Velocity of Money, Equilibrium (In)determinacy and Endogenous Growth*

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Abstract

We show that in a canonical one-sector AK model of endogenous growth with a generalized cash-in-advance constraint, the growth and velocity effects of money are closely related to the local stability properties of the economy's balanced growth paths. When a positive fraction (excluding 100 percent) of gross investment is subject to the liquidity constraint, the economy displays saddle-path stability and negative effects of money on output growth and velocity due to a dominating portfolio substitution effect. By contrast, when the opposing intertemporal substitution effect dominates, the economy exhibits indeterminacy and sunspots, as well as a positive correlation between money, output growth and velocity. Finally, when real balances are required only for the household's consumption purchases, money becomes superneutral in the growth-rate and also in the velocity sense because the equilibrium real rate of return on capital remains constant.

Keywords: Velocity of Money, Indeterminacy, Endogenous Growth, Cash-in-Advance Constraint.

JEL Classification: E52, O42.

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1 Introduction

In an interesting paper, Palivo et al. (1993) present empirical evidence, both across a sample of 20 countries and over time within each individual country, that documents a discernible long-run negative relationship between the rate of nominal money growth and the income velocity of money since 1963. These authors also show that this stylized fact can be obtained in a deterministic, one-sector infinite-horizon representative agent model with fixed labor supply and a generalized cash-in-advance (CIA) constraint *a la* Wang and Yip (1992). Specifically, the entire consumption purchases, together with a positive fraction (but not all) of gross investment, must be financed by the household's existing real money balances. In this no-growth setting, an increase in the money growth rate generates higher inflation, which in turn causes the net rate of return on capital to decline. This will lead to decreases in the steady-state levels of capital, output, consumption and investment. Since the firm's production function exhibits diminishing returns to capital, the capital-to-output ratio falls at the steady state, thereby lowering the economy's money velocity. It is left as a topic of future research to "examine the effects of money growth on velocity in the presence of a non-decreasing returns to scale technology" (Palivo et al. 1993, p. 245). In the current paper, we address this question within a prototypical AK model of endogenous growth.

As it turns out, a recent paper by Suen and Yip (2005) explores the possibility of equilibrium multiplicity and indeterminacy in an AK extension of the representative agent economy in Palivo et al. (1993). However, Suen and Yip (2005) restrict their analysis to the cash-in-advance formulation postulated by Stockman (1981), *i.e.* money is required for all the transactions of consumption and investment goods. It follows that the money velocity always equals one; and hence is independent of changes in the money growth rate. This finding is counterfactual *vis-à-vis* the above-mentioned empirical evidence shown in Palivo et al. (1993). Motivated by this inconsistency, we incorporate a generalized liquidity constraint, as in Wang and Yip (1992) and Palivo et al. (1993), into the Suen-Yip endogenous growth model, and then systematically examine the velocity effects of money growth. Moreover, our modification results in a more complicated dynamical system compared to that in Suen and Yip (2005), and thus allows us to further identify model features and parameters that govern the number and local stability properties of the economy's balanced growth paths (BGP), as well as the associated growth effects of money and inflation.

Our results are summarized as follows. First, when the CIA constraint is applied solely to consumption purchases, as in Clower (1967) and Lucas (1980), the relative price of capital to money (in utility terms) remains unchanged along the unique and locally determinate balanced growth path. This in turn yields a constant real return to investment, hence the economy's output growth rate and velocity are not affected by changes in money and inflation. That is, money becomes superneutral in the growth-rate and also in the velocity sense.

Next, when a positive fraction (including 100 percent) of gross investment is subject to the CIA constraint as well, we find that the sign for the growth effects of money depends crucially on the relative strength of two opposing forces. On the one hand, a rise in the money growth rate leads to a higher inflation, which in turn raises the cost of money holdings. As a result, the representative household substitutes out of real balances and into capital (the portfolio substitution effect). This will cause an increase in the relative shadow price of capital because of a higher demand, thereby reducing its net rate of return and thus the BGP's growth rate. On the other hand, a higher inflation *ceteris paribus* induces the representative household to consume less and invest more today in exchange for higher future consumption (the intertemporal substitution effect). This expands the supply of capital, hence lowering its relative shadow price. It follows that the economy's output growth rate will rise. Our analysis shows that if the intertemporal elasticity of substitution in consumption is smaller than or equal to one, the economy possesses a unique balanced growth path that displays saddle-path stability. Moreover, the BGP's output growth and money/inflation are inversely related in that the portfolio substitution effect outweighs the intertemporal substitution effect. This result is consistent with the empirical findings of Kormendi and Meguire (1985), Grier and Tullock (1989), Levine and Renelt (1992), Roubini and Sala-i-Martin (1992), De Gregorio (1993), and Barro (1995), among many others.

