Concave Consumption Function under Borrowing Constraints

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Abstract

This paper analyzes the optimal consumption behavior of a consumer who faces uninsurable labor income risk and borrowing constraints. In particular, it provides conditions under which the decision rule for consumption is a concave function of existing assets. The current study presents two main findings. First, it is shown that the consumption function is concave if the period utility function is drawn from the HARA class and has either strictly positive or zero third derivative. Second, it is shown that the same result can be obtained for certain period utility functions that are not in the HARA class.

Keywords: Consumption function, borrowing constraints, precautionary saving

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1 Introduction

This paper is concerned with the optimal intertemporal consumption behavior of a risk-averse agent who faces uninsurable labor income risk and has limited borrowing opportunities to finance consumption. In particular, this paper seeks to provide conditions under which the agent's decision rule for consumption at any point in time is a concave function of existing assets.

There are at least two reasons why the concavity of the consumption function deserves attention. First, it is directly supported by empirical evidence. Using data from the Consumption Expenditure Survey, Gourinchas and Parker [6] estimate the consumption functions for households in various age groups and find that these functions are concave in liquid wealth. Second, the concavity of the consumption function is important in understanding precautionary saving behavior. Concavity means that the marginal propensity to consume out of wealth is negatively related to wealth. Thus, when facing the same variation in total resources available for consumption, poor agents would make a larger adjustment in consumption than wealthy agents. This implies that when the total resources at hand are subject to uncertainty, the growth rate of consumption for poor agents is more volatile than that for wealthy agents.[1] This creates an incentive for risk-averse agents to accumulate precautionary wealth. As emphasized by Carroll [2], a concave consumption function is a key building block of the buffer-stock savings model. In this type of models, an impatient consumer (one with rate of time preference greater than the interest rate) chooses to maintain a positive "target" level of wealth because the precautionary saving motive prevails over the desire to dissave induced by impatience when the wealth level is low. The two forces are in balance when the target level of wealth is reached. In another study on precautionary wealth accumulation, Huggett [5] examines the conditions under which the average wealth holding of individuals increases when earnings risk increases. He finds that a concave consumption function (or equivalently, a convex savings function) is a key element in generating this type of behavior.

Despite its theoretical and empirical relevance, only a few studies have examined the theoretical foundations of concave consumption function. In an earlier study, Carroll and Kimball [3] examine this issue in a model similar to the one considered in the present paper but without borrowing

[1] See Zeldes [13] and Carroll [2] for a detailed discussion on this point.

constraints. They show that concavity of the consumption function can be established if the period utility function exhibits hyperbolic absolute risk aversion (HARA) and has either strictly positive or zero third derivative. This result is useful because these two properties are satisfied by a wide range of utility functions, including the constant-relative-risk-aversion (CRRA) utility functions, the constant-absolute-risk-aversion (CARA) utility functions, and the quadratic utility functions. To establish this result, Carroll and Kimball make use of the second and third derivatives of the value function. The existence of these derivatives, however, is not guaranteed in the presence of borrowing constraints. This raises the question of whether the results in Carroll and Kimball [3] remain valid when such constraints are introduced. Two recent studies, Huggett [5] and Carroll and Kimball [4], address this question using a different strategy of proof. These studies are able to establish the concavity of the consumption function in an environment with borrowing constraints, but only for three specific groups of period utility functions: CRRA utility, CARA utility and quadratic utility. The current study is a continuation of these efforts and seeks to generalize their results.

This paper presents two main findings. First, it is shown that in a canonical life-cycle model with borrowing constraints, the decision rule for consumption at any point in time is a concave function of existing assets if the period utility function is drawn from the HARA class and has either strictly positive or zero third derivative. In other words, the current study generalizes the main result in Carroll and Kimball [3] to an environment with borrowing constraints. However, unlike Carroll and Kimball [3], our proof only requires the value function to be once continuously differentiable. This property of the value function is formally established in an intermediate step towards the main result. The results in Huggett [5] and Carroll and Kimball [4] can be viewed as special cases of our first main result. The second main finding of this paper is that concavity of the consumption function can be established *even if the period utility function is not a member of the HARA class*. In other words, hyperbolic absolute risk aversion is not a necessary condition for the consumption function to be concave. To the best of our knowledge, this result is not formerly mentioned in the existing studies. The current study provides specific conditions under which a non-HARA utility function would give rise to concave consumption functions.

The reminder of this paper is organized as follows. Section 2 describes the model environment. Section 3 analyzes the agent's problem and establishes some intermediate results. Section 4 states the main findings mentioned above, discusses their implications and compares them to those in the existing studies. Section 5 presents the proof of the main results. Section 6 concludes.

2 The Model

Consider a consumer who faces a $(T+1)$-period planning horizon, where T is finite. The consumer has preferences over random consumption paths $\{c_t\}_{t=0}^{T}$ which can be represented by

$$E_0\left[\sum_{t=0}^{T}\beta^t u\left(c_t\right)\right], \tag{1}$$

where $\beta \in (0,1)$ is the subjective discount factor and $u(\cdot)$ is the utility function. The domain of the utility function is given by $\mathcal{D} = [\underline{c}, \infty)$, with $\underline{c} \geq 0$.[2] The lower bound $\underline{c}$ is interpreted as a minimum consumption requirement. Throughout this paper, we maintain the following assumptions on the utility function.

Assumption A1 The utility function $u : \mathcal{D} \to [-\infty, \infty)$ is thrice continuously differentiable, strictly increasing and strictly concave.

Note that Assumption A1 does not impose any restriction on the value of $u(\underline{c})$. This means the utility function can be either bounded or unbounded below.

In each period, the consumer receives a random amount of labor income. Let e_t be a random variable which represents labor endowment at time t. Labor income at time t is then given by we_t, where $w > 0$ is a constant wage rate. Let $X \subset \mathbb{R}_+$ be the state space of the random labor endowment, and $(X, \mathcal{X})$ be a measurable space. The random variable e_t is assumed to follow a stationary Markov process with transition function $Q : (X, \mathcal{X}) \to [0,1]$. The following assumption is imposed on the Markov process.

[2] This specification encompasses those utility functions that are not defined at $c = 0$. One example is the Stone-Geary utility function which belongs to the HARA class and features a minimum consumption requirement. All the results in this paper remain valid if we set $\underline{c} = 0$.

Assumption A2 One of the following conditions holds:

(i) X is a countable set in $\mathbb{R}_+$ with minimum element $\underline{e}$ and maximum element $\overline{e}$, $0 < \underline{e} < \overline{e} < \infty$.

(ii) X is a compact interval in $\mathbb{R}_+$, represented by $X = [\underline{e}, \overline{e}]$, $0 < \underline{e} < \overline{e} < \infty$. The transition function Q has the weak Feller property, i.e., for any bounded, continuous function $\phi : X \to \mathbb{R}$, the function $T\phi$ defined by

$$(T\phi)(e) = \int_X \phi(e') \, Q(e, de'),$$

is also continuous.

In light of the uncertainty in labor income, the consumer can only self-insure by borrowing or lending a single risk-free asset. The gross return from the asset is $(1 + r) > 0$. Let a_t be the agent's asset holdings at time t. The consumer is said to be in debt if a_t is negative. In each period t, the consumer is subject to the budget constraint

$$c_t + a_{t+1} = we_t + (1 + r) \, a_t, \tag{2}$$

and the borrowing constraint: $a_{t+1} \geq -\underline{a}_{t+1}$. The parameter $\underline{a}_{t+1} \geq 0$ represents the maximum amount that the consumer can borrow at time t. The borrowing limits are period-specific for the following reason. In life-cycle models, consumers are typically not allowed to die in debt. This means the borrowing limit in the terminal period is $\underline{a}_{T+1} = 0$. But the borrowing limit in all other periods can be different from zero. Throughout this paper, we maintain the following restrictions on the borrowing limits.[3]

Assumption A3 The set of borrowing limits $\{\underline{a}_t\}_{t=0}^{T+1}$ satisfies the following conditions: $\underline{a}_t \geq 0$, for all t, $\underline{a}_{T+1} = 0$ and

$$w\underline{e} - (1 + r) \, \underline{a}_t + \underline{a}_{t+1} > \underline{c}, \qquad \text{for all } t. \tag{3}$$

[3]Similar restrictions are also used in Huggett [5, Section 4.2].

The intuitions of (3) are as follows. Suppose the consumer faces the worst possible state at time t, i.e., $a_t = -\underline{a}_t$ and $e_t = \underline{e}$. The highest attainable consumption in this particular state is $c_t = w\underline{e} - (1+r)\,\underline{a}_t + \underline{a}_{t+1}$. The above condition then ensures that the consumer can meet the minimum consumption requirement even in the worst possible state. The same condition also ensures that any debt at time t can be repaid in the future even if the consumer receives the lowest labor income in all future time periods, i.e.,

$$\sum_{j=0}^{T-t} \frac{w\underline{e} - \underline{c}}{(1+r)^j} - (1+r)\,\underline{a}_t > 0, \qquad \text{for all } t.$$

3 The Agent's Problem

Given the prices w and r, the agent's problem is to choose sequences of consumption and asset holdings, $\{c_t, a_{t+1}\}_{t=0}^{T}$, so as to maximize the expected lifetime utility in (1), subject to the budget constraint in (2), the minimum consumption requirement $c_t \geq \underline{c}$ for all t, the borrowing constraint $a_{t+1} \geq -\underline{a}_{t+1}$ for all t, and an initial condition $a_0 \geq -\underline{a}_0$.

Define a sequence of assets $\{\overline{a}_t\}_{t=0}^{T}$ according to

$$\overline{a}_{t+1} = w\overline{e} + (1+r)\,\overline{a}_t - \underline{c},$$

for $t \in \{0, 1, 2, ..., T-1\}$, and $\overline{a}_0 = a_0$. This sequence specifies the amount of asset available in every period if the consumer receives the highest possible labor income $w\overline{e}$ and consumes only the minimum requirement $\underline{c}$ in every period. Since $(1+r) > 0$, this sequence is monotonically increasing and bounded above by

$$\overline{a}_T \equiv (1+r)^T a_0 + (w\overline{e} - \underline{c}) \sum_{j=0}^{T-1} (1+r)^j,$$

which is finite as T is finite. It is also straightforward to show that any feasible sequence of assets $\{a_t\}_{t=0}^{T}$ must be bounded above by $\{\overline{a}_t\}_{t=0}^{T}$. Hence the state space of asset in every period t can be restricted to the interval $A_t = [-\underline{a}_t, \overline{a}_t]$.

