Copula-based Multivariate GARCH Model with Uncorrelated Dependent Errors*

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ABSTRACT

Multivariate GARCH (MGARCH) models are usually estimated under multivariate normality. In this paper, for non-elliptically distributed financial returns, we propose copula-based multivariate GARCH (C-MGARCH) model with uncorrelated dependent errors, which are generated through a linear combination of dependent random variables. The dependence structure is controlled by a copula function. Our new C-MGARCH model nests a conventional MGARCH model as a special case. The aim of this paper is to model MGARCH for non-normal multivariate distributions using copulas. We model the conditional correlation (by MGARCH) and the remaining dependence (by a copula) separately and simultaneously. We apply this idea to three MGARCH models, namely, the dynamic conditional correlation (DCC) model of Engle (2002), the varying correlation (VC) model of Tse and Tsui (2002), and the BEKK model of Engle and Kroner (1995). Empirical analysis with three foreign exchange rates indicates that the C-MGARCH models outperform DCC, VC, and BEKK in terms of in-sample model selection and out-of-sample multivariate density forecast, and in terms of these criteria the choice of copula functions is more important than the choice of the volatility models.

Key Words: Copula, Density forecast, MGARCH, Non-normal multivariate distribution, Uncorrelated dependent errors.

JEL Classification: C3, C5, G0.

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1 Introduction

Modeling the conditional covariance matrix is in the core of financial econometrics, as it is crucial for the asset allocation, financial risk management, and derivatives pricing. The multivariate generalized autoregressive conditional heteroskedasticity (MGARCH) models in the literature include the BEKK model by Engle and Kroner (1995), the dynamic conditional correlation (DCC) model by Engel (2002), and the varying correlation (VC) model by Tse and Tsui (2002). However, these models have been estimated under the multivariate normality assumption, while this assumption has been rejected in much of the empirical findings – Fama and French (1993), Richardson and Smith (1993), Longin and Solnik (2001), Mashal and Zeevi (2002), among many others.

The aim of this paper is to model MGARCH for non-normal multivariate distributions using copulas. We propose a simple new model named a Copula-based Multivariate GARCH model, or in short C-MGARCH model, which permits modeling conditional correlation (by MGARCH) and dependence (by a copula) separately and simultaneously for non-normal multivariate distributions.

Our approach is based on a transformation, which removes the linear correlation from the dependent variables to form uncorrelated dependent errors. The dependence structure is controlled by a copula while the correlation is modeled by an MGARCH model. The C-MGARCH model can capture the dependence in the uncorrelated errors ignored by all existing MGARCH models. For every MGARCH model, the corresponding C-MGARCH model can be constructed.

Simulation and empirical analysis are conducted to demonstrate the superiority of the new model over existing MGARCH models such as the DCC, the VC, and the BEKK models. The paper takes advantage of both MGARCH models and of copulas. While a number of existing papers have used copulas to model dependence (particularly in the tails) and/or to model non-normality (e.g., skewness, fat tail), the current paper is the first that models MGARCH with copula distributions. The model is therefore able to model the conditional correlations and conditional dependence simultaneously.

The paper is organized as follows. Section 2 provides a brief review on MGARCH models. Section 3 introduces the new C-MGARCH model with uncorrelated dependent errors. In Section 3, we focus on the bivariate case. Section 4 considers the multivariate extensions in several different way. Section 5 conducts empirical analysis for comparison of existing MGARCH models with their corresponding C-MGARCH models in terms of in-sample model selection criteria and out-of-sample (OOS) density predictive ability. The C-MGARCH models outperform corresponding DCC, VC and BEKK models when they are applied to three foreign exchange rates (French Franc, Deutschemark, and Italian Lira). Section 6 concludes. Section 7 is Appendix on copulas.

2 MGARCH Models

We begin with a brief review of three MGARCH models. Suppose a vector of the m return series $\{\mathbf{r}_t\}_{t=1}^n$ with $\mathbb{E}(\mathbf{r}_t|\mathcal{F}_{t-1}) \equiv \boldsymbol{\mu}_t = 0$ and $\mathbb{E}(\mathbf{r}_t\mathbf{r}_t'|\mathcal{F}_{t-1}) \equiv \mathbf{H}_t$ where $\mathcal{F}_{t-1}$ is the information set (σ-field) at time $t-1$. For simplicity, we assume the conditional mean $\boldsymbol{\mu}_t$ is zero. For $\mathbf{H}_t$, many specifications have been proposed.

Engle and Kroner (1995) propose the BEKK model

$$\mathbf{H}_t = \boldsymbol{\Omega} + \mathbf{A}(\mathbf{r}_{t-1}\mathbf{r}_{t-1}')\mathbf{A}' + \mathbf{B}\mathbf{H}_{t-1}\mathbf{B}', \tag{1}$$

With the scalar or diagonal specifications on $\mathbf{A}$ and $\mathbf{B}$, we obtain the scalar BEKK (SBEKK) or the diagonal BEKK. We use the SBEKK in Section 5, which is

$$\mathbf{H}_t = (1-a-b)\bar{\boldsymbol{\Omega}} + a(\mathbf{r}_{t-1}\mathbf{r}_{t-1}') + b\mathbf{H}_{t-1}, \tag{2}$$

where $\bar{\boldsymbol{\Omega}} = n^{-1}\sum_{t=1}^n \mathbf{r}_t\mathbf{r}_t'$ is the sample covariance matrix of $\mathbf{r}_t$.

Instead of modeling $\mathbf{H}_t$ directly, conditional correlation models decompose $\mathbf{H}_t$ into $\mathbf{D}_t\mathbf{R}_t\mathbf{D}_t$, where $\mathbf{D}_t^2 \equiv \operatorname{diag}(\mathbf{H}_t)$. As the conditional covariance matrix for $\boldsymbol{\varepsilon}_t \equiv \mathbf{D}_t^{-1}\mathbf{r}_t$ is the conditional correlation matrix for $\mathbf{r}_t$, The DCC model of Engle (2002) models $\mathbf{Q}_t$, the covariance matrix of $\boldsymbol{\varepsilon}_t$, via a variance-targeting scalar BEKK model:

$$\mathbf{Q}_t = (1-a-b)\,\bar{\mathbf{Q}} + a(\boldsymbol{\varepsilon}_{t-1}\boldsymbol{\varepsilon}_{t-1}') + b\mathbf{Q}_{t-1}, \tag{3}$$

where $\bar{\mathbf{Q}}$ is the sample covariance matrix of $\hat{\boldsymbol{\varepsilon}}_t$. A transformation $\mathbf{R}_t = \operatorname{diag}\mathbf{Q}_t^{-1}\,\mathbf{Q}_t\operatorname{diag}\mathbf{Q}_t^{-1}$ makes the conditional correlation matrix for $\mathbf{r}_t$.

The VC model of Tse and Tsui (2002) uses the following specification

$$\mathbf{R}_t = (1-a-b)\,\bar{\mathbf{R}} + a\tilde{\mathbf{R}}_{t-1} + b\mathbf{R}_{t-1}, \tag{4}$$

where $\bar{\mathbf{R}}$ is the positive definite unconditional correlation matrix with ones in diagonal, and $\tilde{\mathbf{R}}_t = \sum_{i=1}^M \varepsilon_{1,t-i}\varepsilon_{2,t-i}/\left(\sum_{i=1}^M \varepsilon_{1,t-i}^2 \sum_{i=1}^M \varepsilon_{2,t-i}^2\right)^{1/2}$.[1]

[1] In Tse and Tsui (2002), a necessary condition to guarantee $\tilde{\mathbf{R}}_t$ positive definite is $M \geqslant k$. Another necessary condition for non-singularity of $\tilde{\mathbf{R}}_t$, which should be added, is that M should be bigger than the maximum number of observations of consecutive zeros of $\varepsilon_{i,t}$, $i = 1, ..., k$. In the empirical section, we set $M = 5$, which is transaction days in one week.

3 New Model: C-MGARCH

In the vast existing MGARCH literature, the distribution for $\mathbf{r}_t$ is assumed to be a certain bivariate elliptical distribution (e.g., bivariate normal or Student's t) with mean $\boldsymbol{\mu}_t$ $(= \mathbf{0})$ and conditional covariance $\mathbf{H}_t$. The standardized errors $\mathbf{e}_t = \mathbf{H}_t^{-1/2}\mathbf{r}_t$ would then have the same bivariate elliptical distribution with zero mean and identity covariance: $\mathbb{E}(\mathbf{e}_t|\mathcal{F}_{t-1}) = 0$ and $\mathbb{E}(\mathbf{e}_t\mathbf{e}_t'|\mathcal{F}_{t-1}) = \mathbf{I}$. However, Embrechts *et al.* (1999) point out some wide-spread misinterpretations of the correlation, e.g., that no-correlation does not imply independence and a positive correlation does not mean "positive dependence" (Lehmann 1966). Here, the identity conditional covariance matrix of $\mathbf{e}_t$ itself does not imply independence except when $\mathbf{e}_t$ follows an elliptical distribution.

The key point of this paper is that we permit dependence among the elements of $\mathbf{e}_t$ even if they are uncorrelated as shown by $\mathbb{E}(\mathbf{e}_t\mathbf{e}_t'|\mathcal{F}_{t-1}) = \mathbf{I}$. The C-MGARCH model specifies the dependence structure and the conditional correlation separately and simultaneously. The former is controlled by a copula function and the latter is modeled by an MGARCH model for $\mathbf{H}_t$.

We use the (similar) notation of Joe (1996). Let $F_{1,\cdots,m}$ denote an m-variate distribution, with continuous univariate margins $F_1, \ldots, F_m$. Let F_S denote the higher order margins where S is a subset of $\{1, 2, \ldots, m\}$ with cardinality at least 2. The densities, when they exist, are denoted as f_S. For example, $F_{\{1,2\}}$ is the bivariate margin of the variable 1 and variable 2 with $S = \{1, 2\}$. A simplifying notation without braces for the subset S is used, e.g., $F_{12} \equiv F_{\{1,2\}}$. For $j \notin S$, $F_{j|S} \equiv \frac{F_{\{j\}\cup S}}{F_S}$ denotes the conditional distribution of variable j given those whose indices are in S, e.g., $F_{1|2} = \frac{F_{12}}{F_2}$. The corresponding conditional density (if it exists) is denoted as $f_{j|S}$. Denote $u_i \equiv F_i(\eta_i)$ for the probability integral transform of η_i.

In Section 3 we focus on the bivariate case with $m = 2$. In Section 4 the multivariate cases with $m \geq 2$ are considered. Before we introduce our new C-MGARCH model, we first briefly review the copula theory (with some more details in Appendix).

3.1 Copula

Although there are many univariate distributions used in econometrics, for multivariate distribution there are few competitive candidates besides multivariate normal distribution and multivariate Student's t distribution. However, the multivariate normal distribution is not consistent with the well-known asymmetry and excess kurtosis in financial data although it is easy to use. In this paper, we use the recently popular copulas to construct uncorrelated dependent errors. The principle characteristic of a copula function is its ability to decompose the joint distribution into two parts:

marginal distributions and dependence structure. Different dependence structures can combine the same marginal distributions into different joint distributions. Similarly, different marginal distributions under the same dependence structure can also lead to different joint distributions.

Definition (Copula): *A function* $C : [0,1]^2 \to [0,1]$ *is a copula if it satisfies (i)* $C(u_1, u_2) = 0$ *for* $u_1 = 0$ *or* $u_2 = 0$*; (ii)* $\sum_{i=1}^{2} \sum_{j=1}^{2} (-1)^{i+j} C(u_{1,i}, u_{2,j}) \geq 0$ *for all* $(u_{1,i}, u_{2,j})$ *in* $[0,1]^2$ *with* $u_{1,1} < u_{1,2}$ *and* $u_{2,1} < u_{2,2}$*; and (iii)* $C(u_1, 1) = u_1$*,* $C(1, u_2) = u_2$ *for all* u_1, u_2 *in* $[0,1]$*.* ■

The relationship between a copula and joint distribution function is illuminated by Sklar's (1959) theorem.