We also show that there exist two balanced growth paths when the intertemporal elasticity of consumption substitution falls below a critical value that is strictly greater than one. In addition, since the portfolio substitution effect dominates in the low-growth BGP equilibrium, which turns out to be a saddle, the growth effects of money and inflation continue to be negative. On the contrary, due to a stronger intertemporal substitution effect, the high-growth BGP equilibrium is locally indeterminate (a sink), and exhibits a positive correlation between output growth and money. The latter finding provides theoretical support for some of the empirical evidence reported by, for example, Gomme (1993), and Bullard and Keating (1995).

The intuition for the above indeterminacy result is straightforward. When agents expect a higher future return on capital, they will reduce consumption and raise investment today. If the intertemporal substitution effect (from consumption to investment) is sufficiently strong, the rate of return on capital will rise because of a decline in its relative shadow price. As a result, agents' initial optimistic expectations become self-fulfilling. Furthermore, we find that the region of dual BGP equilibria gets larger when there is a fall in the firm's productivity; or when there is an increase in any of the following parameters: the fraction of investment purchases that is subject to the liquidity constraint, the household's subjective discount rate, the rate of nominal money growth and the capital depreciation rate. Each of these parameter changes leads to a higher threshold for the intertemporal elasticity of consumption substitution below which equilibrium multiplicity arises. In this case, households become less risk averse, hence they are more willing to sacrifice today's consumption for the higher returns from belief-driven investment spurts. This will strengthen the intertemporal substitution effect, and thus increase the likelihood of indeterminacy and sunspots.

Finally, we show that when a positive proportion but less than 100 percent of gross investment is subject to the liquidity constraint, as in Wang and Yip (1992) and Palivo et al. (1993), the money velocity along the economy's balanced growth path is negatively correlated with the consumption-to-capital ratio, which turns out to be positively related to the corresponding relative shadow price of capital.[1] Therefore, following up on the earlier discussions, the velocity effects of money and inflation are closely linked with the BGP's local stability properties. In particular, when the BGP equilibrium displays saddle-path stability, an increase in the money growth rate is associated with a dominating portfolio substitution effect. As a result, the net rate of return on capital falls because of a higher relative shadow price, thereby raising the BGP's consumption-capital ratio and thus lowering the money velocity. This result is consistent with the time-series as well as the cross-sectional evidence documented in Palivo et al. (1993). By contrast, since the intertemporal substitution effect is stronger when the BGP equilibrium is a sink, the economy will exhibit a counterfactually positive correlation between money growth and velocity.

The remainder of this paper is organized as follows. Section 2 describes an AK model of endogenous growth with fixed labor supply and a generalized cash-in-advance constraint. Section 3 analyzes the existence and number of the economy's balanced growth paths, together with the associated growth effects of money and local stability properties. Section 4 examines

[1] By contrast, in a one-sector representative agent model which does not allow for sustained economic growth, Palivo et al. (1993) find that the money velocity is positively related to the capital-to-output ratio at the steady state.

the relationship between the rate of nominal money growth and the income velocity of money. Section 5 concludes.

2 The Economy

We incorporate a generalized cash-in-advance constraint into the endogenous growth model of Suen and Yip (2005). In addition, partial capital depreciation is considered for completeness of the analysis.[2] To facilitate comparison, we follow Suen and Yip's notation as much as possible. The economy is populated by a unit measure of identical infinitely-lived households, each has perfect foresight and maximizes a discounted stream of utilities over its lifetime

$$U = \int_0^\infty \frac{c_t^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} dt, \ \ \sigma > 0 \ \text{ and } \ \sigma \neq 1, \tag{1}$$

where c_t is the individual household's consumption, $\rho \in (0, 1)$ denotes the subjective discount rate, and σ is the inverse of the intertemporal elasticity of substitution in consumption. We assume that there are no fundamental uncertainties present in the economy.

The budget constraint faced by the representative household is given by

$$c_t + i_t + \dot{m}_t = y_t - \pi_t m_t + \tau_t, \tag{2}$$

where i_t is investment, π_t is the inflation rate, and τ_t represents lump-sum transfers (expressed in real terms) that households receive from the monetary authority. The variable m_t denotes the real money balances that equal the nominal money supply M_t divided by the price level P_t. In addition, output y_t is produced by

$$y_t = Ak_t, \ \ A > 0\,, \tag{3}$$

where k_t is the household's capital stock. Investment adds to the stock of physical capital according to the law of motion

$$\dot{k}_t = i_t - \delta k_t, \ \ k_0 > 0 \text{ given}, \tag{4}$$

where $\delta \in [0, 1]$ is the capital depreciation rate.