In any given period, the state of the consumer can be summarized by $s = (a, e)$, where a denotes his asset holdings at the beginning of the period, and e is the current realization of labor endowment. The set of all possible states at time t is given by $S_t = A_t \times X$.

Define a sequence of value functions $\{V_t\}_{t=0}^{T}$, $V_t : S_t \to [-\infty, \infty]$ for each t, recursively according to

$$V_t(a, e) = \max_{c \in [\underline{c}, z(a,e) + \underline{a}_{t+1}]} \left\{ u(c) + \beta \int_X V_{t+1}[z(a, e) - c, e'] \, Q(e, de') \right\}, \tag{P1}$$

where $z(a, e) \equiv we + (1 + r) a$. In the terminal period, the value function is given by

$$V_T(a, e) = u[we + (1 + r) a], \qquad \text{for all } (a, e) \in S_T.$$

Define a sequence of optimal policy correspondences for consumption $\{g_t\}_{t=0}^{T}$ according to

$$g_t(a, e) \equiv \operatorname*{arg\,max}_{c \in [\underline{c}, z(a,e) + \underline{a}_{t+1}]} \left\{ u(c) + \beta \int_X V_{t+1}[z(a, e) - c, e'] \, Q(e, de') \right\}, \tag{4}$$

for all $(a, e) \in S_t$ and for each t. Given $g_t(a, e)$, the optimal choices of a_{t+1} are given by

$$h_t(a, e) \equiv \{a' : a' = z(a, e) - c, \text{ for some } c \in g_t(a, e)\}. \tag{5}$$

Our first theorem summarizes the main properties of the value functions. The first part of the theorem states that the value function in every period t is bounded and continuous on S_t. This is true even if the utility function $u(\cdot)$ is unbounded below. Continuity of the objective function in (P1) then ensures that the optimal policy correspondence $g_t(a, e)$ is non-empty and upper hemicontinuous. The second part of the theorem establishes the strict monotonicity and strict concavity of $V_t(\cdot, e)$. Strict concavity of the value function implies that $g_t(a, e)$ is a single-valued function. The last part of Theorem 1 establishes the differentiability of $V_t(\cdot, e)$. Specifically, this result states that $V_t(\cdot, e)$ is not only differentiable in the interior of A_t, it is also (right-hand) differentiable at the endpoint $-\underline{a}_t$.[4] This property is important because, as is well-known in this

[4] Note that the standard result in Stokey, Lucas and Prescott [12, Theorem 9.10] only establishes the differentiability of the value function in the interior of the state space.

literature, the consumer in this problem may choose to exhaust the borrowing limit in certain states.[5] In other words, the agent's problem may have corner solutions in which $h_{t-1}(a,e) = -\underline{a}_t$ for some $(a,e) \in S_{t-1}$. Thus the first-order condition of (P1) has to be valid even when $h_{t-1}(a,e) = -\underline{a}_t$. This requires the value function $V_t(\cdot, e)$ to be once differentiable at $a = -\underline{a}_t$.

Additional conditions are imposed in part (iii) of Theorem 1 to ensure that $g_t(a,e) > \underline{c}$ for all $(a,e) \in S_t$. Specifically, the proof of part (iii) uses an intermediate result which states that if the utility function satisfies the Inada condition $u'(\underline{c}+) \equiv \lim_{c \to \underline{c}+} u'(c) = +\infty$, or the consumer is impatient so that $\beta(1+r) \leq 1$, then it is never optimal to consume only the minimum requirement $\underline{c}$.[6] It follows that $h_t(a,e)$ can never reach the upper bound $\overline{a}_{t+1}$ in any period t. Hence there is no need to consider corner solutions in which $h_{t-1}(a,e) = \overline{a}_t$, and the (left-hand) differentiability of $V_t(\cdot, e)$ at $a = \overline{a}_t$.

Theorem 1 *Suppose Assumptions A1-A3 are satisfied. Then the following results hold for all* $t \in \{0, 1, ..., T\}$.

(i) The value function $V_t(a,e)$ *is bounded and continuous on* S_t.

(ii) For all $e \in X$, $V_t(\cdot, e)$ *is strictly increasing and strictly concave on* A_t.

(iii) Suppose either $u'(\underline{c}+) = +\infty$ *or* $\beta(1+r) \leq 1$. *Then the function* $V_t(\cdot, e)$ *is continuously differentiable on* $[-\underline{a}_t, \overline{a}_t)$ *for all* $e \in X$. *Let* $p_t(a,e)$ *denote the derivative of* $V_t(a,e)$ *with respect to* a. *Then*

$$p_t(a,e) = (1+r)\, u'[g_t(a,e)], \qquad \text{for all } (a,e) \in S_t.$$

Proof. See Appendix A. ■

Our next theorem establishes some basic properties of the policy functions. These results are useful in establishing the concavity property of $g_t(\cdot, e)$. The first part of Theorem 2 states that $g_t(a,e)$ is strictly greater than the minimum consumption requirement $\underline{c}$ for all $(a,e) \in S_t$. As

[5] See, for instance, Schechtman and Escudero [10] and Mendelson and Amihud [8].

[6] The same assumption is also used in Huggett [5, Lemma 1]. In the buffer-stock savings model à la Carroll [2], it is typical to assume $\beta(1+r) \leq 1$.

mentioned above, this follows from the assumption that either $u'(\underline{c}+) = +\infty$ or $\beta(1+r) \leq 1$. The second part of the theorem establishes the Euler equation for consumption. This equation plays a central role in establishing the concavity property of $g_t(\cdot, e)$. The third part of the theorem states that the policy functions for consumption and future asset holdings are increasing functions in current asset holdings.

Theorem 2 *Suppose Assumptions A1-A3 are satisfied. Suppose either* $u'(\underline{c}+) = +\infty$ *or* $\beta(1+r) \leq 1$. *Then the following results hold for all* $t \in \{0, 1, ..., T\}$.

(i) The policy function for consumption is bounded below by $\underline{c}$, *i.e.,* $g_t(a, e) > \underline{c}$ *for all* $(a, e) \in S_t$.

(ii) For all $(a, e) \in S_t$, *the policy functions* $g_t(a, e)$ *and* $h_t(a, e)$ *satisfy the Euler equation*

$$u'[g_t(a, e)] \geq \beta(1+r) \int_X u'\left[g_{t+1}\left(h_t(a, e), e'\right)\right] Q\left(e, de'\right), \tag{6}$$

with equality holds if $h_t(a, e) > -\underline{a}_{t+1}$.

(iii) For all $e \in X$, $g_t(\cdot, e)$ *is strictly increasing whereas* $h_t(\cdot, e)$ *is non-decreasing.*

Proof. See Appendix A. ■

4 Main Results

The main results of this paper are summarized in Theorem 3. This theorem provides a set of conditions on the utility function $u(\cdot)$ such that the policy function for consumption in every period is a concave function in current asset holdings. These conditions cover two classes of utility functions: (i) quadratic utility functions, or those with $u'''(c) = 0$ for all $c \in \mathcal{D}$, and (ii) utility functions with strictly positive third derivative, i.e., $u'''(c) > 0$ for all $c \in \mathcal{D}$. For the latter class of utility functions, an additional condition is needed in order to establish the desired result. It is shown that this condition is satisfied by a wide range of utility functions, including all HARA utility function with strictly positive third derivative, and certain non-HARA utility functions.

Before stating the main theorem, we first introduce some additional notation. For expositional convenience, define $\phi(\cdot)$ as the marginal utility function, i.e., $\phi(c) \equiv u'(c)$ for all $c \in \mathcal{D}$. Under

Assumption A1, $\phi(\cdot)$ is twice continuously differentiable, strictly positive, strictly decreasing and strictly convex. In addition, the inverse function of ϕ, denoted by $\phi^{-1}(\cdot)$, is also twice continuously differentiable and strictly decreasing. If the utility function has strictly positive third derivative, then we define $\eta : \mathcal{D} \rightarrow \mathbb{R}$ according to

$$\eta(c) = \frac{[\phi'(c)]^2}{\phi''(c)} \equiv \frac{[u''(c)]^2}{u'''(c)}, \tag{7}$$

and $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}$ according to

$$\Phi(z) \equiv \eta\left[\phi^{-1}(z)\right]. \tag{8}$$

Both $\eta(\cdot)$ and $\Phi(\cdot)$ are strictly positive as $u'''(\cdot)$ is strictly positive. Within this group of utility functions, we confine our attention to those that satisfy the following assumption.

Assumption A4 Let μ be a probability measure on $\mathbb{R}_+$. For any μ-integrable function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$,

$$\Phi\left[\beta(1+r)\int_{\mathbb{R}_+} \psi d\mu\right] \geq \beta(1+r)\int_{\mathbb{R}_+} \Phi \circ \psi d\mu. \tag{9}$$

If μ is a discrete probability measure with masses $\{\mu_1, ..., \mu_N\}$ on a set of points in $\mathbb{R}_+$, then the condition in (9) can be expressed as

$$\Phi\left[\beta(1+r)\sum_{i=1}^{N} \mu_i \psi_i\right] \geq \beta(1+r)\sum_{i=1}^{N} \mu_i \Phi(\psi_i),$$

for any $\boldsymbol{\psi} = (\psi_1, ..., \psi_N) \in \mathbb{R}_+^N$.

Before proceeding further, it is instructive to examine what kind of functions are consistent with Assumption A4. One important thing to note is that the value of $\beta(1+r)$ plays a crucial role in this issue. If $\beta(1+r) = 1$, then (9) becomes Jensen's inequality. Hence, any concave function $\Phi(\cdot)$ defined on $\mathbb{R}_+$ satisfies Assumption A4 when $\beta(1+r) = 1$. This, however, is not true when $\beta(1+r) \neq 1$. Below are some examples of $\Phi(\cdot)$ that is consistent with Assumption A4 when $\beta(1+r) \neq 1$. In Section 4.1, we show that the first of these examples encompasses all HARA utility functions with strictly positive third derivative. The last of these examples provides the basis for finding non-HARA utility functions that satisfy Assumption A4. The details of this are provided in Section 4.2.

1. If $\Phi(z) \equiv bz$ for some strictly positive real number b, then it satisfies Assumption A4 for any $\beta(1+r) > 0$.

2. If $\Phi : \mathbb{R}_+ \to \mathbb{R}$ is a linear function with $\Phi(0) > 0$, then it satisfies Assumption A4 when $\beta(1+r) \in (0,1)$.