Sklar's Theorem: *Let* F_{12} *be a joint distribution function with margins* F_1 *and* F_2*. Then there exists a copula* C *such that for all* η_1, η_2*,*

$$F_{12}(\eta_1, \eta_2) = C(F_1(\eta_1), F_2(\eta_2)) = C(u_1, u_2). \tag{5}$$

Conversely, if C *is a copula and* F_1 *and* F_2 *are marginal distribution functions, then the function* F_{12} *defined above is a joint distribution function with margins* F_1 *and* F_2*.* ■

The joint density function $f_{12}(\eta_1, \eta_2)$ is

$$\begin{aligned} f_{12}(\eta_1, \eta_2) &= \frac{\partial^2 F_{12}(\eta_1, \eta_2)}{\partial \eta_1 \partial \eta_2} \\ &= \frac{\partial^2 C(u_1, u_2)}{\partial u_1 \partial u_2} \cdot \frac{\partial F_1(\eta_1)}{\partial \eta_1} \cdot \frac{\partial F_2(\eta_2)}{\partial \eta_2} \\ &= c(F_1(\eta_1), F_2(\eta_2)) \cdot f_1(\eta_1) \cdot f_2(\eta_2), \end{aligned} \tag{6}$$

where the copula density is $c(u_1, u_2) = \frac{\partial^2 C(u_1,u_2)}{\partial u_1 \partial u_2}$. For independent copula $C(u_1, u_2) = u_1 u_2$, $c(u_1, u_2) = 1$. An important property of copula function is its invariance under the increasing and continuous transformation, such as log transformation.

The joint survival function $\bar{C}(u_1, u_2)$ is $\bar{C}(u_1, u_2) = \Pr(U_1 > u_2, U_2 > u_2) = 1 - u_1 - u_2 + C(u_1, u_2)$. The survival copula of $C(u_1, u_2)$ is $C_S(u_1, u_2) = u_1 + u_2 - 1 + C(1 - u_1, 1 - u_2)$. The joint survival function and the survival copula are related through $\bar{C}(u_1, u_2) = C_S(1 - u_1, 1 - u_2)$. The density of survival copula can be expressed through the density of original copula as $c_S(u_1, u_2) = c(1 - u_1, 1 - u_2)$.

Upper tail dependence λ_U and lower tail dependence λ_L defined as

$$\begin{aligned} \lambda_U &= \lim_{a \uparrow 1} \Pr[\eta_2 > F_2^{-1}(a) | \eta_1 > F_1^{-1}(a)] = \lim_{a \uparrow 1} \frac{[1 - 2a + C(a,a)]}{1 - a}, \\ \lambda_L &= \lim_{a \downarrow 0} \Pr[\eta_2 \leqslant F_2^{-1}(a) | \eta_1 \leqslant F_1^{-1}(a)] = \lim_{a \downarrow 0} \frac{C(a,a)}{a}, \end{aligned}$$

measure the dependence in extreme cases. The tail dependence of each copula is discussed in Appendix.

In this paper, we use the independent (I) copula, Gumbel (G) copula, Clayton (C) copula, Frank (F) copula, Gumbel survival (GS) copula, and Clayton survival (CS) copula. Their functional forms and properties are discussed in Appendix. From (6), the log-likelihood function for $\{\boldsymbol{\eta}_t\}_{t=1}^n$ is:

$$\begin{aligned}\mathcal{L}^\eta(\boldsymbol{\theta}) &= \sum_{t=1}^n \ln f_{12}(\eta_{1,t}, \eta_{2,t}; \boldsymbol{\theta}) \\ &= \sum_{t=1}^n \ln f_1(\eta_{1,t}; \theta_1) + \ln f_2(\eta_{2,t}; \theta_2) + \ln c\left(F_1(\eta_{1,t}; \theta_1), F_2(\eta_{2,t}; \theta_2); \theta_3\right) \qquad (7)\end{aligned}$$

where n is the number of the observations and $\boldsymbol{\theta} = (\theta_1' \ \theta_2' \ \theta_3')'$ are the parameters in the marginal densities $f_1(\cdot)$ and $f_2(\cdot)$, and the copula shape parameter. The log-likelihood is decomposed into two parts, the first two terms related to the marginals and the last term related to the copula.

3.2 Related literature using copula

We note that the MGARCH models discussed in Section 2 can also be put in the copula framework with elliptical copulas (normal or Student's t). For example, to estimate for $\mathbf{H}_t = \mathbf{D}_t\mathbf{R}_t\mathbf{D}_t$, the DCC model of Engle (2002) assumes the normal margins for elements of $\boldsymbol{\varepsilon}_t = \mathbf{D}_t^{-1}\mathbf{r}_t = (\varepsilon_{1,t} \ \varepsilon_{2,t})'$ and the normal copula for $u_{1,t} = \Phi(\varepsilon_{1,t}; \theta_1)$ and $u_{2,t} = \Phi(\varepsilon_{2,t}; \theta_2)$ (where $\Phi(\cdot)$ is the univariate normal CDF) with the copula shape parameter being the time-varying conditional correlation $\mathbf{R}_t$.[2] This is to assume the bivariate normal distribution. Let $(\theta_1' \ \theta_2')'$ be parameters in $\mathbf{D}_t$, and θ_3 in $\mathbf{R}_t$. The log-likelihood function for the DCC model has the form:

$$\begin{aligned}\mathcal{L}^r(\boldsymbol{\theta}) &= -\frac{1}{2}\sum_{t=1}^n 2\ln(2\pi) + \mathbf{r}_t'\mathbf{H}_t^{-1}\mathbf{r}_t + \ln|\mathbf{H}_t| \qquad (8) \\ &= -\frac{1}{2}\sum_{t=1}^n \left(2\ln(2\pi) + \mathbf{r}_t'\mathbf{D}_t^{-2}\mathbf{r}_t + \ln|\mathbf{D}_t|^2\right) - \frac{1}{2}\sum_{t=1}^n \left(\ln|\mathbf{R}_t| + \boldsymbol{\varepsilon}_t'\mathbf{R}_t^{-1}\boldsymbol{\varepsilon}_t - \boldsymbol{\varepsilon}_t'\boldsymbol{\varepsilon}_t\right),\end{aligned}$$

where the first part corresponds to the normal marginal log-likelihood and the second part corresponds to the normal copula log-likelihood. See (25) for the normal copula function in Appendix. In this case the margins contain $\mathbf{D}_t$ and the copula contains $\mathbf{R}_t$.

To accommodate the deviations from bivariate normality in the financial data, there have been other related attempts in the literature that use copulas. However, these works focus on modeling the conditional dependence instead of the conditional correlation. For example, taking the empirical

[2] However, if non-normal margins are assumed, $\mathbf{R}_t$ is not the the conditional correlation of $\mathbf{r}_t$ and $\mathbf{H}_t$ is not MGARCH of $\mathbf{r}_t$.

distribution functions for the margins and a parametric function for the copula, Breymann *et al.* (2003) and Chen and Fan (2006) estimate $\mathbf{D}_t^2 \equiv \text{diag}(\mathbf{H}_t)$ using univariate realized volatility estimated from high frequency data or using univariate GARCH models. In that framework, estimated are the univariate conditional variances $\mathbf{D}_t^2$ and the conditional dependence, but not the conditional correlation $\mathbf{R}_t$ nor the conditional covariance $\mathbf{H}_t$.

The aim of our paper is different than those of the above mentioned papers. We model the conditional covariance $\mathbf{H}_t$ for non-normal multivariate distributions using copulas. Our model is to separately quantify the conditional correlation (by MGARCH) and the remaining dependence (by copula). We now introduce such a model. The idea is to have the new C-MGARCH model inherited from the existing MGARCH models to model $\mathbf{H}_t$, at the same time it is also to capture the remaining dependence in the uncorrelated dependent standardized errors $\mathbf{e}_t = \mathbf{H}_t^{-1/2}\mathbf{r}_t$.

3.3 Structure of C-MGARCH model

For $m = 2$, let $\mathbf{r}_t = (r_{1,t}\ r_{2,t})'$, $\boldsymbol{\eta}_t = (\eta_{1,t}\ \eta_{2,t})'$, and $\mathbf{e}_t = (e_{1,t}\ e_{2,t})'$. The C-MGARCH model can be formulated as follows:

$$\begin{aligned} \boldsymbol{\eta}_t|\mathcal{F}_{t-1} &\sim F_{12}(\eta_{1,t}, \eta_{2,t}; \boldsymbol{\theta}_t), \\ \mathbf{e}_t &= \boldsymbol{\Sigma}_t^{-1/2}\boldsymbol{\eta}_t, \\ \mathbf{r}_t &= \mathbf{H}_t^{1/2}\mathbf{e}_t, \end{aligned} \tag{9}$$

where $\mathbb{E}(\mathbf{e}_t|\mathcal{F}_{t-1}) = \mathbf{0}$, $\mathbb{E}(\mathbf{e}_t\mathbf{e}_t'|\mathcal{F}_{t-1}) = \mathbf{I}$, $\mathbb{E}(\boldsymbol{\eta}_t|\mathcal{F}_{t-1}) = \mathbf{0}$, and $\mathbb{E}(\boldsymbol{\eta}_t\boldsymbol{\eta}_t'|\mathcal{F}_{t-1}) = \boldsymbol{\Sigma}_t = (\sigma_{ij,t})$. By the Sklar's theorem, $F_{12}(\eta_{1,t}, \eta_{2,t}; \boldsymbol{\theta}_t) = C(F_1(\eta_{1,t}; \theta_{1,t}), F_2(\eta_{2,t}; \theta_{2,t}); \theta_{3,t})$, where $C(\cdot, \cdot)$ is the conditional copula function.

The conventional approach is to assume bivariate independent normality for $\boldsymbol{\eta}_t$ ($C(u_1, u_2) = u_1u_2$, i.e., $\sigma_{12} = 0$), while our approach is to assume a dependent copula for $\boldsymbol{\eta}_t$ keeping $\mathbf{e}_t$ uncorrelated ($C(u_1, u_2) \neq u_1u_2$). The main contribution of our C-MGARCH model is that it permits modeling the conditional correlation and dependence structure, separately and simultaneously.

As the Hoeffding's (1940) lemma shows, the covariance between η_1 and η_2 is a function of marginal distributions $F_1(\cdot)$ and $F_2(\cdot)$, and joint distribution $F_{12}(\cdot)$. See Lehmann (1966), Shea (1983), and Block and Fang (1988).

Hoeffding's Lemma: *Let η_1 and η_2 be random variables with the marginal distributions F_1 and F_2 and the joint distribution F_{12}. If the first and second moments are finite, then*

$$\sigma_{12}(\boldsymbol{\theta}) = \iint_{\mathbb{R}^2} \left[F_{12}(\eta_1, \eta_2; \boldsymbol{\theta}) - F_1(\eta_1; \theta_1)F_2(\eta_2; \theta_2)\right] d\eta_1 d\eta_2. \tag{10}$$

■

By Hoeffding's Lemma and Sklar's Theorem, the off-diagonal element $\sigma_{12,t}$ of the conditional covariance matrix $\mathbf{\Sigma}_t$ between $\eta_{1,t}$ and $\eta_{2,t}$ at time t, can be expressed as

$$\sigma_{12,t}(\boldsymbol{\theta}_t) = \iint_{\mathbb{R}^2} \left[C\left(F_1(\eta_1;\theta_{1,t}), F_2(\eta_2;\theta_{2,t});\theta_{3,t}\right) - F_1(\eta_1;\theta_{1,t})F_2(\eta_2;\theta_{2,t})\right] d\eta_1 d\eta_2. \tag{11}$$

For simplicity, we assume that the marginal standard normal distribution (for which θ_1, θ_2 are known) and the copula parameter θ_3 is not time-varying: $\boldsymbol{\theta}_t \equiv \boldsymbol{\theta} = \theta_3$.[3] This makes $\sigma_{12,t}(\boldsymbol{\theta}_t) \equiv \sigma_{12}(\boldsymbol{\theta})$ and $\mathbf{\Sigma}_t(\boldsymbol{\theta}_t) \equiv \mathbf{\Sigma}(\boldsymbol{\theta})$.