As in Wang and Yip (1992) and Palivo et al. (1993), the representative household also faces the following generalized cash-in-advance (CIA) or liquidity constraint:

[2] For expositional simplicity, Palivo et al. (1993) focus their analysis on the cases with fully depreciated capital, whereas Suen and Yip (2005) examine the model with zero capital depreciation.

$$c_t + \phi i_t \leq m_t, \ \ \phi \in [0,1], \tag{5}$$

that is, all consumption purchases and a fraction ϕ of gross investment must be financed by the household's real balances m_t; and the remaining fraction $(1-\phi)$ of investment goods are acquired through barter.[3] Notice that when $\delta = 0$ and $\phi = 1$, we recover the model of Suen and Yip (2005).

On the monetary side of the economy, nominal money supply is postulated to evolve according to

$$M_t = M_0 e^{\mu t}, \ \ M_0 > 0 \ \text{given}, \tag{6}$$

where $\mu \neq 0$ is the constant money growth rate, and the resulting seigniorage is returned to households as a lump-sum transfer, hence $\tau_t = \mu m_t$.

The first-order conditions for the representative household with respect to the indicated variables and the associated transversality conditions (TVC) are

$$\begin{aligned}
c_t &: \quad c_t^{-\sigma} = \lambda_{mt} + \psi_t, && (7)\\
i_t &: \quad \lambda_{kt} = \lambda_{mt} + \phi\psi_t, && (8)\\
k_t &: \quad \dot{\lambda}_{kt} = (\rho+\delta)\lambda_{kt} - A\lambda_{mt}, && (9)\\
m_t &: \quad \dot{\lambda}_{mt} = (\rho+\pi_t)\lambda_{mt} - \psi_t, && (10)\\
\text{TVC}_1 &: \quad \lim_{t\to\infty} e^{-\rho t}\lambda_{kt}k_t = 0, && (11)\\
\text{TVC}_2 &: \quad \lim_{t\to\infty} e^{-\rho t}\lambda_{mt}m_t = 0, && (12)
\end{aligned}$$

where λ_{mt} and λ_{kt} are the utility values of real money balances and capital, respectively, and ψ_t represents the Lagrange multiplier for the CIA constraint (5). Equation (7) states that the marginal benefit of consumption equals its marginal cost, which is the marginal utility of having an additional real dollar. In addition, equations (8) and (9) together govern the evolution of physical capital over time, and equation (10) equates the marginal values of real money holdings to their marginal costs.

As is common in the literature, we assume that the CIA constraint (5) is strictly binding in equilibrium. Moreover, clearing in the goods and money markets imply that

[3] In contrast to Palivo et al. (1993), we do not take into account the possibility that the fraction ϕ may depend on the inflation rate and on the degree of financial improvement. This simplification streamlines our exposition and maintains comparability with Suen and Yip (2005).

$$c_t + i_t = y_t, \tag{13}$$

and

$$\dot{m}_t = (\mu - \pi_t)\, m_t. \tag{14}$$

3 Balanced Growth Path

We focus on the economy's balanced growth path (BGP) along which output, consumption, capital and real money balances exhibit a common, positive constant growth rate denoted by θ. To facilitate the analysis of perfect-foresight dynamics, we follow Suen and Yip (2005) and make the following transformation of variables: $p_t \equiv \frac{\lambda_{kt}}{\lambda_{mt}}$ and $z_t \equiv \frac{c_t}{k_t}$. With this transformation, the model's equilibrium conditions can be expressed as an autonomous pair of differential equations

$$\frac{\dot{p}_t}{p_t} = \frac{\frac{\sigma(p_t-1)}{\phi} + [\sigma - g_2(z_t)]\left(\delta - \frac{A}{p_t}\right) + \sigma\,[1 - g_2(z_t)]\,(A - \delta - z_t) - \sigma\mu - \rho g_2(z_t)}{\sigma - g_1(p_t) g_2(z_t)}, \tag{15}$$

$$\frac{\dot{z}_t}{z_t} = \frac{1}{\sigma}\left[g_1(p_t)\frac{\dot{p}_t}{p_t} + \frac{A}{p_t} - \rho - \delta\right] - A + \delta + z_t, \tag{16}$$

where

$$g_1(p_t) \equiv \frac{\phi - 1}{p_t - 1 + \phi} \quad \text{and} \quad g_2(z_t) \equiv \frac{(1-\phi)\, z_t}{(1-\phi)\, z_t + \phi A}.$$

