Exact Distribution of the Mean Reversion Estimator in the Ornstein-Uhlenbeck Process*

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Abstract: Econometricians have recently been interested in estimating and testing the mean reversion parameter (κ) in linear diffusion models. It has been documented that the maximum likelihood estimator (MLE) of κ tends to over estimate the true value. Its asymptotic distribution, on the other hand, depends on how the data are sampled (under expanding, infill, or mixed domain) as well as how we spell out the initial condition. This poses a tremendous challenge to practitioners in terms of estimation and inference. In this paper, we provide new and significant results regarding the exact distribution of the MLE of κ in the Ornstein-Uhlenbeck process under different scenarios: known or unknown drift term, fixed or random start-up value, and zero or positive κ. In particular, we employ numerical integration via analytical evaluation of a joint characteristic function. Our numerical calculations demonstrate the remarkably reliable performance of our exact approach. It is found that the true distribution of the MLE can be severely skewed in finite samples and that the asymptotic distributions in general may provide misleading results. Our exact approach indicates clearly the non-mean-reverting behavior of the real federal fund rate.

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1 Introduction

Since the seminal works of Merton (1971) and Black and Scholes (1973), continuous-time models have been used extensively in financial economics, see the excellent survey by Sundaresan (2000). Econometricians have also paid close attention to this line of literature. Maximum likelihood, generalized method of moments, simulated method of moments, and nonparametric approaches have been developed for model estimation, see, for instance, Singleton (2001), Bandi and Phillips (2003), Hong and Li (2005), and Phillips and Yu (2009a).

As shown in the literature, there exists serious bias in estimating the mean reversion parameter (κ) by almost all the methods, especially when the diffusion process has a linear drift function and the speed of mean reversion is slow (i.e., small values of κ). For example, Phillips and Yu (2005) found that the bias of the maximum likelihood estimator (MLE) for κ in the CIR model (Cox, Ingersoll, and Ross, 1985) can be substantial for data sets with very long time spans, regardless of data frequency. Recently, Tang and Chen (2009) showed that the bias of $\widehat{\kappa}$ is up to $O(T^{-1})$ in the stationary Vasicek model (Vasicek, 1977), where T is the time span. They also derived the approximate biases of the diffusion and drift estimators, and their simulations demonstrated that the estimation biases of diffusion and drift parameters are virtually zero, but $\widehat{\kappa}$ could be severely biased. Since the mean reversion parameter κ is of most importance for asset pricing, risk management, and forecasting, considerable attention in the literature has arisen to improve its estimation accuracy. Recent contributions include indirect inference (Phillips and Yu, 2009b), bootstrapping (Tang and Chen, 2009), and analytical bias approximation (Yu, 2012).

In addition to the classical asymptotic analysis under expanding domain ($T \to \infty$), asymptotic results under infill ($n \to \infty$, where n is the number of sample observations within a data span T) and mixed ($n \to \infty$ and $T \to \infty$) domains are also analyzed in the literature. In the context of Vasicek and CIR processes with unknown drift, Tang and Chen (2009) showed that asymptotic distributions of the MLE are quite different under expanding and mixed domains. Aït-Sahalia (2002) derived the asymptotic distribution of his approximate MLE under the expanding domain in diffusions models. A striking observation from his simulations is that under the stationary case, the asymptotic distribution of the estimated mean reversion parameter deviates more seriously from its corresponding finite-sample distribution as the true parameter value deceases from 10 to 1, i.e., as the process is getting closer to a unit root process, even with a very large sample size ($n = 1000$). Under the mixed domain, Brown and Hewitt (1975) obtained the limit normal distribution for the MLE of κ in the Vasicek model with a known drift term, see also Bandi and Phillips (2003, 2007), and Phillips and Yu (2009c) for asymptotic analysis under mixed domain. In a recent paper, Zhou and Yu (2010) derived the asymptotic distributions of the least squares (LS) estimator of κ in a general class of diffusion models under the three different domains. They provided Monte Carlo evidence that the infill asymptotic distribution is much more accurate in approximating the true finite-sample distribution than the asymptotic distributions under the other two domains.

The problems of estimation bias and inaccurate and different distribution approximations floating in the literature are largely due to the absence of exact analytical distribution results. Moreover, in reality, given the discretized data (with a given finite data span T and finite sample size n), we do not really know under which asymptotic domain our inference about $\hat{\kappa}$ shall be, but the asymptotic distribution results can behave quite differently under expanding, infill, and mixed domains. To address these problems, in this paper we investigate the exact distribution of the estimated mean reversion parameter in the Ornstein-Uhlenbeck process. To the best of our knowledge, our paper is the first to examine the exact finite-sample distribution of the estimated κ in continuous-time models. Since the MLE of κ is a simple transformation of the LS estimator of the autoregressive coefficient ϕ in a first-order autoregressive (AR(1)) model with discrete data, our study is intrinsically related to the vast literature studying the finite-sample distribution of the AR(1) coefficient estimator $\hat{\phi}$. The Imhof (1961) technique, in conjunction with Davies (1973, 1980), was typically used to develop the exact distribution of $\hat{\phi}$, see Ullah (2004) for a comprehensive review. Nevertheless, the Imhof (1961) technique is applicable only when the process is strictly stationary with an initial random observation included in formulating $\hat{\phi}$, or when the first observation is discarded. Computational burden of the Imhof (1961) technique also increases tremendously as the sample size of the AR process increases, since it involves computation of eigenvalues of a matrix whose dimension is the same as the sample size. In this paper, we take a different approach by first analytically evaluating the joint characteristic function of the random numerator and denominator in defining $\hat{\phi}$, and then inverting it via Gurland (1948) and Gil-Pelaez (1951) to calculate the exact finite-sample distribution. This approach is in line with Tsui and Ali (1992, 1994) and Ali (2002). However, note that in Tsui and Ali (1992, 1994) and Ali (2002), no intercept term was included in the AR(1) model. This is equivalent to a known drift term in the linear diffusion process. In this paper, we consider explicitly the case when the drift term is unknown. Moreover, Tsui and Ali (1992, 1994) did not include the initial observation in formulating the LS estimator $\hat{\phi}$, whereas Ali (2002) focused on the approximate distribution with the initial observation included. The initial observation does matter in studying the exact distributions in finite samples; in fact, it also matters even for the asymptotic distributions under several scenarios.

The remainder of our paper is as follows. In Section 2, we derive the exact distribution of the MLE of the mean reversion parameter κ. We also presents simulation results and compare our exact distribution results with the asymptotic results under the three different domains. In Section 3 we construct the exact confidence intervals for the mean reversion parameter κ when the linear diffusion model is used to study the real federal fund rate, and we find strong evidence of the non-mean-reverting behavior. Section 4 concludes. Technical details are collected in the appendix.

2 Exact Distribution

We consider the Ornstein-Uhlenbeck (OU) process with the initial value $x(0)$,

$$\mathrm{d}x(t) = \kappa(\mu - x(t))\mathrm{d}t + \sigma \mathrm{d}B(t), \tag{2.1}$$

where $\kappa \in \mathbb{R}$, $\mu \in \mathbb{R}$, $\sigma > 0$, and $B(t)$ is a standard Brownian motion. We are interested in estimating the parameter κ. When $\kappa \neq 0$, the solution to the above process is

$$x(t) = \mu + (x(0) - \mu)\exp(-\kappa t) + \sigma \int_0^t \exp(\kappa(s-t))\,\mathrm{d}B(s),\ t \geq 0. \tag{2.2}$$

Usually $\kappa > 0$ is assumed, and then as $t \to \infty$, the deterministic part of x tends to the mean level μ, so we have a mean-reverting process. When $\kappa = 0$, the process is no longer mean reverting:

$$x(t) = x(0) + \sigma B(t), \tag{2.3}$$

where the parameter μ vanishes.

In practice, the observed data are discretely recorded at $(0, h, 2h, \cdots, nh)$ in the time interval $[0, T]$, where h is the sampling interval and T is the data span. The exact discrete model corresponding to (2.1) is an AR(1) model:

$$x_{ih} = \alpha + \phi x_{(i-1)h} + \varepsilon_{ih},\ i = 0, 1, \cdots, n, \tag{2.4}$$

where $\phi = \exp(-\kappa h)$, $\alpha = \mu[1 - \exp(-\kappa h)]$, and $\varepsilon_{ih} \sim i.i.d.\mathrm{N}(0, \sigma_\varepsilon^2)$. The definition of the error term ε_{ih} depends on whether κ is positive or zero: $\varepsilon_{ih} = \sigma\epsilon_i\sqrt{(1 - \exp(-2\kappa h))/(2\kappa)}$ when $\kappa > 0$, $\varepsilon_{ih} = \sigma\sqrt{h}\epsilon_i$ when $\kappa = 0$, where $\epsilon_i \sim i.i.d.\mathrm{N}(0, 1)$. Correspondingly, $\sigma_\varepsilon^2 = \sigma^2(1 - \exp(-2\kappa h))/(2\kappa)$ when $\kappa > 0$ and $\sigma_\varepsilon^2 = \sigma^2 h$ when $\kappa = 0$. Note that the autoregressive coefficient ϕ is always positive by definition. When $\kappa = 0$, $\phi = 1$, $\alpha = 0$, so (2.4) becomes a random walk (with no drift). In the sequel, we suppress h in x_{ih} and ε_{ih} for notational convenience.

It is well known that the LS/ML estimator of κ based on the discrete data is

$$\hat{\kappa} = -\frac{\ln(\hat{\phi})}{h}, \tag{2.5}$$

where $\hat{\phi}$ is the LS estimator of ϕ in (2.4). When μ is known (without loss of generality, $\mu = 0$), $\hat{\phi} = \sum_{i=1}^{n} x_{i-1}x_i / \sum_{i=1}^{n} x_{i-1}^2$; for the case of unknown μ, $\hat{\phi} = \sum_{i=1}^{n}(x_{i-1} - \bar{x})x_i / \sum_{i=1}^{n}(x_{i-1} - \bar{x})^2$, where $\bar{x} = n^{-1}\sum_{i=1}^{n} x_{i-1}$.[1]

We are interested in studying the properties of $\hat{\kappa}$ estimated from the discrete sample via $\hat{\phi}$. As can be expected, the exact properties of $\hat{\kappa}$ depend on how we spell out the initial observation $x(0) = x_0$. It can be fixed (at zero or a non-zero constant) or random. In this paper, when x_0 is random, we assume that the time series $(x_0, x_1, \cdots, x_n)$ is stationary.

Note that (2.5) is defined only if $\hat{\phi} > 0$. However, $\hat{\phi}$ can be negative with a non-zero probability. Thus, we define exact distribution of $\hat{\kappa} - \kappa$ as $\Pr(\hat{\kappa} - \kappa \le w) \equiv \Pr(\hat{\kappa} - \kappa \le w | \hat{\phi} > 0)$. With this definition, we can show that

$$\Pr\left(\hat{\kappa} - \kappa \le w\right) = \frac{1 - F_{\hat{\phi}}(\phi \exp\left(-hw\right) - \phi)}{1 - F_{\hat{\phi}}(-\phi)}, \tag{2.6}$$

where $F_{\hat{\phi}}(\cdot)$ denotes the cumulative distribution function (CDF) of $\hat{\phi} - \phi$.[2]

As can be seen, with a given sampling frequency h, the distribution of $\hat{\kappa} - \kappa$ follows from the distribution of $\hat{\phi} - \phi$. When $\kappa > 0$ and x_0 is random, we can write $\hat{\phi} - \phi$ as a ratio of quadratic forms in the normal random vector $(x_0, x_1, \cdots, x_n)'$, and the technique of Imhof (1961) can be used to evaluate $F_{\hat{\phi}}$, and thus $F_{\hat{\kappa}}$. For a fixed x_0, it is not obvious how to directly apply Imhof (1961).[3] More fundamentally, Imhof's procedure requires computation of eigenvalues of an $(n+1) \times (n+1)$ matrix, which becomes very cumbersome as the sampling interval h decreases. Therefore, we proceed to derive the distribution function $F_{\hat{\phi}}$ by using the results from Gurland (1948) and Gil-Pelaez (1951) on a ratio of two random variables: Let Y_1 and Y_2 have the joint characteristic function (CF) $\varphi(u, v) = \mathrm{E}(\exp(\mathrm{i}uY_1 + \mathrm{i}vY_2))$. If $\Pr(Y_2 \le 0) = 0$, then the distribution of $\hat{\phi} - \phi = Y_1/Y_2$ is given by

$$F_{\hat{\phi}}(y) = \Pr(\hat{\phi} - \phi \le y) = \frac{1}{2} - \frac{1}{\pi}\int_0^{\infty} \mathrm{Im}\left(\frac{\varphi(u, -uy)}{u}\right) \mathrm{d}u. \tag{2.7}$$

Given the CDF (2.6), one might be tempted to evaluate the probability distribution function (PDF) of $\hat{\kappa} - \kappa$ (again conditional on $\hat{\phi} > 0$), $f_{\hat{\kappa}}(w) = h\phi \exp\left(-hw\right) f_{\hat{\phi}}(\phi \exp\left(-hw\right) - \phi)/[1 - F_{\hat{\phi}}(-\phi)]$, where $f_{\hat{\phi}}(\cdot)$ denotes the PDF of $\hat{\phi} - \phi$ and can be calculated from $F'_{\hat{\phi}}(y) = \pi^{-1}\int_0^{\infty} \mathrm{Im}\left(\partial\varphi(u, v)/\partial v|_{v=-uy}\right) \mathrm{d}u$. As pointed out by Hillier (2001), the density function of a ratio of normal quadratic forms (for instance, in the case when $\kappa > 0$ and x_0 is random), can be nonanayltic at some points. Thus in this paper we focus on the CDF only.