The log-likelihood function for $\{\boldsymbol{\eta}_t\}_{t=1}^n$ is:

$$\mathcal{L}^\eta(\boldsymbol{\theta}) = \sum_{t=1}^n \ln f_1(\eta_{1,t}) + \ln f_2(\eta_{2,t}) + \ln c(F_1(\eta_{1,t}), F_2(\eta_{2,t}); \boldsymbol{\theta}). \tag{12}$$

Because $\mathbf{r}_t = \mathbf{H}_t^{1/2}\mathbf{\Sigma}^{-1/2}\boldsymbol{\eta}_t$, the log-likelihood function for $\{\mathbf{r}_t\}_{t=1}^n$ is:

$$\mathcal{L}^r(\boldsymbol{\theta}, \boldsymbol{\alpha}) = \mathcal{L}^\eta(\boldsymbol{\theta}) + \sum_{t=1}^n \ln \left|\mathbf{\Sigma}^{1/2}(\boldsymbol{\theta})\mathbf{H}_t^{-1/2}(\boldsymbol{\alpha})\right|, \tag{13}$$

where $\left|\mathbf{\Sigma}^{1/2}\mathbf{H}_t^{-1/2}\right|$ is the Jacobian of the transformation from $\boldsymbol{\eta}_t$ to $\mathbf{r}_t$, and $\boldsymbol{\alpha}$ is the parameter vector in the MGARCH model for $\mathbf{H}_t$ (DCC, VC, SBEKK). We maximize $\mathcal{L}^r(\boldsymbol{\theta}, \boldsymbol{\alpha})$ to estimate all parameters in one step, with the diagonal elements of $\mathbf{\Sigma}$ being normalized ($\sigma_{ii} = 1$) for identification.

Remark 1: Because $\mathbf{e}_t = \mathbf{\Sigma}_t^{-1/2}\boldsymbol{\eta}_t$, if $\mathbf{\Sigma}_t^{-1/2} = \mathbf{\Sigma}^{-1/2} \equiv (a_{ij})$, then $e_{1,t} = a_{11}\eta_{1,t} + a_{12}\eta_{2,t}$ and $e_{2,t} = a_{12}\eta_{1,t} + a_{22}\eta_{2,t}$ would be linear combinations of two *dependent* random variables $\eta_{1,t}$ and $\eta_{2,t}$. Even if each of $\eta_{1,t}$ and $\eta_{2,t}$ has the margins of standard normal distribution, the marginal distributions of $e_{1,t}$ and $e_{2,t}$ are not normal because $\eta_{1,t}$ and $\eta_{2,t}$ are not independent. If normal margins of $\boldsymbol{\eta}_t$ are chosen for non-independent copula, then the marginal distributions of $\mathbf{e}_t$ are non-normal. A nice feature of the C-MGARCH model is to allow the non-normal margins of $\mathbf{r}_t$ even if we assume the normality of $\boldsymbol{\eta}_t$. Therefore, the C-MGARCH model not only allows the non-normal joint distribution of $\mathbf{r}_t$ but also allow the (implied) non-normal marginal distributions of the elements of $\mathbf{r}_t$. Note that if the copula for $\boldsymbol{\eta}_t$ is the independence copula, then we obtain the bivariate normality for $\mathbf{r}_t$ if we use the normal margins for the elements of $\boldsymbol{\eta}_t$. Therefore, the better fit of the non-independence copula of $\boldsymbol{\eta}_t$ may be due to the non-normality of the bivariate joint density of $\mathbf{r}_t$ and also due to the non-normality of the margins of the elements of $\mathbf{r}_t$. □

[3] $\theta_{3,t}$ may be modelled to be time-varying. For example, for Gumbel copula, $\theta_{3,t} = 1 + \exp(a + b\theta_{3,t-1} + cu_{1,t-1} + du_{2,t-1})$.

Remark 2: The aim of the paper is to model MGARCH $\mathbf{H}_t$ under the non-normal density of $\mathbf{r}_t$. It is to extend the vast literature on the univariate GARCH with non-normal distribution to the multivariate case.[4] To model MGARCH with non-normal multivariate density, we take an extra step to separate the remaining dependence (not captured by the conditional correlation) from the correlation and model both the conditional correlation and dependence directly. The extra step is a transformation $\mathbf{e}_t = \mathbf{\Sigma}^{-1/2}\boldsymbol{\eta}_t$, to have the conditional second moments $\mathbf{H}_t$ to explicitly enter in the density function, while the non-normal dependent copula is assumed for $\boldsymbol{\eta}_t$. This extra step to separate $\mathbf{e}_t$ and $\boldsymbol{\eta}_t$ is the main innovation of our model. Under normality, uncorrelatedness and independence are equivalent. Under conditional normality, therefore, $\mathbf{e}_t = (e_{1,t}\ e_{2,t})'$ are conditionally uncorrelated ($\mathbb{E}(e_{1,t}e_{2,t}|\mathcal{F}_{t-1}) = 0$) and also independent. Under non-normality, uncorrelatedness and independence are not equivalent. Under non-normality, $\mathbf{e}_t = (e_{1,t}\ e_{2,t})'$ should remain to be conditionally uncorrelated ($\mathbb{E}(e_{1,t}e_{2,t}|\mathcal{F}_{t-1}) = 0$) so that $\mathbf{H}_t$ be the conditional second moment, but $\mathbf{e}_t = (e_{1,t}\ e_{2,t})'$ can be dependent. So we call $\mathbf{e}_t$ uncorrelated dependent errors. To separate the remaining dependence (not captured by the conditional correlation by $\mathbf{H}_t$), we assume that $\boldsymbol{\eta}_t = \left(\eta_{1,t}\ \eta_{2,t}\right)'$ may be correlated ($\mathbb{E}(\eta_{1,t}\eta_{2,t}|\mathcal{F}_{t-1}) \neq 0$) and dependent ($F_{12}(\eta_1, \eta_2) \neq F_1(\eta_1) F_2(\eta_2)$), so that σ_{12} may not be zero (i.e., $\mathbf{e}_t$ and $\boldsymbol{\eta}_t$ are not the same). At the same time, the joint distribution of $\mathbf{r}_t = (r_{1,t}\ r_{2,t})'$ is non-normal, because the joint distribution $F_{12}(\eta_1, \eta_2)$ of $\boldsymbol{\eta}_t$ is not normal and $\mathbf{r}_t = \mathbf{H}_t^{1/2}\mathbf{\Sigma}^{-1/2}\boldsymbol{\eta}_t$. Therefore we are able to model the MGARCH of $\mathbf{r}_t$, using the non-normal distribution of $\mathbf{r}_t$. □

Remark 3: The C-MGARCH model nests all existing MGARCH models. When the copula for $\boldsymbol{\eta}_t$ is independent copula, $\mathbf{\Sigma}$ is diagonal. In addition, if marginal distributions for $\boldsymbol{\eta}_t$ are standard normal, the C-MGARCH model degenerates to the corresponding MGARCH model with bivariate normal distribution for $\mathbf{r}_t$. The C-MGARCH model inherits the dynamics of $\mathbf{H}_t$ from existing MGARCH models. For every MGARCH model, we can construct the corresponding C-MGARCH models with uncorrelated dependent errors. □

Remark 4: The C-MGARCH model permits modeling conditional correlation and dependence separately and simultaneously with non-elliptically distributed dependent errors, and remove correlation from dependence to form the uncorrelated dependent errors. The remaining dependence

[4] On the univariate GARCH literature with univariate non-normal densities, see Bond (2001) for a review and Bao et al (2006) for comparing the density forecasts. There are some papers where MGARCH is modelled under non-normal densities. See Bauwens et al (2006) for a survey. However, unlike in the univariate GARCH models under the non-normal distribution, in the multivariate GARCH models one can not simply replace the multivariate normal density with the multivariate non-normal density as the latter density may not be parametrized in terms of the conditional second moments. Therefore one should carefully formulate a multivariate density for modelling of MGARCH. A nice example is the multivariate skew density of Bauwens and Laurent (2005).

is then captured by a copula. To our knowledge, no previous models incorporate correlation and dependence at the same time. Instead, they focus only on the dependence by modeling shape parameter in copula, Kendall's τ, or Spearman's ϱ. Different from the existing financial applications of copula theory, which focus on (conditional) dependence and ignore the (conditional) correlation, our C-MGARCH models aim to model both dependence and correlation. □

Remark 5: The variance-covariance approach to optimal portfolio allocation is rooted on the assumption of multivariate normality or ellipticality. Without multivariate normality or ellipticality, the variance-covariance approach may not be valid in that we do not consider the higher moments of the joint (non-normal) distribution of the assets for portfolio allocation. The appropriate approach to portfolio allocation under non-normality has been an active research area. For example, Harvey and Siddique (2000), Patton (2004), Krause and Litzenberger (1976), Singleton and Wingender (1986), among others, attempted to incorporate the higher moments (conditional skewness and conditional kurtosis) in asset pricing and portfolio analysis. In this framework, modelling the conditional higher moments (or cumulants) may also be computed from the generalization of the Hoeffding's formula. Heoffding's lemma (Hoeffding 1940) gives an integral representation of the covariance of two or more random variables in terms of the difference between their joint and marginal probability functions. The cumulant generalization of the Hoeffding's formula gives an integral representation of the cumulants of two or more random variables (Block and Fang 1988, Theorem 1). While we only focus on the conditional second moment (MGARCH) in this paper, generalization to the conditional higher moments may be possible using the cumulant generalization of the Hoeffding's formula. Hence, our C-MGARCH model is a simple case that can certainly be generalized to the conditional higher moments of the multivariate non-normal distributions. □

Remark 6: Given the marginal distributions and the copula, one can always work out the implied correlation. In fact, Patton (2006, Footnote 19 and Figure 2) calculated the implied conditional correlations. Our paper is motivated to compute directly the conditional correlation and dependence, simultaneously and separately. What makes our paper different from some of the papers in the MGARCH literature using non-normal multivariate density is that our paper takes advantage of existing MGARCH models and of copulas. By doing this we model the conditional correlation (by MGARCH) directly and at the same time the conditional dependence (by copula) as well.[5] □

[5] The semiparametric copula-based multivariate dynamic (SCOMDY) model of Chen and Fan (2006) is different from our C-MGARCH model. The SCOMDY model is a multivariate model (like ours) to model the non-normal distribution using the copula function for the standardized process by using univariate conditional variance, the diagonal elements of $\mathbf{H}_t$. However, the SCOMDY model does not model the conditional co-variance, the off-diagonal

Remark 7: In the previous version of this paper, Lee and Long (2005), we reported Monte Carlo simulations to see how our model works in estimation, especially because it involves the Hoeffding's lemma with the numerical integration. We generate samples of moderate size $n = 500$ via the method of Nelsen (1999) for bivariate systems simulated from the C-MGARCH models (9) with normal margins, with each of DCC, VC, and SBEKK models for $\mathbf{H}_t$, and with Archimedean copulas. Lee and Long (2005) discuss the details on how to generate $\boldsymbol{\eta}_t$ and $\mathbf{r}_t$. We then estimate C-MGARCH models. In summary, the Monte Carlo simulation confirms that the one-step QML estimation procedure works very well. As a referee pointed out that this can also be seen from the empirical results, to save space we delete the Monte Carlo simulations (available upon request). □

4 m-Variate C-MGARCH

In this section we extend the bivariate C-MGARCH in Section 3.3 to m-variate C-MGARCH with $m \geq 2$. The m-variate C-MGARCH model is formulated as follows

$$\boldsymbol{\eta}_t|\mathcal{F}_{t-1} \sim F_{1,\cdots,m}\left(\boldsymbol{\eta}_t;\boldsymbol{\theta}\right),$$
$$\mathbf{e}_t = \boldsymbol{\Sigma}_t^{-1/2}\boldsymbol{\eta}_t,$$
$$\mathbf{r}_t = \mathbf{H}_t^{1/2}\mathbf{e}_t,$$

where $\mathbb{E}\left(\mathbf{e}_t|\mathcal{F}_{t-1}\right) = 0$, $\mathbb{E}\left(\mathbf{e}_t\mathbf{e}_t'|\mathcal{F}_{t-1}\right) = \mathbf{I}$, $\mathbb{E}\left(\boldsymbol{\eta}_t|\mathcal{F}_{t-1}\right) = 0$, and $\mathbb{E}\left(\boldsymbol{\eta}_t\boldsymbol{\eta}_t'|\mathcal{F}_{t-1}\right) = \Sigma = (\sigma_{ij})$. In order to estimate the C-MGARCH model, we need to construct the joint CDF $F_{1,\cdots,m}\left(\boldsymbol{\eta}_t;\boldsymbol{\theta}\right)$ from which we obtain the bivariate margins $F_{ij}\left(\eta_i,\eta_j\right)$ to compute σ_{ij}, the joint PDF $f_{1,\cdots,m}$, and the log-likelihood $\mathcal{L}^{\eta}(\boldsymbol{\theta})$.