Given the above dynamical system (15) and (16), the BGP equilibrium is characterized by a pair of positive real numbers (p^*, z^*) that satisfy $\dot{p}_t = \dot{z}_t = 0$. It is straightforward to show that p^* is the solution to the following quadratic equation:

$$p^* = 1 + \phi\left[\left(1 - \frac{1}{\sigma}\right)\left(\frac{A}{p^*} - \delta\right) + \mu + \frac{\rho}{\sigma}\right]. \tag{17}$$

We can then obtain the expression of z^* as

$$z^* = \frac{1}{\sigma}\left(\rho + \delta - \frac{A}{p^*}\right) + A - \delta. \tag{18}$$

With (17) and (18), it follows that the common (positive) rate of economic growth θ is

$$\theta = \frac{1}{\sigma}\left(\frac{A}{p^*} - \rho - \delta\right) \quad \text{or} \quad \theta = A - \delta - z^*, \tag{19}$$

which in turn implies that the BGP's growth rate is negatively related to the two transformed state variables, that is

$$\frac{\partial \theta}{\partial p^*} < 0 \text{ and } \frac{\partial \theta}{\partial z^*} < 0. \tag{20}$$

On the other hand, the BGP's inflation rate π^* is *ceteris paribus* positively related to the rate of money growth because equation (14) indicates that $\mu = \pi^* + \theta$.

Next, we take total differentiation on (17) and (19), and find that the growth effect of money (or inflation) is given by

$$\frac{d\theta}{d\mu} = -\frac{A}{\sigma (p^*)^2} \frac{dp^*}{d\mu} = \frac{\phi A}{\sigma \left[\phi A \left(1 - \frac{1}{\sigma}\right) - (p^*)^2\right]}. \tag{21}$$

To examine the existence and number of the economy's balanced growth paths in a transparent manner, we let $f(p^*) \equiv 1 + \phi \left[\left(1 - \frac{1}{\sigma}\right)\left(\frac{A}{p^*} - \delta\right) + \mu + \frac{\rho}{\sigma}\right]$ from (17), and obtain

$$f' = \frac{\phi A \left(\frac{1}{\sigma} - 1\right)}{(p^*)^2} \begin{matrix} > \\ < \end{matrix} 0 \text{ when } \sigma \begin{matrix} < \\ > \end{matrix} 1, \tag{22}$$

and

$$f'' = \frac{-2f'}{p^*} \begin{matrix} > \\ < \end{matrix} 0 \text{ when } f' \begin{matrix} < \\ > \end{matrix} 0. \tag{23}$$

Therefore, the equilibrium p^* can be located from the (possibly more than one) intersection(s) of $f(p^*)$ and the 45-degree line.

In terms of the BGP's local stability properties, we compute the Jacobian matrix J of the dynamical system (15) and (16) evaluated at (p^*, z^*). The trace and determinant of the Jacobian are given by

$$Tr = \frac{[\sigma - g_3(p^*)]\, z^* + \frac{\sigma p^*}{\phi} + \frac{A[\sigma - g_4(z^*)]}{p^*}}{\sigma - g_3(p^*) g_4(z^*)}, \tag{24}$$

$$Det = \frac{z^* \left[\frac{\sigma p^*}{\phi} - \frac{(1-\sigma)A}{p^*}\right]}{\sigma - g_3(p^*) g_4(z^*)}, \tag{25}$$

where[4]

[4]Since $\phi \in [0, 1]$, the numerator of $g_3(p^*)$ is non-positive. Moreover, using $p^* \equiv \frac{\lambda_k^*}{\lambda_m^*}$ and the BGP version of equations (7) and (8), the denominator of $g_3(p^*)$ can be rewritten as $\frac{\phi (c^*)^{-\sigma}}{\lambda_m^*}$, which is non-negative.

$$g_3(p^*) \equiv \frac{\phi - 1}{p^* - 1 + \phi} \leq 0 \quad \text{and} \quad g_4(z^*) \equiv \frac{(1-\phi)\, z^*}{(1-\phi)\, z^* + \phi A} \in [0,1].$$

The stability of a balanced growth path is determined by comparing the eigenvalues of J that have negative real parts to the number of initial conditions in the dynamical system (15)-(16), which is zero because p_t and z_t are both jump variables. As a result, the BGP displays saddle-path stability and equilibrium uniqueness when both eigenvalues have positive real parts. If one or two eigenvalues have negative real parts, then the BGP is locally indeterminate (a sink) and can be exploited to generate endogenous growth fluctuations driven by agents' self-fulfilling expectations or sunspots.

In the remainder of this section, we analyze the existence and number of the model's balanced growth paths, together with the associated comparative statics and local dynamics, in three parametric specifications.