3. If $\Phi : \mathbb{R}_+ \to \mathbb{R}$ is nonincreasing and concave, then it satisfies Assumption A4 when $\beta(1+r) \in (0,1)$.

We are now ready to state the main theorem. Building on the results in Theorem 2, our main theorem states that the policy functions $\{g_t(\cdot, e)\}_{t=0}^T$ are concave if the utility function $u(\cdot)$ belongs to either one of the following categories: (i) quadratic utility functions, or (ii) utility functions with strictly positive third derivative and satisfy Assumption A4.[7] The proof of this theorem is given in Section 5.

Theorem 3 *Suppose the conditions in Theorem 2 are satisfied. For any $e \in X$, the policy functions $\{g_t(\cdot, e)\}_{t=0}^T$ are concave if one of the following conditions holds*

(i) $u'''(c) = 0$ *for all* $c \geq \underline{c}$.

(iii) $u'''(c) > 0$ *for all* $c \geq \underline{c}$ *and Assumption A4 is satisfied.*

4.1 HARA Utility

In this subsection we show that the conditions in Theorem 3 are satisfied by the utility functions considered in Carroll and Kimball [3]. To begin with, a twice continuously differentiable utility function $u : \mathcal{D} \to \mathbb{R}$ is called a HARA utility function if there exists $(\alpha, \gamma) \in \mathbb{R}$ such that $\alpha + \gamma c \geq 0$ and

$$-\frac{u''(c)}{u'(c)} = \frac{1}{\alpha + \gamma c}, \qquad \text{for all } c \in \mathcal{D}. \tag{10}$$

[7] Theorem 3 only establishes the weak concavity of $g_t(\cdot, e)$. In the presence of borrowing constraints, the policy function for consumption may not be strictly concave throughout its domain. In particular, the consumption function is linear in current asset holdings whenever the borrowing constraint is binding. The possibility of a binding borrowing constraint is formally proved in Mendelson and Amibud [8].

This definition implies that all HARA utility functions are at least thrice continuously differentiable in the interior of its domain. The HARA class of utility functions encompasses a wide range of utility functions that are commonly used in economics. For instance, the CARA or exponential utility functions correspond to the case when $\alpha > 0$ and $\gamma = 0$. The standard CRRA utility functions correspond to the case when $\underline{c} = 0$, $\alpha = 0$ and $\gamma > 0$. The more general Stone-Geary utility functions $u(c) = (c - \underline{c})^{1-1/\gamma} / (1 - 1/\gamma)$ correspond to the case when $\underline{c} > 0$, $\alpha = -\underline{c}\gamma$, and $\gamma > 0$. Finally, quadratic utility functions of the form

$$u(c) = \varphi_0 + \varphi_1 (c - \underline{c}) + \varphi_2 (c - \underline{c})^2, \qquad \text{with } \varphi_2 < 0,$$

correspond to the case when $\alpha = \underline{c} - \varphi_1/\varphi_2 > 0$ and $\gamma = -1$. Except for the quadratic utility functions, all the utility functions mentioned above have strictly positive third derivative. However, not all of them satisfy the Inada condition $u'(\underline{c}+) = +\infty$. For instance, $u'(\underline{c}+)$ is finite under the CARA class and the quadratic class.

An alternative characterization of the HARA utility functions can be obtained by differentiating (10) with respect to c, which yields

$$\frac{u'(c)\, u'''(c)}{[u''(c)]^2} = 1 + \gamma, \qquad \text{for all } c \in \mathcal{D}.$$

Carroll and Kimball [3] consider the subclass of HARA utility functions with $\gamma \geq -1$, which implies a nonnegative $u'''(\cdot)$. When $\gamma > -1$, the above expression implies

$$\eta(c) = \frac{[u''(c)]^2}{u'''(c)} = \frac{u'(c)}{1+\gamma} = \frac{\phi(c)}{1+\gamma},$$

$$\Rightarrow \Phi(z) = \frac{z}{1+\gamma}.$$

In other words, the subclass of HARA utility functions with $\gamma > -1$ corresponds to the case when $\Phi(z) \equiv bz$ for some $b > 0$. Hence any HARA utility function with $\gamma > -1$ satisfies Assumption A4 for any $\beta(1+r) > 0$.

The following corollary summarizes what we have learned about the policy functions $\{g_t(\cdot, e)\}_{t=0}^{T}$

when the utility function is of the HARA class. These results generalize those obtained in Huggett [5, Lemma 1]. Specifically, Huggett proves that $\{g_t(\cdot, e)\}_{t=0}^T$ are concave in two particular cases: (i) when the utility function exhibits CRRA [hence $u'(0+) = +\infty$], and (ii) when the utility function exhibits CARA [hence $u'(0+) < +\infty$] and $\beta(1+r) \leq 1$. The following corollary generalizes the first case to any HARA utility functions with $\gamma \geq -1$ and $u'(\underline{c}+) = +\infty$. It generalizes the second case to any HARA utility functions with $\gamma \geq -1$ and $u'(\underline{c}+) < \infty$, and $\beta(1+r) \leq 1$.

Corollary 4 *Suppose the utility function $u(\cdot)$ is of the HARA class and has either strictly positive or zero third derivative. Suppose Assumptions A2 and A3 are satisfied, and $\beta(1+r) > 0$. Then for each $e \in X$, the policy functions $\{g_t(\cdot, e)\}_{t=0}^T$ are strictly increasing and concave if one of the following conditions hold:*

(i) Marginal utility is infinite at $c = \underline{c}$, i.e., $u'(\underline{c}+) = +\infty$.

(ii) Marginal utility is finite at $c = \underline{c}$ and $\beta(1+r) \leq 1$.

4.2 Non-HARA Utility

In the previous discussion, we show that Assumption A4 can be satisfied by several forms of the function $\Phi(\cdot)$. In particular, the linear form $\Phi(z) \equiv bz$, which encompasses all the HARA utility functions with strictly positive third derivative, is only one of the admissible forms. This suggests that Assumption A4 can also be satisfied by utility functions that are *not* in the HARA class. The main objective of this subsection is to illustrate this possibility for the case when $\beta(1+r) \leq 1$. In particular, we provide conditions on the period utility function such that $\Phi(\cdot)$ is strictly decreasing and strictly concave.

Suppose the utility function $u(\cdot)$ is sufficiently smooth so that the function $\Phi(\cdot)$ is twice continuously differentiable. Then it is straightforward to show that

$$\Phi'(z) = 1 + \frac{d}{dc}\left[\frac{u''(c)}{u'''(c)}\right] \qquad \text{and} \qquad \Phi''(z) = \frac{d^2}{dc^2}\left[\frac{u''(c)}{u'''(c)}\right],$$

for $z = u'(c)$ and for all $c \in \mathcal{D}$. Hence $\Phi(\cdot)$ is strictly decreasing and strictly concave if and only if

$$\frac{d}{dc}\left[\frac{u''(c)}{u'''(c)}\right] < -1 \qquad \text{and} \qquad \frac{d^2}{dc^2}\left[\frac{u''(c)}{u'''(c)}\right] < 0. \tag{11}$$

Following Kimball [7], we define $\pi(c) \equiv -u'''(c)/u''(c)$ as a measure of absolute prudence. Then the conditions in (11) hold if and only if

$$\pi'(c) < -[\pi(c)]^2, \qquad \text{and} \qquad \pi''(c)\,\pi(c) < -2\left[\pi'(c)\right]^2,$$

for all $c \in \mathcal{D}$. These results suggest that a strictly decreasing and strictly convex absolute prudence is a necessary condition for $\Phi(\cdot)$ to be strictly decreasing and strictly concave.

5 Proof of Main Theorem

The main ideas of the proof are as follows. For each $e \in X$ and for all $t \in \{0, 1, ..., T\}$, the function $g_t(\cdot, e)$ is concave if and only if its hypograph $\Omega_t(e)$, defined by

$$\Omega_t(e) \equiv \{(c, a) \in \mathcal{D} \times A_t : c \leq g_t(a, e)\},$$

is a convex set. Theorem 3 essentially provides a set of sufficient conditions under which $\Omega_t(e)$ is convex.

The first step of the proof is to derive an alternative but equivalent expression for $\Omega_t(e)$.[8] The main advantage of this alternative expression is that it is analytically tractable. For each $e \in X$, define a set $\Lambda_t(e)$ according to

$$\Lambda_t(e) \equiv \left\{(c, a) \in \mathcal{D} \times A_t : c \in \Phi_t(a, e) \text{ and } u'(c) \geq \beta \int_X p_{t+1}\left[z(a, e) - c, e'\right] Q\left(e, de'\right)\right\}, \tag{12}$$

where $\Phi_t(a, e)$ is the constraint set at state (a, e) in period t, i.e.,

$$\Phi_t(a, e) \equiv \left\{c : \underline{c} \leq c \leq z(a, e) + \underline{a}_{t+1}\right\},$$

[8] The same step is also used in the proof of Lemma 1 in Huggett [5].

and $p_{t+1}(a',e') \equiv (1+r)\,\phi\,[g_{t+1}(a',e')]$ for all $(a',e') \in S_{t+1}$ by Theorem 1. We now show that $\Omega_t(e)$ and $\Lambda_t(e)$ are equivalent. For any $(c,a) \in \Omega_t(e)$, it must be the case that $c \in \Phi_t(a,e)$ and $z(a,e) - c \geq z(a,e) - g_t(a,e)$. Since $u'(\cdot)$ and $p_{t+1}(\cdot,e')$ are both decreasing functions, we have

$$\begin{aligned} u'(c) &\geq u'[g_t(a,e)] \geq \beta \int_X p_{t+1}\left[z(a,e) - g_t(a,e), e'\right] Q\left(e, de'\right) \\ &\geq \beta \int_X p_{t+1}\left[z(a,e) - c, e'\right] Q\left(e, de'\right). \end{aligned}$$

The second inequality is the Euler equation in (6). This shows that $\Omega_t(e) \subseteq \Lambda_t(e)$. Next, consider any $(c,a) \in \Lambda_t(e)$ and suppose the contrary that $c > g_t(a,e)$. If $g_t(a,e) = z(a,e) + \underline{a}_{t+1}$, then any feasible consumption must be no greater than $g_t(a,e)$ and hence there is a contradiction. So consider the case when $z(a,e) + \underline{a}_{t+1} \geq c > g_t(a,e)$. This has two implications: (i) $h_t(a,e) > -\underline{a}_{t+1}$, and (ii) $h_t(a,e) > z(a,e) - c$. The first inequality implies that the Euler equation in (6) holds with equality. Thus we have

$$\begin{aligned} u'(c) &< u'[g_t(a,e)] = \beta \int_X p_{t+1}\left[h_t(a,e), e'\right] Q\left(e, de'\right) \\ &< \beta \int_X p_{t+1}\left[z(a,e) - c, e'\right] Q\left(e, de'\right). \end{aligned}$$

This means $(c,a) \notin \Lambda_t(e)$ which gives rise to a contradiction. Hence $\Lambda_t(e) \subseteq \Omega_t(e)$. This establishes the equivalence between $\Lambda_t(e)$ and $\Omega_t(e)$.