Let C_{ij} denote the bivariate copula associated with the bivariate margin F_{ij}

$$C_{ij}(F_i(\eta_i), F_j(\eta_j); \theta_{ij}) = F_{ij}(\eta_i,\eta_j), \tag{14}$$

where θ_{ij} is the copula parameter. Once F_{ij} $(i,j = 1,\ldots,m)$ is obtained, σ_{ij} is determined by Hoeffding's Lemma. (If we assume the standard normality on the margin F_i of η_i, then the diagonal elements of Σ are normalized at $\sigma_{ii} = 1$ for identification.)

4.1 How to construct $F_{1,\cdots,m}\left(\boldsymbol{\eta}_t;\boldsymbol{\theta}\right)$

To obtain the joint density $F_{1,\cdots,m}$ with the bivariate margins F_{ij}, we consider two methods. They are different in construction and have some advantages and disadvantages.

elements of $\mathbf{H}_t$ (unlike ours) and it only models the conditional dependence.

4.1.1 Method 1

The first method is based on Joe (1996), who derives a class of m-variate distributions with $m(m-1)/2$ dependence parameters from given univariate margins and bivariate copula margins.

For $m = 3$, given univariate margins (F_1, F_2, F_3), bivariate marginal copulas (F_{12}, F_{23}), and a bivariate conditional copula ($C_{13|2}$), the trivariate joint distribution is

$$F_{123}\left(\eta_1, \eta_2, \eta_3; \theta_{12}, \theta_{13}, \theta_{23}\right) = \int_{-\infty}^{y_2} C_{13|2}\left(F_{1|2}\left(\eta_1|z_2; \theta_{12}\right), F_{3|2}\left(\eta_3|z_2; \theta_{23}\right)\right) F_2\left(dz_2\right) \tag{15}$$

where $F_{1|2}$, $F_{3|2}$ are conditional CDF's obtained from F_{12}/F_2, F_{23}/F_2, respectively.[6] By construction, (15) is a proper trivariate distribution with univariate margins F_1, F_2, F_3, and bivariate margins F_{12}, F_{23}. The $(1,3)$ bivariate margin of F_{123} can be obtained as

$$F_{13}\left(\eta_1, \eta_3; \theta_{12}, \theta_{23}, \theta_{13}\right) = F_{123}\left(\eta_1, \infty, \eta_3; \theta_{12}, \theta_{23}, \theta_{13}\right). \tag{16}$$

Note that F_{13} depends on all three dependence parameters $\theta_{12}, \theta_{23}, \theta_{13}$. In general, it will not be the same as $C_{13}\left(F_1, F_3; \theta\right)$ for some θ. The different copula functions can be chosen for C_{12}, C_{23}, and $C_{13|2}$.

For $m = 4$, we first obtain F_{234} in the same way to get F_{123} in (15). Given the bivariate margin F_{23}, we obtain the conditional CDFs $F_{1|23} = F_{123}/F_{23}$ and $F_{4|23} = F_{234}/F_{23}$. Then following Joe (1996), the 4-variate distribution is

$$\begin{aligned} & F_{1234}\left(\eta_1, \eta_2, \eta_3, \eta_4; \theta_{12}, \cdots, \theta_{34}\right) \\ = & \int_{-\infty}^{\eta_2}\int_{-\infty}^{\eta_3} C_{14|23}(F_{1|23}\left(\eta_1|z_2, z_3; \theta_{12}, \theta_{13}, \theta_{23}\right), F_{4|23}\left(\eta_4|z_2, z_3; \theta_{23}, \theta_{24}, \theta_{34}\right))F_{23}\left(dz_2 dz_3; \theta_{23}\right) \end{aligned} \tag{17}$$

where $F_{23}\left(dz_2 dz_3; \theta_{23}\right) = c_{23}\left(F_2\left(z_2\right), F_3\left(z_3\right); \theta_{23}\right) \cdot f_2\left(z_2\right) \cdot f_3\left(z_3\right) dz_2 dz_3$ by applying the chain rule to (14) for F_{23}. This can be extended recursively and inductively to obtain higher dimensional distributions. See Joe (1996, p. 122).

4.1.2 Method 2

The second method is to use the multivariate Archimedean m-copulas, that can be obtained from the symmetricity and associativity properties of Archimedean copulas. See Joe (1997, p. 87),

[6] This equation (15) holds because $f_{123}\left(\eta_1, \eta_2, \eta_3\right) = f_{13|2}\left(\eta_1, \eta_3|\eta_2\right) f_2(\eta_2)$ implies

$$F_{123}\left(\eta_1, \eta_2, \eta_3\right) = \int_{-\infty}^{\eta_2} F_{13|2}\left(\eta_1, \eta_3|\eta_2\right) f_2(z_2) dz_2,$$

and $F_{13|2}\left(\eta_1, \eta_3|\eta_2\right) = C_{13|2}\left(F_{1|2}, F_{3|2}\right)$ by Sklar theorem.

Nelsen (1999, p. 121), and Embrechts et al (2003, p. 373) for derivation of

$$\begin{aligned} F_{1,\cdots,m}(\boldsymbol{\eta};\theta) &= C_{1,\cdots,m}(\mathbf{u};\theta) \\ &= \varphi^{-1}(\varphi(u_1)+\varphi(u_2)+\cdots+\varphi(u_m)), \end{aligned}$$

where φ is the generator of an Archimedean copula. We discuss the m-variate Archimedean copulas in Appendix. From $C_{1,\cdots,m}(\mathbf{u})$, we derive the bivariate copula function $C_{ij}(u_i,u_j)$ between u_i and u_j by setting all $u_k=1$, $\forall k\neq i,j$. Once $F_{ij}(\eta_i,\eta_j)=C_{ij}(u_i,u_j)$ $(i,j=1,\ldots,m)$ is obtained, σ_{ij} is determined by Hoeffding's Lemma.

Remark 8: The associativity property of Archimedean copulas is not shared by other copulas in general and thus Method 2 is generally for Archimedean copulas. Another disadvantage of Method 2 is that it assumes the same dependence structure for all pairs of (u_i,u_j), i.e., the same copula function $C_{ij}=C$ with the same parameter $\theta_{ij}=\theta$.[7] This is to set all the off-diagonal elements of $\boldsymbol{\Sigma}$ to take the same value $(\sigma_{ij}=\sigma)$, which is obviously restrictive as can be seen from estimated values of θ_{ij} and σ_{ij} in our empirical results (Tables 1-3) for different pairs of three foreign exchange series. While this may be restrictive for in-sample estimation especially for a large m, our empirical analysis for $m=3$ shows this may not be a serious problem in OOS forecasting as the resulted parsimony with only one parameter can reduce the effect of parameter estimation uncertainty in OOS forecasting.[8] □

Remark 9: A different method other than the above two methods may also be possible. For example, Joe (1997, p. 156) provides a three variable extension of the Frank copula with multiple shape parameters. For more discussion of the multivariate copula functions, see Joe (1997, Sections 5.3 and 5.5). We do not use this method in our empirical analysis in the next section, where we only consider Method 1 and Method 2. □

4.2 Joint density $f_{1,\cdots,m}(\boldsymbol{\eta}_t;\boldsymbol{\theta})$ and the likelihood

Let $\boldsymbol{\theta}$ be the copula parameters and $\boldsymbol{\alpha}$ be the MGARCH parameters to parameterize $\mathbf{H}_t$, e.g., DCC, VC, and BEKK as discussed in Section 2. The log-likelihood of $\{\mathbf{r}_t\}_{t=1}^n$ is

$$\mathcal{L}^r_{1,\cdots,m}(\boldsymbol{\theta},\boldsymbol{\alpha})=\mathcal{L}^\eta_{1,\cdots,m}(\boldsymbol{\theta})+\ln|\boldsymbol{\Sigma}^{1/2}(\boldsymbol{\theta})\mathbf{H}_t^{-1/2}(\boldsymbol{\alpha})| \tag{18}$$

[7] In Section 5, we use the 3-variate Frank copula as shown in (27) with $m=3$. It can be easily seen that $C_{123}(u_1,u_2,1;\theta)=C_{12}(u_1,u_2;\theta)$, and they have the same shape parameter θ.

[8] The effects of parameter estimation on prediction densities have been studied in recent literature, e.g., Pascual *et al.* (2001).

where

$$\mathcal{L}^{\eta}_{1,\cdots,m}(\boldsymbol{\theta}) = \sum_{t=1}^{n} \ln f_{1,\cdots,m}(\boldsymbol{\eta}_t;\boldsymbol{\theta}). \tag{19}$$

Therefore, in order to get the log-likelihood, we need to get the joint density $f_{1,\cdots,m}(\boldsymbol{\eta}_t;\boldsymbol{\theta})$.

4.2.1 Likelihood for Method 1

For $m = 3$, from (15), we obtain the joint 3-density as follows:

$$\begin{aligned}
& f_{123}\left(\eta_1,\eta_2,\eta_3;\theta_{12},\theta_{13},\theta_{23}\right) \qquad (20)\\
\equiv\; & \frac{\partial^3 F_{123}\left(\eta_1,\eta_2,\eta_3;\theta_{12},\theta_{13},\theta_{23}\right)}{\partial\eta_1\partial\eta_2\partial\eta_3}\\
=\; & \frac{\partial^2}{\partial\eta_1\partial\eta_3} C_{13|2}\left(F_{1|2}\left(\eta_1|\eta_2;\theta_{12}\right), F_{3|2}\left(\eta_3|\eta_2;\theta_{23}\right)\right)\cdot f_2\left(\eta_2\right)\\
=\; & \frac{\partial^2}{\partial u_{1|2}\partial u_{3|2}} C_{13|2}\left(F_{1|2}\left(\eta_1|\eta_2;\theta_{12}\right), F_{3|2}\left(\eta_3|\eta_2;\theta_{23}\right)\right)\cdot \frac{\partial F_{1|2}\left(\eta_1|\eta_2;\theta_{12}\right)}{\partial\eta_1}\cdot\frac{\partial F_{3|2}\left(\eta_3|\eta_2;\theta_{23}\right)}{\partial\eta_3}\cdot f_2\left(\eta_2\right)\\
=\; & c_{13|2}(u_{1|2},u_{3|2})\cdot f_{1|2}\left(\eta_1|\eta_2;\theta_{12}\right)\cdot f_{3|2}\left(\eta_2|\eta_2;\theta_{23}\right)\cdot f_2\left(\eta_2\right),
\end{aligned}$$

where $u_{1|2} \equiv F_{1|2}\left(\eta_1|\eta_2;\theta_{12}\right)$ and $u_{3|2} \equiv F_{3|2}\left(\eta_3|\eta_2;\theta_{23}\right)$. This generalizes Equation (4) of Patton (2006). Note that even for 3-variate distribution we only need a bivariate copula function for Method 1. Then the log-likelihood function for $\{\boldsymbol{\eta}_t\}_{t=1}^n$ is

$$\mathcal{L}^{\eta}_{123}(\boldsymbol{\theta}) = \sum_{t=1}^{n} \ln c_{13|2}(u_{1|2,t},u_{3|2,t}) + \ln f_{1|2}\left(\eta_{1,t}|\eta_{2,t};\theta_{12,t}\right) + \ln f_{3|2}\left(\eta_{3,t}|\eta_{2,t};\theta_{23,t}\right) + \ln f_2\left(\eta_{2,t}\right).$$

The log-likelihood function for $\{\mathbf{r}_t\}_{t=1}^n$ is $\mathcal{L}^{r}_{123}(\boldsymbol{\theta},\boldsymbol{\alpha}) = \mathcal{L}^{\eta}_{123}(\boldsymbol{\theta}) + \sum_{t=1}^{n}\ln\left|\boldsymbol{\Sigma}^{1/2}(\boldsymbol{\theta})\mathbf{H}_t^{-1/2}(\boldsymbol{\alpha})\right|$.

