Rational Evaluation of Actions Under Complete Uncertainty

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Abstract. This work analyzes the problem of individual choice of actions under complete uncertainty. In this context, each action consists of a set of di®erent possible outcomes with no probability distribution associated with them. The work examines and de¯nes a class of choice procedures in which: a): the evaluation of sets (actions) is element-induced; and b): certain assumption of rationality, which is an adaptation of Sen's ® condition, is satis¯ed. Some results of characterization show that different well-known rules can be reinterpreted as particular cases within the de¯ned class, each of them responding to di®erent attitudes towards uncertainty by the agent.

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1 Introduction

Theories of individual choice under uncertainty assume that the decision maker knows the possible consequences of each alternative action, but he cannot assign a probability to their occurrence under each particular choice.

One possible approach to such problems of choice takes into account a set of possible states of the world, so that a feasible action by the agent is represented by a vector of di®erent outcomes contingent on the possible states (see Arrow and Hurwicz [2], Maskin [28], or Cohen and Ja®ray [14]). Another approach represents actions as sets of possible outcomes without specifying any states of nature. These sets are also called uncertain prospects, or prospects. That is, for any action (or prospect) it only matters which outcomes may result. Examples of this approach are Barberà, Barret and Pattanaik [5], Barberà and Pattanaik [6], Nitzan and Pattanaik [29], Pattanaik and Peleg [32] Bossert [9], [10], and Bossert, Pattanaik and Xu [11].

Di®erent arguments in favour of the set-based approach can be found in Pattanaik and Peleg [32], or Bossert, Pattanaik and Xu [11]: Compared with the vector-based approach, the set-based one involves a loss of information as long as it does not allow the decision maker to take into account the number of states in which an action leads to a certain outcome. However, the set-based approach overcomes certain problems of speci¯cation of the set of the states of the world which sometimes arise. For example, the set of states may be so large that it can hardly be assumed that the decision maker is able to perceive the actions as if they were vectors of contingent outcomes. Furthermore, it is sometimes argued that the way of partitioning the set of all possible contingencies in a concrete number of states is arbitrary to a large extent. Also, the set-based approach is appropriate for formalizing the Rawlsian formulation of individual values under the veil of ignorance (Rawls [33]), where presumably no states of the world are considered.

Within the set-based approach, models usually consist of extending axiomatically an ordering R, de¯ned over a universal space X of outcomes, to another preference < over the possible subsets of X. Those subsets are interpreted as feasible actions, represented by their associated uncertain consequences (or outcomes). Axioms are imposed on <, and try to capture reasonable properties of it taking into account the information given by R. Usually the axioms display plausible attitudes towards uncertainty by the agents, as well as certain conditions of consistency. Well-known rules -such as the maximin rule, the maximax rule, or their lexicographic extensions,- have actually been characterized by means of this methodology.[1] In the same methodological line, Bossert, Pattanaik and Xu [?] propose the min-max and max-min criteria, which look ¯rst at the worst (best) element of each prospect, and secondly at the best (worst) one, and only when both elements are equal, the prospects are considered indi®erent. In the same work these authors also characterize lexicographic extensions of the min-max and the max-min rules.

It is interesting to point out that almost all of the criteria considered by the related literature are element-induced. That is, given an ordering R over the universe of outcomes, the comparison of feasible actions (sets of outcomes) is always induced from the comparison by means of R of certain elements (outcomes) within the respective sets, such as the worst element, the best one, or the second worst if the worst coincide, and so on.

However, element-induction is not the only possible way to compare and evaluate prospects. For example, if R is an ordering, then a utility function can be de¯ned over X, and additive processings could be applied. Also, it is

[1] Fishburn [15], Heiner and Packard [20], Holzman [21] [22], Kannai and Peleg [24], Bandyopadhyay [4], or Bossert [8], analyze this formal problem of extension of an ordering R to a relation < over sets, but without devoting expressely to a problem of choice under uncertainty. However, these works can be perfectly interpreted in such a context.

possible to adapt to our framework the satisficing rule of Simon [39], consisting of establishing two indiference classes among the prospects: on the one hand those in which all the outcomes surpasses certain critical value, and on the other hand those in which at least one outcome does not surpasse that critical value. Prospects in the former class would then be preferred to those in the latter one. Russo and Dosher [34] propose the majority of confirming dimensions rule, according to which, in order to compare a pair of prospects, the decision maker compares, two-by-two, the possible respective dimensions (outcomes). Then, the prospect with a majority of better outcomes would be declared better.[2] Finally, nothing at this stage prevents us from considering even an entirely random rule.

Nonetheless, seeing as how published work focuses on element-induced rules suggests that these have some natural virtue, at least for a context of choice under complete uncertainty. The primary goal of this work is to open up some discussion about the suitability of element-induced processes themselves. Also, as long as these rules presumably belong to a certain common class, it then becomes necessary to formally describe and define such a class.

On the other hand, a basic premise of this work is that, independently of the axiomatic structure which leads to different particular rules, each of these rules can be intuitively explained as the direct result of different reasonable and basic patterns of behavior. For example, the maximin rule could be interpreted as the result of a risk-averse behavior by an agent who never looks further than one alternative; the lexicographic extension of the maximin as the result of a risk-averse behavior by an agent who is in some way recurrent or iterative; and analogous explanations can be figured out for the maximax and its lexicographic extension. Even the max-min and min-max rules of Bossert, Pattanaik and Xu [11] are interpretable as following certain patterns of behavior based on "focal" or "conspicuous" characteristics of the prospects.

[2] This rule is not well defined for prospects of different size but plausible extensions for different-sized prospects could be defined.

Those patterns have to do with the decision maker's internal attitudes relating to the problem of choice, such as his risk aversion, his willingness to iterate when ties appear, or the tendency to focus on particular characteristics of the sets. Now the question is: Is it possible to ¯nd such plausible explanations for any kind of element-induced processes of choice? Let us consider the case of an individual (call him \Gage") who, in order to evaluate feasible actions, takes into account the worst possible outcome when the number in the prospect is even, and the best one when the number of outcomes is odd, and then extends the preference relation obtained over these elements to rank the corresponding actions.

Undoubtedly, his behavior seeks some procedural logic, but which is arbitrary to a large extent, as long as his pattern does not display reasonable attitudes when facing the problem of choice under uncertainty. Actually, models of decision usually try to avoid these kinds of pathological behavior. The standard approach consists of impossing axioms on **<**, for example: \given a prospect A, if a new possible outcome is added to A and that outcome is worse than all the outcomes in A, then the enlarged prospect should be worse". Thus, Gage's behavior would be implicitely rejected by imposing such an axiom.

But conditions could be impossed on the mere process of evaluation too. In this example, the inconsistencies lay in the process itself, more than in the resultant inconsistencies when comparing actions as a product of that process. Actually, in Herbert Simon's terms, Gage's irrationality is procedural rather than substantive.[3] From an epistemological point of view, the procedural approach

[3] According to Simon [41], \behavior is procedurally rational when it is the outcome of appropriate deliberation", while \behavior is substantively rational when it is appropriate to the achievement of given goals within the limits imposed by given conditions and constraints". Also, behavior is procedurally irrational when it simply \represents impulsive response to a®ective mechanisms without an adequate intervention of thought".

seems to be more apropriate than the substantive one. The following sections try to justify the view that, in a context of complete uncertainty, element-induction processes are, a priori, procedurally plausible as a natural way of gathering information and the evaluation of the alternatives. But even under this assumption, we would like to somehow restrict the kind of element-inductive processes to those which obey certain procedural coherence.