[1] Since we are interested in studying the finite sample properties of $\hat{\kappa}$, the initial condition x_0 matters and we include it in the estimation procedure. This stands in contrast to the convention of Hurwicz (1950). For the case of known μ, if $\mu \neq 0$, one can simply define $y_i = x_i - \mu$ and work with y_i.

[2] Note that (2.6) holds regardless of the distribution assumption.

[3] If we discard x_0 in formulating $\hat{\phi}$, then the Imhof (1961) technique is still applicable, as we can define $\hat{\phi}$ in terms of quadratic forms in the random vector $(x_1, x_2, \cdots, x_n)'$.

2.1 Joint Characteristic Function

For us to be able to use (2.7) to derive (2.6) for $\hat{\kappa}-\kappa$, an essential task is to evaluate the joint characteristic function of the numerator and denominator in defining $\hat{\phi}-\phi$. To facilitate presentation, we first introduce some notation. Let $\mathbf{0}_n$ be an $n\times 1$ vector of zeros, $\boldsymbol{I}_n$ be the identity matrix of size n, $\boldsymbol{\iota}_n$ be an $n\times 1$ vector of ones, $\boldsymbol{M}_n=\boldsymbol{I}_n-n^{-1}\boldsymbol{\iota}_n\boldsymbol{\iota}_n'$, $\boldsymbol{e}_{i,n}$ be a unit/elementary vector in the n-dimensional Euclidean space with its i-th element being 1. Given an $(n-1)\times(n-1)$ matrix $\boldsymbol{C}_{n-1}$, we define $\boldsymbol{A}_n^{\boldsymbol{C}}$ as an $n\times n$ matrix with its lower left block being $\boldsymbol{C}_{n-1}$ and all other elements being zero, and $\boldsymbol{B}_n^{\boldsymbol{C}}$ as an $n\times n$ matrix with its upper left block being $\boldsymbol{C}_{n-1}$ and all other elements being zero. When $\boldsymbol{C}_{n-1}=\boldsymbol{I}_{n-1}$, we simply put $\boldsymbol{A}_n=\boldsymbol{A}_n^{\boldsymbol{I}}$ and $\boldsymbol{B}_n=\boldsymbol{B}_n^{\boldsymbol{I}}$. For an $n\times n$ matrix $\boldsymbol{C}_n$, we use $c_{n,ij}$ to denote its ij-th element, and $c_n^{(ij)}$ to denote the ij-th element of $\boldsymbol{C}_n^{-1}$, whenever it exists. Throughout, $\boldsymbol{V}_n$ is an $n\times n$ matrix with its ij-th element $\phi^{|i-j|}$.

When $\kappa>0$, define $z_i=(x_i-\mu)/\sigma_\varepsilon$, $i=0,\cdots,n$, and $\bar{z}=n^{-1}\sum_{i=1}^n z_{i-1}$. Obviously, $\hat{\phi}=\sum_{i=1}^n z_{i-1}z_i/\sum_{i=1}^n z_{i-1}^2$ for the case of known μ, and $\hat{\phi}=\sum_{i=1}^n(z_{i-1}-\bar{z})z_i/\sum_{i=1}^n(z_{i-1}-\bar{z})^2$ for the case of unknown μ.

In the case of $\kappa=0$, the parameter μ vanishes, and we define $\tilde{z}_i=x_i/\sigma_\varepsilon$ and $\bar{\tilde{z}}=n^{-1}\sum_{i=1}^n\tilde{z}_{i-1}$. In practice, we may still proceed to estimate ϕ from the discrete AR(1) model without or with an intercept even when the parameter μ is not defined. Correspondingly, $\hat{\phi}=\sum_{i=1}^n\tilde{z}_{i-1}\tilde{z}_i/\sum_{i=1}^n\tilde{z}_{i-1}^2$ or $\hat{\phi}=\sum_{i=1}^n(\tilde{z}_{i-1}-\bar{\tilde{z}})\tilde{z}_i/\sum_{i=1}^n(\tilde{z}_{i-1}-\bar{\tilde{z}})^2$.

Denote $\boldsymbol{z}_n=(z_1,\cdots,z_n)'$ and $\boldsymbol{\zeta}_{n+1}=(z_0,\boldsymbol{z}_n)'$. Note that $(z_2,\cdots,z_n)'=(\mathbf{0}_{n-1},\boldsymbol{I}_{n-1})\,\boldsymbol{z}_n$, $(z_1,\cdots,z_{n-1})'=(\boldsymbol{I}_{n-1},\mathbf{0}_{n-1})\,\boldsymbol{z}_n$, and $(z_0-\bar{z},\cdots,z_{n-1}-\bar{z})'=\boldsymbol{M}_n\boldsymbol{A}_n\boldsymbol{z}_n+z_0\boldsymbol{M}_n\boldsymbol{e}_{1,n}$. (Results pertaining to $\tilde{z}$ can be obviously defined.) Thus we can express the following sums in matrix notation:

$\sum_{i=1}^n z_{i-1}z_i=z_0\boldsymbol{z}_n'\boldsymbol{e}_{1,n}+\boldsymbol{z}_n'\boldsymbol{A}_n\boldsymbol{z}_n=\boldsymbol{\zeta}_{n+1}'\boldsymbol{A}_{n+1}\boldsymbol{\zeta}_{n+1}$,

$\sum_{i=1}^n z_{i-1}^2=z_0^2+\boldsymbol{z}_n'\boldsymbol{B}_n\boldsymbol{z}_n=\boldsymbol{\zeta}_{n+1}'\boldsymbol{B}_{n+1}\boldsymbol{\zeta}_{n+1}$,

$\sum_{i=1}^n(z_{i-1}-\bar{z})z_i=\boldsymbol{z}_n'\boldsymbol{M}_n\boldsymbol{A}_n\boldsymbol{z}_n+z_0\boldsymbol{z}_n'\boldsymbol{M}_n\boldsymbol{e}_{1,n}=\boldsymbol{\zeta}_{n+1}'\boldsymbol{A}_{n+1}^{\boldsymbol{M}}\boldsymbol{\zeta}_{n+1}$,

$\sum_{i=1}^n(z_{i-1}-\bar{z})^2=\boldsymbol{z}_n'\boldsymbol{A}_n'\boldsymbol{M}_n\boldsymbol{A}_n\boldsymbol{z}_n+z_0^2\boldsymbol{e}_{1,n}'\boldsymbol{M}_n\boldsymbol{e}_{1,n}+2z_0\boldsymbol{z}_n'\boldsymbol{A}_n'\boldsymbol{M}_n\boldsymbol{e}_{1,n}=\boldsymbol{\zeta}_{n+1}'\boldsymbol{B}_{n+1}^{\boldsymbol{M}}\boldsymbol{\zeta}_{n+1}$.

Consequently, when $\kappa>0$, we can write the numerator and denominator (Y_1 and Y_2) in formulating $\hat{\phi}-\phi$ in terms of quadratic forms in either $\boldsymbol{z}_n$ (when x_0 is fixed) or $\boldsymbol{\zeta}_{n+1}$ (when x_0 is random). Define the following matrices/vectors (note that these matrices and vectors depend on u,v, and model parameters; we have suppressed their arguments):

$\boldsymbol{R}_n=\boldsymbol{I}_n+\left(\phi^2+2\mathrm{i}u\phi-2\mathrm{i}v\right)\boldsymbol{B}_n-(\phi+\mathrm{i}u)\left(\boldsymbol{A}_n+\boldsymbol{A}_n'\right)$,

$\boldsymbol{\delta}_n=\mathrm{i}u(\boldsymbol{I}_n-2\phi\boldsymbol{A}_n')\boldsymbol{M}_n\boldsymbol{e}_{1,n}+2\mathrm{i}v\boldsymbol{A}_n'\boldsymbol{M}_n\boldsymbol{e}_{1,n}+\phi\boldsymbol{e}_{1,n}$,

$\boldsymbol{S}_n=\boldsymbol{I}_n+\phi^2\boldsymbol{B}_n-\phi(\boldsymbol{A}_n+\boldsymbol{A}_n')-\mathrm{i}u(\boldsymbol{M}_n\boldsymbol{A}_n+\boldsymbol{A}_n'\boldsymbol{M}_n)+2\mathrm{i}(u\phi-v)\boldsymbol{A}_n'\boldsymbol{M}_n\boldsymbol{A}_n$,

$\boldsymbol{T}_{n+1}=(1-\phi^2)\boldsymbol{V}_{n+1}^{-1}-\mathrm{i}u(\boldsymbol{A}_{n+1}^{\boldsymbol{M}}+\boldsymbol{A}_{n+1}^{\boldsymbol{M}\prime})+2\mathrm{i}(u\phi-v)\boldsymbol{B}_{n+1}^{\boldsymbol{M}}$.

Then we can present the characteristic function $\varphi(u,v)$ under the different scenarios as follows (see the appendix for detailed derivation; when $\kappa = 0$, $\boldsymbol{R}_n$, $\boldsymbol{\delta}_n$, and $\boldsymbol{S}_n$ are defined with $\phi = 1$ in their expressions).

Characteristic Function of $\hat{\phi} - \phi$

κ	μ (intercept in AR(1) or not)	x_0	$\varphi(u,v)$
> 0	Known (no intercept)	Fixed	$\exp\left[\frac{z_0^2}{2}\left(1 - \frac{\lvert\boldsymbol{R}_{n+1}\rvert}{\lvert\boldsymbol{R}_n\rvert}\right)\right]\lvert\boldsymbol{R}_n\rvert^{-1/2}$
		Random	$\sqrt{1-\phi^2}\left(\lvert\boldsymbol{R}_{n+1}\rvert - \phi^2\lvert\boldsymbol{R}_n\rvert\right)^{-1/2}$
	Unknown (with intercept)	Fixed	$\exp\left[-\frac{\phi^2 z_0^2}{2} - \mathrm{i}(u\phi - v)z_0^2\left(1 - \frac{1}{n}\right) + \frac{1}{2}z_0^2\boldsymbol{\delta}_n'\boldsymbol{S}_n^{-1}\boldsymbol{\delta}_n\right]\lvert\boldsymbol{S}_n\rvert^{-1/2}$
		Random	$\sqrt{1-\phi^2}\,\lvert\boldsymbol{T}_{n+1}\rvert^{-1/2}$
$= 0$	No intercept	Fixed	$\exp\left[\frac{\tilde{z}_0^2}{2}\left(1 - \frac{\lvert\boldsymbol{R}_{n+1}\rvert}{\lvert\boldsymbol{R}_n\rvert}\right)\right]\lvert\boldsymbol{R}_n\rvert^{-1/2}$
	With intercept		$\exp\left[-\frac{\phi^2 \tilde{z}_0^2}{2} - \mathrm{i}(u - v)\tilde{z}_0^2\left(1 - \frac{1}{n}\right) + \frac{1}{2}\tilde{z}_0^2\boldsymbol{\delta}_n'\boldsymbol{S}_n^{-1}\boldsymbol{\delta}_n\right]\lvert\boldsymbol{S}_n\rvert^{-1/2}$

With $\varphi(u,v)$ in hand, we can directly use (2.7) to derive the distribution function of $\hat{\phi}$, which in turn can be plugged into (2.6) to obtain the distribution function of $\hat{\kappa}$. Clearly, in this procedure we need to find determinants ($\lvert\boldsymbol{R}_n\rvert$, $\lvert\boldsymbol{R}_{n+1}\rvert$, $\lvert\boldsymbol{S}_n\rvert$, and $\lvert\boldsymbol{T}_{n+1}\rvert$) and inverses ($\boldsymbol{S}_n^{-1}$ and $\boldsymbol{T}_{n+1}^{-1}$). When n is small, this can be done easily numerically. But for a moderately large n, for example, $T = 10$, $h = 1/252$, $n = 2520$ (daily data over 10 years), this can be quite time consuming. Instead of numerically evaluating these determinants and inverses, we proceed to derive direct analytical expressions, presented in Appendix B, by utilizing the special structures of these matrices.

The essential merit of our approach is computational efficiency based on analytical, instead of numerical, determinants and inverses. We make no attempt to control computational accuracy when applying numerical integration in (2.7); instead we rely on Matlab's adaptive Gauss-Kronrod quadrature integral procedure `quadgk` by setting numerical absolute error tolerance ($1e-12$). The eigenvalue-based approach via Imhof (1961), when applicable, might be able to manually control the integration error and truncation error, see Davies (1973, 1980), Ansley et al. (1992), and Lu and King (2002). Our numerical exercise in the next subsection demonstrates that our numeral integration approach in fact produces very accurate results. Also, as emphasized before, the Imhof (1961) approach is not always applicable.

2.2 Numerical Results

In this section, we conduct Monte Carlo simulations to illustrate the finite sample performance of our exact distribution in comparison with the "true" distribution and the asymptotic distribution. The data generating process follows the OU model in (2.4), and the error term is generated from a normal distribution. The asymptotic distribution results are available from Zhou and Yu (2010).[4] Note that under the infill asymptotics, the results are conditional on the initial x_0.

We set $T = 1, 2, 5, 10,\ h = 1/12, 1/52, 1/252,\ \kappa = 0.01, 0.1, 1,\ \mu = 0,\ 0.1,\ \sigma = 0.1,\ x_0 = \mu$ or $x_0 \sim \mathrm{N}(\mu, \sigma^2/(2\kappa))$. Compared with Zhou and Yu (2010), we have a more comprehensive experiment design, so as to have a better understanding of the finite-sample distributions. For the fixed start-up case ($x_0 = 0$), we also consider $\kappa = 0$. As pointed out in Zhou and Yu (2010), the values of 0.01 and 0.1 for κ are empirically realistic for interest rate data while the value of 1 is empirically realistic for volatility.