Rewrite the inequality in (12) as

$$\phi(c) \geq \beta(1+r) \int_X \phi\left[g_{t+1}\left(z(a,e) - c, e'\right)\right] Q\left(e, de'\right).$$

Since the marginal utility function $\phi(\cdot)$ is strictly decreasing, this inequality is equivalent to

$$c \leq \phi^{-1}\left\{\beta(1+r) \int_X \phi\left[g_{t+1}\left(z(a,e) - c, e'\right)\right] Q\left(e, de'\right)\right\}.$$

For each $e \in X$, define a function $\Psi_{t+1}(\cdot; e) : A_{t+1} \to \mathcal{D}$ according to

$$\Psi_{t+1}(a'; e) \equiv \phi^{-1}\left\{\beta(1+r)\int_X \phi\left[g_{t+1}(a', e')\right] Q(e, de')\right\}. \tag{13}$$

Then the set $\Lambda_t(e)$ can be rewritten as

$$\Lambda_t(e) \equiv \{(c; a) \in \mathcal{D} \times A_t : c \in \Phi_t(a, e) \text{ and } c \leq \Psi_{t+1}(z(a, e) - c; e)\}.$$

This set is convex if $\Psi_{t+1}(\cdot; e)$ is a concave function. The converse, however, is not necessarily true.

Part 1

Suppose $u'''(c) = 0$ for all $c \geq \underline{c}$. In other words, the utility function $u(\cdot)$ is quadratic and the marginal utility function can be expressed as

$$\phi(c) = \gamma_1 + \gamma_2 c,$$

with $\gamma_2 < 0$ and $\gamma_1 + \gamma_2 \underline{c} > 0$. It is then straightforward to show that

$$\Psi_{t+1}(a'; e) \equiv \frac{[\beta(1+r) - 1]\gamma_1}{\gamma_2} + \beta(1+r)\int_X g_{t+1}(a', e') Q(e, de'),$$

for all $a' \in A_{t+1}$ and for all $e \in X$. The concavity of $\Psi_{t+1}(\cdot; e)$ follows immediately from an inductive argument. In the terminal period, the policy function is $g_T(a, e) \equiv we + (1+r)a$, for all $(a, e) \in S_T$. Suppose $g_{t+1}(\cdot, e')$ is concave for all $e' \in X$. Since concavity is preserved under integration, the function $\Psi_{t+1}(\cdot; e)$ is also concave for all $e \in X$. Hence $\Lambda_t(e)$ is a convex set for all $e \in X$.

Part 2

Suppose now $u'''(c) > 0$ for all $c \geq \underline{c}$. Again we use an inductive argument to establish the concavity of $\Psi_{t+1}(\cdot; e)$. Suppose $g_{t+1}(\cdot, e')$ is concave for all $e' \in X$. In the following proof, we first establish the concavity of $\Psi_{t+1}(\cdot; e)$ for the case when X is a finite point set. We then extend this result to more general state spaces.

Suppose the labor endowment process $\{e_t\}$ follows a discrete state-space Markov process. Formally, let $X = \{\widetilde{e}_1, \widetilde{e}_2, ..., \widetilde{e}_N\}$ be the state space of the process and $Q = [q_{i,j}]$ be the transition probability matrix. The (i, j)th entry of Q denotes the probability of moving from state $\widetilde{e}_i$ to state $\widetilde{e}_j$ in one period, hence $q_{i,j} \in [0, 1]$ for all i and j.

Suppose the current state of the Markov process is $\widetilde{e}_i$. Then

$$\Psi_{t+1}\left(a'; \widetilde{e}_i\right) \equiv \phi^{-1}\left\{\beta\left(1+r\right)\sum_{j=1}^{N} q_{i,j}\phi\left[g_{t+1}\left(a', \widetilde{e}_j\right)\right]\right\}. \tag{14}$$

Pick any a'_1 and a'_2 in A_{t+1}. Define $a'_\lambda \equiv \lambda a'_1 + (1-\lambda)\, a'_2$ for any $\lambda \in (0, 1)$. Concavity of $g_{t+1}(\cdot, e')$ implies

$$g_{t+1}\left(a'_\lambda, e'\right) \geq \lambda g_{t+1}\left(a'_1, e'\right) + (1-\lambda)\, g_{t+1}\left(a'_2, e'\right),$$

for all $e' \in X$. Since the marginal utility function $\phi(\cdot)$ is strictly decreasing, we have

$$\phi\left[g_{t+1}\left(a'_\lambda, e'\right)\right] \leq \phi\left[\lambda g_{t+1}\left(a'_1, e'\right) + (1-\lambda)\, g_{t+1}\left(a'_2, e'\right)\right],$$

and hence

$$\beta\left(1+r\right)\sum_{j=1}^{N} q_{i,j}\phi\left[g_{t+1}\left(a'_\lambda, \widetilde{e}_j\right)\right] \leq \beta\left(1+r\right)\sum_{j=1}^{N} q_{i,j}\phi\left[\lambda g_{t+1}\left(a'_1, \widetilde{e}_j\right) + (1-\lambda)\, g_{t+1}\left(a'_2, \widetilde{e}_j\right)\right].$$

Since the inverse function $\phi^{-1}(\cdot)$ is strictly decreasing, we can write

$$\begin{aligned} \Psi_{t+1}\left(a'_\lambda; \widetilde{e}_i\right) &\equiv \phi^{-1}\left\{\beta\left(1+r\right)\sum_{j=1}^{N} q_{i,j}\phi\left[g_{t+1}\left(a'_\lambda, \widetilde{e}_j\right)\right]\right\} \\ &\geq \phi^{-1}\left\{\beta\left(1+r\right)\sum_{j=1}^{N} q_{i,j}\phi\left[\lambda g_{t+1}\left(a'_1, \widetilde{e}_j\right) + (1-\lambda)\, g_{t+1}\left(a'_2, \widetilde{e}_j\right)\right]\right\}. \end{aligned}$$

Define two sequences of positive real numbers $\{x_j\}_{j=1}^{N}$ and $\{y_j\}_{j=1}^{N}$ by $x_j \equiv g_{t+1}(a'_1, \widetilde{e}_j)$ and

$y_j \equiv g_{t+1}(a_2', \widetilde{e}_j)$ for all j. The function $\Psi_{t+1}(\cdot; \widetilde{e}_i)$ is concave if

$$\begin{aligned} & \phi^{-1}\left\{\beta(1+r)\sum_{j=1}^{N} q_{i,j}\phi[\lambda x_j + (1-\lambda) y_j]\right\} \\ \geq\ & \lambda \Psi_{t+1}(a_1'; \widetilde{e}_i) + (1-\lambda)\Psi_{t+1}(a_2'; \widetilde{e}_i) \\ \equiv\ & \lambda\phi^{-1}\left\{\beta(1+r)\sum_{j=1}^{N} q_{i,j}\phi(x_j)\right\} + (1-\lambda)\phi^{-1}\left\{\beta(1+r)\sum_{j=1}^{N} q_{i,j}\phi(y_j)\right\}. \end{aligned}$$

In other words, if the function $\theta : (\underline{c}, \infty)^N \to \mathcal{D}$ defined by

$$\theta(\mathbf{x}) \equiv \phi^{-1}\left\{\beta(1+r)\sum_{j=1}^{N} q_{i,j}\phi(x_j)\right\}, \tag{15}$$

is concave, then $\Psi_{t+1}(\cdot; \widetilde{e}_i)$ is also concave and the hypograph of $g_t(\cdot, ; \widetilde{e}_i)$ is a convex set for all $\widetilde{e}_i \in X$.

The function $\theta(\mathbf{x})$ is concave if and only if its Hessian matrix is negative semi-definite. Let $H(\mathbf{x}) = [h_{m,n}(\mathbf{x})]$ be the Hessian matrix of $\theta(\cdot)$ evaluated at a point $\mathbf{x}$. Then for any column vector $\mathbf{z} \in \mathbb{R}^N$, $\mathbf{z}^T \cdot H(\mathbf{x})\mathbf{z} \leq 0$ if and only if [9]

$$\frac{\{\phi'[\theta(\mathbf{x})]\}^2}{\phi''[\theta(\mathbf{x})]} \geq \beta(1+r)\frac{\left[\sum_{m=1}^{N} q_{i,m} z_m \phi'(x_m)\right]^2}{\left[\sum_{m=1}^{N} q_{i,m} z_m^2 \phi''(x_m)\right]}. \tag{16}$$

Using the definitions in (7) and (8), we can rewrite the left-hand side of this inequality as

$$\begin{aligned} \frac{\{\phi'[\theta(\mathbf{x})]\}^2}{\phi''[\theta(\mathbf{x})]} &\equiv \eta[\theta(\mathbf{x})] = \eta\left[\phi^{-1}\left\{\beta(1+r)\sum_{m=1}^{N} q_{i,m}\phi(x_m)\right\}\right] \\ &= \Phi\left[\beta(1+r)\sum_{j=1}^{N} q_{i,m}\phi(x_m)\right]. \end{aligned}$$

[9] The mathematical derivation of this result can be found in Appendix B.