3.1 When $\phi = 0$

When $\phi = 0$, money holdings are required only for real purchases of the consumption good, as in Clower (1967) and Lucas (1980), among many others. Using (17) and (19), it is immediately clear that the economy possesses a unique balanced growth path along which output, consumption, capital and real money balances all grow at the (positive) rate of $\theta = \frac{1}{\sigma}(A - \rho - \delta)$. It follows that money is superneutral in the growth-rate sense, $\frac{d\theta}{d\mu} = 0$. The intuition for this well-known result is straightforward. When the CIA constraint applies exclusively to consumption, the BGP's relative price of capital to money (in utility terms) p^* equals one and remains unchanged, which in turn yields a constant real rate of return to investment. As a result, the economy's output growth rate is independent of money and inflation. Moreover, it can be shown that both eigenvalues of this specification ($\sigma(\mu + 1 + z^*)$ and z^*) are positive real numbers, thus the unique BGP equilibrium is a saddle.

3.2 When $\phi \in (0, 1]$ and $\sigma \geq 1$

When $\sigma > 1$, we note that $f(p^*) \to \infty$ as $p^* \to 0$. Figure 1 shows that this feature, together with (22) and (23), implies that $f(p^*)$ is a downward-sloping and concave curve that intersects the 45-degree line once in the positive quadrant. Figure 1 also shows that a higher nominal money growth shifts the $f(p^*)$ locus to the right such that $\frac{dp^*}{d\mu} > 0$, which then results in a lower BGP's growth rate since $\frac{d\theta}{d\mu} < 0$ (see equations 20 and 21). Therefore, the economy exhibits a unique balanced growth path in which money and inflation are inversely related to the rate of economic growth. Regarding local dynamics, it can be shown that in this formulation, the model's Jacobian matrix exhibits a positive trace and a positive determinant, indicating the BGP's saddle-path stability. On the other hand, when the household utility (1) is logarithmic in consumption $\sigma = 1$, we find that $\theta = \frac{A}{1+\phi(\mu+\rho)} - \rho - \delta$ and obtain qualitatively identical results (BGP's uniqueness and local determinacy, and the negative growth effects of money and inflation) to those under $\sigma > 1$.

As is pointed out by Suen and Yip (2005), when real purchases of both consumption and investment goods are subject to the CIA constraint ($\phi \neq 0$), there are two opposing forces which interact to determine how money and inflation affect the BGP's growth rate. First, a higher inflation causes the representative household to substitute out of real money balances and into capital (the portfolio substitution effect). This in turn raises the relative shadow price of capital p^* because of a higher demand, thereby reducing the net (after-inflation) rate of return on capital and thus the economy's output growth rate (see equation 20). Second, other things being equal, an increase in the money growth rate induces the representative household to consume less and invest more today in exchange for higher future consumption (the intertemporal substitution effect). This in turn expands the supply of capital, hence lowering its relative shadow price p^*. As a result, the BGP's growth rate will rise. Our preceding analysis illustrates that the portfolio substitution effect dominates in the current specification where the intertemporal elasticity of substitution in consumption is not strictly greater than one ($\frac{1}{\sigma} \leq 1$). Consequently, the economy displays a negative relationship between the BGP's output growth and money/inflation $\frac{d\theta}{d\mu} < 0$. This finding is consistent with the empirical evidence reported in Kormendi and Meguire (1985), Grier and Tullock (1989), Levine and Renelt (1992), Roubini and Sala-i-Martin (1992), De Gregorio (1993), and Barro (1995), among many others.

3.3 When $\phi \in (0, 1]$ and $\sigma < 1$

Figure 2 shows that when $\sigma < 1$, $f(p^*) \to -\infty$ as $p^* \to 0$ and $f(p^*)$ is a upward-sloping concave curve. Therefore, the number of intersections between $f(p^*)$ and the 45-degree line in the positive quadrant can be zero, one or two. We proceed with first deriving the critical value of σ, denoted as $\hat{\sigma}$, at which $f(p^*)$ is tangent to the 45-degree line ($f' = 1$) so that there exists a unique BGP equilibrium characterized by $\hat{p}$ and thus $\theta(\hat{p})$. Using (22), it is straightforward to show that the expression for $\hat{\sigma}$ is given by

$$\hat{\sigma} = \frac{\phi A}{\phi A + (\hat{p})^2}, \tag{26}$$

which lies in the interval $(0, 1)$ since $\phi \in (0, 1]$ and A, $\hat{p} > 0$. Next, we note that a higher σ shifts the locus of $f(p^*)$ upwards because

$$\frac{\partial f(p^*)}{\partial \sigma} = \frac{\phi}{\sigma^2} \left(\frac{A}{p^*} - \rho - \delta \right) = \frac{\phi \theta}{\sigma} > 0. \tag{27}$$

It follows that the economy possesses no (two) balanced growth path(s) provided $\sigma <$ $(>)$ $\hat{\sigma}$. Moreover, starting from the balanced growth path with $\theta(\hat{p})$, (26) and (27) together imply that any small changes in the intertemporal elasticity of substitution in consumption will lead to the BGP's disappearance or the emergence of dual BGP equilibria. This indicates that the economy undergoes a saddle-node bifurcation whereby the number of balanced growth paths changes as the parameter σ passes through some critical value.