Tables 1–4 report the cumulative distributions of $T(\hat{\kappa} - \kappa)$, where the "true" distribution results come from 1,000,000 replications, and we make comparison of the exact (p), true (p_{edf}), and asymptotic results under the three asymptotics (p_{exp}, p_{mix}, p_{inf}).[5] In calculating the exact distribution with the analytical characteristic function, we still need to implement numerical integration in (2.7). Appendix C discusses how to overcome the problem of discontinuity of the square root function in the complex domain. The complete experiment results are available from the corresponding author upon request. To save space, we report only the results for $h = 1/12, 1/252$ in Tables 1–4 (each with two panels corresponding to $T = 1, 10$, respectively). Tables 1 and 2 report the cumulative distributions of $T(\hat{\kappa} - \kappa)$ under a fixed start-up when $\kappa = 0.01$, with $x_0 = \mu = 0, 0.1$, respectively, and Tables 3 and 4 report the results when x_0 is random.

Several striking features are present in these tables. First, the exact distribution results match to at least the third decimal place with those obtained by 1 million simulations, in all the cases considered. This indicates high accuracy of the exact results calculated by our numerical integration algorithm. In consistent with the asymptotic results in Zhou and Yu (2010), there is no much difference between the results under the expanding and mixed domains, and the infill asymptotics provide relatively better performance. Yet, the asymptotic distribution under the infill domain may still provide poor approximation to the true distribution when the data span is short, especially so in the left tails. While increasing data frequency does not affect much the

[4] Zhou and Yu (2010) did not give the expanding and infill asymptotic distribution results when $\kappa = 0$ and $\mu \neq 0$. This corresponds to the scenario, in a discrete framework, when no intercept is present in the true model, but a constant term is included in the regression. The expanding and infill asymptotic distribution results easily follow via the generalized delta method.

[5] In simulating the asymptotic (non-normal) results, 10,000 replications are used and a sample size of 5,000 is used to approximate the integrals involving the Brownian motion by the discrete Riemann sums, with the exception that the infill asymptotic results when x_0 is random are calculated as averaging over 2,000 replications, where in each replication, $x_0 \sim \mathrm{N}(\mu, \sigma^2/(2\kappa))$ and the discrete AR(1) process is of sample size of 2,000.

asymptotic distributions, it does affect the true distribution, and the remarkable performance of the exact distribution is robust to data frequency, as well as to data span and other aspects of model specification.

Second, the true distribution of $\hat{\kappa}$ is highly skewed to the right. Normality is a terrible approximation of the finite-sample distribution of $\hat{\kappa}$. Moreover, we can infer from these tables the exact/true median of $T(\hat{\kappa}-\kappa)$ in all cases are substantially positive. (A direct calculation of the median is also possible, see the last paragraph in this section to follow.) This suggests that $\hat{\kappa}$ can significantly over estimate κ in finite samples. This degree of overestimation does not decrease with a higher data frequency (given a fixed data span). This is in line with the observations made by Phillips and Yu (2005) and Tang and Chen (2009). On the other hand, increasing data span might help alleviate this problem, though somewhat marginally.

Third, how the initial observation is spelled out affects significantly the exact distribution of $\hat{\kappa}$. For example, for the fixed start-up case, the exact distribution is less skewed to the right when $x_0 = 0$ compared with when $x_0 \neq 0$. Comparing the random start-up case versus the fixed start-up case with a known drift term (Table 3 versus Table 1), we see quite a difference in the exact distributions across the two cases and the former one is less skewed; on the other hand, with a unknown drift term (Table 4 versus Table 2), there is virtually no significant difference in the exact distributions across the two cases.[6] This feature is related to the role of initial observation in the unit-root test literature, see, for example, Müller and Elliott (2003) and Elliot and Müller (2006). It also suggests that the conclusions in Tsui and Ali (1992, 1994) with x_0 discarded should be examined with more scrutiny.

Given the CDF function (2.7), one might be tempted to calculate the quantile function $F_{\hat{\kappa}}^{-1}(t)$, $t \in [0,1]$ by Newton's method of interpolation. However, this involves calculation of the PDF function, which requires another round of numerical integration, in addition to the possible problem pointed out by Hillier (2001). Instead, we suggest employing a very simple bisection search algorithm. Since it is relatively cheap to simulate the asymptotic results and we have observed that the infill asymptotic results are more reliable compared with the expanding and mixed asymptotic results, we start with the t-th empirical quantile of the simulated sample for approximating the in-fill asymptotic results, say c_0. If $F_{\hat{\kappa}}(c_0) < t$, we set c_1 as the $\min\{2t, 1\}$-th empirical quantile of the simulated sample. (Typically, $F_{\hat{\kappa}}(c_1) > t$. If not, one can set c_1 as the $\min\{ct, 1\}$-th empirical quantile of the simulated sample, $c = 3, 4, \cdots$, until one finds $F_{\hat{\kappa}}(c_1) > t$.) If $F_{\hat{\kappa}}(c_0) > t$, we set c_1 as the $t/2$-th empirical quantile of the simulated sample. (Typically, $F_{\hat{\kappa}}(c_1) < t$. If not, one can set c_1 as the ct-th empirical quantile of the simulated sample, $c = 1/3, 1/4, \cdots$, until one finds $F_{\hat{\kappa}}(c_1) < t$.) Given the two initial points c_0 and c_1, a bisection search can then be straightforwardly applied to search numerically for $F_{\hat{\kappa}}^{-1}(t)$. This

[6] The effects of the initial observation x_0 can be examined more carefully by looking at the characteristic functions presented in Section 2.1. For fixed x_0, the characteristic functions behave differently under zero and non-zero x_0. When the drift term is unknown, we see that in the characteristic function, the exponent has terms involving α that dominate the initial value x_0.

algorithm is in a similar spirit of the algorithm in Lu and King (2002).

3 An Empirical Example

The linear diffusion model has been used to study the short-term interest rate in the literature. Even though the term structure literature has documented that the short-term interest rate is highly persistent, an agreement has yet to reach among economists regarding whether there is a unit root in the time series.

Figure 1: Real Federal Fund Rate

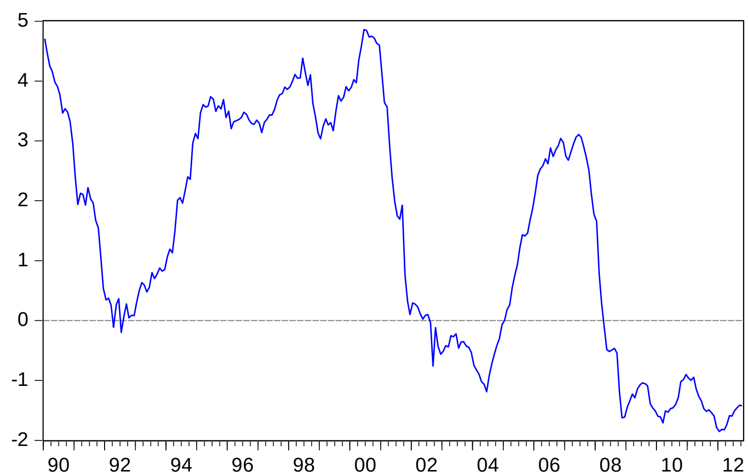

Figure 1 displays the monthly real federal fund rate from January 1990 to October 2012.[7] If we use the augmented Dickey-Fuller test or Phillips-Perron test, we fail to reject the null of unit root at any conventional level. If we use the KPSS test, on the other hand, we fail to reject the null of stationarity at any conventional level. The mixed results are in line with the observation from Bierens (2000).[8]

If we are willing to use the linear diffusion model for the real federal fund rate, then based on our exact distribution approach, we can construct straightforwardly the exact confidence intervals of the mean reversion parameter κ.[9] For comparison, we also report the confidence intervals under the infill sampling scheme.

[7] The real federal fund rate is calculated as the effective H-15 federal fund rate adjusted for the core PCE inflation rate. The former is retrieved from www.federalreserve.gov and the latter is retrieved from www.bea.gov.

[8] One possibility, as discussed in Bierens (2000), is that the series is nonlinear trend stationary.

[9] The distribution of $\hat{\kappa}$ depends on the diffusion and drift parameters. We set them equal to the estimated values from the sample. (Recall from Tang and Chen (2009) that estimation biases of the diffusions and drift parameters are virtually zero.) Also, we regard the first sample observation as fixed and treat the drift term as unknown.

Confidence Intervals of κ for Real Federal Fund Rate

	99%	95%	90%
Exact	$[-0.1054, 1.1005]$	$[-0.0531, 0.7940]$	$[-0.0213, 0.6607]$
Infill	$[-0.1032, 1.0799]$	$[-0.0492, 0.7671]$	$[-0.0210, 0.6327]$

It can be seen that the null value of $\kappa = 0$ is contained in the confidence intervals. The exact confidence intervals are wider compared with those from the infill asymptotic distribution. Based on the infill confidence intervals alone we might not be confident to conclude that $\kappa = 0$, since we have already seen in the previous section that the asymptotic distribution under the infill domain in general is good but may still provide poor approximation to the true distribution in the left tail. (Note that the value of $k = 0$ lies on the left tail.) With the exact confidence intervals, we are more confident to believe the non-mean-reverting hypothesis regarding the federal fund rate.

4 Conclusions

We have investigated the exact finite-sample distribution of the estimated mean-reversion parameter in the Ornstein-Uhlenbeck diffusion process. We have considered several different set-ups: known or unknown drift term, fixed or random start-up value, and zero or positive mean-reversion parameter. In particular, we employ numerical integration via analytical evaluation of a joint characteristic function. Our numerical calculations demonstrate the remarkably reliable performance of the exact approach. It is found that the true distribution of the maximum likelihood estimator of the mean-reversion parameter can be severely skewed in finite samples. The asymptotic results under expanding and mixed domains in general perform worse than those under the infill domain, though the latter may still perform poorly in the left tails. Our exact approach provides distribution results of high accuracy, and thus it could be useful for conducting hypothesis testing and constructing confidence intervals.

We note that the linear diffusion model, although extensively studied in the literature, is simple and restrictive in modeling financial time series. Nonlinear diffusion, diffusion-jump, and self-exiting jump models have been proposed in the recent literature to accommodate hectic scenarios with large drops and recurring crises. The estimation strategy and inference procedure in these models are more complicated. Typically, no closed-form solution might be available for the estimator of interest. Therefore, studying the exact distribution theory for such estimators for both Gaussian and non-Gaussian cases are extremely difficult and challenging, and they are beyond the scope of this paper. Our efforts in this paper can be regarded as a starting point in studying the exact finite-sample distribution theory in continuous-time models. By no means a simple extension of our methodology can deliver finite-sample results in the more general nonlinear and non-Gaussian models. However, this paper provides a message that the crude asymptotics (expanding, mixed, or infill) may give misleading inferences in finite or even moderately large samples. Thus this calls for more caution in interpreting the existing results, and more need for developing the exact finite-sample results for the more general models. When the exact finite-sample results are not available, Edgeworth approximations, see Zhang, Mykland and Aït-Sahalia (2011), might provide better approximations than the asymptotics, and they might be regarded as standing in between the exact and asymptotic results.

Appendix A: Derivation of $\varphi(u,v)$

When μ is known (0) and $\kappa > 0$ (with x_0 fixed or random), or when there is no intercept in the discrete AR(1) model and $\phi = 1$ ($\kappa = 0$), $\varphi(u,v)$ can follow from Tsui and Ali (1994) with slight modification. (Recall that we have the initial observation x_0 included in formulating the LS estimator $\hat{\phi}$, whereas in Tsui and Ali (1994) the initial observation was discarded.)

When μ is unknown, $\kappa > 0$, and x_0 is fixed,

$$\hat{\phi} = \frac{\boldsymbol{z}_n'\boldsymbol{M}_n\boldsymbol{A}_n\boldsymbol{z}_n + z_0\boldsymbol{z}_n'\boldsymbol{M}_n\boldsymbol{e}_{1,n}}{\boldsymbol{z}_n'\boldsymbol{A}_n'\boldsymbol{M}_n\boldsymbol{A}_n\boldsymbol{z}_n + 2z_0\boldsymbol{z}_n'\boldsymbol{A}_n'\boldsymbol{M}_n\boldsymbol{e}_{1,n} + z_0^2\boldsymbol{e}_{1,n}'\boldsymbol{M}_n\boldsymbol{e}_{1,n}}.$$

The density function of $\boldsymbol{z}_n$ (conditional on z_0) is

$$\begin{aligned} f(\boldsymbol{z}_n) &= (2\pi)^{-\frac{n}{2}}\exp\left[-\frac{\sum_{i=1}^n (z_i - \phi z_{i-1})^2}{2}\right] \\ &= (2\pi)^{-\frac{n}{2}}\exp\left\{-\frac{1}{2}[\boldsymbol{z}_n'\left(\boldsymbol{I}_n + \phi^2\boldsymbol{B}_n - 2\phi\boldsymbol{A}_n\right)\boldsymbol{z}_n - 2\phi z_0\boldsymbol{z}_n'\boldsymbol{e}_{1,n} + \phi^2 z_0^2]\right\}, \end{aligned}$$

and the joint CF of the numerator and denominator in defining $\hat{\phi} - \phi$ is

$$\begin{aligned} \varphi(u,v) &= (2\pi)^{-\frac{n}{2}}\exp\left(-\frac{\phi^2 z_0^2}{2} - \mathrm{i}(u\phi - v)z_0^2\boldsymbol{e}_{1,n}'\boldsymbol{M}_n\boldsymbol{e}_{1,n}\right) \\ &\quad\cdot\int_{-\infty}^{+\infty}\exp\left(-\frac{1}{2}\boldsymbol{z}_n'\boldsymbol{S}_n\boldsymbol{z}_n + z_0\boldsymbol{z}_n'\boldsymbol{\delta}_n\right)\mathrm{d}\boldsymbol{z}_n. \end{aligned}$$

Note that $\boldsymbol{e}_{1,n}'\boldsymbol{M}_n\boldsymbol{e}_{1,n} = 1 - n^{-1}$. Thus we have

$$\varphi(u,v) = |\boldsymbol{S}_n|^{-1/2}\exp\left[-\frac{\phi^2 z_0^2}{2} - \mathrm{i}(u\phi - v)z_0^2\left(1 - \frac{1}{n}\right) + \frac{1}{2}z_0^2\boldsymbol{\delta}_n'\boldsymbol{S}_n^{-1}\boldsymbol{\delta}_n\right]. \tag{A.1}$$

If $\kappa = 0$ and there is an intercept in the AR(1) model, with x_0 fixed, one can inspect that setting $\phi = 1$ in (A.1) and replacing z_0 with $\tilde{z}_0$ yield the corresponding CF.