Using Assumption A4, we have

$$
\begin{aligned}
\frac{\left\{\phi'\left[\theta\left(\mathbf{x}\right)\right]\right\}^2}{\phi''\left[\theta\left(\mathbf{x}\right)\right]} &= \Phi\left[\beta\left(1+r\right)\sum_{j=1}^{N} q_{i,m}\phi\left(x_m\right)\right] \\
&\geq \beta\left(1+r\right)\sum_{j=1}^{N} q_{i,m}\Phi\left[\phi\left(x_m\right)\right] = \beta\left(1+r\right)\sum_{j=1}^{N} q_{i,m}\frac{\left[\phi'\left(x_m\right)\right]^2}{\phi''\left(x_m\right)}. \qquad (17)
\end{aligned}
$$

Finally, we use the Cauchy-Schwartz inequality to show that (17) implies (16). Define two sequences of real numbers $\{b_m\}_{m=1}^{N}$ and $\{d_m\}_{m=1}^{N}$ according to $b_m \equiv \left[q_{i,m}\phi''\left(x_m\right)\right]^{\frac{1}{2}} z_m$ and $d_m \equiv \left\{q_{i,m}\left[\phi'\left(x_m\right)\right]^2/\phi''\left(x_m\right)\right\}^{\frac{1}{2}}$. Then by the Cauchy-Schwartz inequality,

$$
\begin{aligned}
\left(\sum_{m=1}^{N} b_m^2\right)\left(\sum_{m=1}^{N} d_m^2\right) &= \left[\sum_{m=1}^{N} q_{i,m}z_m^2\phi''\left(x_m\right)\right]\left[\sum_{m=1}^{N} q_{i,m}\frac{\left[\phi'\left(x_m\right)\right]^2}{\phi''\left(x_m\right)}\right] \\
&\geq \left(\sum_{m=1}^{N} b_m d_m\right)^2 \\
&= \left[\sum_{m=1}^{N} q_{i,m}z_m\phi'\left(x_m\right)\right]^2.
\end{aligned}
$$

Since the marginal utility function is strictly convex, i.e., $\phi''\left(\cdot\right) > 0$, this can be rearranged to become

$$
\sum_{m=1}^{N} q_{i,m}\frac{\left[\phi'\left(x_m\right)\right]^2}{\phi''\left(x_m\right)} \geq \frac{\left[\sum_{m=1}^{N} q_{i,m}z_m\phi'\left(x_m\right)\right]^2}{\left[\sum_{m=1}^{N} q_{i,m}z_m^2\phi''\left(x_m\right)\right]}.
$$

Hence (17) implies (16). This establishes the concavity of $\theta\left(\mathbf{x}\right)$ which implies the concavity of $\Psi_{t+1}\left(\cdot;\widetilde{e}_i\right)$ for each $\widetilde{e}_i \in X$. As a result, the hypograph of $g_t\left(\cdot,\widetilde{e}_i\right)$ is a convex set and $g_t\left(\cdot,\widetilde{e}_i\right)$ is concave for each $\widetilde{e}_i \in X$. This completes the inductive argument for the case when X is a finite point set.

The above result can be readily extended to more general state spaces. Suppose now X is a countably infinite set of positive real numbers. For each state i, the transition probabilities $\{q_{i,j}\}_{j=1}^{\infty}$ is a sequence of positive real numbers that satisfies $\sum_{j=1}^{\infty} q_{i,j} = 1$. For each $\widetilde{e}_i \in X$, the function $\Psi_{t+1}(\cdot;\widetilde{e}_i) : A_{t+1} \to \mathcal{D}$ is now given by

$$\Psi_{t+1}\left(a';\widetilde{e}_i\right) \equiv \phi^{-1}\left\{\beta(1+r)\sum_{j=1}^{\infty} q_{i,j}\phi\left[g_{t+1}\left(a',\widetilde{e}_j\right)\right]\right\}.$$

Since $g_{t+1}(a',e') > \underline{c}$ for all $(a',e') \in S_{t+1}$, $\phi[g_{t+1}(a',e')]$ is bounded above for all possible states in S_{t+1}. Hence the infinite series in the above expression is convergent. For each positive integer N, define $\Psi^N_{t+1}(\cdot;\widetilde{e}_i)$ according to

$$\Psi^N_{t+1}\left(a';\widetilde{e}_i\right) \equiv \phi^{-1}\left\{\beta(1+r)\sum_{j=1}^{N} \widetilde{q}_{i,j}\phi\left[g_{t+1}\left(a',\widetilde{e}_j\right)\right]\right\},$$

where $\widetilde{q}_{i,j} \equiv q_{i,j}/\sum_{j=1}^{N} q_{i,j}$. Then $\left\{\Psi^N_{t+1}(\cdot;\widetilde{e}_i)\right\}$ forms a sequence of finite concave function which converges pointwise to $\Psi_{t+1}(\cdot;\widetilde{e}_i)$ as $\phi^{-1}(\cdot)$ is continuous. By Theorem 10.8 in Rockafellar [9], the limiting function $\Psi_{t+1}(\cdot;\widetilde{e}_i)$ is also a concave function on A_{t+1}.

We now extend this result to the case when X is a compact interval in $\mathbb{R}_+$. For each $a' \in A_{t+1}$, $\phi[g_{t+1}(a',\cdot)]$ is bounded and continuous on $X = [\underline{e},\overline{e}]$. Hence it is Riemann integrable on X. In the following proof it is more convenient to deal with distribution function rather than probability measure. For each $e \in X$, define the conditional distribution function $F(\cdot|e) : X \to [0,1]$ according to $F(e'|e) \equiv Q(e,[\underline{e},e'])$ for all $e' \in X$. For each $e \in X$, the function $\Psi_{t+1}(\cdot;e) : A_{t+1} \to \mathcal{D}$ is now defined as

$$\Psi_{t+1}\left(a';e\right) \equiv \phi^{-1}\left\{\beta(1+r)\int_X \phi\left[g_{t+1}\left(a',e'\right)\right]dF\left(e'|e\right)\right\},$$

which is equivalent to (13).

Let N be a positive integer and $\{\widetilde{e}_i\}_{i=0}^N$ be an arbitrary partition of X so that

$$\underline{e} = \widetilde{e}_0 \leq \widetilde{e}_1 \leq ... \leq \widetilde{e}_N = \overline{e}.$$

Using this partition, define a set of real numbers $\{p_i(e)\}_{i=1}^N$ according to $p_i(e) = F(\widetilde{e}_i|e) -$

$F(\widetilde{e}_{i-1}|e) \geq 0$, for $i = 1, ..., N$ and a step function

$$F_N(e'|e) = \sum_{i=1}^{N} \chi_i(e') F(\widetilde{e}_i|e),$$

where $\chi_i(e') = 1$ if $e' \in [\widetilde{e}_{i-1}, \widetilde{e}_i]$ and zero elsewhere. The function $F_N(e'|e)$ can be interpreted as the distribution function of a discrete random variable with masses $\{p_i(e)\}_{i=1}^{N}$ on the set of points $\{\widetilde{e}_i\}_{i=1}^{N}$. Since $F_N(\cdot|e)$ converges to $F(\cdot|e)$ pointwise on X and $\phi[g_{t+1}(a', \cdot)]$ is bounded and continuous on X, we have

$$\sum_{i=1}^{N} p_i(e) \phi\left[g_{t+1}\left(a', \widetilde{e}_i\right)\right] \to \int_X \phi\left[g_{t+1}\left(a', e'\right)\right] dF\left(e'|e\right), \quad \text{for all } a' \in A_{t+1},$$

as N tends to infinity.[10] For each $e \in X$, define a function $\Psi_{t+1}^{N}(\cdot; e)$ according to

$$\Psi_{t+1}^{N}\left(a'; e\right) \equiv \phi^{-1}\left\{\beta(1+r) \sum_{i=1}^{N} p_i(e) \phi\left[g_{t+1}\left(a', \widetilde{e}_i\right)\right]\right\}.$$

Based on our previous result, $\Psi_{t+1}^{N}(\cdot; e)$ is a concave function for all $e \in X$ and for all N. By the continuity of $\phi^{-1}(\cdot)$, we have $\Psi_{t+1}^{N}(a'; e) \to \Psi_{t+1}(a'; e)$ for all $a' \in A_{t+1}$. Hence $\left\{\Psi_{t+1}^{N}(\cdot; e)\right\}$ forms a sequence of finite concave functions that converges pointwise to $\Psi_{t+1}(\cdot; e)$. The limiting function $\Psi_{t+1}(\cdot; e)$ is also a concave function on A_{t+1}.This completes the proof of Theorem 3.

6 Concluding Remarks

The main contribution of this paper is twofold. First, we formally prove that, in the presence of borrowing constraints, the policy function for consumption at any point in time is a concave function of current asset holdings if the period utility function exhibits hyperbolic absolute risk aversion and has a nonnegative third derivative. This result essentially generalizes the main result in Carroll and Kimball [3] to an environment with borrowing constraints. The second contribution

[10] This uses the following result: Let $\{F_n\}$ be a sequence of distribution functions defined on a set X. If $\lim_{n\to\infty} F_n(x) = F(x)$ for all $x \in X$, then $\int_X h(x)\, dF_n(x) \to \int_X h(x)\, dF(x)$ for all bounded, continuous, real-valued function h defined on X. For a proof of this statement, see Severini [11, p.325-328].

of this paper is to show that hyperbolic absolute risk aversion is not necessary for the consumption function to be concave. The current study also provides specific conditions under which a non-HARA utility function would satisfy the conditions for concave consumption function.

Appendix A

Proof of Theorem 1

The proof of this theorem is divided into two parts. The first part of the proof establishes the boundedness and the continuity of the value functions. Once these properties are established, the proofs of strict monotonicity and strict concavity are straightforward and is thus omitted. The second part of the proof establishes the differentiability of $V_t(\cdot, e)$ for each t and for all $e \in X$. An inductive argument is used in each part. For each $t \in \{0, 1, ..., T\}$, define $q_t \equiv w\underline{e} - (1+r)\underline{a}_t + \underline{a}_{t+1}$.

Part 1: Boundedness and Continuity

In the terminal period, the value function is given by

$$V_T(a, e) = u[we + (1+r)a], \qquad \text{for all } (a, e) \in S_T.$$

This function is bounded above by $V_T(\overline{a}_T, \overline{e}) = u[w\overline{e} + (1+r)\overline{a}_T] < \infty$, bounded below by

$$V_T(-\underline{a}_T, \underline{e}) = u[w\underline{e} - (1+r)\underline{a}_T] > u(\underline{c}) \geq -\infty, \tag{18}$$

and continuous on S_T. The first inequality in (18) follows from Assumption A3.