For $m = 4$, from (17), we obtain the joint 4-density as follows:

$$\begin{aligned}
& f_{1234}\left(\eta_1,\eta_2,\eta_3,\eta_4;\theta_{12},\cdots,\theta_{34}\right) \qquad (21)\\
=\; & \frac{\partial^2}{\partial\eta_1\partial\eta_4}\left(\frac{\partial^2 F_{1234}\left(\eta_1,\eta_2,\eta_3,\eta_4;\theta_{12},\cdots,\theta_{34}\right)}{\partial\eta_2\partial\eta_3}\right)\\
=\; & c_{14|23}(u_{1|23},u_{4|23})\cdot f_{1|23}\left(y_1|y_2,y_3\right)\cdot f_{4|23}\left(y_4|y_2,y_3\right)\cdot c_{23}\left(u_2,u_3\right)\cdot f_2\left(y_2\right)\cdot f_3\left(y_3\right).
\end{aligned}$$

Note that for the 4-variate distribution we only need bivariate copula functions for Method 1. Then the log-likelihood function can be obtained similarly, from (18), (19), and (21).

4.2.2 Likelihood for Method 2

For $m = 3$, the (conditional) joint PDF function of $\boldsymbol{\eta}_t$ is

$$\begin{aligned} f_{123}(\eta_1, \eta_2, \eta_3) &\equiv \frac{\partial^3 F_{123}(\eta_1, \eta_2, \eta_3)}{\partial \eta_1 \partial \eta_2 \partial \eta_3} \\ &= \frac{\partial^3 C(F_1(\eta_1), F_2(\eta_2), F_3(\eta_3))}{\partial \eta_1 \partial \eta_2 \partial \eta_3} \\ &= \frac{\partial^3 C(F_1(\eta_1), F_2(\eta_2), F_3(\eta_3))}{\partial u_1 \partial u_2 \partial u_3} \cdot \frac{\partial F_1(\eta_1)}{\partial \eta_1} \cdot \frac{\partial F_2(\eta_2)}{\partial \eta_2} \cdot \frac{\partial F_3(\eta_3)}{\partial \eta_3} \\ &= c(u_1, u_2, u_3) \cdot f_1(\eta_1) \cdot f_2(\eta_2) \cdot f_3(\eta_3). \end{aligned} \tag{22}$$

Note that for the 3-variate distribution we need the 3-variate copula function for Method 2 (while we only need bivariate copulas for Method 1). The log-likelihood function can be obtained similarly, from (18), (19), and (22).

5 Empirical Analysis

The objective of this section is to compare the C-MGARCH models with the conventional MGARCH models in terms of in-sample estimation and OOS forecasting. To elucidate the effect of the distinct feature of the C-MGARCH model, we adopt the same normal marginal distribution so that the difference arises only from the copula density.

We examine three foreign exchange (FX) rate series (in U.S. dollars) – French Franc (FF), Deutschemark (DM), and Italian Lira (IL). The return series are 100 times the log-difference of the exchange rates. The daily spot FX series are from the Federal Reserve Statistical Release. The entire sample period that we consider is daily from 1993 to 1997 with $T = 1256$ observations. We split the sample in two parts. The in-sample estimation period is from January 4, 1993 to December 31, 1996 ($R \equiv 1005$ observations). The OOS forecast validation period is from January 5, 1997 to December 31, 1997 ($P \equiv T - R = 251$ observations).

We use the "fixed scheme" for which we estimate the parameters at $t = R$ (December 31, 1996) using the sample of size $n = R$, and fix the parameters at these estimated values (no updating) to make one-day-ahead density forecasts throughout the total $P = 251$ days in 1997.[9] The results are presented in Tables 1-4, with in-sample estimation results in Panel A and the OOS forecasting results in Panel B.

[9] The parameters β^*_{t-1} can be estimated based on the whole subsample $\{\mathbf{r}_{t-1}, \ldots, \mathbf{r}_1\}$, a rolling sample $\{\mathbf{r}_{t-1}, \ldots, \mathbf{r}_{t-R}\}$ of size R, or a fixed sample $\{\mathbf{r}_R, \ldots, \mathbf{r}_1\}$ at the end of the in-sample. We use the fixed scheme in this paper. See West and McCracken (1998, p. 819) for more discussion on these three forecasting schemes.

5.1 In-sample estimation

For the in-sample comparison between our C-MGARCH models and MGARCH models, we present in Panel A of each table the following three model selection criteria:

$$
\begin{aligned}
\text{logL} &= \mathcal{L}^r(\hat{\boldsymbol{\theta}}_R, , \hat{\boldsymbol{\alpha}}_R)/R \\
\text{AIC} &= -2\log L(\hat{\boldsymbol{\theta}}_R, \hat{\boldsymbol{\alpha}}_R) + 2k/R \\
\text{SIC} &= -2\log L(\hat{\boldsymbol{\theta}}_R, \hat{\boldsymbol{\alpha}}_R) + k\ln(R)/R
\end{aligned}
$$

where k is the number of parameters in each model and

$$(\hat{\boldsymbol{\theta}}_R, \hat{\boldsymbol{\alpha}}_R) = \arg\max_{\boldsymbol{\theta},\boldsymbol{\alpha}} \mathcal{L}^r(\boldsymbol{\theta}, \boldsymbol{\alpha}) = \sum_{t=1}^{R} \ln f(\boldsymbol{\eta}_t; \boldsymbol{\theta}) + \ln\left|\boldsymbol{\Sigma}^{1/2}(\boldsymbol{\theta})\mathbf{H}_t^{-1/2}(\boldsymbol{\alpha})\right|. \tag{23}$$

To test for the null hypothesis that Σ is an identity matrix, it can be tested by the likelihood ratio (LR) statistic

$$LR_R = -2 \times R \times [(\text{logL of MGARCH}) \; - \; (\text{logL of C-MGARCH})],$$

which is asymptotically χ^2-distributed (with d.f. equal to $m(m-1)/2$ for Method 1 and d.f. equal to one for Method 2). The log-likelihood ratio of two models is the entropy gain in the sense of Vuong (1989). The superiority of Gumbel Survival copula for FF-DM (Table 1A), Gumbel copula for DM-IL (Table 2A) and Frank copula for FF-IL (Table 3A) over the independent copula indicates that the conditional joint distributions of these three pairs of FX return series are not bivariate normal. For example, for FF-DM, $LR_R = 176.88$ to compare I-DCC and GS-DCC (Table 1A). For all three bivariate pairs of FF-DM, DM-IL, FF-IL, the LR statistics obtained from Panel A of each tables confirm the conditional multivariate non-normality. For all three families (DCC, VC, BEKK), the independent copula yields the smallest (worst) logL. The ranking of the different copula functions are robust to the different MGARCH models, implying that the choice of the copula function is more important than the choice of the MGARCH (DCC, VC, BEKK). The best models are GS-MGARCH model for FF-DM (Table 1A), G-MGARCH for DM-IL (Table 2A), and F-MGARCH for FF-IL (Table 3A), for all three specifications of $\mathbf{H}_t$.[10]

It is interesting to note the tail dependence properties implied by the best selected C-MGARCH model. Gumbel Survival copula chosen for FF-DM has the asymmetric tail dependence with positive

[10] We use a symbol G for Gumbel copula, C for Clayton copula, F for Frank copula, GS for Gumbel survival copula, CS for Clayton survival copula, and I for independent copula. Then a C-MGARCH model with a particular copula function and a particular MGARCH specification of $\mathbf{H}_t$ will be denoted like G-DCC, C-VC, GS-SBEKK, I-DCC. In trivariate case it will be denoted as (e.g.) I-I-I-MGARCH for the benchmark model and GS-G-F-MGARCH with GS for the (1, 2) pair FF-DM, G copula for (2, 3) pair DM-IL, and Frank copula for (1, 3) pair FF-IL.

lower tail dependence ($\lambda_L > 0$) and zero upper tail dependence ($\lambda_U = 0$). For DM-IL, Gumbel copula is selected, which has the asymmetric tail dependence with positive upper tail dependence ($\lambda_U > 0$) and zero lower tail dependence ($\lambda_L = 0$). For the FF-IL pair, the model selection criteria select Frank copula, which has symmetric tail dependence.[11]

For the trivariate system of FF-DM-IL (Table 4A), we compare three C-MGARCH models (GS-G-F-MGARCH using Method 1, F-F-F-MGARCH using Method 1, and F-F-F-MGARCH using Method 2) with the benchmark MGARCH (I-I-I-MGARCH). The benchmark MGARCH yields the worst values for logL, AIC, and SIC. The best model in terms of these three model selection criteria is the F-F-F-MGARCH with Method 1 for DCC and VC families, and the GS-G-F-MGARCH with Method 1 for SBEKK family. While Method 1 works better than Method 2 in terms of the three model selection criteria, the performance of Method 2 is quite encouraging as it is only slightly worse but much simpler to use than Method 1.

Panel A in each of the three tables also presents the estimated copula shape parameters $\hat{\theta}_R$, their robust standard errors, and the corresponding $\hat{\sigma}_{ij}$ obtained from the Hoeffding's lemma. For FF-DM (Table 1A), the estimated Gumbel Survival copula parameter $\hat{\theta}_R = 1.10$ gives $\hat{\sigma}_{12} = 0.14$ for GS-DCC. GS-VC and GS-SBEKK have the similar values for $\hat{\theta}_R = 1.09$, 1.08, and for $\hat{\sigma}_{12} = 0.13$, 0.12, respectively. For DM-IL (Table 2A), the estimated Gumbel copula parameters $\hat{\theta}_R$ (= 1.16, 1.19, 1.15) and $\hat{\sigma}_{12}$ (= 0.22, 0.25, 0.20) are bigger than those of FF-DM. For FF-IL (Table 3A), the estimated Frank copula shape parameter $\hat{\theta}_R$ is 3.11 in F-DCC, 3.51 in F-VC, and 3.17 in F-SBEKK, and all significantly positive, and the corresponding values of $\hat{\sigma}_{12}$ are 0.44, 0.48, 0.44, indicating the remaining dependence in the standardized uncorrelated errors $\mathbf{e}_t$ is substantial.