When μ is unknown and x_0 is random (and $\kappa > 0$),

$$\hat{\phi} = \frac{\boldsymbol{\zeta}_{n+1}' \boldsymbol{A}_{n+1}^{\boldsymbol{M}} \boldsymbol{\zeta}_{n+1}}{\boldsymbol{\zeta}_{n+1}' \boldsymbol{B}_{n+1}^{\boldsymbol{M}} \boldsymbol{\zeta}_{n+1}},$$

which is invariant to μ and σ_ε^2.[10] Without loss of generality, assume $\mu = 0$ and $\sigma^2 = 2\kappa/(1-\phi^2)$, so that $\boldsymbol{\zeta}_{n+1} \sim \mathrm{N}(\mathbf{0}_{n+1}, \boldsymbol{V}_{n+1}/(1-\phi^2))$.[11] Then the density of $\boldsymbol{\zeta}_{n+1}$ is

$$f\left(\boldsymbol{\zeta}_{n+1}\right) = (2\pi)^{-\frac{n+1}{2}} \left| \frac{\boldsymbol{V}_{n+1}}{1-\phi^2} \right|^{-1/2} \exp\left[-\frac{1}{2} \boldsymbol{\zeta}_{n+1}' \left(\frac{\boldsymbol{V}_{n+1}}{1-\phi^2} \right)^{-1} \boldsymbol{\zeta}_{n+1} \right],$$

and the joint CF of the numerator and denominator in defining $\hat{\phi} - \phi$ is

$$\varphi\left(u, v\right) = (2\pi)^{-\frac{n+1}{2}} \left| \frac{\boldsymbol{V}_{n+1}}{1-\phi^2} \right|^{-1/2} \cdot \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2} \boldsymbol{\zeta}_{n+1}' \boldsymbol{T}_{n+1} \boldsymbol{\zeta}_{n+1} \right) \mathrm{d}\boldsymbol{\chi}_{n+1}.$$

Therefore,

$$\varphi\left(u, v\right) = \sqrt{1-\phi^2} \left| \boldsymbol{T}_{n+1} \right|^{-1/2}. \tag{A.2}$$

Appendix B: Analytical Determinants and Inverses

This appendix first gives the analytical determinants and inverses of various matrices involved in evaluating $\varphi(u, v)$ and then verifies that these analytical expressions are valid.

First we introduce the determinant and inverse of a tridiagonal matrix. Let $\boldsymbol{C}_n$ be an $n \times n$ tridiagonal matrix with c_0 on its main diagonal and $c_1 \neq 0$ on its super- and sub-diagonals. (When $c_1 = 0$, the results are trivial.)

From Bernstein (2009, page 235), the determinant of $|\boldsymbol{C}_n|$ is as follows:

$$|\boldsymbol{C}_n| = \begin{cases} \frac{(n+1)c_0^n}{2^n} & \frac{c_0}{2c_1} = \pm 1 \\ \frac{c_1^n \sin((n+1)\theta)}{\sin\theta} & \frac{c_0}{2c_1} \neq \pm 1 \end{cases}, \tag{B.1}$$

where $\theta = \arccos(c_0/(2c_1))$.[12] From Hu and O'Connell (1996), the symmetric $\boldsymbol{C}_n^{-1}$ has elements $c_n^{(ij)} =$

[10] $\hat{\phi}$ is independent of μ because $\boldsymbol{A}_{n+1}^{\boldsymbol{M}} \mathbf{1}_{n+1} = \mathbf{0}_{n+1}$, $\boldsymbol{A}_{n+1}^{\boldsymbol{M}\prime} \mathbf{1}_{n+1} = \mathbf{0}_{n+1}$ and $\boldsymbol{B}_{n+1}^{\boldsymbol{M}} \mathbf{1}_{n+1} = \mathbf{0}_{n+1}$.

[11] This is different from the case when no intercept is included in the AR(1) model.

[12] When $c_0 \neq \pm 2c_1$, Bernstein (2009, page 235) gives $|\boldsymbol{C}_n| = (b_1^{n+1} - b_2^{n+1})/(b_1 - b_2)$, where $b_1 = (c_0 + \sqrt{c_0^2 - 4c_1^2})/2$ and

$(-1)^{i+j}\left|\boldsymbol{C}_{i-1}\right|\left|\boldsymbol{C}_{n-j}\right|/(c_1^{i-j}\left|\boldsymbol{C}_n\right|)$ for $i \leq j$. Substituting the determinant result (B.1) yields

$$c_n^{(ij)} = \begin{cases} \frac{2i(n-j+1)}{(n+1)c_0}\left(-\frac{c_0}{2c_1}\right)^{i-j} & \frac{c_0}{2c_1} = \pm 1 \\ \frac{(-1)^{i+j}\sin(i\theta)\sin((n-j+1)\theta)}{c_1\sin\theta\sin((n+1)\theta)} & \frac{c_0}{2c_1} \neq \pm 1 \end{cases}, \quad i \leq j, \tag{B.2}$$

With the explicit expression of $c_n^{(ij)}$, we can derive the follow results:

$$\boldsymbol{\iota}_n'\boldsymbol{C}_n^{-1}\boldsymbol{\iota}_n = \begin{cases} \frac{n(n+1)(n+2)}{6c_0} & \frac{c_0}{2c_1} = -1 \\ \frac{2n^2+4n+1-(-1)^n}{4c_0(n+1)} & \frac{c_0}{2c_1} = 1 \\ \frac{n+1}{4c_1\cos^2(\theta/2)} - \frac{\tan^2(\theta/2)[(-1)^n+\cos((n+1)\theta)]}{2c_1\sin\theta\sin((n+1)\theta)} & \frac{c_0}{2c_1} \neq \pm 1 \end{cases}, \tag{B.3}$$

$$\boldsymbol{e}_{1,n}'\boldsymbol{C}_n^{-1}\boldsymbol{\iota}_n = \begin{cases} \frac{n}{c_0} & \frac{c_0}{2c_1} = -1 \\ \frac{2n+1-(-1)^n}{2c_0(n+1)} & \frac{c_0}{2c_1} = 1 \\ \frac{\sin((2n+1)\theta/2)-(-1)^n\sin(\theta/2)}{2c_1\sin((n+1)\theta)\cos(\theta/2)} & \frac{c_0}{2c_1} \neq \pm 1 \end{cases}, \tag{B.4}$$

$$\boldsymbol{e}_{1,n}'\boldsymbol{C}_n^{-1}\boldsymbol{e}_{1,n} = \begin{cases} \frac{2n}{c_0(n+1)} & \frac{c_0}{2c_1} = \pm 1 \\ \frac{\sin(n\theta)}{c_1\sin((n+1)\theta)} & \frac{c_0}{2c_1} \neq \pm 1 \end{cases}, \tag{B.5}$$

$$\boldsymbol{e}_{1,n}'\boldsymbol{C}_n^{-1}\boldsymbol{e}_{n,n} = \begin{cases} \frac{2(-c_0/(2c_1))^{n+1}}{c_0(n+1)} & \frac{c_0}{2c_1} = \pm 1 \\ \frac{(-1)^{n+1}\sin(\theta)}{c_1\sin((n+1)\theta)} & \frac{c_0}{2c_1} \neq \pm 1 \end{cases}. \tag{B.6}$$

In the sequel, let $c_0 = 1+\phi^2+2\mathrm{i}(u\phi - v)$, $c_1 = -(\phi+\mathrm{i}u)$, $c_2 = 2\mathrm{i}(u+v-u\phi)$, and $c_3 = \mathrm{i}(2\phi u - 2v - u)$. Since $\phi > 0$, the case of $c_0 = 2c_1$, corresponding to $\phi = -1$ and $v = 0$, is ruled out.

Determinant of $\boldsymbol{R}_n$

Note that $\boldsymbol{R}_n$ is a tridiagonal matrix with its main diagonal elements $r_{n,ii} = c_0$, $i = 1, \cdots\ n-1$, $r_{n,ii} = 1$, $i = n$, and sub- and super-diagonal elements c_1. Expanding along the last row of $\boldsymbol{R}_n$ leads to

$$\left|\boldsymbol{R}_n\right| = \left|\boldsymbol{C}_{n-1}\right| - c_1^2\left|\boldsymbol{C}_{n-2}\right|, \tag{B.7}$$

where the determinants of $\boldsymbol{C}_{n-1}$ and $\boldsymbol{C}_{n-2}$ follow from (B.1).

$b_2 = (c_0 - \sqrt{c_0^2 - 4c_1^2})/2$. This can be numerically unstable when b_1 is close to b_2. Using polar representation, we write $b_1 = c_1(\cos\theta + \mathrm{i}\sin\theta)$, $b_2 = c_1(\cos\theta - \mathrm{i}\sin\theta)$. Then it follows that when $c_0 \neq \pm 2c_1$, $\left|\boldsymbol{C}_n\right| = [c_1^n\sin((n+1)\theta)]/\sin\theta$. We thank Raymond Kan for pointing out this result and the results (B.3)–(B.6) to follow.

Determinant of $\boldsymbol{S}_n$

Note that

$$\boldsymbol{A}_n'\boldsymbol{\iota}_n\boldsymbol{\iota}_n'\boldsymbol{A}_n = \begin{pmatrix} \boldsymbol{\iota}_{n-1}\boldsymbol{\iota}_{n-1}' & \boldsymbol{0}_{n-1} \\ \boldsymbol{0}_{n-1}' & 0 \end{pmatrix}, \quad \boldsymbol{\iota}_n\boldsymbol{\iota}_n'\boldsymbol{A}_n = \begin{pmatrix} \boldsymbol{\iota}_{n-1}\boldsymbol{\iota}_{n-1}' & \boldsymbol{0}_{n-1} \\ \boldsymbol{\iota}_{n-1}' & 0 \end{pmatrix}.$$

So

$$\begin{aligned}
\boldsymbol{S}_n &= \boldsymbol{I}_n + \phi^2\boldsymbol{B}_n - \phi\left(\boldsymbol{A}_n + \boldsymbol{A}_n'\right) - \mathrm{i}u(\boldsymbol{M}_n\boldsymbol{A}_n + \boldsymbol{A}_n'\boldsymbol{M}_n) + 2\mathrm{i}(u\phi - v)\boldsymbol{A}_n'\boldsymbol{M}_n\boldsymbol{A}_n \\
&= \boldsymbol{I}_n + \phi^2\boldsymbol{B}_n - \phi\left(\boldsymbol{A}_n + \boldsymbol{A}_n'\right) - \mathrm{i}u(\boldsymbol{A}_n + \boldsymbol{A}_n') + 2\mathrm{i}(u\phi - v)\boldsymbol{A}_n'\boldsymbol{A}_n \\
&\quad + n^{-1}\mathrm{i}u\boldsymbol{\iota}_n\boldsymbol{\iota}_n'\boldsymbol{A}_n + n^{-1}\mathrm{i}u\boldsymbol{A}_n'\boldsymbol{\iota}_n\boldsymbol{\iota}_n' - 2n^{-1}\mathrm{i}(u\phi - v)\boldsymbol{A}_n'\boldsymbol{\iota}_n\boldsymbol{\iota}_n'\boldsymbol{A}_n \\
&= \boldsymbol{R}_n + \frac{\mathrm{i}}{n}\left[u\boldsymbol{\iota}_n\boldsymbol{\iota}_n'\boldsymbol{A}_n + u\boldsymbol{A}_n'\boldsymbol{\iota}_n\boldsymbol{\iota}_n' - 2(u\phi - v)\boldsymbol{A}_n'\boldsymbol{\iota}_n\boldsymbol{\iota}_n'\boldsymbol{A}_n\right] \\
&= \boldsymbol{R}_n + \frac{\mathrm{i}}{n}\begin{pmatrix} 2(u + v - u\phi)\boldsymbol{\iota}_{n-1}\boldsymbol{\iota}_{n-1}' & u\boldsymbol{\iota}_{n-1} \\ u\boldsymbol{\iota}_{n-1}' & 0 \end{pmatrix} \\
&\equiv \begin{pmatrix} \boldsymbol{\Delta}_{n-1} & \boldsymbol{a}_{n-1} \\ \boldsymbol{a}_{n-1}' & 1 \end{pmatrix},
\end{aligned}$$

where

$$\boldsymbol{\Delta}_{n-1} = \boldsymbol{C}_{n-1} + \frac{c_2}{n}\boldsymbol{\iota}_{n-1}\boldsymbol{\iota}_{n-1}', \quad \boldsymbol{a}_{n-1} = \frac{\mathrm{i}u}{n}\boldsymbol{\iota}_{n-1} + c_1\boldsymbol{e}_{n-1,n-1}.$$

Then immediately,

$$|\boldsymbol{S}_n| = |\boldsymbol{\Delta}_{n-1}|\left(1 - \boldsymbol{a}_{n-1}'\boldsymbol{\Delta}_{n-1}^{-1}\boldsymbol{a}_{n-1}\right), \tag{B.8}$$

where

$$|\boldsymbol{\Delta}_{n-1}| = |\boldsymbol{C}_{n-1}|\left(1 + \frac{c_2}{n}\boldsymbol{\iota}_{n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1}\right), \tag{B.9}$$

and

$$\boldsymbol{\Delta}_{n-1}^{-1} = \boldsymbol{C}_{n-1}^{-1} - \frac{c_2}{n + c_2\boldsymbol{\iota}_{n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1}}\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1}\boldsymbol{\iota}_{n-1}'\boldsymbol{C}_{n-1}^{-1}. \tag{B.10}$$

Keep in mind that (B.8) is valid only if $\boldsymbol{\Delta}_{n-1}$ is nonsingular; (B.9) is valid only if $\boldsymbol{C}_{n-1}$ is nonsingular; (B.10) is valid only if $\boldsymbol{C}_{n-1}$ is nonsingular and $n + c_2\boldsymbol{\iota}_{n-1}'\boldsymbol{C}_{n-1}\boldsymbol{\iota}_{n-1} \neq 0$.