Suppose $V_{t+1}(a, e)$ is bounded and continuous on S_{t+1} for some $t \leq T-1$. For each $(a, e) \in S_t$, define the constraint correspondence Φ_t according to

$$\Phi_t(a, e) \equiv \left\{c : \underline{c} \leq c \leq z(a, e) + \underline{a}_{t+1}\right\},$$

where $z(a, e) \equiv we + (1+r)a$. Define the objective function at time t as

$$W_t(c; a, e) \equiv u(c) + \beta \int_X V_{t+1}\left[z(a, e) - c, e'\right] Q\left(e, de'\right).$$

Since $V_{t+1}(a, e)$ is bounded and continuous on S_{t+1}, the conditional expectation in the above expression is well-defined. By Assumptions A1-A2 and the induction hypothesis, W_t is continuous whenever it is finite. If $u(\underline{c}) > -\infty$, then the objective function $W_t(c; a, e)$ is bounded and continuous on $\Phi_t(a, e)$ for all $(a, e) \in S_t$. By the Theorem of the Maximum, the value function V_t is continuous and the optimal policy correspondence g_t defined in (4) is non-empty, compact-valued and upper hemicontinuous. Since $W_t(c; a, e)$ is bounded for all $c \in \Phi_t(a, e)$ and for all $(a, e) \in S_t$, the value function V_t is also bounded.

Suppose now $u(\underline{c}) = -\infty$. In this case, $W_t(\underline{c}; a, e) = -\infty$ for all $(a, e) \in S_t$. This means for any possible states at time t, the objective function in (P1) is not continuous on the constraint set. Thus we cannot apply the Theorem of the Maximum directly. However, the same results can be obtained with some additional effort. The following argument is similar to Lemma 2 in Alvarez and Stokey [1]. Under Assumption A3, we have $z(a, e) + \underline{a}_{t+1} \geq q_t > \underline{c}$, for all $(a, e) \in S_t$ and for all t. Define a modified constraint correspondence Φ_t^* according to

$$\Phi_t^*(a, e) \equiv \left\{c : q_t \leq c \leq z(a, e) + \underline{a}_{t+1}\right\}.$$

This correspondence is non-empty, compact-valued and continuous. Most importantly, the objective function $W_t(c; a, e)$ is finite and continuous on $\Phi_t^*(a, e)$ for all $(a, e) \in S_t$. Define the set of maximizers of $W_t(c; a, e)$ on $\Phi_t^*(a, e)$ as

$$g_t^*(a, e) \equiv \underset{c \in \Phi_t^*(a,e)}{\arg\max} \{W_t(c; a, e)\}.$$

Then $g_t^*(a, e)$ is non-empty and $W_t(c^*; a, e) > -\infty$ for any $c^* \in g_t^*(a, e)$. Let $c^* \in g_t^*(a, e)$. If $W_t(c^*; a, e) \geq W_t(c; a, e)$ for all $c \in [\underline{c}, q_t)$, then $c^* \in g_t(a, e)$. Suppose there exists $\widetilde{c} \in [\underline{c}, q_t)$ such that $W_t(\widetilde{c}; a, e) > W_t(c^*; a, e) > -\infty$, then $\widetilde{c} \in g_t(a, e)$. In either case, the optimal policy correspondence $g_t(a, e)$ is non-empty. Note that in the latter case, $\widetilde{c}$ must be strictly greater than $\underline{c}$ because $W_t(\widetilde{c}; a, e) > -\infty$. It follows that $c > \underline{c}$ whenever $c \in g_t(a, e)$. Hence $V_t(a, e) > -\infty$. Since $W_t(c; a, e)$ is still bounded above for all $c \in \Phi_t(a, e)$ and for all $(a, e) \in S_t$, $V_t(a, e)$ is also bounded above.

We now establish the continuity of V_t under the assumption that $u(\underline{c}) = -\infty$. Since V_t is single-valued, it suffice to show that it is upper hemicontinuous. Let $\{(a_n, e_n)\}$ be a sequence in S_t

that converges to some $(a,e) \in S_t$. Pick a sequence of consumption $\{c_n\}$ such that $c_n \in g_t(a_n, e_n)$ for each n. Such a sequence can always be drawn because $g_t(a_n, e_n)$ is non-empty for all n. Since $c_n \in \Phi_t(a_n, e_n)$ and Φ_t is compact-valued and upper hemicontinuous, there exists a subsequence of $\{c_n\}$, denoted by $\{c_{n_k}\}$, such that c_{n_k} converges to some $c \in \Phi_t(a,e)$. Since $c_{n_k} \in g_t(a_{n_k}, e_{n_k})$, it follows that $c_{n_k} > \underline{c}$ and $W_t(c_{n_k}; a_{n_k}, e_{n_k}) > -\infty$ for all n_k. By the continuity of W_t, we have $V_t(a_{n_k}, e_{n_k}) = W_t(c_{n_k}; a_{n_k}, e_{n_k}) \to W_t(c; a, e)$. If we can show that $W_t(c; a, e) = V_t(a,e)$, then this establishes (i) the upper hemicontinuity of V_t at (a,e), and (ii) $c \in g_t(a,e)$ which implies the upper hemicontinuity of g_t.

Suppose the contrary that there exists $\widehat{c} \in \Phi_t(a,e)$ such that $W_t(\widehat{c}; a, e) > W_t(c; a, e) > -\infty$. This implies $\widehat{c} > \underline{c}$. Since Φ_t is lower hemicontinuous and (a_{n_k}, e_{n_k}) converges to (a,e), there exists a sequence $\{\widehat{c}_{n_k}\}$ such that $\widehat{c}_{n_k} \in \Phi_t(a_{n_k}, e_{n_k})$ for all n_k and $\widehat{c}_{n_k}$ converges to $\widehat{c}$. Since $\widehat{c} > \underline{c}$, it follows that $\widehat{c}_{n_k} > \underline{c}$ when n_k is sufficiently large. Then by the continuity of W_t,

$$\lim_{n_k \to \infty} W_t(\widehat{c}_{n_k}; a_{n_k}, e_{n_k}) = W_t(\widehat{c}; a, e) > W_t(c; a, e) = \lim_{n_k \to \infty} W_t(c_{n_k}; a_{n_k}, e_{n_k}).$$

This means when n_k is sufficiently large, we have $W_t(\widehat{c}_{n_k}; a_{n_k}, e_{n_k}) > W_t(c_{n_k}; a_{n_k}, e_{n_k})$ which contradicts the fact that $c_{n_k} \in g_t(a_{n_k}, e_{n_k})$. Hence $W_t(c; a, e) = V_t(a,e)$ for any $(a,e) \in S_t$.

From (18), it is obvious that $V_T(\cdot, e)$ is strictly increasing and strictly concave for all $e \in X$. A straightforward inductive argument can be used to establish these properties for all $t \leq T-1$. Given the strict concavity of $V_t(\cdot, e)$, the optimal policy correspondences $g_t(a,e)$ and $h_t(a,e)$ are both single-valued continuous functions for all t. If $u(\underline{c}) = -\infty$, then the above argument shows that it is never optimal to choose $c_t = \underline{c}$. Hence $g_t(a,e) > \underline{c}$ for all $(a,e) \in S_t$. However, when $u(\underline{c}) > -\infty$ it is still possible to have $g_t(a,e) = \underline{c}$ for some (a,e).

Part 2: Differentiability

Fix $e \in X$. Let $V_t^+(a,e)$ be the right-hand derivative of $V_t(\cdot, e)$ at any $a \in [-\underline{a}_t, \overline{a}_t)$ and $V_t^-(a,e)$ be the left-hand derivative of $V_t(\cdot, e)$ at $a \in (-\underline{a}_t, \overline{a}_t)$. Since $V_t(\cdot, e)$ is strictly concave, both $V_t^+(a,e)$ and $V_t^-(a,e)$ exist and are finite for all $a \in (-\underline{a}_t, \overline{a}_t)$. To show that $V_t(a,e)$ is differentiable on $[-\underline{a}_t, \overline{a}_t)$, we need to establish two properties: (i) it is differentiable in the interior of A_t, and (ii)

$V_t^+(-\underline{a}_t, e)$ exists and is finite. To establish the first property, we will appeal to Theorem 25.1 in Rockafellar [9]. This theorem states that if the set of supergradients of $V_t(\cdot, e)$ at point a is a singleton, then $V_t(\cdot, e)$ is differentiable at a. Recall that a real number $\lambda(a)$ is a supergradient of $V_t(\cdot, e)$ at $a \in A_t$ if it satisfies the following condition

$$V_t(\widetilde{a}, e) - V_t(a, e) \leq \lambda(a) \cdot (\widetilde{a} - a), \qquad \text{for every } \widetilde{a} \in A_t.$$

Both $V_t^+(a, e)$ and $V_t^-(a, e)$ are supergradients at a. Any supergradient at $a \in (-\underline{a}_t, \overline{a}_t)$ must also satisfy $V_t^+(a, e) \leq \lambda(a) \leq V_t^-(a, e) < \infty$.

Again we use an inductive argument in the following proof. In the terminal period, we have

$$g_T(a, e) = we + (1 + r)a \geq q_T > \underline{c},$$

and $V_T(a, e) = u[we + (1 + r)a]$, for all $(a, e) \in S_T$. Given Assumption A1, $V_T(\cdot, e)$ is continuously differentiable in the interior of A_T. The derivative of $V_T(\cdot, e)$ in the interior of A_T is given by $p_T(a, e) = (1 + r)u'[g_T(a, e)]$. Also, the right-hand derivative of $V_T(\cdot, e)$ at $a = -\underline{a}_T$ exists and is given by $(1 + r)u'[g_T(-\underline{a}_T, e)]$ which is finite.

Suppose the desired result is true for some $t + 1 \leq T$ and $g_{t+1}(a, e) > \underline{c}$ for all $(a, e) \in S_{t+1}$. The remaining proof is divided into several steps. Steps 1-4 essentially establish all the results in Theorem 2.

Step 1 First, we show that if $g_t(a, e) > \underline{c}$, then $g_t(a, e)$ and $h_t(a, e)$ satisfies the following condition

$$u'[g_t(a, e)] \geq \beta \int_X p_{t+1}[h_t(a, e), e'] Q(e, de'). \tag{19}$$

If in addition $h_t(a, e) > -\underline{a}_{t+1}$, then (19) holds with equality.