When there are two BGP equilibria, the equilibrium path with a lower relative shadow price of capital, denoted as p_1^* in Figure 2, will grow faster than the other that is associated with p_2^*, that is $\theta(p_1^*) > \theta(p_2^*)$. Figure 3 shows that in this case, $f(p^*)$ shifts up in response to a rise in the money growth rate. Hence, if the economy starts at the low-growth BGP equilibrium, the negative growth effects of money and inflation ($\frac{d\theta(p_2^*)}{d\mu} < 0$), as in the previous case with $\sigma \geq 1$, continue to hold because of a stronger portfolio substitution effect. Conversely, the intertemporal substitution effect outweighs the portfolio substitution effect for the high-growth equilibrium path. As a consequence, the BGP's output growth and money/inflation are positively correlated ($\frac{d\theta(p_1^*)}{d\mu} > 0$). This result provides theoretical support for some of the empirical findings in, for example, Gomme (1993), and Bullard and Keating (1995).

We also find that around the balanced growth path associated with p_2^*, the model's Jacobian matrix possesses a positive trace and a positive determinant. Thus, this low-growth equilibrium path is a saddle. On the other hand, in the neighborhood of the BGP equilibrium associated with p_1^*, the determinant of the Jacobian is negative, indicating that one of the eigenvalues has negative real part. It follows that the high-growth equilibrium exhibits indeterminacy and sunspots. The intuition for this indeterminacy result can be understood as follows. When agents expect a higher future return on capital, they will reduce consumption and increase investment today. If the intertemporal substitution effect (from consumption to investment) is sufficiently strong, the rate of return on capital will rise because of a fall in its relative shadow price p^*. As a result, agents' initial optimistic expectations become self-fulfilling. On the contrary, equilibrium indeterminacy does not occur when the portfolio substitution effect dominates. In this case, agents' optimism leads to a dominating portfolio substitution from real money balances to capital. This in turn raises the relative shadow price of capital p^* and lowers its rate of return, thus preventing agents' expectations from becoming self-fulfilling. The same intuition is applicable to other specifications (when $\phi = 0$; or when $\phi \in (0,1]$ and $\sigma \geq 1$) in which the economy displays saddle-path stability and equilibrium uniqueness.

Finally, equation (26) implies that the critical value for the inverse of the intertemporal elasticity of consumption substitution $\hat{\sigma}$ is a complicated function of the model's parameters A, ϕ, ρ, μ and δ. We can then derive the following comparative-statics results:

$$\frac{\partial\hat{\sigma}}{\partial\phi}, \frac{\partial\hat{\sigma}}{\partial\rho}, \frac{\partial\hat{\sigma}}{\partial\mu}, \frac{\partial\hat{\sigma}}{\partial\delta} < 0 \quad \text{and} \quad \frac{\partial\hat{\sigma}}{\partial A} > 0. \tag{28}$$

Recall that the economy exhibits dual BGP equilibria, one is a saddle and the other is a sink, whenever $\hat{\sigma} < \sigma < 1$. Therefore, (28) shows that, keeping other parameters unchanged, a smaller A or an increase in ϕ, ρ, μ or δ will lower $\hat{\sigma}$, and thus enlarge the region of BGP-equilibrium multiplicity. In this case, households become less risk averse, hence they are more willing to give up today's consumption in exchange for the higher returns from belief-driven investment spurts. This will generate a stronger intertemporal substitution effect, and thus raise the likelihood of indeterminacy and sunspots.

4 Velocity of Money

The income velocity of money is defined as the number of times a nominal dollar turns over each period. Using (3), (4), (5) and (13), it can be shown that the money velocity along the economy's balanced growth path V^* is given by

$$V^* = \frac{A}{\phi A + (1-\phi)z^*}. \tag{29}$$

Furthermore, notice that the BGP's consumption-capital ratio z^* is positively related to the corresponding relative shadow price of capital p^* (see equation 18). The intuition for this positive correlation is straightforward. As discussed earlier, a higher relative shadow price of capital p^* reduces its net rate of return and thus investment. It follows that output and consumption will fall as well. Given the aggregate resource constraint (13), the decline in consumption is smaller than that in output, hence the consuming-output ratio rises. This in turn leads to an increase in the consumption-capital ratio z^* because of the AK production technology.