From (B.1), we see that $|\boldsymbol{C}_{n-1}| \neq 0$. From (B.3), we also see that $n + c_2\boldsymbol{\iota}_{n-1}'\boldsymbol{C}_{n-1}\boldsymbol{\iota}_{n-1} \neq 0$ for any positive integer n. Further, these two conditions ensure that $|\boldsymbol{\Delta}_{n-1}| \neq 0$ so that the expression for $|\boldsymbol{S}_n|$ given by (B.8) is valid. With $|\boldsymbol{C}_{n-1}|$ from (B.1) and $\boldsymbol{\iota}_{n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1}$ from (B.3), we can easily calculate $|\boldsymbol{\Delta}_{n-1}|$ in (B.8) via

(B.9). Note that the remaining term

$$\begin{aligned}
\boldsymbol{a}_{n-1}'\boldsymbol{\Delta}_{n-1}^{-1}\boldsymbol{a}_{n-1} &= -\frac{u^2}{n^2}\boldsymbol{\iota}_{n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1} + c_1^2\boldsymbol{e}_{1,n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{e}_{1,n-1} \\
&\quad + \frac{2\mathrm{i}c_1 u}{n}\boldsymbol{e}_{1,n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1} \\
&\quad - \frac{2\mathrm{i}c_1 c_2 u}{n(n + c_2\boldsymbol{\iota}_{n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1})}\boldsymbol{\iota}_{n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1}\boldsymbol{e}_{1,n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1} \\
&\quad + \frac{c_2 u^2}{n^2(n + c_2\boldsymbol{\iota}_{n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1})}(\boldsymbol{\iota}_{n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1})^2 \\
&\quad - \frac{c_1^2 c_2}{n + c_2\boldsymbol{\iota}_{n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1}}(\boldsymbol{e}_{1,n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1})^2, \qquad \text{(B.11)}
\end{aligned}$$

where $\boldsymbol{\iota}_{n-1}'\boldsymbol{\Delta}_{n-1}^{-1}\boldsymbol{\iota}_{n-1}$, $\boldsymbol{e}_{1,n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1}$, and $\boldsymbol{e}_{1,n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{e}_{1,n-1}$ follow directly from (B.3), (B.4), and (B.5), respectively.

Inverse of $\boldsymbol{S}_n$

With $\boldsymbol{\Delta}_{n-1}^{-1}$ given by (B.10), the inverse of $\boldsymbol{S}_n$ is straightforward:

$$\boldsymbol{S}_n^{-1} = \begin{pmatrix} \boldsymbol{\Delta}_{n-1}^{-1} & \boldsymbol{0}_{n-1} \\ \boldsymbol{0}_{n-1}' & 0 \end{pmatrix} + \frac{\begin{pmatrix} \boldsymbol{\Delta}_{n-1}^{-1}\boldsymbol{a}_{n-1} \\ -1 \end{pmatrix}\begin{pmatrix} \boldsymbol{a}_{n-1}'\boldsymbol{\Delta}_{n-1}^{-1} & -1 \end{pmatrix}}{1 - \boldsymbol{a}_{n-1}'\boldsymbol{\Delta}_{n-1}^{-1}\boldsymbol{a}_{n-1}}, \qquad \text{(B.12)}$$

if $\boldsymbol{\Delta}_{n-1}$ is nonsingular and $1 - \boldsymbol{a}_{n-1}'\boldsymbol{\Delta}_{n-1}^{-1}\boldsymbol{a}_{n-1} \neq 0$. We have already showed that $\boldsymbol{\Delta}_{n-1}$ is nonsingular. From (B.11), we can verify that $1 - \boldsymbol{a}_{n-1}'\boldsymbol{\Delta}_{n-1}^{-1}\boldsymbol{a}_{n-1} \neq 0$ for any positive integer n.

Note that we need $\boldsymbol{S}_n^{-1}$ as $\boldsymbol{\delta}_n'\boldsymbol{S}_n^{-1}\boldsymbol{\delta}_n$ appears in the CF (A.1). Given $\boldsymbol{\delta}_n = \mathrm{i}u(\boldsymbol{I}_n - 2\phi\boldsymbol{A}_n')\boldsymbol{M}_n\boldsymbol{e}_{1,n} + 2\mathrm{i}v\boldsymbol{A}_n'\boldsymbol{M}_n\boldsymbol{e}_{1,n} + \phi\boldsymbol{e}_{1,n}$, we partition it as follows:

$$\begin{aligned}
\boldsymbol{\delta}_n &= \begin{pmatrix} \mathrm{i}u(\boldsymbol{e}_{1,n-1} - \frac{1}{n}\boldsymbol{\iota}_{n-1}) + \frac{2\mathrm{i}(\phi u - v)}{n}\boldsymbol{\iota}_{n-1} + \phi\boldsymbol{e}_{1,n-1} \\ -\frac{\mathrm{i}u}{n} \end{pmatrix} \\
&\equiv \begin{pmatrix} \boldsymbol{\delta}_{1:n-1,n} \\ -\frac{\mathrm{i}u}{n} \end{pmatrix}.
\end{aligned}$$

So

$$
\begin{aligned}
\boldsymbol{\delta}_n' \boldsymbol{S}_n^{-1} \boldsymbol{\delta}_n &= \boldsymbol{\delta}_n' \begin{pmatrix} \boldsymbol{\Delta}_{n-1}^{-1} & \mathbf{0}_{n-1} \\ \mathbf{0}_{n-1}' & 0 \end{pmatrix} \boldsymbol{\delta}_n \\
&\quad + \frac{\boldsymbol{\delta}_n' \begin{pmatrix} \boldsymbol{\Delta}_{n-1}^{-1} \boldsymbol{a}_{n-1} \\ -1 \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_{n-1}' \boldsymbol{\Delta}_{n-1}^{-1} & -1 \end{pmatrix} \boldsymbol{\delta}_n}{1 - \boldsymbol{a}_{n-1}' \boldsymbol{\Delta}_{n-1}^{-1} \boldsymbol{a}_{n-1}} \\
&= \boldsymbol{\delta}_{1:n-1,n}' \boldsymbol{\Delta}_{n-1}^{-1} \boldsymbol{\delta}_{1:n-1,n} \\
&\quad + \frac{(\boldsymbol{\delta}_{1:n-1,n}' \boldsymbol{\Delta}_{n-1}^{-1} \boldsymbol{a}_{n-1})^2 + \frac{2\mathrm{i}u}{n} \boldsymbol{\delta}_{1:n-1,n}' \boldsymbol{\Delta}_{n-1}^{-1} \boldsymbol{a}_{n-1} - \frac{u^2}{n^2}}{1 - \boldsymbol{a}_{n-1}' \boldsymbol{\Delta}_{n-1}^{-1} \boldsymbol{a}_{n-1}},
\end{aligned} \tag{B.13}
$$

where $\boldsymbol{a}_{n-1}' \boldsymbol{\Delta}_{n-1}^{-1} \boldsymbol{a}_{n-1}$ follows from (B.11) and

$$
\begin{aligned}
\boldsymbol{\delta}_{1:n-1,n}' \boldsymbol{\Delta}_{n-1}^{-1} \boldsymbol{\delta}_{1:n-1,n} &= \frac{c_3^2}{n^2} \boldsymbol{\iota}_{n-1}' \boldsymbol{C}_{n-1}^{-1} \boldsymbol{\iota}_{n-1} + \frac{2c_1c_3}{n} \boldsymbol{e}_{1,n-1}' \boldsymbol{C}_{n-1}^{-1} \boldsymbol{\iota}_{n-1} \\
&\quad + c_1^2 \boldsymbol{e}_{1,n-1}' \boldsymbol{C}_{n-1}^{-1} \boldsymbol{e}_{1,n-1} \\
&\quad - \frac{c_2 c_3^2}{n^2(n + c_2 \boldsymbol{\iota}_{n-1}' \boldsymbol{C}_{n-1}^{-1} \boldsymbol{\iota}_{n-1})} \left(\boldsymbol{\iota}_{n-1}' \boldsymbol{C}_{n-1}^{-1} \boldsymbol{\iota}_{n-1} \right)^2 \\
&\quad - \frac{c_1^2 c_2}{n + c_2 \boldsymbol{\iota}_{n-1}' \boldsymbol{C}_{n-1}^{-1} \boldsymbol{\iota}_{n-1}} (\boldsymbol{e}_{1,n-1}' \boldsymbol{C}_{n-1}^{-1} \boldsymbol{\iota}_{n-1})^2 \\
&\quad - \frac{2c_1c_2c_3}{n(n + c_2 \boldsymbol{\iota}_{n-1}' \boldsymbol{C}_{n-1}^{-1} \boldsymbol{\iota}_{n-1})} \boldsymbol{e}_{1,n-1}' \boldsymbol{C}_{n-1}^{-1} \boldsymbol{\iota}_{n-1} \boldsymbol{\iota}_{n-1}' \boldsymbol{C}_{n-1}^{-1} \boldsymbol{\iota}_{n-1},
\end{aligned} \tag{B.14}
$$

$$
\begin{aligned}
\boldsymbol{\delta}_{1:n-1,n}' \boldsymbol{\Delta}_{n-1}^{-1} \boldsymbol{a}_{n-1} &= \frac{\mathrm{i}uc_3}{n^2} \boldsymbol{\iota}_{n-1}' \boldsymbol{C}_{n-1}^{-1} \boldsymbol{\iota}_{n-1} - c_1^2 \boldsymbol{e}_{1,n-1}' \boldsymbol{C}_{n-1}^{-1} \boldsymbol{e}_{n-1,n-1} \\
&\quad - \frac{c_1c_2}{n} \boldsymbol{e}_{1,n-1}' \boldsymbol{C}_{n-1}^{-1} \boldsymbol{\iota}_{n-1} \\
&\quad + \frac{c_1^2 c_2}{n + c_2 \boldsymbol{\iota}_{n-1}' \boldsymbol{C}_{n-1}^{-1} \boldsymbol{\iota}_{n-1}} (\boldsymbol{e}_{1,n-1}' \boldsymbol{C}_{n-1}^{-1} \boldsymbol{\iota}_{n-1})^2 \\
&\quad - \frac{\mathrm{i}uc_2c_3}{n^2(n + c_2 \boldsymbol{\iota}_{n-1}' \boldsymbol{C}_{n-1}^{-1} \boldsymbol{\iota}_{n-1})} (\boldsymbol{\iota}_{n-1}' \boldsymbol{C}_{n-1}^{-1} \boldsymbol{\iota}_{n-1})^2 \\
&\quad + \frac{c_1 c_2^2}{n(n + c_2 \boldsymbol{\iota}_{n-1}' \boldsymbol{C}_{n-1}^{-1} \boldsymbol{\iota}_{n-1})} \boldsymbol{e}_{1,n-1}' \boldsymbol{C}_{n-1}^{-1} \boldsymbol{\iota}_{n-1} \boldsymbol{\iota}_{n-1}' \boldsymbol{C}_{n-1}^{-1} \boldsymbol{\iota}_{n-1}.
\end{aligned} \tag{B.15}
$$

which can be calculated by substituting (B.3)–(B.6).