In Gage's example, let us consider an action fx; y; zg such that x is better than y, and y is better than z. Gage takes x as a representative, focal, or paradigmatic element of the prospect; and if it shrank to fx; yg, then Gage would concentrate on y as a representative element. However, if x is representative for fx; y; zg, it would be reasonable to impose that x should be also representative for fx; yg. That is, a property close to the condition ® of rational choice (Sen [35]) could be applied to the way the agent evaluates each prospect. The justi¯cations for imposing such a condition would not be far from the justi¯cations argued for imposing it in the standard framework of rational choice. In fact, deciding which attribute to evaluate ¯rst within a certain prospect, is not a much di®erent mental exercise than is the standard one of choosing an alternative among a universe of alternatives. But, in our context, the former problem is simpler than, and previous to, that of choosing among prospects (each of which consisting of multiple possible outcomes). Thus, the demand for rationality of the procedure is, in some way, weaker then the standard one, based on the ¯nal results of that procedure.

In summary, this paper is a contribution to the set-based approach to the problem of choice under uncertainty, but a di®erent formalization of the problem is presented. Conditions are imposed at three distinct levels: ¯rst, the model takes as an assumption that rankings over actions are element-induced, and the suitability of such an assumption is discussed. Second, a condition of rationality, which is an adaptation of Sen's ® condition , is imposed on the evaluation process of each set, and some results are proposed. That is, a model of rational evaluation of the sets, in contrast with models of rational choice among sets, is proposed. Assumptions at these first two levels represent procedural conditions on the decision problem. Third, some additional axioms are imposed on the binary relation over set. These axioms display, at a very basical level, different possible attitudes of the agent towards uncertainty, and allow us to characterize some known rules in an alternative fashion as particular cases of element-induced and rational rules. That is, unlike the rest of the set-based literature, these axioms do not play the role of introducing the "rationality" in the agent's behavior. They simply explain different ways of behaving rationally according to the condition imposed on the procedure of evaluation.

The paper is organized as follows. In Section 2 the notation and preliminar definitions are posed. Section 3 examines and formally defines the class of element-induced rules. In Section 4 the basic condition of rationality for element-induced processes is proposed. Section 5 contains some axioms which capture different possible attitudes towards uncertainty, and some results of characterization are presented. Section 6 presents some additional properties which allow us to characterize the max-min rule, the min-max rule and their lexicographic extensions as element-induced rational rules. In Section 7 some final remarks are noted.

2 Notation and Definitions

Let X be a finite set of outcomes ($\#X = n$). Let Z denote $2^X \setminus \emptyset$, and let R be a linear preference ordering defined over X (that is, R is a reflexive, transitive, complete and antisymmetric binary relation). The interpretation of R is the common one: $\forall x, y \in X, xRy$ is read as "x is at least as desirable as y". P denotes the asymmetric factor of R, reading xPy as "x is more desirable than y". For all $A \in Z$, $\underline{a}$ and $\bar{a}$ denote, respectively, the worst and bests element of A according to R, and $\underline{\underline{a}}$ and $\bar{\bar{a}}$ denote, respectively the second worst and second best element of A according to R. Since R is a linear ordering, hence, of all these elements are well-defined and unique for all $A \in Z$.

The formal concern of this work is the extension of the linear ordering R over X to an ordering $\succcurlyeq$ over Z (an ordering is a reflexive, complete and transitive binary relation). $\succ$ and $\sim$ denote, respectively, the asymmetric and symmetric parts of $\succcurlyeq$. This formal problem of extension is interpreted in a context of choice under complete uncertainty, where each element A of Z is interpreted as a set of possible outcomes of a certain action (or prospect), such that the decision maker does not assign any probability nor any likelihood ranking to any of the possible outcomes. Therefore, $\succcurlyeq$ is interpreted as reflecting the agent's preference over the possible actions.

Given a finite set X and certain relation R defined on it, the following rules, standard in the field, are going to be analyzed:

- The maximin relation $\succcurlyeq_m$ is defined by: $\forall A, B \in Z; A \succcurlyeq_m B$ iff $\underline{a} R \underline{b}$
- The maximax relation $\succcurlyeq_M$ is defined by: $\forall A, B \in Z; A \succcurlyeq_M B$ iff $\overline{a} R \overline{b}$
- The leximin relation $\succcurlyeq_{lm}$ is defined by: $\forall A \in Z$; $\#A = r$, let $A = \{a_1; a_2; \dots; a_r\}$ s.t. $a_r R a_{r-1} R \dots R a_2 R a_1$. Then, $\forall A, B \in Z$, $A \succcurlyeq_{lm} B$ iff $\exists l \in N; l \leq \max\{\#A; \#B\}$ s.t. $a_i = b_i \forall i < l$, and $[(a_l R b_l)$ or (a_l exists and b_l does not exist)]
- The leximax relation $\succcurlyeq_{LM}$ is defined by: $\forall A \in Z$; $\#A = r$, let $A = \{a_1; a_2; \dots; a_r\}$ s.t. $a_1 R a_2 R \dots R a_r$. Then, $\forall A; B \in Z; A \succcurlyeq_{LM} B$ iff $\exists l \in N; l \leq \max\{\#A; \#B\}$ s.t. $a_i = b_i \forall i < l$, and $[(a_l R b_l)$ or (a_l exists and b_l does not exist)]

Also, the following rules, which appear in Bossert, Pattanaik and Xu [11] will be considered:

- The min-max relation $\succcurlyeq_{mM}$ is defined by: $\forall A, B \in Z; A \succcurlyeq_{mM} B$ iff $(\underline{a} P \underline{b})$ or ($\underline{a} = \underline{b}$ and $\overline{a} R \overline{b}$)
- The max-min relation $\succcurlyeq_{Mm}$ is defined by: $\forall A, B \in Z; A \succcurlyeq_{mM} B$ iff $(\overline{a} P \overline{b})$ or ($\overline{a} = \overline{b}$ and $\underline{a} R \underline{b}$)

– The lexicographic min-max relation $\succsim_{lmM}$, and lexicographic max-min relation $\succsim_{LMm}$ are defined by (see Bossert, Pattanaik and Xu [11]):

$\forall A \in Z$, let $A_0 = A$ and

$$n_A = \begin{cases} \#A/2 & \text{if } \#A \text{ is even} \\ (\#A - 1)/2 & \text{if } \#A \text{ is odd} \end{cases}$$

If $n_A > 0$, let, for all $t = 1; \ldots; n_A$, $A_t = A_{t-1} \setminus \{\underline{a}_{t-1}; \overline{a}_{t-1}\}$. For all $A; B \in Z$, let $n_{AB} = \min(\{n_A; n_B\})$.