Fix $(a, e) \in S_t$. Define $\widetilde{c} = g_t(a, e)$ and $\widetilde{a}' = h_t(a, e)$. If $\widetilde{c} > \underline{c}$, then $\widetilde{a}' < \overline{a}_{t+1}$. Suppose now we increase $\widetilde{a}'$ by $\varepsilon > 0$, reduce $\widetilde{c}$ by $\varepsilon > 0$ but maintain $\widetilde{c} - \varepsilon > \underline{c}$. The utility loss generated by this is $u(\widetilde{c}) - u(\widetilde{c} - \varepsilon)$. The utility gain generated by this is

$$\beta \int_X \left[V_{t+1}\left(\widetilde{a}' + \varepsilon, e'\right) - V_{t+1}\left(\widetilde{a}', e'\right)\right] Q\left(e, de'\right).$$

If the borrowing constraint is binding originally, i.e., $\widetilde{a}' = -\underline{a}_{t+1}$, then any reduction in consumption would lower the value of the objective function. This means the loss in utility is no less than the gain for any $\varepsilon > 0$ so that

$$\frac{u(\widetilde{c}) - u(\widetilde{c} - \varepsilon)}{\varepsilon} \geq \beta \int_X \left[\frac{V_{t+1}(\widetilde{a}' + \varepsilon, e') - V_{t+1}(\widetilde{a}', e')}{\varepsilon}\right] Q\left(e, de'\right).$$

By taking the limit $\varepsilon \to 0+$, we get

$$u'(\widetilde{c}) \geq \lim_{\varepsilon \to 0+} \left\{\beta \int_X \left[\frac{V_{t+1}(\widetilde{a}' + \varepsilon, e') - V_{t+1}(\widetilde{a}', e')}{\varepsilon}\right] Q\left(e, de'\right)\right\}. \tag{20}$$

Since $V_{t+1}(a, e)$ is strictly concave in a, the function

$$\Gamma(\eta; a, e) \equiv \frac{V_{t+1}(a + \eta, e) - V_{t+1}(a, e)}{\eta} > 0$$

is strictly decreasing in η for any $a \in \left[-\underline{a}_{t+1}, \overline{a}_{t+1}\right)$. Hence, it is bounded above by $p_{t+1}(a, e)$ which is finite as $V_{t+1}(a, e)$ is differentiable on $\left[-\underline{a}_{t+1}, \overline{a}_{t+1}\right)$. By the Lebesgue Convergence Theorem, the limit in (20) can be expressed as

$$\int_X \lim_{\varepsilon \to 0+} \left[\frac{V_{t+1}(\widetilde{a}' + \varepsilon, e') - V_{t+1}(\widetilde{a}', e')}{\varepsilon}\right] Q\left(e, de'\right) = \int_X p_{t+1}(\widetilde{a}', e')\, Q\left(e, de'\right).$$

Substituting this into (20) gives (19) for the case when $\widetilde{a}' = h_t(a, e) = -\underline{a}_{t+1}$. If $\widetilde{a}' = h_t(a, e) > -\underline{a}_{t+1}$, then any infinitesimal change in consumption would not affect the maximized value of the objective. This means (20) will hold with equality. It follows from the above argument that (19) holds with equality when $\widetilde{a}' = h_t(a, e) > -\underline{a}_{t+1}$.

Step 2 Using a similar perturbation argument as in Step 1, we can show that if $g_t(a,e) = \underline{c}$, then the following condition must be satisfied

$$u'(\underline{c}+) \leq \beta \int_X p_{t+1}\left[h_t(a,e), e'\right] Q\left(e, de'\right), \tag{21}$$

where $u'(\underline{c}+)$ denote the right-hand derivative of $u(\cdot)$ at $\underline{c}$. Suppose $\widetilde{c} = g_t(a,e) = \underline{c}$ for some $(a,e) \in S_t$. Suppose now we increase $\widetilde{c}$ by $\varepsilon > 0$, reduce $\widetilde{a}' = h_t(a,e)$ by $\varepsilon > 0$, but maintain $\widetilde{a}' - \varepsilon \geq -\underline{a}_{t+1}$. If it is optimal to consume $\underline{c}$, then any infinitesimal increase in consumption would either lower or have no effect on the value of the objective function, i.e.,

$$\lim_{\varepsilon \to 0+}\left[\frac{u(\underline{c}+\varepsilon) - u(\underline{c})}{\varepsilon}\right] = u'(\underline{c}+) \leq \beta \int_X \lim_{\varepsilon \to 0+}\left[\frac{V_{t+1}(\widetilde{a}', e') - V_{t+1}(\widetilde{a}' - \varepsilon, e')}{\varepsilon}\right] Q\left(e, de'\right).$$

Since $V_{t+1}(a,e)$ is differentiable on $\left[-\underline{a}_{t+1}, \overline{a}_{t+1}\right)$, the left-hand derivative in the above expression exists and is given by $p_{t+1}\left[h_t(a,e), e'\right]$.

Step 3 We now show that $g_t(a,e) > \underline{c}$ if either $u'(\underline{c}+) = +\infty$ or $\beta(1+r) \leq 1$. If $u'(\underline{c}+) = +\infty$, then (21) cannot be satisfied and hence it is never optimal to consume the minimum level $\underline{c}$. Consider the case when $u'(\underline{c}+) < +\infty$ and $\beta(1+r) \leq 1$. By the induction hypothesis, we have $g_{t+1}(a', e') > \underline{c}$ and $p_{t+1}(a', e') = (1+r)\, u'\left[g_{t+1}(a', e')\right]$ for all $(a', e') \in S_{t+1}$. Fix $(a,e) \in S_t$. Suppose the contrary that $g_t(a,e) = \underline{c}$. Then by Step 2, it must be the case that

$$\begin{aligned} u'(\underline{c}+) &\leq \beta(1+r) \int_X u'\left[g_{t+1}\left(h_t(a,e), e'\right)\right] Q\left(e, de'\right) \\ &\leq \int_X u'\left[g_{t+1}\left(h_t(a,e), e'\right)\right] Q\left(e, de'\right) \\ &< u'(\underline{c}+). \end{aligned}$$

The second inequality uses the assumption that $\beta(1+r) \leq 1$. The third inequality uses the fact that $g_{t+1}(a', e') > \underline{c}$ for all $(a', e') \in S_{t+1}$. This gives rise to a contradiction. Hence $g_t(a,e) > \underline{c}$ for all $(a,e) \in S_t$. This means the optimal policy functions would only satisfy (19), but not (21).

Step 4 We now show that $g_t(\cdot,e)$ is strictly increasing and $h_t(\cdot,e)$ is non-decreasing in a for all $e \in X$. Fix $e \in X$. Pick any $a_2 > a_1 \geq -\underline{a}_t$. Suppose the contrary that $g_t(a_2,e) \leq g_t(a_1,e)$. This has two implications: $h_t(a_2,e) > h_t(a_1,e) \geq -\underline{a}_{t+1}$ and $u'[g_t(a_2,e)] \geq u'[g_t(a_1,e)]$. Since $p_{t+1}(a',e')$ is strictly decreasing in a',

$$p_{t+1}\left[h_t(a_2,e),e'\right] < p_{t+1}\left[h_t(a_1,e),e'\right],$$

for all $e' \in X$. Taking the expectation of this and applying (19) gives

$$\begin{aligned} u'[g_t(a_2,e)] &= \beta \int_X p_{t+1}\left[h_t(a_2,e),e'\right] Q\left(e,de'\right) \\ &< \beta \int_X p_{t+1}\left[h_t(a_1,e),e'\right] Q\left(e,de'\right) \leq u'[g_t(a_1,e)]. \end{aligned}$$

The first equality follows from the fact that $h_t(a_2,e) > -\underline{a}_{t+1}$. This contradicts $u'[g_t(a_2,e)] \geq u'[g_t(a_1,e)]$. Hence $g_t(\cdot,e)$ is strictly increasing. A similar argument can be used to show that $h_t(\cdot,e)$ is a non-decreasing function.

Step 5 We now show that $V_t(\cdot,e)$ is differentiable in the interior of A_t and $p_t(\cdot,e) = (1+r)u'[g_t(\cdot,e)]$ for each $e \in X$. Fix $e \in X$. Let $\lambda_t(a,e)$ be a supergradient of $V_t(a,e)$ at $a \in (-\underline{a}_t, \overline{a}_t)$. Since $V_t(\cdot,e)$ is strictly increasing and strictly concave on A_t, $\lambda_t(a,e)$ is strictly positive and finite. The main idea of the proof is to show that for any $a \in (-\underline{a}_t, \overline{a}_t)$, there exists a neighborhood $N(a)$ of $g_t(a,e)$ such that $g_t(a,e)$ is an interior solution of the following problem

$$\max_{c \in N(a)} \left\{(1+r)\,u(c) - \lambda_t(a,e)\,c\right\}. \tag{P4}$$

This problem is well-posed as $\lambda_t(a,e)$ is finite and the objective function is strictly concave. Thus, its solution must satisfy the first-order condition

$$\lambda_t(a,e) = (1+r)\,u'[g_t(a,e)].$$

Since this is true for *any* supergradient $\lambda_t(a,e)$, this means $(1+r)\,u'[g_t(a,e)]$ is the unique supergradient at $a \in (-\underline{a}_t, \overline{a}_t)$ and so $V_t(\cdot,e)$ is differentiable at a. We now establish the key steps of this argument. Fix $a \in (-\underline{a}_t, \overline{a}_t)$. Since $g_t(a,e) > \underline{c} \geq 0$ and $(1+r) > 0$, we can find an $\varepsilon > 0$ such that

$$a - \varepsilon \geq \max\left\{-\underline{a}_t, a - \frac{g_t(a,e) - \underline{c}}{1+r}\right\}.$$

For any $\widetilde{a} \in (a-\varepsilon, a+\varepsilon)$, define c according to

$$c = we + (1+r)\,\widetilde{a} - h_t(a,e) = (1+r)(\widetilde{a} - a) + g_t(a,e). \tag{22}$$

which is strictly greater than $\underline{c}$ as $\widetilde{a} > a - \varepsilon \geq a - [g_t(a,e) - \underline{c}]/(1+r)$. In other words, both c and $h_t(a,e)$ are feasible when the current state is $(\widetilde{a}, e)$. In addition, $\widetilde{a} \in (a-\varepsilon, a+\varepsilon)$ implies c is within a certain neighborhood of $g_t(a,e)$, i.e.,

$$N(a) = \{c : g_t(a,e) - (1+r)\,\varepsilon < c < g_t(a,e) + (1+r)\,\varepsilon\}.$$

Recall that $\lambda_t(a,e)$ is a supergradient of $V_t(a,e)$ at $a \in (-\underline{a}_t, \overline{a}_t)$. Then for any $\widetilde{a} \in A_t$, we have

$$V_t(\widetilde{a}, e) - V_t(a,e) \leq \lambda_t(a,e) \cdot (\widetilde{a} - a).$$