Determinant of $\boldsymbol{T}_{n+1}$

Note that

$$
\boldsymbol{T}_{n+1} = (1 - \phi^2) \boldsymbol{V}_{n+1}^{-1} - \mathrm{i}u(\boldsymbol{A}_{n+1}^{\boldsymbol{M}} + \boldsymbol{A}_{n+1}^{\boldsymbol{M}\prime}) + 2\mathrm{i}(u\phi - v) \boldsymbol{B}_{n+1}^{\boldsymbol{M}}
$$

$$
\begin{aligned}
&= (1-\phi^2)\boldsymbol{V}_{n+1}^{-1} - \mathrm{i}u(\boldsymbol{A}_{n+1}+\boldsymbol{A}'_{n+1}) + 2\mathrm{i}(u\phi - v)\boldsymbol{B}_{n+1} \\
&\quad + \frac{\mathrm{i}u}{n}\left[\begin{pmatrix} \mathbf{0}'_n & 0 \\ \boldsymbol{\iota}_n\boldsymbol{\iota}'_n & \mathbf{0}_n \end{pmatrix} + \begin{pmatrix} \mathbf{0}_n & \boldsymbol{\iota}_n\boldsymbol{\iota}'_n \\ 0 & \mathbf{0}'_n \end{pmatrix}\right] - \frac{2\mathrm{i}(u\phi - v)}{n}\begin{pmatrix} \boldsymbol{\iota}_n\boldsymbol{\iota}'_n & \mathbf{0}_n \\ \mathbf{0}'_n & 0 \end{pmatrix} \\
&= \boldsymbol{R}_{n+1} - \phi^2 \boldsymbol{e}_{1,n+1}\boldsymbol{e}'_{1,n+1} \\
&\quad + \frac{\mathrm{i}}{n}\begin{pmatrix} 2(v-u\phi) & (u+2v-2u\phi)\boldsymbol{\iota}'_{n-1} & u \\ (u+2v-2u\phi)\boldsymbol{\iota}_{n-1} & 2(v+u-u\phi)\boldsymbol{\iota}_{n-1}\boldsymbol{\iota}'_{n-1} & u\boldsymbol{\iota}_{n-1} \\ u & u\boldsymbol{\iota}'_{n-1} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1+2\mathrm{i}(u\phi - v) - \frac{2\mathrm{i}(u\phi-v)}{n} & \boldsymbol{b}'_{n-1} & \frac{\mathrm{i}u}{n} \\ \boldsymbol{b}_{n-1} & \boldsymbol{\Delta}_{n-1} & \boldsymbol{a}_{n-1} \\ \frac{\mathrm{i}u}{n} & \boldsymbol{a}'_{n-1} & 1 \end{pmatrix} \\
&\equiv \begin{pmatrix} \boldsymbol{\Delta}^*_n & \boldsymbol{a}^*_n \\ \boldsymbol{a}^{*\prime}_n & 1 \end{pmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
\boldsymbol{\Delta}^*_n &= \begin{pmatrix} 1+2\mathrm{i}(u\phi - v) - \frac{2\mathrm{i}(u\phi-v)}{n} & \boldsymbol{b}'_{n-1} \\ \boldsymbol{b}_{n-1} & \boldsymbol{\Delta}_{n-1} \end{pmatrix}, \\
\boldsymbol{a}^*_n &= (\frac{\mathrm{i}u}{n},\ \boldsymbol{a}'_{n-1})', \\
\boldsymbol{b}_{n-1} &= c_1\boldsymbol{e}_{1,n-1} - \frac{c_3}{n}\boldsymbol{\iota}_{n-1}.
\end{aligned}
$$

Following the same strategy as before,

$$
|\boldsymbol{T}_{n+1}| = |\boldsymbol{\Delta}^*_n|\left(1 - \boldsymbol{a}^{*\prime}_n \boldsymbol{\Delta}^{*-1}_n \boldsymbol{a}^*_n\right), \tag{B.16}
$$

where

$$
|\boldsymbol{\Delta}^*_n| = |\boldsymbol{\Delta}_{n-1}|\left(1+2\mathrm{i}(u\phi - v) - \frac{2\mathrm{i}(u\phi - v)}{n} - \boldsymbol{b}'_{n-1}\boldsymbol{\Delta}^{-1}_{n-1}\boldsymbol{b}_{n-1}\right), \tag{B.17}
$$

and

$$
\boldsymbol{\Delta}^{*-1}_n = \begin{pmatrix} 0 & \mathbf{0}'_{n-1} \\ \mathbf{0}_{n-1} & \boldsymbol{\Delta}^{-1}_{n-1} \end{pmatrix}
$$

$$
+\frac{\begin{pmatrix} -1 \\ \mathbf{\Delta}_{n-1}^{-1}\boldsymbol{b}_{n-1} \end{pmatrix}\begin{pmatrix} -1 & \boldsymbol{b}_{n-1}'\mathbf{\Delta}_{n-1}^{-1} \end{pmatrix}}{1+2\mathrm{i}(u\phi-v)-\frac{2\mathrm{i}(u\phi-v)}{n}-\boldsymbol{b}_{n-1}'\mathbf{\Delta}_{n-1}^{-1}\boldsymbol{b}_{n-1}}. \tag{B.18}
$$

For (B.16) to be valid, $\mathbf{\Delta}_n^*$ needs to be nonsingular; for (B.17) to be valid, $\mathbf{\Delta}_{n-1}$ needs to be nonsingular; for (B.18) to be valid, $\mathbf{\Delta}_{n-1}$ needs to be nonsingular and $1+2\mathrm{i}(u\phi-v)-2\mathrm{i}(u\phi-v)/n-\boldsymbol{b}_{n-1}'\mathbf{\Delta}_{n-1}^{-1}\boldsymbol{b}_{n-1}\neq 0$. We have already shown that $\mathbf{\Delta}_{n-1}$ is nonsingular. Note that

$$
\begin{aligned}
\boldsymbol{b}_{n-1}'\mathbf{\Delta}_{n-1}^{-1}\boldsymbol{b}_{n-1} \;=\;& \frac{c_3^2}{n^2}\boldsymbol{\iota}_{n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1}-\frac{2c_1c_3}{n}\boldsymbol{e}_{1,n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1} \\
&+c_1^2\boldsymbol{e}_{1,n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{e}_{1,n-1} \\
&-\frac{c_2c_3^2}{n^2(n+c_2\boldsymbol{\iota}_{n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1})}(\boldsymbol{\iota}_{n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1})^2 \\
&-\frac{c_1^2c_2}{n+c_2\boldsymbol{\iota}_{n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1}}(\boldsymbol{e}_{1,n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1})^2 \\
&+\frac{2c_1c_2c_3}{n(n+c_2\boldsymbol{\iota}_{n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1})}\boldsymbol{e}_{1,n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1}\boldsymbol{\iota}_{n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1},
\end{aligned} \tag{B.19}
$$

which can be evaluated with analytical expressions from (B.3)–(B.5), and we can verify that $1+2\mathrm{i}(u\phi-v)-2\mathrm{i}(u\phi-v)/n-\boldsymbol{b}_{n-1}'\mathbf{\Delta}_{n-1}^{-1}\boldsymbol{b}_{n-1}\neq 0$ for any positive integer n.

For us to use (B.16), we still need expression for $\boldsymbol{a}_n^{*\prime}\mathbf{\Delta}_n^{*-1}\boldsymbol{a}_n^*$. By substitution,

$$
\begin{aligned}
\boldsymbol{a}_n^{*\prime}\mathbf{\Delta}_n^{*-1}\boldsymbol{a}_n^* \;=\;& \boldsymbol{a}_{n-1}'\mathbf{\Delta}_{n-1}^{-1}\boldsymbol{a}_{n-1} \\
&+\frac{-u^2-2\mathrm{i}nu\boldsymbol{a}_{n-1}'\mathbf{\Delta}_{n-1}^{-1}\boldsymbol{b}_{n-1}+n^2(\boldsymbol{a}_{n-1}'\mathbf{\Delta}_{n-1}^{-1}\boldsymbol{b}_{n-1})^2}{n^2+2\mathrm{i}n^2(u\phi-v)-2\mathrm{i}n(u\phi-v)-n^2\boldsymbol{b}_{n-1}'\mathbf{\Delta}_{n-1}^{-1}\boldsymbol{b}_{n-1}},
\end{aligned} \tag{B.20}
$$

where $\boldsymbol{a}_{n-1}'\mathbf{\Delta}_{n-1}^{-1}\boldsymbol{a}_{n-1}$ is given by (B.11), $\boldsymbol{b}_{n-1}'\mathbf{\Delta}_{n-1}^{-1}\boldsymbol{b}_{n-1}$ is given by (B.19), and

$$
\begin{aligned}
\boldsymbol{a}_{n-1}'\mathbf{\Delta}_{n-1}^{-1}\boldsymbol{b}_{n-1} \;=\;& -\frac{\mathrm{i}uc_3}{n^2}\boldsymbol{\iota}_{n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1}+\frac{c_1c_2}{n}\boldsymbol{e}_{1,n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1} \\
&+c_1^2\boldsymbol{e}_{1,n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{e}_{n-1,n-1} \\
&+\frac{\mathrm{i}uc_2c_3}{n^2(n+c_2\boldsymbol{\iota}_{n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1})}(\boldsymbol{\iota}_{n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1})^2 \\
&-\frac{c_1^2c_2}{n+c_2\boldsymbol{\iota}_{n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1}}(\boldsymbol{e}_{1,n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1})^2 \\
&-\frac{c_1c_2^2}{n(n+c_2\boldsymbol{\iota}_{n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1})}\boldsymbol{e}_{1,n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1}\boldsymbol{\iota}_{n-1}'\boldsymbol{C}_{n-1}^{-1}\boldsymbol{\iota}_{n-1},
\end{aligned} \tag{B.21}
$$

which can be evaluated with analytical expressions from (B.3)–(B.6).

Appendix C: Numerical Integration

Given the characteristic functions in Section 2.1, we need to implement numerical integration to calculate (2.6) via (2.7). This can be straightforwardly implemented using Matlab's `quadgk` command. One caveat to note is that the square root function in the complex domain is not continuous. One choice is to follow Perron (1989) to identify explicitly the discontinuous points by grid search and then integrate by parts. The search, however, might be inefficient and time-consuming. Instead, we use the following algorithm so that the integrand function for `quadgk` is always continuous. Let $g(t) = \sqrt{a(t) + \mathrm{i}b(t)}$ denote the integrand function in question with $t \in [l, u]$. `quadgk` requires the integrand function to accept a vector $(t_1, t_2, \cdots, t_n)$ and returns a vector of output. Let $\theta_i = \arg(a(t_i) + \mathrm{i}b(t_i)) \in [-\pi, \pi]$ and denote $a_i = a(t_i)$, $b_i = b(t_i)$, and $g_i = g(t_i)$.

1. Start with t_1 and set $g_1 = \texttt{sqrt}(a_1 + \mathrm{i}b_1)$. Set $k = 0$.

2. Beginning with t_2, if $a_i < 0$, $a_{i-1} < 0$, and $b_i b_{i-1} <= 0$, set $k = k + 1$; otherwise, k is unchanged. Set $g_i = \sqrt{a_i^2 + b_i^2}\left(\cos(\theta_i^*/2) + \mathrm{i}\sin(\theta_i^*/2)\right)$, where $\theta_i^* = \theta_i + 2k\pi$.

Table 1, Panel A ($T = 1$): $\Pr(T(\hat{\kappa} - \kappa) \leq w)$, Fixed x_0, $\kappa = 0.01$, $x_0 = \mu = 0$, $\sigma = 0.1$

	Monthly					Daily				
w	p	p_{edf}	p_{exp}	p_{mix}	p_{inf}	p	p_{edf}	p_{exp}	p_{mix}	p_{inf}
-5	0.0002	0.0002	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000
-3	0.0037	0.0036	0.0000	0.0000	0.0013	0.0013	0.0013	0.0000	0.0000	0.0014
-2	0.0201	0.0202	0.0000	0.0000	0.0109	0.0110	0.0112	0.0000	0.0000	0.0105
-1.5	0.0487	0.0485	0.0000	0.0000	0.0317	0.0323	0.0326	0.0000	0.0000	0.0310
-1	0.1126	0.1129	0.0000	0.0000	0.0884	0.0892	0.0888	0.0000	0.0000	0.0876
-0.8	0.1511	0.1515	0.0000	0.0000	0.1249	0.1261	0.1253	0.0000	0.0000	0.1245
-0.6	0.1958	0.1963	0.0000	0.0000	0.1679	0.1691	0.1697	0.0000	0.0000	0.1688
-0.4	0.2439	0.2444	0.0023	0.0023	0.2173	0.2190	0.2186	0.0023	0.0023	0.2171
-0.2	0.2928	0.2931	0.0787	0.0786	0.2682	0.2688	0.2686	0.0787	0.0786	0.2682
-0.1	0.3169	0.3174	0.2398	0.2398	0.2933	0.2946	0.2932	0.2398	0.2398	0.2934
-0.01	0.3380	0.3384	0.4718	0.4718	0.3153	0.3154	0.3153	0.4718	0.4718	0.3157
-0.001	0.3401	0.3405	0.4972	0.4972	0.3175	0.3180	0.3175	0.4972	0.4972	0.3179
0	0.3403	0.3408	0.5000	0.5000	0.3177	0.3183	0.3177	0.5000	0.5000	0.3181
0.001	0.3406	0.3410	0.5028	0.5028	0.3180	0.3185	0.3180	0.5028	0.5028	0.3184
0.01	0.3427	0.3432	0.5282	0.5282	0.3202	0.3207	0.3202	0.5282	0.5282	0.3203
0.1	0.3632	0.3638	0.7602	0.7602	0.3411	0.3421	0.3414	0.7602	0.7602	0.3414
0.2	0.3853	0.3856	0.9213	0.9214	0.3644	0.3651	0.3644	0.9213	0.9214	0.3649
0.4	0.4271	0.4271	0.9977	0.9977	0.4081	0.4095	0.4088	0.9977	0.9977	0.4086
0.6	0.4660	0.4660	1.0000	1.0000	0.4492	0.4512	0.4507	1.0000	1.0000	0.4516
0.8	0.5024	0.5026	1.0000	1.0000	0.4889	0.4904	0.4899	1.0000	1.0000	0.4907
1	0.5366	0.5372	1.0000	1.0000	0.5255	0.5267	0.5263	1.0000	1.0000	0.5273
1.5	0.6111	0.6115	1.0000	1.0000	0.6050	0.6053	0.6051	1.0000	1.0000	0.6055
2	0.6711	0.6713	1.0000	1.0000	0.6700	0.6691	0.6687	1.0000	1.0000	0.6695
3	0.7592	0.7594	1.0000	1.0000	0.7628	0.7638	0.7634	1.0000	1.0000	0.7647
5	0.8617	0.8618	1.0000	1.0000	0.8757	0.8746	0.8742	1.0000	1.0000	0.8755
8	0.9318	0.9318	1.0000	1.0000	0.9490	0.9485	0.9484	1.0000	1.0000	0.9498
20	0.9906	0.9904	1.0000	1.0000	0.9980	0.9980	0.9981	1.0000	1.0000	0.9982
50	0.9995	0.9995	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 1, Panel B ($T = 10$): $\Pr(T(\hat{\kappa} - \kappa) \leq w)$, Fixed x_0, $\kappa = 0.01$, $x_0 = \mu = 0$, $\sigma = 0.1$