When money holding is required only for the transactions of consumption goods ($\phi = 0$), V^* is equal to $\frac{A}{z^*}$. In this case, as discussed in section 3.1, money becomes superneutral in the growth-rate sense because of a time-invariant relative shadow price of capital p^*. Therefore, the BGP's consumption-capital ratio z^* is also a constant, which in turn implies that the money velocity is independent of money and inflation $\frac{dV^*}{d\mu} = 0$. Next, when the CIA constraint applies to all the investment purchases as well ($\phi = 1$), as in Stockman (1981) and Suen and Yip (2005), V^* always stays at one. Consequently, changes in nominal money growth will not affect the velocity of money $\frac{dV^*}{d\mu} = 0$.

It turns out that the velocity effects of money and inflation are closely related to the BGP's local stability properties when $\phi \in (0, 1)$. If the BGP equilibrium is a saddle and locally determinate (when $\sigma \geq 1$; or the low-growth equilibrium when $\hat{\sigma} < \sigma < 1$), the portfolio substitution effect outweighs the intertemporal substitution effect in response to a higher money growth rate. As a result, the net rate of return on capital falls because of a higher relative shadow price p^*, thereby raising the consumption-capital ratio z^* and reducing the BGP's money velocity, thus $\frac{dV^*}{d\mu} < 0$. This result is consistent with the empirical evidence, both over time and across countries, documented in Palivo et al. (1993). By contrast, the intertemporal substitution effect dominates if the BGP equilibrium is a sink and locally indeterminate (the high-growth equilibrium when $\hat{\sigma} < \sigma < 1$). In this configuration, an increase in the nominal money growth leads to decreases in the relative shadow price of capital p^* and the consumption-capital ratio z^*. It follows that the BGP's velocity of money will rise, hence $\frac{dV^*}{d\mu} > 0$.

5 Conclusion

In response to the research question posed by Palivo et al. (1993), this paper systematically examines the impact of changing money growth on the velocity in a one-sector endogenous growth model with an AK production technology and a generalized cash-in-advance constraint. Our analysis shows that the velocity effects of money and inflation can be negative, positive, or zero, depending on the fraction of gross investment that is subject to the liquidity constraint, as well as the relative strength of the portfolio substitution effect versus the intertemporal substitution effect. Moreover, this finding turns out to be closely linked with the local stability properties of the economy's balanced growth paths and the associated growth effects of money. In particular, when the cash-in-advance constraint is applicable to a positive proportion (excluding 100 percent) of the household's investment purchases, the economy displays saddle-path stability and negative effects of money on output growth and velocity because of a dominating portfolio substitution effect. In addition, these results are reversed, *i.e.* the economy exhibits indeterminacy and sunspots, and positive growth and velocity effects of money, when the opposing intertemporal substitution effect dominates. However, a positive correlation between money and velocity is not consistent with the empirical estimates that Palivo et al. (1993) have obtained from an international data set during the $1963 - 1985$ sample period.

Finally, regarding possible extensions of our analysis, it would be worthwhile to analyze a monetary endogenous growth model which includes features like variable labor supply, human capital accumulation, multiple production sectors, or endogenous technological change, among others . This will allow us to examine the robustness of our results, and further identify other channels that can affect the interrelations between the growth and velocity effects of money and the local stability properties of the economy's balanced growth paths. We plan to pursue these research projects in the near future.

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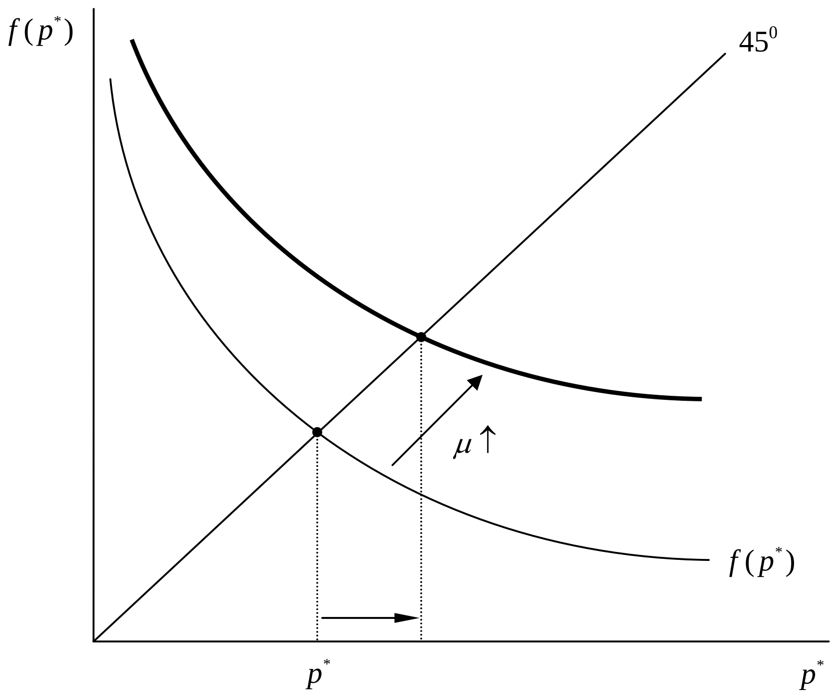