5.2 Out-of-sample predictive ability

Let $\boldsymbol{\beta} = (\boldsymbol{\theta}' \ \boldsymbol{\alpha}')'$ and its estimate $\hat{\boldsymbol{\beta}}_R = (\hat{\boldsymbol{\theta}}_R' \ \hat{\boldsymbol{\alpha}}_R')'$ be obtained from (23). Suppose there are $l+1$ models in a set of the competing density forecast models, possibly misspecified. We compare $l = 5$ bivariate C-MGARCH models and $l = 3$ trivariate C-MGARCH models with the corresponding benchmark MGARCH model. Let these models be indexed by j ($j = 0, 1, \ldots, l$) with the jth density forecast model denoted by $\psi_t^j(\mathbf{r}_t; \boldsymbol{\beta}_{t-1}^j)$ for $t = R+1, \ldots, T$. The benchmark model is indexed with $j = 0$. If a density forecast model $\psi_t(\mathbf{r}_t; \boldsymbol{\beta}_0)$ coincides with the true density $\varphi_t(\mathbf{r}_t)$ almost surely for

[11] If the copula of $\boldsymbol{\eta}_t$ is independent, the copula of $\mathbf{e}_t$ is also independent. If the copula $C(u_1, u_2)$ of $\boldsymbol{\eta}_t$ is the independent copula, $\boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}^{-1}$ are diagonal. If $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^{-1} = \mathbf{I}$, then $\mathbf{e}_t = \boldsymbol{\eta}_t$. Therefore, if the copula $C(u_1, u_2)$ of $\boldsymbol{\eta}_t$ is the independent copula, the copula for $\mathbf{e}_t$ is also independent copula. However, as pointed out by an anonymous referee, we should note that $\mathbf{e}_t$ and $\boldsymbol{\eta}_t$ may not necessarily share the same important features. For example, the tail dependence for $\boldsymbol{\eta}_t$ may not necessarily indicates the same direction of the tail dependence for $\mathbf{e}_t$ or $\mathbf{r}_t$. Unfortunately, these are unknown properties unless $\boldsymbol{\Sigma}$ is diagonal even for the linear transform $\mathbf{e}_t = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{\eta}_t$.

some $\boldsymbol{\theta}_0 \in \boldsymbol{\Theta}$, then the one-step-ahead density forecast is said to be optimal because it dominates all other density forecasts for any loss functions (Diebold *et al.* 1998, Granger and Pesaran 2000). As in practice it is rarely the case that we can find an optimal model, our task is to investigate which density forecast model approximates the true conditional density most closely. If a metric is defined to measure the distance of a given model to the truth, we then compare different models in terms of this distance.

Following Bao *et al.* (2006), we compare C-MGARCH models by comparing the *conditional* Kullback-Leibler (1951) information criterion (KLIC), $\mathbb{I}_t\left(\varphi : \psi^j, \boldsymbol{\beta}_{t-1}\right) = \mathbb{E}_{\varphi_t} \ln\left[\varphi_t\left(\mathbf{r}_t\right)/\psi_t^j\left(\mathbf{r}_t; \boldsymbol{\beta}_{t-1}^j\right)\right]$, where the expectation is with respect to the true conditional density $\varphi_t\left(\cdot|\mathcal{F}_{t-1}\right)$. While KLIC is not a metric as noted in White (1994, p. 9), KLIC can be used as if it is a metric as long as the benchmark model is fixed and all the other models are compared against the benchmark.[12] Following White (1994), we define the distance between a density model and the true density as the minimum KLIC, $\mathbb{I}_t(\varphi : \psi^j, \boldsymbol{\beta}_{t-1}^{*j}) = \mathbb{E}_{\varphi_t} \ln[\varphi_t\left(\mathbf{r}_t\right)/\psi_t^j(\mathbf{r}_t; \boldsymbol{\beta}_{t-1}^{*j})]$, where $\boldsymbol{\beta}_{t-1}^{*j} = \arg\min \mathbb{I}_t\left(\varphi : \psi^j, \boldsymbol{\beta}_{t-1}^j\right)$ is the pseudo-true value of $\boldsymbol{\beta}_{t-1}^j$. To estimate $\boldsymbol{\beta}_{t-1}^{*j}$, we split the data into two parts – one for the estimation and the other for OOS validation. We use the "fixed scheme" for which we estimate the parameters only once at $t = R$ (December 31, 1996) using the sample of size $R = 1005$, and fix the parameters at these estimated values (no updating) to make one-day-ahead density forecasts $\psi_t^j(\mathbf{r}_t; \hat{\boldsymbol{\beta}}_R^j)$ throughout $t = R+1, \ldots, T$ for the total $P = 251$ days in year 1997. We use the observations $\{\mathbf{r}_R, \ldots, \mathbf{r}_1\}$ to estimate the unknown parameter vector $\boldsymbol{\beta}_R^{*j}$ and denote the estimate as $\hat{\boldsymbol{\beta}}_R^j$. Under some regularity conditions, we can consistently estimate $\boldsymbol{\beta}_R^{*j}$ by $\hat{\boldsymbol{\beta}}_R^j = \arg\max R^{-1}\sum_{t=1}^R \ln \psi_t^j\left(\mathbf{r}_t; \boldsymbol{\beta}^j\right)$, i.e., it is obtained from (23). See White (1994) for the sets of conditions for the existence and consistency of $\hat{\boldsymbol{\beta}}_R^j$.

Using $\hat{\boldsymbol{\beta}}_R$, we can obtain the OOS estimate of $\mathbb{E}\mathbb{I}_t(\varphi : \psi^j, \boldsymbol{\beta}_R^{*j})$ by

$$\mathbb{I}_{R,P}(\varphi : \psi) \equiv P^{-1}\sum_{t=R+1}^{T} \ln\left[\varphi_t(\mathbf{r}_t)/\psi_t^j(\mathbf{r}_t; \hat{\boldsymbol{\beta}}_R^j)\right],$$

where $P \equiv T - R$ is the size of the OOS period. Since the KLIC takes a smaller value when a model is closer to the truth, we can regard it as a loss function. Note that the OOS average of the KLIC-differential between the benchmark model 0 and model j is then simply the log-ratio of the

[12] KLIC does not satisfy some basic properties of a metric, $\mathbb{I}\left(\psi_1 : \psi_2\right) \neq \mathbb{I}\left(\psi_2 : \psi_1\right)$ and KLIC does not satisfy a triangle inequality. However, as noted in Bao et al (2006), as we use the KLIC in comparing various C-MGARCH models with a fixed benchmark model (i.e., MGARCH), KLIC can serve as a distance metric with respect to the fixed benchmark.

predicted likelihoods

$$\mathbb{I}_{R,P}(\varphi:\psi^0)-\mathbb{I}_{R,P}(\varphi:\psi^j)=\frac{1}{P}\sum_{t=R+1}^{T}\ln[\psi_t^j(\mathbf{r}_t;\hat{\boldsymbol{\beta}}_R^j)/\psi_t^0(\mathbf{r}_t;\hat{\boldsymbol{\beta}}_R^0)]. \tag{24}$$

When we compare multiple l C-MGARCH models using various copulas against a benchmark MGARCH model, the null hypothesis of interest is that no C-MGARCH model is better than the benchmark MGARCH. White (2000) proposes a test statistic (so called "Reality Check") and the bootstrap procedure to compute its p-value.[13]

Panel B in each of Tables 1-4 reports the density forecast comparison in terms of the OOS KLIC together with the Reality Check p-values. For OOS forecasting for $P = 251$ days (from Jan/05/1997 to Dec/31/1997), we report the OOS average of the predicted log-likelihood

$$\frac{1}{P}\sum_{t=R+1}^{T}\ln\psi_t(\mathbf{r}_t;\hat{\boldsymbol{\beta}}_R)\equiv\frac{1}{P}\sum_{t=R+1}^{T}\ln f_{123}\left(\boldsymbol{\eta}_t;\hat{\boldsymbol{\theta}}_R\right)+\frac{1}{P}\sum_{t=R+1}^{T}\ln\left|\boldsymbol{\Sigma}^{1/2}(\hat{\boldsymbol{\theta}}_R)\mathbf{H}_t^{-1/2}(\hat{\boldsymbol{\alpha}}_R)\right|,$$

which is reported under "logL" in Panel B. In addition, we also report the OOS standard deviation of the predicted log-likelihood, which is reported under "std(logL)". In general, the in-sample results in Panel A and the OOS results in Panel B are consistent, in that the in-sample ranking across the C-MGARCH models is generally carried over to the OOS predictive ranking of the models.

For bivariate pairs of FF-DM, DM-IL, FF-IL, the density forecast comparison confirms the non-normality. For all three families (DCC, VC, BEKK), the independent copula yields the smallest (worst) predictive likelihood. For FF-DM, Table 1B shows the predictive superiority of C-MGARCH models based on Gumbel Survival copula over the MGARCH model with the bivariate normal distribution is significant (with Reality Check p-values 0.000, 0.000, 0.001, respectively for DCC, VC, SBEKK families). For DM-IL (Table 2B), the best density forecast copula function is Gumbel

[13] For Reality Check test in our paper, it may worth making the following two remarks on the parameter estimation uncertainty and the nestedness problem: (1) In general, the distribution of test statistic may be rather complicated because it depends on parameter estimation uncertainty (West, 1996). However, we note that one of the significant merits of using the KLIC as a loss function in comparing density forecast models is that parameter estimation uncertainty does not affect asymptotic inference. This is due to the use of the same objective function for the in-sample parameter estimation (minimizing the KLIC to get $\hat{\boldsymbol{\beta}}_R$) and for the out-of-sample forecast validation (using the KLIC as a forecast evaluation criterion). See West (1996) and Bao et al (2006) for more details. (2) While the Reality Check permits some of the competiting C-MGARCH models to nest the benchmark, it requires that at least one of them not nest the benchmark (White 2000, p. 1105-1106). In our case, some C-MGARCH models nest the benchmark when the parameter space includes the value of θ to make a copula independent. For example, Gumbel-MGARCH with $\theta = 1$ makes MGARCH. But Clayton-MGARCH and Frank-MGARCH do not nest the benchmark as the parameter space of θ does not make them independent. In addition, treating the test conditional on the estimated parameter values $\hat{\theta}_R^j$ in line with Giacomini and White (2006), the OOS reality check inference may handle the nested cases as well.

for all DCC, VC, BEKK families and is significantly better than MGARCH (with Reality Check p-values 0.036, 0.000, 0.022, respectively for DCC, VC, SBEKK families). For FF-IL (Table 3B), the predictive superiority of C-MGARCH models based on Frank copula over the MGARCH model is also very significant (with Reality Check p-values 0.001, 0.000, 0.000, respectively for DCC, VC, SBEKK families). The C-MGARCH models are significantly better than the benchmark MGARCH model even after accounting for potential data-snooping bias due to the specification search over five different copula functions.

For trivariate case of FF-DM-IL (Table 4B), the independent copula yields the worst predictive likelihood. The largest (best) predicted likelihood is obtained from F-F-F-MGARCH with Method 2 for DCC family, from F-F-F-MGARCH with Method 1 for VC family, and from GS-G-F-MGARCH with Method 1 for SBEKK family (with Reality Check p-values 0.000, 0.000, 0.000, respectively for DCC, VC, SBEKK families). The results are strong in favor of the trivariate C-MGARCH in terms of both the in-sample fit and out-of-sample forecasting.

Our generalization of MGARCH models using copula is analogous to the efforts in the literature to use non-normal densities in estimation of the univariate GARCH models. Bao et al (2006) find that the choice of the density (e.g., skewness and fat tails, or different departures from normality) is more important than the choice of the volatility model (e.g., symmetric vs. asymmetric conditional variance) in modeling financial time series. Our empirical results show that this univariate conclusion is carried over to the multivariate case. Comparing the KLIC based on the in-sample fitted log-likelihood and the OOS predicted log-likelihood, we find that the choice of the density (different copula functions for different tail dependence or different departures from normality) is more important than the choice of the volatility model (e.g., DCC, VC, and BEKK) because once a right copula function has been chosen the ranking of the C-MGARCH models does not change with the different dynamic specifications of $\mathbf{H}_t$.