Since $\lambda_t(a,e)$ is finite, we can rewrite this as

$$V_t(\widetilde{a}, e) - \lambda_t(a,e)\,\widetilde{a} \leq V_t(a,e) - \lambda_t(a,e)\,a. \tag{23}$$

This inequality has to be true for any $\widetilde{a} \in A_t$. So pick $\widetilde{a} \in (a-\varepsilon, a+\varepsilon)$, define c as in (22). Then it follows from the definition of the value function and (23) that

$$
\begin{aligned}
& u(c) + \beta \int_X V_{t+1}\left[h_t(a,e), e'\right] Q\left(e, de'\right) - \lambda_t(a,e) \cdot \left[\frac{h_t(a,e) + c - we}{1+r}\right] \\
\leq\ & V_t(\widetilde{a}, e) - \lambda_t(a,e) \cdot \left[\frac{h_t(a,e) + c - we}{1+r}\right] \\
\leq\ & V_t(a, e) - \lambda_t(a,e) \cdot \left[\frac{h_t(a,e) + g_t(a,e) - we}{1+r}\right] \\
=\ & u\left[g_t(a,e)\right] + \beta \int_X V_{t+1}\left[h_t(a,e), e'\right] Q\left(e, de'\right) - \lambda_t(a,e) \cdot \left[\frac{h_t(a,e) + g_t(a,e) - we}{1+r}\right].
\end{aligned}
$$

This can be simplified to become

$$
(1+r)\, u(c) - \lambda_t(a,e)\, c \leq (1+r)\, u\left[g_t(a,e)\right] - \lambda_t(a,e)\, g_t(a,e),
$$

which is true for all c in $N(a)$. In other words, $g_t(a,e)$ is an interior solution of the problem (P4). This establishes the desired results.

Step 6 We now show that $V_t(a,e)$ is right-hand differentiable at $a = -\underline{a}_t$ and the right-hand derivative is given by $(1+r)\, u'\left[g_t(-\underline{a}_t, e)\right]$. Fix $e \in X$ and $a \in (-\underline{a}_t, \overline{a}_t)$. By the concavity of $u(\cdot)$, we have

$$
u\left[g_t(a,e)\right] - u\left[g_t(-\underline{a}_t, e)\right] \leq u'\left[g_t(-\underline{a}_t, e)\right]\left[g_t(a,e) - g_t(-\underline{a}, e)\right].
$$

By the result in Step 3, we know that $u'\left[g_t(-\underline{a}_t, e)\right] < u'(\underline{c}) \leq +\infty$. Similarly, by the concavity and differentiability of $V_{t+1}(\cdot, e')$ on $\left[-\underline{a}_{t+1}, \overline{a}_{t+1}\right)$, we have

$$
V_{t+1}\left(h_t(a,e), e'\right) - V_{t+1}(h_t(-\underline{a}_t, e), e') \leq p_{t+1}\left(h_t(-\underline{a}_t, e), e'\right) \cdot \left[h_t(a,e) - h_t(-\underline{a}_t, e)\right],
$$

for all $e' \in X$. Using these, we can write

$$\begin{aligned}
& V_t(a,e) - V_t(-\underline{a}_t, e) \\
= \; & u[g_t(a,e)] - u[g_t(-\underline{a}_t, e)] \\
& + \beta \int_X \left[V_{t+1}\left(h_t(a,e), e'\right) - V_{t+1}(h_t(-\underline{a}_t, e), e') \right] Q\left(e, de'\right) \\
\leq \; & u'[g_t(-\underline{a}_t, e)] [g_t(a,e) - g_t(-\underline{a}, e)] \\
& + \beta \left[\int_X p_{t+1}\left(h_t(-\underline{a}_t, e), e'\right) Q\left(e, de'\right) \right] [h_t(a,e) - h_t(-\underline{a}_t, e)] \\
\leq \; & u'[g_t(-\underline{a}_t, e)] [g_t(a,e) + h_t(a,e) - g_t(-\underline{a}_t, e) - h_t(-\underline{a}_t, e)] = u'[g_t(-\underline{a}_t, e)] (1+r)(a + \underline{a}_t).
\end{aligned}$$

The second line follows from the definition of $V_t(a,e)$ and $V_t(-\underline{a}_t, e)$. The fourth line is obtained by using condition (19) and the fact that $h_t(a,e)$ is non-decreasing in a. The last equality follows from the budget constraint. Thus we have

$$\frac{V_t(a,e) - V_t(-\underline{a}_t, e)}{a - (-\underline{a}_t)} \leq (1+r) u'[g_t(-\underline{a}_t, e)] < +\infty.$$

By taking the limit $a \to -\underline{a}_t+$, we can establish that $V_t^+(-\underline{a}_t, e)$ exists and is bounded above by $(1+r) u'[g_t(-\underline{a}_t, e)]$. Hence $V_t^+(-\underline{a}_t, e)$ is finite. Since $V_t^+(a,e)$ is strictly decreasing in a, we have

$$(1+r) u'[g_t(a,e)] = V_t^+(a,e) < V_t^+(-\underline{a}_t, e) \leq (1+r) u'[g_t(-\underline{a}_t, e)],$$

for all $a \in (-\underline{a}_t, \overline{a}_t)$. Since both $u'(c)$ and $g_t(\cdot, e)$ are continuous, we have $u'[g_t(a,e)] \to u'[g_t(-\underline{a}_t, e)]$ as $a \to -\underline{a}_t$. Hence $V_t^+(-\underline{a}_t, e) = (1+r) u'[g_t(-\underline{a}_t, e)]$. This completes the proof of Theorem 1.

Proof of Theorem 2

Part (i) of this theorem is proved in Step 3 of the second part of the above proof. Part (ii) of Theorem 2 is established in Step 1 of that proof. Part (iii) is established in Step 4 of that proof.

Appendix B

This section contains the technical details on how to derive the Hessian matrix $H(\mathbf{x})$ and the expression of $\mathbf{z}^T \cdot H(\mathbf{x})\,\mathbf{z}$ for any column vector $\mathbf{z} \in \mathbb{R}^N$. First, rewrite (15) as

$$\phi[\theta(\mathbf{x})] = \beta(1+r)\sum_{j=1}^{N} q_{i,j}\phi(x_j).$$

Differentiating this with respect to x_m gives

$$\phi'[\theta(\mathbf{x})]\,h_m(\mathbf{x}) = \beta(1+r)\,q_{i,m}\phi'(x_m), \tag{24}$$

$$\Rightarrow h_m(\mathbf{x}) = \beta(1+r)\,q_{i,m}\frac{\phi'(x_m)}{\phi'[\theta(\mathbf{x})]}, \tag{25}$$

where $h_m(\mathbf{x}) \equiv \partial\theta(\mathbf{x})/\partial x_m$. Differentiating (24) with respect to x_n gives

$$\phi''[\theta(\mathbf{x})]\,h_m(\mathbf{x})\,h_n(\mathbf{x}) + \phi'[\theta(\mathbf{x})]\,h_{m,n}(\mathbf{x}) = 0, \tag{26}$$

if $m \neq n$, and

$$\phi''[\theta(\mathbf{x})]\,[h_m(\mathbf{x})]^2 + \phi'[\theta(\mathbf{x})]\,h_{m,m}(\mathbf{x}) = \beta(1+r)\,q_{i,m}\phi''(x_m), \tag{27}$$

if $m = n$. Combining (25) and (26) gives

$$\begin{aligned} h_{m,n}(\mathbf{x}) &= -\frac{\phi''[\theta(\mathbf{x})]}{\phi'[\theta(\mathbf{x})]}h_m(\mathbf{x})\,h_n(\mathbf{x}) \\ &= -[\beta(1+r)]^2\,q_{i,m}q_{i,n}\phi'(x_m)\,\phi'(x_n)\,\frac{\phi''[\theta(\mathbf{x})]}{\{\phi'[\theta(\mathbf{x})]\}^3}, \end{aligned}$$

for $m \neq n$. Similarly, combining (25) and (27) gives

$$h_{m,m}(\mathbf{x}) = \beta(1+r)\,q_{i,m}\frac{\phi''(x_m)}{\phi'[\theta(\mathbf{x})]} - \left[\beta(1+r)\,q_{i,m}\phi'(x_m)\right]^2\frac{\phi''[\theta(\mathbf{x})]}{\{\phi'[\theta(\mathbf{x})]\}^3}.$$

For any $\mathbf{z} \in \mathbb{R}^N$, we have

$$\begin{aligned}
\mathbf{z}^T \cdot H(\mathbf{x})\,\mathbf{z} &= \beta(1+r)\frac{\sum_{m=1}^{N} q_{i,m} z_m^2 \phi''(x_m)}{\phi'[\theta(\mathbf{x})]} - [\beta(1+r)]^2 \left[\sum_{m=1}^{N} q_{i,m}\phi'(x_m)\right]^2 \frac{\phi''[\theta(\mathbf{x})]}{\left\{\phi'[\theta(\mathbf{x})]\right\}^3} \\
&= \beta(1+r)\frac{\phi''[\theta(\mathbf{x})]}{\left\{\phi'[\theta(\mathbf{x})]\right\}^3}\left[\sum_{m=1}^{N} q_{i,m} z_m^2 \phi''(x_m)\right]\left\{\frac{\left\{\phi'[\theta(\mathbf{x})]\right\}^2}{\phi''[\theta(\mathbf{x})]} - \beta(1+r)\frac{\left[\sum_{m=1}^{N} q_{i,m} z_m \phi'(x_m)\right]^2}{\left[\sum_{m=1}^{N} q_{i,m} z_m^2 \phi''(x_m)\right]}\right\}
\end{aligned}$$

Since $\beta(1+r) > 0$, $\phi'(\cdot) < 0$ and $\phi''(\cdot) > 0$, $\mathbf{z}^T \cdot H(\mathbf{x})\,\mathbf{z} \leq 0$ if and only if

$$\frac{\left\{\phi'[\theta(\mathbf{x})]\right\}^2}{\phi''[\theta(\mathbf{x})]} \geq \beta(1+r)\frac{\left[\sum_{m=1}^{N} q_{i,m} z_m \phi'(x_m)\right]^2}{\left[\sum_{m=1}^{N} q_{i,m} z_m^2 \phi''(x_m)\right]},$$

for all $\mathbf{z} \in \mathbb{R}^N$.

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