Figure 1: $0<\phi\leq 1$ and $\sigma>1$

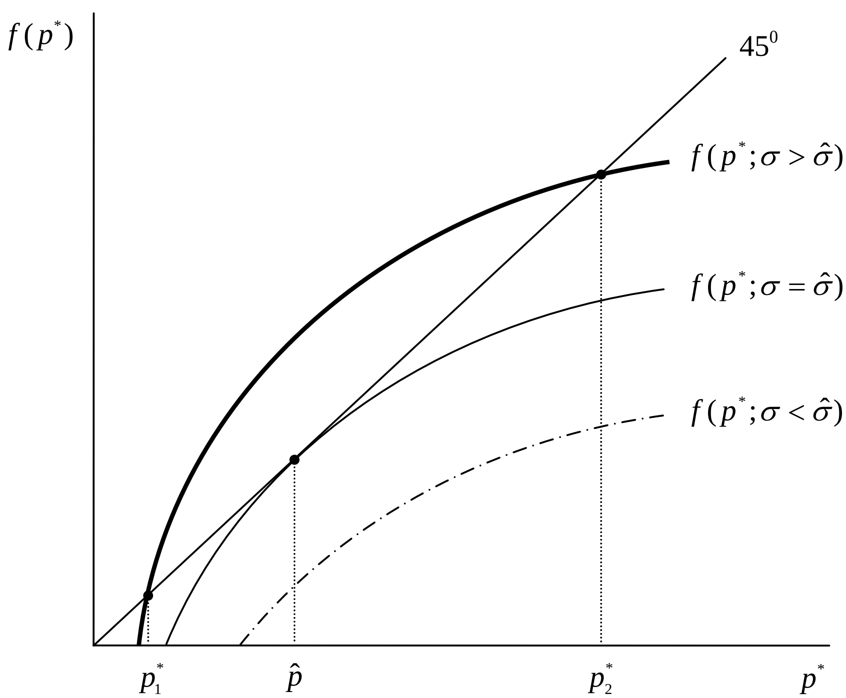

Figure 2: $0<\phi\leq 1$ and $\sigma<1$

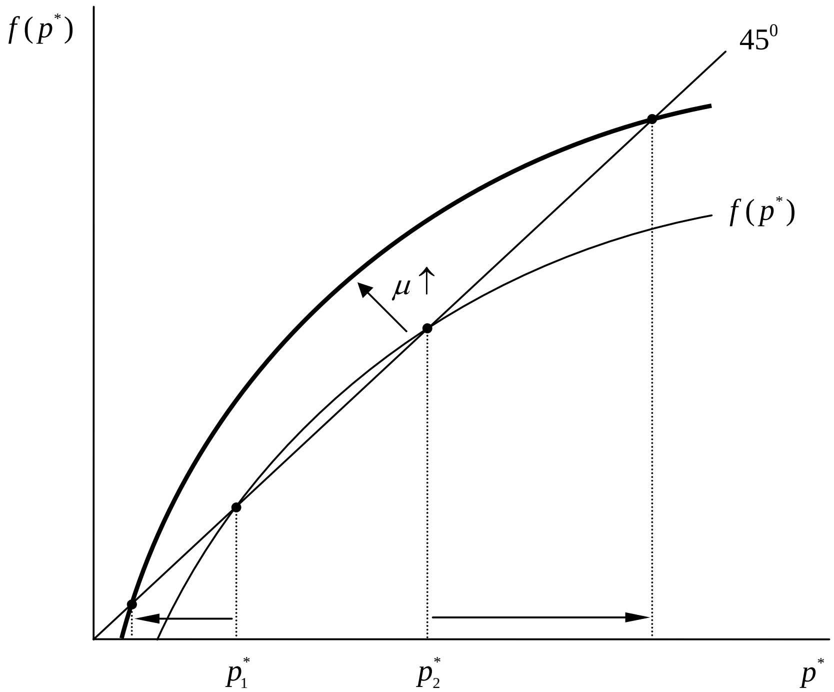

Figure 3: $0 < \phi \le 1$ and $\hat{\sigma} < \sigma < 1$