then,

$\forall A; B \in Z; A \succsim_{lmM} B$ iff $\exists t \in \{0; \ldots; n_{AB}\}$ such that $(A_s \sim_{mM} B_s \forall s < t)$ and $(A_t \succ_{mM} B_t$ or $B_t = \emptyset)$

$\forall A; B \in Z; A \succsim_{LMm} B$ iff $\exists t \in \{0; \ldots; n_{AB}\}$ such that $(A_s \sim_{Mm} B_s \forall s < t)$ and $(A_t \succ_{Mm} B_t$ or $B_t = \emptyset)$

3 Bounded rationality and element-induced rules

As has been pointed out in Section 1, in many decisional contexts the decision maker compares sets by comparing certain elements within the sets. The most natural way to understand this behavior is by assuming that the decision maker concentrates in one element of the set which for him is representative or focal, and which, for some reason, constitutes a good proxy of the value of the set, perhaps because it represents a key feature of the set in the decisional context where the comparison of sets is being made. This behavior can be formalized by assuming that there exists a certain function $f : Z \longrightarrow X$ which determines for each prospect, the outcome in the set which is focal or representative for the agent, and such that $\forall A; B \in Z$, $A \succsim B \Leftrightarrow f(A) R f(B)$. For example, in a context of choice over opportunity sets, the standard indirect-utility criterion is a clear case of an element-induced rule, where, $\forall A \in Z$, $f(A) = \overline{a}$. In the context of complete uncertainty -in which decision maker does not control the final result of the set-, the maximin and maximax rules are also examples of this.

However, the only information obtained from the first-focal elements may be insufficient to declare a strict preference between a pair of sets. One possible cause is that the preference relation over the basic elements is incomplete, or, in the case of our formal framework, that the respective focal elements may coincide. In this situation the decision maker can directly declare both sets as indifferent, or otherwise look for another feature in the set which helps him to establish a preference. If the agent looks for more information, he will plausibly repeat the inductive procedure concentrating on other element of each set (if it exists). In other words, there exists another function $f' : Z \longrightarrow X$ which, $\forall A \in Z$, determines the second-focal element in A, and such that: $\forall A, B \in Z$, $f(A) P f(B)$ implies $A \succ B$, but if $f(A) = f(B)$, then $A \succcurlyeq B \leftrightarrow [(f'(A) R f'(B))$ or ($f'(A)$ exists and $f'(B)$ does not exist)].[4] If, at this second step a new tie arises, then the again agent faces the same dilemma: to continue with another step in this sequential process, or to declare the sets indifferent. Again, if the agent decides to continue, a new similar tie, and therefore a new similar dilemma might arise, and so on. Thus, we could formalize the process of evaluation of prospects and comparison between pairs of prospects by considering succesive functions $f^* : Z \longrightarrow X$, whose number would depend on the number of iterations the agent is willing to make before declaring an indifference. Although interpretable in an "iterative" or "sequential" way, this kind of decision procedure is also element-induced: It is the value of one of the elements in the set which finally determines the preference relation between sets.

If the evaluation procedure over sets is of this type, then different degrees of "iterativeness" can be established, depending on the maximal number of times the agent is willing to iterate before declaring an indifference. In this sense, lexicographic rankings of prospects are n-times-iterative; the maximax or the maximin are once-iterative, or the max-min and the min-max rules are twice

[4] In the subsequent formalization of the decision procedure it is assumed that, for all prospects, the first focal element always exists.

iterative. We could imagine other examples of criteria which evaluate lexicographically the two best (or two worst) elements of the sets, or many other combinations.

The literature on bounded rationality, as well as experimental psychology, provide intuitive justification for this element-inductive and sequential behavior, and also provide an explanation for the eventual existence of a limit in the sequential process, which implies that the decision maker might ignore potentially relevant information concerning alternatives. In particular, that evaluation strategy is an example of what in psychology is called attribute-based processing of alternatives, where the values of the alternatives on a single attribute are processed before information about a second attribute is processed, the second atribute is analyzed before the third one, and so on. Russo and Dosher [34] suggest that attribute-based processing is cognitively easier than a holistic processing where all the dimensions of the alternatives (possible outcomes of the different prospects in our case) are taken into account. In the same line of thought, Payne, Bettman and Johnson [31] provide experimental evidence that under time pressure and in complex decisional environments, agents tend to choose lexicographic prodecures, and that these procedures perform better. It is also sometimes argued that, like computers, man's ways of thinking are serial in organization; one step in thought follows another, and solving problems requires the execution of a certain amount of steps in sequence (Simon [41]).

The basic assumption behind this simplifying behavior is that there is a background computational effort for evaluating the alternatives. As a consequence of this computational effort (or limited computational abilities) of the agent, the decision maker tends to substitute the complex reality by a handly simplification of it, consisting of its main features. Then, the computational limits of the agent leads then to a behavior based on a satisfactory performance rather than on a maximizer pattern. These kinds of arguments have been well-developed in works on bounded rationality (see for example pioneer works of Simon [39][40], or March and Simon [27]).

Thus, when choosing which elements to consider, the decision maker presumably concentrates on those which: a) contain an important characteristic of the set, and b) display features easily identi¯able, such as the maximal element or the minimal element.[5] The agent may be satis¯ed with the information given by the evaluation of one or two representative outcomes if this information allows him to establish a strict preference at a low cost. On the other hand the agent may declare, at a certain point, a relation of indi®erence, ignoring potentially relevant information if the marginal computational cost is expected to be high. One could argue that assuming the existence of a linear ordering R on X is in contradiction with the assumption of bounded rationality based on limited computational abilities of the agent. However, the existence of R means that the agent is able to order all the alternatives in X, which is compatible with the idea that certain e®ort could be necessary for: a) identifying and ordering all of the possible outcomes of a given prospect; and b) ¯nding out how to compare a given pair of prospects, especially if they consist of a large number of possible outcomes.

Obviously this kind of heuristics implies a potential cost in terms of less accurate choices. Thus a trade-o® between accurate choices and computational savings arises. The mere adoption of an iterative process is a consequence of the need to simplify the decision problem. But even among iterative processes, computational e®ort acts as a deterrent against the inde¯nite repetition of the sequential process, while the desire for accurate choices compells the decision maker to iterate more. As Beach and Mitchell [7], Payne [30], or Russo and Dosher [34] argue, \the selection strategy is the result of the costs derived from the e®ort required to use a rule", (in our case to iterate inde¯nitely), \and bene¯ts from selecting the best alternative". That is, the number of iterations

[5] Clearly, median element(s) are representative, but in the presence of computational e®ort it is hardly defendable that they are easily identi¯able.

applied by the agent may narrowly depend on the particular environment where the decision problem is considered. It will depend upon internal characteristics of the decision maker, such as his persistance, and upon such external factors as complexity of the alternatives, time pressure, the similarity of the alternatives, or their overall attractiveness (see Payne [30]).

In sum, although non-element induced criteria are plausible too, element-induced rules seem to provide a good equilibrium between the high computational effort required by holistic rules, such as arithmetic operations, and the lack of accuracy of the other rules, such as random rules. Also, within the class of element-induced rules, the different possible degrees of iterativeness allow us to display diverse combinations of the trade-offs between both kinds of factors.