	Monthly					Daily				
w	p	p_{edf}	p_{exp}	p_{mix}	p_{inf}	p	p_{edf}	p_{exp}	p_{mix}	p_{inf}
-5	0.0000	0.0000	0.1319	0.1318	0.0000	0.0000	0.0000	0.1318	0.1318	0.0000
-3	0.0015	0.0015	0.2513	0.2512	0.0014	0.0013	0.0013	0.2512	0.2512	0.0014
-2	0.0124	0.0124	0.3274	0.3274	0.0114	0.0116	0.0115	0.3274	0.3274	0.0120
-1.5	0.0355	0.0356	0.3687	0.3687	0.0347	0.0341	0.0342	0.3687	0.3687	0.0348
-1	0.0940	0.0938	0.4116	0.4115	0.0924	0.0918	0.0919	0.4115	0.4115	0.0929
-0.8	0.1310	0.1308	0.4290	0.4290	0.1291	0.1288	0.1289	0.4290	0.4290	0.1284
-0.6	0.1749	0.1745	0.4467	0.4466	0.1734	0.1725	0.1729	0.4466	0.4466	0.1715
-0.4	0.2226	0.2221	0.4644	0.4644	0.2207	0.2194	0.2206	0.4644	0.4644	0.2192
-0.2	0.2715	0.2710	0.4822	0.4822	0.2703	0.2677	0.2698	0.4822	0.4822	0.2694
-0.1	0.2957	0.2952	0.4911	0.4911	0.2958	0.2928	0.2941	0.4911	0.4911	0.2935
-0.01	0.3171	0.3166	0.4991	0.4991	0.3169	0.3152	0.3155	0.4991	0.4991	0.3153
-0.001	0.3192	0.3188	0.4999	0.4999	0.3191	0.3163	0.3176	0.4999	0.4999	0.3175
0	0.3194	0.3190	0.5000	0.5000	0.3194	0.3166	0.3178	0.5000	0.5000	0.3177
0.001	0.3197	0.3193	0.5001	0.5001	0.3195	0.3168	0.3181	0.5001	0.5001	0.3179
0.01	0.3218	0.3213	0.5009	0.5009	0.3216	0.3190	0.3201	0.5009	0.5009	0.3199
0.1	0.3427	0.3422	0.5089	0.5089	0.3424	0.3404	0.3410	0.5089	0.5089	0.3415
0.2	0.3654	0.3647	0.5178	0.5178	0.3655	0.3637	0.3639	0.5178	0.5178	0.3641
0.4	0.4107	0.4082	0.5356	0.5356	0.4101	0.4077	0.4077	0.5356	0.5356	0.4077
0.6	0.4505	0.4494	0.5533	0.5534	0.4518	0.4493	0.4493	0.5534	0.5534	0.4500
0.8	0.4892	0.4882	0.5710	0.5710	0.4904	0.4883	0.4882	0.5710	0.5710	0.4891
1	0.5204	0.5239	0.5884	0.5885	0.5264	0.5244	0.5244	0.5885	0.5885	0.5252
1.5	0.6030	0.6017	0.6313	0.6313	0.6059	0.6027	0.6033	0.6313	0.6313	0.6044
2	0.6663	0.6653	0.6726	0.6726	0.6683	0.6664	0.6674	0.6726	0.6726	0.6681
3	0.7607	0.7602	0.7487	0.7488	0.7636	0.7616	0.7627	0.7488	0.7488	0.7629
5	0.8720	0.8717	0.8681	0.8682	0.8735	0.8735	0.8740	0.8682	0.8682	0.8739
8	0.9467	0.9468	0.9631	0.9632	0.9484	0.9484	0.9487	0.9632	0.9632	0.9489
20	0.9977	0.9977	1.0000	1.0000	0.9981	0.9982	0.9981	1.0000	1.0000	0.9982
50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 2, Panel A ($T = 1$): $\Pr(T(\hat{\kappa} - \kappa) \leq w)$, Fixed x_0, $\kappa = 0.01$, $x_0 = \mu = 0.1$, $\sigma = 0.1$

	Monthly					Daily				
w	p	p_{edf}	p_{exp}	p_{mix}	p_{inf}	p	p_{edf}	p_{exp}	p_{mix}	p_{inf}
-5	0.0001	0.0001	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000
-3	0.0016	0.0016	0.0000	0.0000	0.0003	0.0004	0.0004	0.0000	0.0000	0.0004
-2	0.0060	0.0060	0.0000	0.0000	0.0021	0.0020	0.0022	0.0000	0.0000	0.0021
-1.5	0.0114	0.0114	0.0000	0.0000	0.0049	0.0045	0.0051	0.0000	0.0000	0.0049
-1	0.0209	0.0209	0.0000	0.0000	0.0108	0.0113	0.0112	0.0000	0.0000	0.0108
-0.8	0.0263	0.0264	0.0000	0.0000	0.0146	0.0152	0.0151	0.0000	0.0000	0.0144
-0.6	0.0329	0.0330	0.0000	0.0000	0.0194	0.0188	0.0199	0.0000	0.0000	0.0192
-0.4	0.0408	0.0409	0.0023	0.0023	0.0258	0.0263	0.0262	0.0023	0.0023	0.0251
-0.2	0.0501	0.0501	0.0787	0.0787	0.0332	0.0339	0.0339	0.0787	0.0787	0.0324
-0.1	0.0553	0.0554	0.2398	0.2398	0.0373	0.0379	0.0383	0.2398	0.2398	0.0369
-0.01	0.0603	0.0603	0.4718	0.4718	0.0417	0.0425	0.0425	0.4718	0.4718	0.0412
-0.001	0.0609	0.0608	0.4972	0.4972	0.0421	0.0429	0.0430	0.4972	0.4972	0.0418
0	0.0609	0.0609	0.5000	0.5000	0.0422	0.0430	0.0430	0.5000	0.5000	0.0418
0.001	0.0610	0.0609	0.5028	0.5028	0.0423	0.0430	0.0431	0.5028	0.5028	0.0419
0.01	0.0615	0.0615	0.5282	0.5282	0.0427	0.0435	0.0436	0.5282	0.5282	0.0423
0.1	0.0669	0.0669	0.7602	0.7603	0.0474	0.0484	0.0483	0.7603	0.7603	0.0470
0.2	0.0732	0.0732	0.9213	0.9214	0.0532	0.0541	0.0539	0.9214	0.9214	0.0525
0.4	0.0871	0.0869	0.9977	0.9977	0.0655	0.0665	0.0665	0.9977	0.9977	0.0649
0.6	0.1023	0.1021	1.0000	1.0000	0.0798	0.0809	0.0809	1.0000	1.0000	0.0791
0.8	0.1190	0.1187	1.0000	1.0000	0.0952	0.0399	0.0967	1.0000	1.0000	0.0950
1	0.1368	0.1366	1.0000	1.0000	0.1130	0.1145	0.1143	1.0000	1.0000	0.1125
1.5	0.1854	0.1851	1.0000	1.0000	0.1620	0.1643	0.1641	1.0000	1.0000	0.1635
2	0.2380	0.2380	1.0000	1.0000	0.2187	0.2207	0.2206	1.0000	1.0000	0.2186
3	0.3477	0.3475	1.0000	1.0000	0.3423	0.3423	0.3419	1.0000	1.0000	0.3410
5	0.5460	0.5456	1.0000	1.0000	0.5641	0.5634	0.5629	1.0000	1.0000	0.5658
8	0.7392	0.7386	1.0000	1.0000	0.7840	0.7792	0.7787	1.0000	1.0000	0.7812
20	0.9562	0.9562	1.0000	1.0000	0.9879	0.9873	0.9872	1.0000	1.0000	0.9877
50	0.9977	0.9977	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 2, Panel B ($T = 10$): $\Pr(T(\hat{\kappa} - \kappa) \leq w)$, Fixed x_0, $\kappa = 0.01$, $x_0 = \mu = 0.1$, $\sigma = 0.1$

	Monthly					Daily				
w	p	p_{edf}	p_{exp}	p_{mix}	p_{inf}	p	p_{edf}	p_{exp}	p_{mix}	p_{inf}
-5	0.0000	0.0000	0.1319	0.1318	0.0000	0.0000	0.0000	0.1318	0.1318	0.0000
-3	0.0005	0.0005	0.2513	0.2512	0.0004	0.0004	0.0004	0.2512	0.2512	0.0004
-2	0.0025	0.0026	0.3274	0.3274	0.0023	0.0022	0.0022	0.3274	0.3274	0.0024
-1.5	0.0056	0.0058	0.3687	0.3687	0.0052	0.0052	0.0051	0.3687	0.3687	0.0054
-1	0.0121	0.0124	0.4116	0.4115	0.0115	0.0114	0.0114	0.4115	0.4115	0.0120
-0.8	0.0160	0.0165	0.4290	0.4290	0.0152	0.0153	0.0152	0.4290	0.4290	0.0158
-0.6	0.0213	0.0215	0.4467	0.4466	0.0199	0.0202	0.0203	0.4466	0.4466	0.0210
-0.4	0.0276	0.0279	0.4644	0.4644	0.0260	0.0257	0.0265	0.4644	0.4644	0.0274
-0.2	0.0354	0.0357	0.4822	0.4822	0.0333	0.0341	0.0341	0.4822	0.4822	0.0351
-0.1	0.0398	0.0401	0.4911	0.4911	0.0376	0.0384	0.0385	0.4911	0.4911	0.0393
-0.01	0.0441	0.0445	0.4991	0.4991	0.0420	0.0425	0.0426	0.4991	0.4991	0.0436
-0.001	0.0446	0.0449	0.4999	0.4999	0.0424	0.0430	0.0431	0.4999	0.4999	0.0442
0	0.0446	0.0450	0.5000	0.5000	0.0425	0.0430	0.0432	0.5000	0.5000	0.0443
0.001	0.0447	0.0450	0.5001	0.5001	0.0426	0.0431	0.0432	0.5001	0.5001	0.0444
0.01	0.0451	0.0455	0.5009	0.5009	0.0430	0.0434	0.0437	0.5009	0.5009	0.0447
0.1	0.0499	0.0503	0.5089	0.5089	0.0475	0.0484	0.0484	0.5089	0.5089	0.0492
0.2	0.0555	0.0560	0.5178	0.5178	0.0534	0.0540	0.0540	0.5178	0.5178	0.0547
0.4	0.0680	0.0685	0.5356	0.5356	0.0662	0.0664	0.0665	0.5356	0.5356	0.0671
0.6	0.0822	0.0829	0.5533	0.5534	0.0797	0.0805	0.0807	0.5534	0.5534	0.0819
0.8	0.0979	0.0988	0.5710	0.5710	0.0954	0.0962	0.0965	0.5710	0.5710	0.0978
1	0.1151	0.1162	0.5884	0.5885	0.1123	0.1135	0.1137	0.5885	0.5885	0.1148
1.5	0.1640	0.1651	0.6313	0.6313	0.1613	0.1627	0.1629	0.6313	0.6313	0.1632
2	0.2195	0.2207	0.6726	0.6726	0.2175	0.2186	0.2190	0.6726	0.6726	0.2172
3	0.3394	0.3406	0.7487	0.7488	0.3393	0.3398	0.3398	0.7488	0.7488	0.3383
5	0.5585	0.5597	0.8681	0.8682	0.5624	0.5613	0.5613	0.8682	0.8682	0.5584
8	0.7740	0.7742	0.9631	0.9632	0.7822	0.7789	0.7796	0.9632	0.9632	0.7781
20	0.9857	0.9856	1.0000	1.0000	0.9884	0.9880	0.9881	1.0000	1.0000	0.9885
50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 3, Panel A ($T = 1$): $\Pr(T(\hat{\kappa} - \kappa) \leq w)$, Random x_0, $\kappa = 0.01$, $\mu = 0$, $\sigma = 0.1$

	Monthly					Daily				
w	p	p_{edf}	p_{exp}	p_{mix}	p_{inf}	p	p_{edf}	p_{exp}	p_{mix}	p_{inf}
-5	0.0000	0.0000	0.0000	0.0000	0.0099	0.0000	0.0000	0.0000	0.0000	0.0099
-3	0.0001	0.0001	0.0000	0.0000	0.0158	0.0001	0.0001	0.0000	0.0000	0.0155
-2	0.0008	0.0009	0.0000	0.0000	0.0210	0.0005	0.0005	0.0000	0.0000	0.0203
-1.5	0.0025	0.0025	0.0000	0.0000	0.0250	0.0018	0.0018	0.0000	0.0000	0.0242
-1	0.0084	0.0084	0.0000	0.0000	0.0339	0.0071	0.0070	0.0000	0.0000	0.0328
-0.8	0.0144	0.0145	0.0000	0.0000	0.0423	0.0127	0.0127	0.0000	0.0000	0.0412
-0.6	0.0264	0.0265	0.0000	0.0000	0.0581	0.0241	0.0242	0.0000	0.0000	0.0580
-0.4	0.0536	0.0536	0.0023	0.0023	0.0919	0.0506	0.0508	0.0023	0.0023	0.0892
-0.2	0.1343	0.1340	0.0787	0.0787	0.1715	0.1302	0.1297	0.0787	0.0787	0.1651
-0.1	0.2430	0.2432	0.2398	0.2398	0.2658	0.2382	0.2378	0.2398	0.2398	0.2613
-0.01	0.4202	0.4208	0.4718	0.4718	0.4232	0.4152	0.4145	0.4718	0.4718	0.4227
-0.001	0.4406	0.4411	0.4972	0.4972	0.4427	0.4355	0.4348	0.4972	0.4972	0.4427
0	0.4428	0.4434	0.5000	0.5000	0.4449	0.4377	0.4370	0.5000	0.5000	0.4450
0.001	0.4451	0.4456	0.5028	0.5028	0.4471	0.4400	0.4393	0.5028	0.5028	0.4472
0.01	0.4653	0.4658	0.5282	0.5282	0.4669	0.4602	0.4596	0.5282	0.5282	0.4676
0.1	0.6361	0.6365	0.7602	0.7603	0.6472	0.6315	0.6310	0.7603	0.7603	0.6526
0.2	0.7445	0.7454	0.9213	0.9214	0.7733	0.7404	0.7403	0.9214	0.9214	0.7814
0.4	0.8390	0.8394	0.9977	0.9977	0.8816	0.8360	0.8360	0.9977	0.9977	0.8822
0.6	0.8803	0.8807	1.0000	1.0000	0.9218	0.8781	0.8780	1.0000	1.0000	0.9199
0.8	0.9041	0.9045	1.0000	1.0000	0.9414	0.9025	0.9024	1.0000	1.0000	0.9405
1	0.9200	0.9205	1.0000	1.0000	0.9530	0.9189	0.9189	1.0000	1.0000	0.9526
1.5	0.9446	0.9451	1.0000	1.0000	0.9684	0.9441	0.9443	1.0000	1.0000	0.9676
2	0.9590	0.9593	1.0000	1.0000	0.9760	0.9590	0.9591	1.0000	1.0000	0.9748
3	0.9750	0.9751	1.0000	1.0000	0.9837	0.9756	0.9757	1.0000	1.0000	0.9826
5	0.9886	0.9886	1.0000	1.0000	0.9901	0.9897	0.9897	1.0000	1.0000	0.9895
8	0.9954	0.9952	1.0000	1.0000	0.9937	0.9966	0.9965	1.0000	1.0000	0.9935
20	0.9995	0.9995	1.0000	1.0000	0.9975	0.9999	0.9999	1.0000	1.0000	0.9974
50	1.0000	1.0000	1.0000	1.0000	0.9990	1.0000	1.0000	1.0000	1.0000	0.9990
100	1.0000	1.0000	1.0000	1.0000	0.9995	1.0000	1.0000	1.0000	1.0000	0.9995