6 Conclusions

In this paper we propose a new MGARCH model, namely, the C-MGARCH model. The C-MGARCH model includes a conventional MGARCH model as a special case. The C-MGARCH model is to exploit the fact that the uncorrelated errors are not necessarily independent. The C-MGARCH model permits modeling the conditional covariance for the non-elliptically distributed financial returns, and at the same time separately modeling the dependence structure beyond the conditional covariance. We compare the C-MGARCH models with the corresponding MGARCH models using the three foreign exchange rates. The empirical results from the in-sample and OOS analysis clearly demonstrate the advantages of the new model.

7 Appendix

We present here some details on copula functions for two widely used copula families – elliptical copula family and Archimedean copula family. The former includes the Gaussian copula and the Student's t copula. The latter includes Gumbel copula, Clayton copula and Frank copula. We also discuss the survival copulas of Archimedean copulas and m-variate Archimedean copulas.

7.1 Elliptical copulas

Gaussian copula: Let $\mathbf{R}$ be the symmetric, positive definite correlation matrix and $\mathbf{\Phi_R}(\cdot,\cdot)$ be the standard bivariate normal distribution with correlation matrix $\mathbf{R}$. The density function of bivariate Gaussian copula is:

$$c^{\text{Gaussian}}(u_1,u_2) = \frac{1}{|\mathbf{R}|^{1/2}}\exp(-\frac{1}{2}\boldsymbol{\eta}'(\mathbf{R}^{-1}-\mathbf{I})\boldsymbol{\eta}), \tag{25}$$

where $\boldsymbol{\eta} = (\Phi^{-1}(u_1)\ \Phi^{-1}(u_2))'$ and $\Phi^{-1}(\cdot)$ is the inverse of the univariate normal CDF. The bivariate Gaussian copula is:

$$C^{\text{Gaussian}}(u_1,u_2;\mathbf{R}) = \mathbf{\Phi_R}\left(\Phi^{-1}(u_1),\Phi^{-1}(u_2)\right).$$

Hu (2003) shows the bivariate Gaussian copula can be approximated by Taylor expansion:

$$C^{\text{Gaussian}}(u_1,u_2;\theta) \approx u_1u_2 + \theta\cdot\phi(\Phi^{-1}(u_1))\phi(\Phi^{-1}(u_2)),$$

where ϕ is the density function of univariate Gaussian distribution and θ is the correlation coefficient between η_1 and η_2. Both the upper tail dependence λ_U and the lower tail dependence λ_L are zero, reflecting the asymptotic tail independence of Gaussian copula.

Student's t **copula**: Let ω_c be the degree of freedom, and $\mathbf{T}_{\mathbf{R},\omega_c}(\cdot,\cdot)$ be the standard bivariate Student's t distribution with degree of freedom ω_c and correlation matrix $\mathbf{R}$. The density function of bivariate Student's t copula is:

$$c^{\text{Student's t}}(u_1,u_2;\mathbf{R},\omega_c) = |\mathbf{R}|^{-\frac{1}{2}}\frac{\Gamma(\frac{\omega_c+2}{2})\Gamma(\frac{\omega_c}{2})}{\Gamma(\frac{\omega_c+1}{2})^2}\frac{(1+\frac{\boldsymbol{\eta}'R^{-1}\boldsymbol{\eta}}{\omega_c})^{-\frac{\omega_c+2}{2}}}{\Pi_{i=1}^2(1+\frac{\eta_i^2}{\omega_c})^{-\frac{\omega_c+1}{2}}}$$

where $\boldsymbol{\eta} = (t_{\omega_c}^{-1}(u_1), t_{\omega_c}^{-1}(u_2))'$, $u_1 = t_{\omega_1}(x)$, $u_2 = t_{\omega_2}(y)$, and $t_{\omega_i}(\cdot)$ is the univariate Student's t CDF with degree of freedom ω_i. The bivariate Student's t copula is

$$C^{\text{Student's t}}(u_1,u_2;\mathbf{R},\omega_c) = \mathbf{T}_{\mathbf{R},\omega_c}\left(t_{\omega_c}^{-1}(u_1), t_{\omega_c}^{-1}(u_2)\right).$$

It has two copula parameters $\boldsymbol{\theta} = (\mathbf{R}\ \omega_c)'$. The upper tail dependence λ_U of Student's t copula is $\lambda_U = 2 - 2t_{\omega_c+1}\left(\sqrt{\omega_c+1}\sqrt{1-\rho}/\sqrt{1+\rho}\right)$, where ρ is the off-diagonal element of $\mathbf{R}$. Because of the symmetry property, the lower tail dependence λ_L can be obtained easily.

7.2 Archimedean copulas

Archimedean copula can be expressed as

$$C(u_1, u_2) = \varphi^{-1}(\varphi(u_1) + \varphi(u_2)),$$

where φ is a convex decreasing function, called generator. Different generator will induce different copula in the family of Archimedean copula. The Kendall's $\tau = 1 + 4\int_0^1 \frac{\varphi(u)}{\varphi'(u)} du$.

Gumbel copula: The generator for Gumbel copula is $\varphi_\theta(x) = (-\ln x)^\theta$. For $\theta \geq 1$ ($\theta = 1$ for independence and $\theta \to \infty$ for more dependence), the CDF and PDF for Gumbel copula are

$$\begin{aligned} C^{\text{Gumbel}}(u_1, u_2; \theta) &= \exp\{-[(-\ln u_1)^\theta + (-\ln u_2)^\theta]^{1/\theta}\}, \\ c^{\text{Gumbel}}(u_1, u_2; \theta) &= \frac{C^{\text{Gumbel}}(u_1, u_2; \theta)(\ln u_1 \ln u_2)^{\theta-1}\{[(-\ln u_1)^\theta + (-\ln u_2)^\theta]^{1/\theta} + \theta - 1\}}{u_1 u_2[(-\ln u_1)^\theta + (-\ln u_2)^\theta]^{2-1/\theta}}. \end{aligned}$$

The Kendall's τ for Gumbel copula is $\tau = 1 - \frac{1}{\theta}$. This one-to-one mapping relationship between τ and θ clearly shows the copula shape parameter θ controlling the dependence structure. The dependence structure of Gumbel copula are asymmetric: $\lambda_U = 2 - 2^{1/\theta}$ and $\lambda_L = 0$.

The survival copula of Gumbel copula has mirror image to Gumbel copula. Its CDF and PDF are

$$\begin{aligned} C^{\text{GS}}(u_1, u_2; \theta) &= u_1 + u_2 - 1 + \exp\{-[(-\ln(1-u_1))^\theta + (-\ln(1-u_2))^\theta]^{1/\theta}\}, \quad \theta \geq 1 \\ c^{\text{GS}}(u_1, u_2; \theta) &= c^{\text{Gumbel}}(1-u_1, 1-u_2; \theta). \end{aligned}$$

The Kendall's τ for Gumbel survival copula is $\tau = 1 - \frac{1}{\theta}$. Gumbel Survival copula has the positive lower tail dependence: $\lambda_U = 0$ and $\lambda_L = 2 - 2^{1/\theta}$.

Clayton copula: The generator for Clayton copula is $\varphi_\theta(x) = \frac{x^{-\theta}-1}{\theta}$. For $\theta > 0$, the CDF and the PDF for Clayton copula are

$$\begin{aligned} C^{\text{Clayton}}(u_1, u_2; \theta) &= (u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta}, \\ c^{\text{Clayton}}(u_1, u_2; \theta) &= \frac{(1+\theta)(u_1^{-\theta} + u_2^{-\theta} - 1)^{-\frac{1}{\theta}-2}}{(u_1 u_2)^{\theta+1}}. \end{aligned}$$

The Kendall's τ for Clayton copula is $\frac{\theta}{\theta+2}$. The upper tail dependence $\lambda_U = 0$ and the lower tail dependence is $\lambda_L = 2^{-1/\theta}$.

Frank copula: The generator for Frank copula is $\varphi_\theta(x) = -\ln(\frac{e^{-\theta x}-1}{e^{-\theta}-1})$. For $\theta \in \mathbb{R}\backslash\{0\}$, the CDF and PDF for Frank copula are

$$\begin{aligned} C^{\text{Frank}}(u_1, u_2; \theta) &= -\frac{1}{\theta} \ln \left[1 + \frac{(e^{-\theta u_1} - 1)(e^{-\theta u_2} - 1)}{(e^{-\theta} - 1)} \right], \\ c^{\text{Frank}}(u_1, u_2; \theta) &= \frac{-\theta(e^{-\theta} - 1)e^{-\theta(u_1+u_2)}}{[(e^{-\theta} - 1) + (e^{-\theta u_1} - 1)(1 - e^{-\theta u_2} - 1)]^2}. \end{aligned} \tag{26}$$

The dependence structure described by Frank copula is symmetric: $\theta > 0$ for positive dependence, $\theta \to 0$ for independence, and $\theta < 0$ for negative dependence.

7.3 Multivariate Archimedean copulas

The multivariate Archimedean m-copulas can be obtained from the symmetry and associativity properties of Archimedean copulas. See Joe (1997, p. 87), Nelsen (1999, p. 121), and Embrechts et al (2003, p. 373) for derivation of

$$C_{1,\cdots,m}(\mathbf{u}; \theta) = \varphi^{-1}(\varphi(u_1) + \varphi(u_2) + \cdots + \varphi(u_m)),$$

where φ is the generator of an Archimedean copula.

The m-variate Clayton copula function with $\theta > 0$ (Nelsen 1999, Example 4.21) is:

$$C_{1,\cdots,m}^{\text{Clayton}}(\mathbf{u}; \theta) = \left(u_1^{-\theta} + \cdots + u_m^{-\theta} - m + 1 \right)^{-1/\theta}.$$

The m-variate Gumbel copula function with $\theta \geq 1$ (Nelsen 1999, Example 4.23) is:

$$C_{1,\cdots,m}^{\text{Gumbel}}(\mathbf{u}; \theta) = \exp\left\{ -\left[(-\ln u_1)^\theta + (-\ln u_2)^\theta + \cdots + (-\ln u_n)^\theta \right]^{1/\theta} \right\}.$$

The m-variate Frank copula function is:

$$C_{1,\cdots,m}^{\text{Frank}}(\mathbf{u}; \theta) = -\frac{1}{\theta} \ln \left[1 + \frac{\left(e^{-\theta u_1} - 1\right)\left(e^{-\theta u_2} - 1\right) \cdots \left(e^{-\theta u_n} - 1\right)}{\left(e^{-\theta} - 1\right)^{n-1}} \right]. \tag{27}$$

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Table 1. Bivariate C-MGARCH (FF-DM)

Panel A. In-sample Results

		MGARCH	C-MGARCH				
		I	G	GS	C	CS	F
DCC	θ		1.0507	1.0970	0.1199	0.0030	1.7024
	se(θ)		0.0048	0.0401	0.0019	0.0010	0.2045
	σ		0.0789	0.1429	0.0917	0.0024	0.2600
	logL	-0.5893	-0.5081	**-0.5013**	-0.5038	-0.5093	-0.5074
	AIC	1.1945	1.0341	**1.0206**	1.0254	1.0366	1.0326
	SIC	1.2336	1.0781	**1.0646**	1.0694	1.0806	1.0766
VC	θ		1.0419	1.0887	0.1218	0.0021	0.8741
	se(θ)		0.0028	0.0030	0.0016	0.0151	0.0778
	σ		0.0659	0.1320	0.0930	0.0017	0.1373
	logL	-0.6032	-0.5205	**-0.5124**	-0.5153	-0.5217	-0.5212
	AIC	1.2222	1.0590	**1.0427**	1.0484	1.0612	1.0602
	SIC	1.2614	1.1030	**1.0867**	1.0924	1.1052	1.1042
SBEKK	θ		1.0391	1.0788	0.1146	0.0021	0.7356
	se(θ)		0.0021	0.0020	0.0015	0.0031	0.0643
	σ		0.0617	0.1186	0.0879	0.0017	0.1159
	logL	-0.5736	-0.5008	**-0.4938**	-0.4957	-0.5017	-0.5013
	AIC	1.1511	1.0076	**0.9936**	0.9975	1.0094	1.0086
	SIC	1.1609	1.0223	**1.0083**	1.0121	1.0241	1.0233

Notes: Daily changes in log exchange rates for French Franc and Deutschemark from January 4, 1993 to December 31, 1996 (R = 1005) are used. MGARCH is the C-MGARCH with the Independent copula (denoted as I). Estimated are five other C-MGARCH models with Gumbel copula (G), Gumbel Survival copula (GS), Clayton copula (C), Clayton Survival copula (CS), and Frank copula (F). For space, only copula shape parameter estimates and their robust standard errors are reported. The parameter estimates are from the one-step QMLE. The numbers in parentheses are the standard errors calculated from the robust QMLE covariance matrix of the parameters. θ is the copula shape parameter. Each copula function has only one shape parameter. σ is the off-diagonal element of Σ. logL, AIC, and SIC are as defined in Section 5. The largest in-sample average of the estimated log-likelihood (logL) and smallest AIC and SIC for each family are in bold font to indicate the best C-MARCH model of each family.