We are ready now to provide the following formal definition:

Definition 1. Let $\succcurlyeq$ be an ordering defined on Z, and let R be a linear ordering defined on X. $\succcurlyeq$ is said to be an element-induced rule if there exists a natural number k, $k \leq n$, and $\forall A \in Z$ there exists a mapping $F : Z \longrightarrow Z$, $F(A) = \{f_1(A), \ldots, f_{j_A(A)}\}$ $\forall A \in Z$ such that:

1. $f_i(A) \in A$ $\forall A \in Z$, $\forall i \leq j_A$
2. $j_A = k$ if $\#A \geq k$ and $j_A \in [\#A, k]$ if $\#A \leq k$
3. $\forall A, B \in Z$, $A \succcurlyeq B \Leftrightarrow \exists l \leq k$ such that $\forall i \in N$, $i < l$, $f_i(A) = f_i(B)$ and $[(f_l(A) R f_l(B)$ or $(f_l(A)$ exists and $f_l(B)$ does not exist$)]$

For all $A \in Z$, $F(A) = \{f_1(A), \ldots, f_{j_A}(A)\}$ will be said to be "the evaluation sequence of set A," and F a "mapping of evaluation sequences". For any set, the evaluation sequence identifies both the elements and the order in which they are successively evaluated by the agent. In the definition, k represents the agent's willingness to iterate in order to find successive focal elements: The agent is never willing to iterate more than k times in any set whose cardinal is greater than k, and, on the other hand, for sets whose cardinal is smaller than k, the agent is willing to iterate at least as many times as elements are in the set, but never more than k times. Therefore the pair (k; F) describes an evaluation procedure: the number of elements that might be considered in each set, the elements considered, an in which order they are going to be considered. The de¯nition of element-induced rules simply states that it is possible to ¯nd an evaluation procedure (k; F) which \explains" or \rationalizes" a given ordering in terms of the representation statment 3 in the de¯nition. When, given an ordering <, it is possible to ¯nd a certain pair (k; F) satifying the conditions in the previous de¯nition, it will be said that \< is element-induced in relation to (k; F)," or simply that \< is element-induced in relation with F" when the particular value of k is not meaningful for the discourse.

The idea of evaluation sequence is related to, but di®erent from, the standard concept of choice function in Arrow [1] or Sen [35]). For any A 2 Z, F(A) is a sequence of functions, while a choice function is unique. Also, choice functions determine, among a set of available alternatives, the subset of those which are choosen by the agent. In contrast, F determines, in a recurrent way, and for each set of possible uncertain outcomes, the outcome that attracts the attention of the agent, but which does not necessarily happen as long as the ¯nal result is out of the control of the decision maker.

4 Rational Evaluation of Actions

De¯nition 1 provides the formal tools to allow us to establish an alternative theory of choice over prospects based on the procedural aspects, that is, based on particular properties of the evaluation procedure.

For example, despite the many conceptual and formal di®erences between choice functions over alternatives on the one hand, and evaluation sequences over sets of outcomes on the other, it makes sense to extrapolate the standard properties of rationality from the former to the latter. In particular, F will be assumed to satisfy the following condition:

Rationality: $\forall A, B \in Z$, s.t. $B \subseteq A$, $\forall i \in N$, $i \leq n$,
if $\{f_1(A), \ldots, f_{i-1}(A)\} = \{f_1(B), \ldots, f_{i-1}(B)\}$, $f_i(A) = a$, and $a \in B$, then $f_i(B) = a$.

That is, given a set A and a subset B of A, if a set of successive "representative" elements of A coincide with those in B, and if the next representative element in A belongs to B, then that element should be the next representative one in B as well.

Note that if F is rational, and for certain $i \in N$, $\{f_1(A), \ldots, f_{i-1}(A)\} = \{f_1(B), \ldots, f_{i-1}(B)\}$, then necessarily, for all $j < i$, $f_j(A) = f_j(B)$.

When $i = 1$, the Rationality assumption is even closer to the classical postulate of rationality in Chernoff [13]) and α-property of choice functions in Sen [35], but, as mentioned, it is now extended to the context of the sequential evaluation of prospects.

The assumption of Rationality for F is not in conflict with the general motivation of bounded rationality underlying this work. The decision maker is allowed to have a limited ability to compute all of the possible outcomes of the prospects compelling him to concentrate only on a limited number of outcomes. It is plausible to assume that if the agent has been able to identify certain outcomes as representative in a given prospect A, he should then be able to identify them if they are present in a subset of A. Hence, Rationality of F simply imposes that, once the agent has decided what to concentrate on, he maintains what March [26] calls a selective or calculated rationality. Similar arguments can be found in the Prospect Theory of Kahneman and Tversky [23][42]): once the decision maker completes the phase of simplification of the decision problem, certain rationality in his simplified analysis is maintained.[6] For example, let us suppose

[6] Actually, some authors in the field of Organization Theory argue that it is precisely the human necessity of being coherent and following clear goals which motivates satisfactory-performance-based behavior, that is, behavior based on the satisfaction of those clear and simplified goals (see Friedman [16], or Krulee [25]).

that, instead of evaluating all of the possible outcomes of a certain action A, the agent concentrates on what he identifies as the worst possible element, x. If the agent is rational in the chosen proxy method for the value of prospects, for any subset of A containing x, he should not concentrate on other elements different from x.

At this point the following definitions can be posed:

Definition 2. Let $\succsim$ an ordering. We will say that "$\succsim$ is an element-induced rational rule" if there exist a natural number k and a rational mapping of evaluation sequences F such that $\succsim$ is an element-induced rule in relation with $(k; F)$.

Definition 3. Given an ordering $\succsim$, the minimal number k such that there exists F such that F is rational and $\succsim$ is element-induced in relation to $(k; F)$ will be said to be "the degree of iterativeness of $\succsim$."

When, for a given element-induced rule $\succsim$, the degree of iterativeness is k, then it will be said that "$\succsim$ is k-times iterative". As long as labeling an ordering $\succsim$ as k-times iterative only makes sense if $\succsim$ is an element-induced rational rule, whenever that expression is used it will be understood that $\succsim$ is also an element-induced and rational rule.

The previous definitions allow us to present the following results: Lemma 4 establishes that the rules defined in Section 2 are particular cases of element-induced rational rules. Lemma 5 states that any rational rule which is a linear ordering must be n-times iterative:

Lemma 4. $\succsim_m; \succsim_M; \succsim_{lm}; \succsim_{LM}; \succsim_{mM}; \succsim_{Mm}; \succsim_{lmM}$ and $\succsim_{LMm}$ are element-induced rational rules.

Proof. :

– $\succsim_m$: Let $k = 1$. For all $A \in Z$, let $F(A) = \{f_1(A)\} = \underline{a}$. Then, for all $A; B \in Z$, $A \succsim_m B \Leftrightarrow f_1(A) R f_1(B)$, which proves that $\succsim$ is once-iterative. Also, F is rational: $\forall A; B$ s.t. $B \subseteq A$ implies $f_1(A) = \overline{a} = \overline{b} = f_1(B)$.