Table 3, Panel B ($T = 10$): $\Pr(T(\hat{\kappa} - \kappa) \leq w)$, Random x_0, $\kappa = 0.01$, $\mu = 0$, $\sigma = 0.1$

	Monthly					Daily				
w	p	p_{edf}	p_{exp}	p_{mix}	p_{inf}	p	p_{edf}	p_{exp}	p_{mix}	p_{inf}
-5	0.0000	0.0000	0.1319	0.1318	0.0265	0.0000	0.0000	0.1318	0.1318	0.0260
-3	0.0002	0.0002	0.2513	0.2512	0.0418	0.0002	0.0001	0.2512	0.2512	0.0407
-2	0.0019	0.0019	0.3274	0.3274	0.0548	0.0019	0.0017	0.3274	0.3274	0.0527
-1.5	0.0063	0.0065	0.3687	0.3687	0.0639	0.0062	0.0062	0.3687	0.3687	0.0611
-1	0.0235	0.0237	0.4116	0.4115	0.0850	0.0231	0.0230	0.4115	0.4115	0.0811
-0.8	0.0411	0.0410	0.4290	0.4290	0.1053	0.0405	0.0406	0.4290	0.4290	0.1013
-0.6	0.0736	0.0731	0.4467	0.4466	0.1432	0.0729	0.0729	0.4466	0.4466	0.1404
-0.4	0.1342	0.1338	0.4644	0.4644	0.2081	0.1334	0.1332	0.4644	0.4644	0.2077
-0.2	0.2394	0.2393	0.4822	0.4822	0.2931	0.2385	0.2385	0.4822	0.4822	0.2944
-0.1	0.3102	0.3084	0.4911	0.4911	0.3441	0.3073	0.3076	0.4911	0.4911	0.3458
-0.01	0.3760	0.3740	0.4991	0.4991	0.3991	0.3730	0.3730	0.4991	0.4991	0.4016
-0.001	0.3805	0.3804	0.4999	0.4999	0.4052	0.3795	0.3797	0.4999	0.4999	0.4077
0	0.3812	0.3812	0.5000	0.5000	0.4059	0.3803	0.3804	0.5000	0.5000	0.4084
0.001	0.3819	0.3819	0.5001	0.5001	0.4065	0.3810	0.3811	0.5001	0.5001	0.4091
0.01	0.3885	0.3884	0.5009	0.5009	0.4127	0.3875	0.3878	0.5009	0.5009	0.4153
0.1	0.4512	0.4510	0.5089	0.5089	0.4779	0.4502	0.4505	0.5089	0.5089	0.4813
0.2	0.5135	0.5134	0.5178	0.5178	0.5553	0.5126	0.5132	0.5178	0.5178	0.5588
0.4	0.6113	0.6108	0.5356	0.5356	0.7052	0.6106	0.6111	0.5356	0.5356	0.7094
0.6	0.6804	0.6803	0.5533	0.5534	0.7949	0.6799	0.6801	0.5534	0.5534	0.7980
0.8	0.7309	0.7308	0.5710	0.5710	0.8479	0.7305	0.7311	0.5710	0.5710	0.8509
1	0.7694	0.7693	0.5884	0.5885	0.8805	0.7691	0.7695	0.5885	0.5885	0.8821
1.5	0.8349	0.8349	0.6313	0.6313	0.9208	0.8348	0.8349	0.6313	0.6313	0.9206
2	0.8763	0.8759	0.6726	0.6726	0.9393	0.8763	0.8765	0.6726	0.6726	0.9389
3	0.9249	0.9249	0.7487	0.7488	0.9585	0.9251	0.9251	0.7488	0.7488	0.9576
5	0.9677	0.9676	0.8681	0.8682	0.9744	0.9680	0.9680	0.8682	0.8682	0.9738
8	0.9891	0.9890	0.9631	0.9632	0.9838	0.9894	0.9896	0.9632	0.9632	0.9834
20	0.9997	0.9997	1.0000	1.0000	0.9934	0.9998	0.9998	1.0000	1.0000	0.9933
50	1.0000	1.0000	1.0000	1.0000	0.9973	1.0000	1.0000	1.0000	1.0000	0.9973
100	1.0000	1.0000	1.0000	1.0000	0.9987	1.0000	1.0000	1.0000	1.0000	0.9987

Table 4, Panel A ($T = 1$): $\Pr(T(\hat{\kappa} - \kappa) \leq w)$, Random x_0, $\kappa = 0.01$, $\mu = 0.1$, $\sigma = 0.1$

	Monthly					Daily				
w	p	p_{edf}	p_{exp}	p_{mix}	p_{inf}	p	p_{edf}	p_{exp}	p_{mix}	p_{inf}
-5	0.0001	0.0001	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000
-3	0.0016	0.0016	0.0000	0.0000	0.0003	0.0004	0.0004	0.0000	0.0000	0.0003
-2	0.0061	0.0060	0.0000	0.0000	0.0021	0.0020	0.0023	0.0000	0.0000	0.0021
-1.5	0.0114	0.0114	0.0000	0.0000	0.0049	0.0045	0.0053	0.0000	0.0000	0.0049
-1	0.0210	0.0210	0.0000	0.0000	0.0109	0.0113	0.0115	0.0000	0.0000	0.0109
-0.8	0.0264	0.0264	0.0000	0.0000	0.0148	0.0152	0.0155	0.0000	0.0000	0.0147
-0.6	0.0331	0.0330	0.0000	0.0000	0.0197	0.0189	0.0205	0.0000	0.0000	0.0197
-0.4	0.0410	0.0410	0.0023	0.0023	0.0258	0.0265	0.0269	0.0023	0.0023	0.0258
-0.2	0.0503	0.0504	0.0787	0.0787	0.0334	0.0341	0.0344	0.0787	0.0787	0.0335
-0.1	0.0556	0.0556	0.2398	0.2398	0.0378	0.0384	0.0389	0.2398	0.2398	0.0379
-0.01	0.0606	0.0605	0.4718	0.4718	0.0421	0.0427	0.0433	0.4718	0.4718	0.0422
-0.001	0.0611	0.0610	0.4972	0.4972	0.0426	0.0432	0.0437	0.4972	0.4972	0.0426
0	0.0612	0.0611	0.5000	0.5000	0.0426	0.0434	0.0438	0.5000	0.5000	0.0427
0.001	0.0612	0.0611	0.5028	0.5028	0.0427	0.0435	0.0438	0.5028	0.5028	0.0427
0.01	0.0618	0.0617	0.5282	0.5282	0.0432	0.0437	0.0443	0.5282	0.5282	0.0432
0.1	0.0672	0.0671	0.7602	0.7603	0.0479	0.0487	0.0491	0.7603	0.7603	0.0479
0.2	0.0735	0.0736	0.9213	0.9214	0.0535	0.0544	0.0548	0.9214	0.9214	0.0536
0.4	0.0874	0.0873	0.9977	0.9977	0.0661	0.0668	0.0674	0.9977	0.9977	0.0662
0.6	0.1028	0.1026	1.0000	1.0000	0.0805	0.0813	0.0818	1.0000	1.0000	0.0805
0.8	0.1194	0.1194	1.0000	1.0000	0.0965	0.0974	0.0979	1.0000	1.0000	0.0966
1	0.1373	0.1374	1.0000	1.0000	0.1141	0.1150	0.1154	1.0000	1.0000	0.1142
1.5	0.1861	0.1862	1.0000	1.0000	0.1642	0.1649	0.1653	1.0000	1.0000	0.1643
2	0.2387	0.2388	1.0000	1.0000	0.2210	0.2214	0.2217	1.0000	1.0000	0.2211
3	0.3485	0.3485	1.0000	1.0000	0.3433	0.3430	0.3436	1.0000	1.0000	0.3434
5	0.5467	0.5463	1.0000	1.0000	0.5658	0.5641	0.5648	1.0000	1.0000	0.5659
8	0.7397	0.7395	1.0000	1.0000	0.7825	0.7796	0.7800	1.0000	1.0000	0.7826
20	0.9562	0.9561	1.0000	1.0000	0.9885	0.9873	0.9872	1.0000	1.0000	0.9885
50	0.9977	0.9977	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 4, Panel B ($T = 10$): $\Pr(T(\hat{\kappa} - \kappa) \leq w)$, Random x_0, $\kappa = 0.01$, $\mu = 0.1$, $\sigma = 0.1$

	Monthly					Daily				
w	p	p_{edf}	p_{exp}	p_{mix}	p_{inf}	p	p_{edf}	p_{exp}	p_{mix}	p_{inf}
-5	0.0000	0.0000	0.1319	0.1318	0.0000	0.0000	0.0000	0.1318	0.1318	0.0000
-3	0.0005	0.0005	0.2513	0.2512	0.0004	0.0004	0.0004	0.2512	0.2512	0.0004
-2	0.0026	0.0026	0.3274	0.3274	0.0023	0.0023	0.0024	0.3274	0.3274	0.0023
-1.5	0.0076	0.0060	0.3687	0.3687	0.0054	0.0055	0.0054	0.3687	0.3687	0.0054
-1	0.0127	0.0129	0.4116	0.4115	0.0118	0.0120	0.0118	0.4115	0.4115	0.0119
-0.8	0.0168	0.0172	0.4290	0.4290	0.0159	0.0161	0.0158	0.4290	0.4290	0.0160
-0.6	0.0223	0.0226	0.4467	0.4466	0.0211	0.0213	0.0210	0.4466	0.4466	0.0212
-0.4	0.0290	0.0292	0.4644	0.4644	0.0276	0.0271	0.0275	0.4644	0.4644	0.0277
-0.2	0.0371	0.0373	0.4822	0.4822	0.0355	0.0358	0.0353	0.4822	0.4822	0.0357
-0.1	0.0417	0.0419	0.4911	0.4911	0.0401	0.0403	0.0398	0.4911	0.4911	0.0402
-0.01	0.0462	0.0465	0.4991	0.4991	0.0445	0.0335	0.0443	0.4991	0.4991	0.0447
-0.001	0.0467	0.0470	0.4999	0.4999	0.0450	0.0450	0.0447	0.4999	0.4999	0.0452
0	0.0467	0.0470	0.5000	0.5000	0.0450	0.0341	0.0448	0.5000	0.5000	0.0452
0.001	0.0468	0.0471	0.5001	0.5001	0.0451	0.0451	0.0448	0.5001	0.5001	0.0453
0.01	0.0473	0.0475	0.5009	0.5009	0.0456	0.0455	0.0453	0.5009	0.5009	0.0457
0.1	0.0522	0.0525	0.5089	0.5089	0.0504	0.0411	0.0501	0.5089	0.5089	0.0506
0.2	0.0580	0.0584	0.5178	0.5178	0.0562	0.0524	0.0559	0.5178	0.5178	0.0564
0.4	0.0710	0.0712	0.5356	0.5356	0.0691	0.0799	0.0689	0.5356	0.5356	0.0694
0.6	0.0857	0.0859	0.5533	0.5534	0.0837	0.0840	0.0834	0.5534	0.5534	0.0840
0.8	0.1019	0.1022	0.5710	0.5710	0.0999	0.1003	0.0994	0.5710	0.5710	0.1002
1	0.1196	0.1200	0.5884	0.5885	0.1176	0.1180	0.1172	0.5885	0.5885	0.1180
1.5	0.1697	0.1704	0.6313	0.6313	0.1679	0.1683	0.1674	0.6313	0.6313	0.1684
2	0.2260	0.2267	0.6726	0.6726	0.2248	0.2251	0.2244	0.6726	0.6726	0.2253
3	0.3466	0.3468	0.7487	0.7488	0.3468	0.3470	0.3459	0.7488	0.7488	0.3473
5	0.5649	0.5644	0.8681	0.8682	0.5678	0.5677	0.5670	0.8682	0.8682	0.5683
8	0.7778	0.7774	0.9631	0.9632	0.7833	0.7827	0.7818	0.9632	0.9632	0.7835
20	0.9860	0.9859	1.0000	1.0000	0.9885	0.9883	0.9882	1.0000	1.0000	0.9885
50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

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