Panel B. Out-of-sample Results

		I	G	GS	C	CS	F	
DCC	logL	-0.1785	-0.1050	-0.1071	-0.1099	-0.1109	**-0.1032**	RC = 0.000
	std(logL)	0.9776	1.0810	1.0869	1.0945	1.0869	1.1216	
VC	logL	-0.2217	-0.1227	**-0.1225**	-0.1266	-0.1286	-0.1245	RC = 0.000
	std(logL)	0.9650	1.1816	1.1899	1.1962	1.1927	1.2044	
SBEKK	logL	-0.1299	-0.1020	**-0.0995**	-0.1032	-0.1061	-0.1039	RC = 0.001
	std(logL)	1.0159	1.0776	1.0899	1.0916	1.0853	1.0960	

Notes: The out-of-sample forecast period is daily from January 5, 1997 to December 31, 1997 (with P = 251 days). logL denotes the out-of-sample average of the predicted log-likelihood. std(logL) denotes the out-of-sample standard deviation of the predicted log-likelihood. The largest logL for each family is in bold font to indicate the best C-MARCH model of each family. The reality check p-values (denoted as RC) are reported to compare the five C-MGARCH models with the benchmark MGARCH model with I copula. We use 1000 bootstrap samples of the "stationary bootstrap" with the mean block size equal to 5 days (a week) (i.e., with the stationary bootstrap smoothing parameter (1/5)). The benchmark model in each family is MGARCH with Independent (I) copula. We also computed the modified reality check p-values of Hansen (2005) which all turn out to be exactly the same as those of White (2000) in all four tables in our paper.

Table 2. Bivariate C-MGARCH (DM-IL)

Panel A. In-sample Results

		MGARCH	C-MGARCH				
		I	G	GS	C	CS	F
DCC	θ		1.1587	1.0502	0.0608	0.0762	1.3689
	se(θ)		0.0039	0.0018	0.0013	0.0019	0.0472
	σ		0.2183	0.0781	0.0479	0.0595	0.2118
	logL	-1.4001	**-1.3859**	-1.3877	-1.3889	-1.3893	-1.3896
	AIC	2.8162	**2.7897**	2.7934	2.7957	2.7964	2.7972
	SIC	2.8553	**2.8337**	2.8374	2.8397	2.8404	2.8412
VC	θ		1.1884	1.0474	0.0502	0.0976	1.7717
	se(θ)		0.0047	0.0020	0.0013	0.0020	0.0436
	σ		0.2510	0.0740	0.0397	0.0755	0.2697
	logL	-1.4300	**-1.3948**	-1.3984	-1.3997	-1.3990	-1.3977
	AIC	2.8760	**2.8076**	2.8148	2.8174	2.8159	2.8133
	SIC	2.9151	**2.8516**	2.8588	2.8614	2.8599	2.8573
SBEKK	θ		1.1456	1.0457	0.0609	0.0789	1.1843
	se(θ)		0.0036	0.0017	0.0012	0.0020	0.0457
	σ		0.2031	0.0716	0.0479	0.0616	0.1844
	logL	-1.4032	**-1.3871**	-1.3892	-1.3898	-1.3902	-1.3901
	AIC	2.8103	**2.7802**	2.7844	2.7856	2.7863	2.7861
	SIC	2.8201	**2.7948**	2.7991	2.8002	2.8010	2.8008

Notes: Daily changes in log exchange rates for Deutschemark and Italian Lira from January 4, 1993 to December 31, 1996 (R = 1005) are used. Also see notes for Table 1A.

Panel B. Out-of-sample Results

		I	G	GS	C	CS	F	
DCC	logL	-0.9849	-0.9750	-0.9875	-0.9844	-0.9786	**-0.9630**	RC = 0.036
	std(logL)	1.0360	1.1215	1.0992	1.0919	1.0910	1.1233	
VC	logL	-1.0152	-0.9532	-0.9630	-0.9621	-0.9563	**-0.9481**	RC = 0.000
	std(logL)	1.0500	1.1625	1.1369	1.1309	1.1357	1.1563	
SBEKK	logL	-0.9043	-0.8910	-0.8884	-0.8909	-0.8931	**-0.8840**	RC = 0.022
	std(logL)	1.1106	1.1614	1.1469	1.1346	1.1328	1.1454	

Notes: See notes for Table 1B.

Table 3. Bivariate C-MGARCH (FF-IL)

Panel A. In-sample Results

		MGARCH	C-MGARCH				
		I	G	GS	C	CS	F
DCC	θ		1.2033	1.0342	0.0501	0.0771	3.1094
	se(θ)		0.0061	0.0019	0.0012	0.0020	0.0500
	σ		0.2667	0.0542	0.0396	0.0602	0.4381
	logL	-1.3011	-1.2799	-1.2826	-1.2826	-1.2826	**-1.2752**
	AIC	2.6182	2.5777	2.5830	2.5832	2.5832	**2.5682**
	SIC	2.6573	2.6217	2.6270	2.6272	2.6272	**2.6122**
VC	θ		1.1984	1.0378	0.0507	0.0660	3.5098
	se(θ)		0.0073	0.0019	0.0012	0.0020	0.0827
	σ		0.2616	0.0597	0.0401	0.0518	0.4807
	logL	-1.3223	-1.2858	-1.2877	-1.2879	-1.2882	**-1.2801**
	AIC	2.6604	2.5896	2.5932	2.5937	2.5943	**2.5782**
	SIC	2.6995	2.6336	2.6372	2.6377	2.6383	**2.6222**
SBEKK	θ		1.2144	1.0318	0.0495	0.0809	3.1688
	se(θ)		0.0060	0.0019	0.0011	0.0021	0.0400
	σ		0.2780	0.0507	0.0392	0.0631	0.4447
	logL	-1.3059	-1.2824	-1.2855	-1.2853	-1.2853	**-1.2767**
	AIC	2.6159	2.5707	2.5769	2.5766	2.5765	**2.5593**
	SIC	2.6256	2.5854	2.5915	2.5912	2.5912	**2.5740**

Notes: Daily changes in log exchange rates for French Franc and Italian Lira from January 4, 1993 to December 31, 1996 (R = 1005) are used. Also see notes for Table 1A.

Panel B. Out-of-sample Results

		I	G	GS	C	CS	F	
DCC	logL	-0.9142	-0.8841	-0.9049	-0.8994	-0.8957	**-0.8613**	RC = 0.001
	std(logL)	1.0935	1.1504	1.1536	1.1478	1.1460	1.2047	
VC	logL	-1.0133	-0.9687	-0.9817	-0.9812	-0.9792	**-0.9613**	RC = 0.000
	std(logL)	1.0686	1.1090	1.1355	1.1328	1.1192	1.2082	
SBEKK	logL	-0.8440	-0.8027	-0.8055	-0.8062	-0.8069	**-0.7802**	RC = 0.000
	std(logL)	1.1390	1.2230	1.2169	1.2079	1.2050	1.2475	

Notes: See notes for Table 1A.

Table 4. Trivariate C-MGARCH (FF-DM-IL)

Panel A. In-sample Results

		MGARCH	C-MGARCH (Method 1)			C-MGARCH (Method 1)			C-MGARCH (Method 2)
			C12	C23	C13|2	C12	C23	C13|2	
		I-I-I	GS	G	F	F	F	F	F-F-F
DCC	θ		1.0045	1.0020	1.4980	-0.0460	0.0132	6.0080	1.6320
	se(θ)		0.0013	0.0015	0.1423	0.0027	0.0026	0.0525	0.0264
	σ		0.0075	0.0033	0.2380	-0.0073	0.0021	0.6332	0.2500
	logL	-1.0978		-0.9893			**-0.9754**		-0.9917
	AIC	2.2175		2.0065			**1.9786**		2.0072
	SIC	2.2175		2.0065			**1.9786**		2.0072
VC	θ		1.0041	1.0029	1.4980	-0.0908	0.0550	6.4788	1.3453
	se(θ)		0.0007	0.0013	0.1102	0.0023	0.0023	0.0524	0.0242
	σ		0.0067	0.0049	0.2388	-0.0144	0.0087	0.6564	0.2084
	logL	-1.1296		-1.0059			**-0.9926**		-1.0095
	AIC	2.2810		2.0397			**2.0131**		2.0428
	SIC	2.2810		2.0397			**2.0131**		2.0428
SBEKK	θ		1.0020	1.0020	5.3495	0.0439	-0.0418	1.4980	1.7040
	se(θ)		0.0003	0.0005	0.0528	0.0126	0.0106	0.0732	0.0239
	σ		0.0033	0.0033	0.6259	0.0070	-0.0066	0.2066	0.2602
	logL	-1.0893		**-0.9716**			-0.9866		-0.9863
	AIC	2.1825		**1.9532**			1.9831		1.9785
	SIC	2.1825		**1.9532**			1.9831		1.9785

Notes: Daily changes in logarithms of three foreign exchange rates for FF, DM, and IL from January 4, 1993 to December 31, 1996 (R = 1005) are used. The 3-variate distributions are obtained from Method 1 and Method 2 as discussed in Section 4. When Method 1 is used there are three copula parameters (for three pair-wise copulas C12, C23, and C13|2). We consider four models: MGARCH with the 3-variate Independent copula (denoted as I-I-I), C-MGARCH with GS-G-F copula using Method 1, C-MGARCH with F-F-F copula using Method 1, and C-MGARCH with 3-variate Frank copula using Method 2. θ is the copula shape parameter(s). Method 1 using GS-G-F copula or F-F-F copula has three copula parameters. Method 1 using the trivariate Frank copula has only one copula shape parameter. The notation σ is to denote the off-diagonal elements of **Σ**. Method 1 has the three distinct off-diagonal elements of **Σ**. Method 2 has the three identical off-diagonal elements of **Σ** because the three pairwise bivariate copula functions have the same shape parameter θ. The largest average log-likelihood (logL) and smallest AIC and SIC for each family are in bold font to indicate the best C-MARCH model of each family. Also see notes for Table 1A.

Panel B. Out-of-sample Results

		I-I-I	GS-G-F	FFF1	FFF2	
DCC	logL	-0.2448	-0.0842	-0.0681	**-0.0550**	RC = 0.000
	std(logL)	1.1929	1.3824	1.4116	1.4782	
VC	logL	-0.3220	-0.0673	**-0.0565**	-0.0641	RC = 0.000
	std(logL)	1.2048	1.5290	1.5190	1.5849	
SBEKK	logL	-0.0727	**0.0103**	0.0011	-0.0070	RC = 0.000
	std(logL)	1.2917	1.4836	1.4111	1.5016	

Notes: The reality check p-values are reported to compare the three C-MGARCH models with the benchmark MGARCH model with I-I-I copula. The three trivariate C-MGARCH models, denoted as GS-G-F (using Method 1), FFF1 (F-F-F using Method 1), and FFF2 (F-F-F using Method 2), are the same models as considered in Panel A. See also notes for Table 1B.