– $<_M$: The proof is analogous to the one of $<_m$.

– $<_{lm}$: Let $k = \#X (= n)$. For all $A \in Z$, $\#A = j$, let $F(A) = \{a_1, \ldots, a_j\}$ such that $a_j R a_{j-1} R \ldots a_1$. Then, $\forall A, B \in Z$, $A < B \Leftrightarrow \exists l \leq \max(\#A, \#B)$ $(\leq n)$ such that $\forall i \in N$, $i < l$, $f_i(A) = f_i(B)$ and [$f_l(A) R f_l(B)$ or ($f_l(A)$ exists and $f_l(B)$ does not exist)], which demostrates that $<$ is an element-induced rule in relation to $(k = n, F)$. To see that F is also rational note that, as for all A, $F(A)$ is A inversely ordered according to R, then, for all $B \subseteq A$, $\{f_1(A), \ldots, f_{i-1}(A)\} = \{f_1(B), \ldots, f_{i-1}(B)\}$, $f_i(A) = a$, and $a \in B$, imply $f_i(B) = a$.

– $<_{LM}$: The proof is analogous to the one of $<_{lm}$.

– $<_{mM}$: Let $k = 2$. For all $A \in Z$, let $F(A) = \{f_1(A), f_2(A)\} = \{\underline{a}, \overline{a}\}$. Then, $\forall A, B \in Z$, $A <_{mM} B \Leftrightarrow f_1(A) R f_1(B)$ or $(f_1(A) = f_1(B)$ and $f_2(A) P f_2(B))$ To demostrate that F is also rational note that, $\forall A, B \in Z$, $B \subseteq A$, $f_1(A)(= \underline{a}) \in B$ implies $f_1(B) = f_1(A)$, and also, if $f_1(A) = f_1(B)$ and $f_2(A)(= \overline{a}) \in B$, that implies $f_2(B) = f_2(A)$.

– $<_{Mm}$: The proof is analogous to the one of $<_{mM}$.

– $<_{lmM}$: Let $k = n$. For all $A \in Z$, let $A_0 = A$ and

$$m_A = \begin{cases} \frac{\#A}{2} - 1 & \text{if } \#A \text{ is even} \\ \frac{\#A+1}{2} - 1 & \text{if } \#A \text{ is odd} \end{cases}$$

Let, for all $t = 1, \ldots, m_A$, $A_t = A_{t-1} \setminus \{\underline{a_{t-1}}, \overline{a_{t-1}}\}$.

Let, $\forall A \in Z$, $F(A) = \{\underline{a_0}, \overline{a_0}, \underline{a_1}, \overline{a_1}, \ldots, \underline{a_t}, \overline{a_t}\}$

Then, $\forall A, B \in Z$, $A <_{lmM} B \Leftrightarrow \exists l \leq n$ such that $\forall i < l$, $f_i(A) = f_i(B)$ and [$(f_l(A) R f_l(B))$ or $((f_l(A)$ exists and $f_l(B)$ does not exist)], which proves that $<$ is an element-induced rule in relation to $(k = n, F)$. Now, rationality of F should be proved:

Clearly, for all $A, B \in Z$, if $B \subseteq A$ and $f_1(A)(= \underline{a}) \in B$, then $f_1(B)(= \underline{b}) = f_1(A)$. Also, if $f_1(A) = f_1(B)$ and $f_2(A)(= \overline{a}) \in B$, then $f_2(B)(= \overline{b}) = f_2(A)$. If $f_1(A) = f_1(B)$, $f_2(A) = f_2(B)$, and $f_3(A)(= \min(A \setminus \{\underline{a}, \overline{a}\})$ belongs to B, then $f_3(B)(= \min(B \setminus \{\underline{b}, \overline{b}\})$ must be equal to $f_3(A)$. Due to the manner by which F is constructed, the argument can be repeated to assert that $\forall i \in N$, $i \leq n$, $\forall A, B \in Z; B \subseteq A$, $\{f_1(A), \ldots, f_{i-1}(A)\} = \{f_1(B), \ldots, f_{i-1}(B)\}; f_i(A) = a$; and $a \in B$, implies $f_i(B) = a$.

– $\Re_{LMm}$: The proof is similar to the one of $\Re_{lmM}$.

$\square$

Lemma 5. Let $\Re$ be a rational rule defined on Z. If $\Re$ is a linear ordering, then it is n-times iterative.

Proof. Let $\Re$ be a rational rule defined on Z such that $\Re$ is also a linear ordering. Let us suppose that $\Re$ is not n-times iterative. As $\Re$ is rational, that implies that there exists a rational mapping F and a natural number k, $k < n$, such that $\Re$ is element-induced in relation to $(k; F)$. Take $F(X) = \{f_1(X), \ldots, f_m(X)\}$. By the definition of element-induced rule and by hypothesis $m = k$. Consider $X^* = \{x \in X \text{ s.t. } \exists i \leq k \text{ s.t. } x = f_i(X)\}$. As $k < n$, $X^* \subset X$. Therefore, by Rationality, $f_i(X^*) = f_i(X)\ \forall i \leq k$. As $\Re$ is element-induced in relation to $(k; F)$, this implies $X \sim X^*$. Therefore $\Re$ is not a linear ordering, reaching a contradiction

$\square$

Finally, an additional property of F will be considered in some cases, but unlike the Rationality condition, it will not be maintained as a general assumption throughout the paper:

Iteration Independence: $\forall A \in Z; \forall i \in N, i \leq n;$
$f_i(A) = f_1(A \backslash \{f_1(A), \ldots, f_{i-1}(A)\})$

Iteration Independence establishes that, given a set A, then the element considered by the agent in a certain iteration i for A is the same as the one he would have considered in a first iteration for a set consisting of A after removing those elements considered in previous iterations. For example, consider a set A consisting on the alternatives $\{a, b, c, d, e\}$, and $f_1(A) = a$, $f_2(A) = b$. Then

Iteration Independence requires that $f_3(A)$ should be the same as $f_1(\{c, d, e\})$. This property establishes a kind of coherency in the evaluation process. We can also take the equality in the inverse sense in order to appreciate the meaning of this axiom from other perspective. Returning to the previous example, the first focal element of $\{c, d, e\}$ should be the same as the i-focal element of any set where, after removing the previous focal elements, the remaining elements are c, d, and e.

A direct implication of Iteration Independence is that, in the sequential process of evaluation, the agent always concentrates sucessively on new elements of the set. That is, $\forall A \in Z$, $\forall i \neq j$, $f_i(A) \neq f_j(A)$. Another consequence of Iteration Independence is that, $\forall A \in Z$, $F(A)$ is a permutation of A.

5 Attitudes Towards Uncertainty: Some Axioms and Characterization Results

The class of element-induced rational rules contains a wide range of possible criteria. The agent's different possible attitudes towards risk play an important role at this stage. Some of these attitudes will be expressed by means of the following simple axioms:

Simple Risk Aversion (SRAV) $\forall x, y \in X$, xPy implies $\{x\} \succ \{x, y\}$

Simple Risk Neutrality (SRN) $\forall x, y \in X$, xPy implies $\{x\} \sim \{x, y\}$

Simple Risk Appeal (SRAP) $\forall x, y \in X$, xPy implies $\{x, y\} \succ \{x\}$

(SRAV) is very natural in a context of choice under complete uncertainty. (SRN) and (SRAP) are not so plausible in that context, but as long as they are satisfied by some rules in the field they will be considered.

