mathbf{x}^1, \mathbf{p}^1\mathbf{x}^1 \geq \mathbf{p}^1\mathbf{x}^2, ..., \mathbf{p}^{k-1}\mathbf{x}^{k-1} \geq \mathbf{p}^{k-1}\mathbf{x}^k \text{ and } \mathbf{p}^k\mathbf{x}^k \geq \mathbf{p}^k\mathbf{z}. \tag{20}$$

Crucially, we may choose $\mathbf{p}^1, \ldots, \mathbf{p}^k$ to be distinct. Indeed, generalized cyclical consistency requires that if $\mathbf{p}^r = \mathbf{p}^s$ for some $r < s$, then $I^r \geq I^s$; thus $B(\mathbf{p}^s, I^s) \subseteq B(\mathbf{p}^r, I^r)$ and it follows that $\mathbf{p}^r\mathbf{x}^r \geq \mathbf{p}^r\mathbf{x}^{s+1}$, i.e., we may 'snip off' the part of the sequence in (20) between $r + 1$ and s.

Suppose there are sequences $\mathbf{y}_m \to \bar{\mathbf{y}}$ and $\mathbf{z}_m \to \bar{\mathbf{z}}$ such that $\mathbf{y}_m$ tran($R(c) \cup \geq$) $\mathbf{z}_m$ for each m. We claim that $\bar{\mathbf{y}}$ tran($R(c) \cup \geq$) $\bar{\mathbf{z}}$. Indeed, each pair of $\mathbf{y}_m$ and $\mathbf{z}_m$ is linked by a sequence of inequalities like

(20), where the price vectors in that sequence are distinct. This property, together with the finiteness of P, guarantees that we can find subsequences $\mathbf{y}_{m_t}$ and $\mathbf{z}_{m_t}$ and distinct vectors $\mathbf{p}^1, \ldots, \mathbf{p}^k$ in P such that

$$\mathbf{y}_{m_t} \geq \mathbf{x}^1_{m_t}, \mathbf{p}^1\mathbf{x}^1_{m_t} \geq \mathbf{p}^1\mathbf{x}^2_{m_t}, ..., \mathbf{p}^{k-1}\mathbf{x}^{k-1}_{m_t} \geq \mathbf{p}^{k-1}\mathbf{x}^k_{m_t} \text{ and } \mathbf{p}^k\mathbf{x}^k_{m_t} \geq \mathbf{p}^k\mathbf{z}_{m_t}.$$

By the upper hemicontinuity of d, and after taking further subsequences if necessary, we obtain $I^l_{m_t} \to \bar{I}^l > 0$ and $\mathbf{x}^l_{m_t} \to \bar{\mathbf{x}}^l \in d(\mathbf{p}^l, \bar{I}^l)$. Thus,

$$\bar{\mathbf{y}} \geq \bar{\mathbf{x}}^1, \mathbf{p}^1\bar{\mathbf{x}}^1 \geq \mathbf{p}^1\bar{\mathbf{x}}^2, ..., \mathbf{p}^{k-1}\bar{\mathbf{x}}^{k-1} \geq \mathbf{p}^{k-1}\bar{\mathbf{x}}^k \text{ and } \mathbf{p}^k\bar{\mathbf{x}}^k \geq \mathbf{p}^k\bar{\mathbf{z}},$$

which means that $\bar{\mathbf{y}} \, \text{tran}(R(c) \cup \geq) \, \bar{\mathbf{z}}$, as we sought.

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