The following theorems provide characterizations of some of the rules defined in Section 2, but from a different perspective. The results show that these rules can be characterized as particular cases of element-induced rational rules which respond to different possible attitudes towards uncertainty, and to different particular properties of the evaluation procedure.

Theorem 6. Let $\succeq$ be an ordering defined on Z. $\succeq = \succeq_m$ if and only if $\succeq$ is a once-iterative rule and satisfies (SRAV).

Theorem 7. Let $\succeq$ be an ordering defined on Z. $\succeq = \succeq_M$ if and only if $\succeq$ is a once-iterative rule and satisfies (SRN).

Theorem 8. Let $\succeq$ be an ordering defined on Z. $\succeq = \succeq_{lm}$ if and only if $\succeq$ satisfies (SRAV) and there exists a rational and Iteration Independent mapping F in relation to which $\succeq$ is element-induced.

Theorem 9. Let $\succeq$ be an ordering defined on Z. $\succeq = \succeq_{LM}$ if and only if $\succeq$ satisfies (SRAP) and there exists a rational and Iteration Independent mapping F in relation to which $\succeq$ is element-induced.

Proof of Theorem 6:

If $\succeq$ is once-iterative, that implies that there exists $F : Z \rightarrow Z$ such that: F is rational; $\forall A \in Z\ j_A = 1$; and $\forall A, B \in Z$, $A \succeq B \Leftrightarrow f_1(A) R f_1(B)$. First, we will prove that if $\succeq$ satisfies (SRAV) and if F is rational, then $f_1(A) = \underline{a}\ \forall A$: This is obvious when $\#A = 1$, so we assume $\#A > 1$. Let us suppose that there exists $A \in Z$ such that $f_1(A) = a^* \neq \underline{a}$. Then, by (SRAV), $\{a^*\} \succ \{a^*, \underline{a}\}$. As $k = 1$ that implies $f_1(\{a^*\}) P f_1(\{a^*, \underline{a}\})$, that is, $f_1(\{a^*\}) = a^*$ and $f_1(\{a^*, \underline{a}\}) = \underline{a}$. By Rationality $f_1(\{a^*, \underline{a}\}) = \underline{a}$ implies $f_1(A) \neq a^*$, which results in a contradiction.

To prove that there exists a rational mapping F such that $\succeq_m$ is element-induced in relation to $(k = 1; F)$, see the corresponding part of the proof of Lemma 4. That the degree of iterativeness is 1 is obvious because, by definition of element-induced rule, $k < 1$ is impossible. That $\succeq_m$ satisfies (SRAV) is easily proven.

Proof of Theorem 7: The proof is analogous to that for $\succeq_m$, but instead of using (SRAV), (SRN) needs to be applied.

Proof of Theorem 8:

If $\succcurlyeq$ is element-induced, then all of the conditions of Definition 1 are satisfied. First, we prove that if F is rational, Iteration Independent, and if $\succcurlyeq$ satisfies (SRAV), then $\forall A \in Z$, $f_1(A) = \underline{a}$: Let us suppose that there exists $A \in Z$ such that $f_1(A) = a^* \neq \underline{a}$. By (SRAV) $\{a^*\} \succ \{a^*; \underline{a}\}$. As F is Iteration Independent, $f_2(\{a^*\})$ does not exist and $f_2(\{a^*; \underline{a}\})$ does exist. Therefore, as $\succcurlyeq$ is element-induced in relation to F, $\{a^*\} \succ \{a^*; \underline{a}\}$ is only possible if $f_1(\{a^*\}) P f_1(\{a^*; \underline{a}\})$, in which case, proceeding as in the proof of Theorem 6, we reach a contradiction.

Now, $\forall A \in Z$, $\#A = r$, let $\{a_1; a_2; \ldots; a_r\}$ such that $a_r R a_{r-1} R \ldots R a_1$. Then, if $\forall A \in Z$, $f_1(A) = \underline{a}$, by Iteration Independence, $\forall A \in Z$, $\forall i \leq \#A$, $f_i(A) = a_i$ and $\forall j > \#A$, $f_j(A)$ does not exist. Therefore, as $\succcurlyeq$ is n-times iterative, $\forall A; B \in Z$, $A \succcurlyeq B \Leftrightarrow \exists l \leq n$ such that $\forall i \in N$, $i < l$, $a_i = b_i$ and $[a_l R b_l$ or (a_l exists and b_l does not exist)]. That is, $A \succcurlyeq B \Leftrightarrow A \succcurlyeq_{lm} B$.

To prove that there exists an Iteration Independent and rational mapping F, and certain natural number k such that $\succcurlyeq_{lm}$ is element-induced in relation to $(k; F)$, consider the corresponding part of the proof in Lemma 4. That $\succcurlyeq_{lm}$ satisfies (SRAV) is easily proven.

Proof of Theorem 9:

If $\succcurlyeq$ is element-induced then all of the conditions of Definition 1 are satisfied. First, we prove that, if F is rational, Iteration Independent and if $\succcurlyeq$ satisfies (SRAP), then $\forall A \in Z$, $f_1(A) = \overline{a}$: Let us suppose that there exists $A \in Z$ such that $f_1(A) = a^* \neq \overline{a}$. By (SRAP) $\{\overline{a}; a^*\} \succ \{\overline{a}\}$. As $\succcurlyeq$ is element-induced in relation to F this implies that:

(i): $f_1(\{\overline{a}; a^*\}) P f_1(\{\overline{a}\})$, which is impossible, or

(ii): $f_1(\{\overline{a}; a^*\}) = f_1(\{\overline{a}\})$, that is, $f_1(\{\overline{a}; a^*\}) = \overline{a}$. Then, by Rationality of F, $f_1(A) \neq a^*$, which is a contradiction.

Now, $\forall A \in Z$, $\#A = r$, let $\{a_1; a_2; \ldots; a_r\}$ such that $a_1 R a_2 R \ldots R a_r$. At this stage, the proof is similar to the proof of $\succcurlyeq_{lm}$.

To prove that there exists an Iteration Independent and rational mapping

F, and certain natural number k such that $\Re_{LM}$ is element-induced in relation to $(k; F)$, consider the corresponding part of the proof in Lemma 4. That $\Re_{LM}$ satisfies (SRAP) is easily proven. $\square$

Next we show the pertinent examples to prove the independence of the conditions used in Theorems 6, 7, 8 and 9. For all these examples we will assume that sets are ordered according to R from the best to the worst element.

1. $\Re_{m}$
   - $\Re_{lm}$ satisfies (SRAV), but it is not once-iterative, that is, it is impossible to find a pair $(k; F)$ such that F is rational and $\Re_{lm}$ is once-iterative in relation to F.
   - $\Re_{M}$ is once-iterative, but it does not satisfy (SRAV).
2. $\Re_{M}$
   - Let $\Re$ be defined on Z such that $\forall A; B \in Z$, $A \sim B$. Then $\Re$ satisfies (SRN), but it is not once-iterative (actually it is not element-induced)
   - $\Re_{m}$ is once-iterative, but does not satisfy (SRN).
3. $\Re_{lm}$
   - $\Re_{LM}$ is an element-induced rational rule in relation to certain Iteration Independent mapping F, but it does not satisfy (SRAV).
   - Let $\{x; y; z\} \succ \{x\} \succ \{x; y\} \succ \{y\} \succ \{x; z\} \succ \{y; z\} \succ \{z\}$. Then $\Re$ satisfies (SRAV) and there exists an Iteration Independent F with which $\Re$ is element-induced, but $\Re$ is not rational.
   - $\Re_{m}$ satisfies (SRAV) and is rational, but there does not exist any Iteration Independent mapping F with which $\Re$ is element-induced.
4. $\Re_{LM}$
   - $\Re_{lm}$ is element-induced in relation to certain Iteration Independent mapping F, but $\Re_{lm}$ does not satisfy (SRAP).
   - Let $\{x; y\} \succ \{x; z\} \succ \{x\} \succ \{y; z\} \succ \{y\} \succ \{x; y; z\} \succ \{z\}$. Then $\Re$ satisfies (SRAP) and there exists an Iteration Independent F with

which < is element-induced, but < is not rational. (It can be proved that, for # X < 3, if an ordering < is element-induced with an Iteration Independent F, then F must be rational).

{ Let fx;y;zg » fx;yg Â fx;zg Â fxg Â fy;zg Â fyg Â fzg. Then < satis¯es (SRAP) and is rational, but there does not exist any Iteration Independent mapping F with which < is element-induced.

6 The min-max rule, the max-min rule and their lexicographic extensions

The main object of this section is the analysis, as element-induced rules, of $<_{mM}$, $<_{Mm}$, $<_{lmM}$ and $<_{LMm}$. These constitute plausible rules in the context of choice under complete uncertainty if we believe in the existence of computational costs as far as they represent situations where the decision maker tends to concentrate on focal aspects of the prospects. The hypothesis that under uncertainty, the agent focusess on certain outcomes is initially due to Shackle [37]. According to Shackle's theory, in the context of choice under complete uncertainty, the agent evaluates actions taking into account only two outcomes: the one that the agent most intensively desires and the one that he less intensively desires. The desirability function depends directly upon the value of the outcome, and inversely upon the potential surprise its ocurrence would cause. However, Shackle's elegant explanation of his conjecture, unlike our approach, has nothing to do with any kind of bounded rationality assumption, but with the non-probabilistic nature of the context of choice under complete uncertainty.[7] (For further detail see Shackle [37, pp.37-42 and 109-114)], [38]).

The analysis of these four rules is done under this separate section because

[7] Carter [12] remarks on Shackle's theory considering some intuitions about the simplifying behavior to argue in favour of the hypothesis that the focalizing tendency might not concentrate on the extreme values of the set.

additional conditions on F and $\succcurlyeq$ need to be considered, and because of the length of the proofs.

As for F, two new conditions are considered:

Elimination in Uncertain Prospects

$\forall A \in Z$, $\#A \geq 2$, $\forall i, j \in N$ $(i, j \leq n)$, $f_i(A) \neq f_j(A)$

Alternate Iteration Independence

$\forall A \in Z$, $\forall i \in N$, $2 < i \leq \#A$,

$$f_i(A) = \begin{cases} f_1(A \setminus \{f_1(A), \ldots, f_{i-2}(A)\}) & \text{when } i \text{ is odd} \\ f_2(A \setminus \{f_1(A), \ldots, f_{i-2}(A)\}) & \text{when } i \text{ is even} \end{cases}$$

Elimination in Uncertain Prospects establishes that, for all those prospects which are uncertain (that is, those sets which contain at least two possible outcomes), the agent concentrates sucessively in different representative elements.

Alterante Iteration Independence is close to, but different from, the simple Iteration Independence condition defined in the previous Section: The intuition behind the original Iteration Independence was that any iteration constitutes an independent step in the sequential process of evaluation, and that it does not matter in which moment the step is made if the evaluated set is the same. In contrast, under Alternate Iteration Independence, we can interprete the evaluation process as if each of those steps were made by two successive iterations, and as if the independence were required at the level of steps, not at the level of simple iterations.

When F satisfies Elimination in Uncertain Prospects (Alternate Iteration Independence) it will be said that F is Eliminative (Alternate Iteration Independent)

The following additional axioms, concerning $\succcurlyeq$ will be also considered:

Potential Benefit Appeal (PBAP) $\forall A \in Z$, $\forall x \in X \setminus A$ s.t. $\forall a \in A$ xPa, $A \cup \{x\} \succ A$

Risk Aversion (RAV) $\forall A \in Z$, $\forall x \in X \setminus A$ s.t. $\forall a \in A$ aPx, $A \succ A \cup \{x\}$

Simple Uncertainty Aversion (SUAV) $\forall x, y, z \in X$ s.t. $xPyPz$, $\{y\} \succ \{x, y, z\}$

Simple Uncertainty Appeal (SUAP) $\forall x, y, z \in X$ s.t. $xPyPz$, $\{x, y, z\} \succ \{y\}$

Simple Richness Appeal (SRICH): $\forall x, y, z \in X$ s.t. $xPyPz$, $\{x, y, z\} \succ \{x, z\}$

(PBAP) ensures that by adding to a certain prospect A a new outcome which is better than all of the possible outcomes in A, then the enlarged prospect becomes strictly better. (RAV) extends condition (SRAV) to prospects of any size. (RAV) and (PBAP) together are equivalent to the Gärdenfors Principle (see Gardenfors [19] or Kannai and Peleg [24]).

(SUAV) ((SUAP)) establishes that adding to a secure prospect $\{y\}$ a better and a worse possible outcome leads to a better (worse) new prospect. Close axioms are considered and widely discussed by Bossert [10] and Bossert, Pattanaik and Xu [11]. Both are plausible in the context of choice under complete uncertainty, and simply display different attitudes towards uncertainty.

(SRICH) establishes that any set A with at least three elements is strictly better than another set consisting only of the best and worst elements of A. (SRICH) can be interpreted also as an attitude towards uncertainty: the decision maker prefers to diversify the possible outcomes obtainable within the range of possible results, rather than being constrained to the two extremes.

Lemma 10. Let $\succcurlyeq$ be an ordering defined on Z. If $\succcurlyeq$ is an n-times iterative rule in relation to a certain Eliminative mapping F, then $\succcurlyeq$ satisfies (PBAP).

Proof. Let $A \in Z$ such that $\exists x \in X \setminus A$ s.t. xPa_i $\forall a_i \in A$. Let the rational and Eliminative mapping F in relation to which $\succcurlyeq$ is element-induced. Two possibilities are considered: $\# A = 1$ and $\# A > 1$:

If $\# A = 1$ ($A = \{a\}$), let $x \in X \setminus A$ s.t. xPa. If $f_1(\{x; a\}) = x$ then, as $\succcurlyeq$

is element-induced, $\{x, a\} \succ \{a\}$. If $f_1(\{x, a\}) = a$, as $\succsim$ is n-times iterative and F is Eliminative, $f_2(\{x, a\})$ exists and is equal to x. Therefore, as $\succsim$ is element-induced $\{a\} \succsim \{x, a\}$ is impossible.

If $\#A > 1$, let $F(A) = \{a_1, a_2, \ldots, a_m\}$. By n-times iterativeness and Elimination in Uncertain Prospects, $m = \#A$. Let $x \in X \setminus A$ such that $xPa_i \; \forall i \leq n$. If $f_1(A \cup \{x\}) = x$, then, as $\succsim$ is element-induced, $A \cup \{x\} \succ A$. If $f_1(A \cup \{x\}) \neq x$, then, by Rationality, $f_1(A \cup \{x\}) = a_1 = f_1(A)$. By Elimination in Uncertain Prospects $f_2(A \cup \{x\}) \neq a_1$. If $f_2(A \cup \{x\}) = x$, then, as $\succsim$ is n-times iterative, $A \cup \{x\} \succ A$. If $f_2(A \cup \{x\}) \neq x$, then, by Rationality, $f_2(A \cup \{x\}) = a_2 = f_2(A)$. We can repeat analogously these steps to assert that only two circumstances are possible:

(i): There exists $k \leq m$ such that $f_k(A \cup \{x\}) = x$ and $f_i(A \cup \{x\}) = f_i(A)$ $\forall i < k$, in which case, as $\succsim$ is n-times iterative, $A \cup \{x\} \succ A$.

or (ii): $f_i(A \cup \{x\}) = f_i(A)$ for all $i \leq m$, in which case, as $\succsim$ is n-times iterative and Eliminative, $f_{m+1}(A) = x$, and therefore, $A \cup \{x\} \succ A$. ∎

Theorem 11. Let $\succsim$ be an ordering defined on Z. $\succsim = \succsim_{mM}$ if and only if $\succsim$ is a twice-iterative rule and satisfies (SRAV), (SUAV) and (PBAP).

Theorem 12. Let $\succsim$ be an ordering defined on Z. $\succsim = \succsim_{Mm}$ if and only if $\succsim$ is a twice-iterative rule and satisfies (RAV), (SUAP) and (PBAP).

Theorem 13. Let $\succsim$ be an ordering defined on Z. $\succsim = \succsim_{lmM}$ if and only if $\succsim$ satisfies (SRAV), (SUAV) and (SRICH), and it is n-times iterative in relation to a mapping F which is Alternate Iteration Independent and Eliminative.

Theorem 14. Let $\succsim$ be an ordering defined on Z. $\succsim = \succsim_{LMm}$ if and only if $\succsim$ satisfies (RAV) and (SUAP), and it is n-times iterative in relation to a mapping F which is Alternate Iteration Independent and Eliminative

Proof of Theorem 11:

The following proof is made provided that X contains at least three elements. The case $\#X = 1$ is degenerate, and in the case $\#X = 2$, if $\succsim$ satisfies (SRAV) and (PBAP), then directly $\succsim = \succsim_{mM}$.

If $\succsim$ is twice-iterative, then it is element-induced by definition, and that implies that all of the conditions of Definition 1 are satisfied.

Step 1: We will prove that $\forall A \in Z$ such that $\#A = 3$ ($A = \{\overline{a}, a, \underline{a}\}$), $f_1(A) = \underline{a}$:

By (SUAV) $\{a\} \succ A$, which implies $f_1(A) \neq \overline{a}$. Let us suppose $f_1(A) \neq \underline{a}$. Then $f_1(A) = a = f_1(\{a\})$. As $\{a\} \succ A$ and $\succsim$ is twice-iterative, this implies that $f_2(A) = \underline{a}$ and $f_2(\{a\}) = a$. By (PBAP) $A \succ \{a, \underline{a}\}$. As $f_1(A) = a$, $A \succ \{a, \underline{a}\}$ is only possible if (i): $f_1(\{a, \underline{a}\}) = \underline{a}$ or (ii): $f_1(\{a, \underline{a}\}) = a$ and $f_2(A) P f_2(\{a, \underline{a}\})$. If (i), by Rationality, $f_1(A) \neq a$, which results in a contradiction. If (ii), as $A \succ \{a, \underline{a}\}$ and $f_2(A) = \underline{a}$, then $f_2(\{a, \underline{a}\})$ must be strictly worse than $\underline{a}$, which is impossible. In sum, $\forall A \in Z$ such that $\#A = 3$, $f_1(A) = \underline{a}$.

Step 2: $\forall A \in Z$ such that $\#A = 2$ and $\underline{x} \in A$, $f_1(A) = \underline{x}$:

Let $A \in Z$ ($A = \{a, \underline{x}\}$). As $\#X \geq 3$, there exists $b \in X \setminus A$, $b \neq a, \underline{x}$. By Step 1 $f_1(\{a, b, \underline{x}\}) = \underline{x}$. Therefore, by Rationality, $f_1(A) = \underline{x}$.

Step 3: Let $A \in Z$ such that $\#A = 2$. If $f_1(A) = \overline{a}$ then $f_2(A) = \underline{a}$:

By (SRAV) $\{\overline{a}\} \succ \{\overline{a}, \underline{a}\}$. As $\succsim$ is element-induced and twice-iterative, if $f_1(A) = \overline{a}$, then $f_2(A)$ exists and is equal to $\underline{a}$; $f_1(\{\overline{a}\}) = \overline{a}$ and $f_2(\{\overline{a}\})$ exists and is equal to $\overline{a}$.

Step 4: $\forall A \in Z$ such that $\#A > 3$, $f_1(A) = \underline{a}$.

Let us suppose that $f_1(A) = a^* \neq \underline{a}$. Take any $a \in A$; $a \neq a^*, \underline{a}$. By Step 1, $f_1(\{a^*, a, \underline{a}\}) \neq a^*$. By Rationality $f_1(A) \neq a^*$, reaching a contradiction.

Step 5: $\forall A \in Z$ such that $\#A \geq 3$, $f_2(A) = \overline{a}$:

By (PBAP) $A \succ A \setminus \{\overline{a}\}$. As $\succsim$ is element-induced and twice-iterative, this is only possible if (i): $f_1(A) P f_1(A \setminus \overline{a})$ or (ii): $f_1(A) = f_1(A \setminus \{\overline{a}\})$ and $f_2(A) P f_2(A \setminus \{\overline{a}\})$. By Step 1 and Step 4, $f_1(A) = \underline{a}$. That implies that case (i) is impossible, and in case (ii), by Rationality implies $f_1(A \setminus \overline{a}) = \underline{a}$. By twice-iterativeness we know that $f_2(A)$ exists. Let us suppose $f_2(A) = a^* \neq \overline{a}$. Then, by Rationality, $f_2(A \backslash \overline{a}) = a^*$, which is in contradiction with (i). Therefore $f_2(A) = \overline{a}$.

Step 6: Let $A \in Z$ such that $\#A = 2$. Then, $f_1(A) = \underline{a}$ implies $f_2(A) = \overline{a}$

As $\Re$ is twice-iterative, this implies, by the definition of element-induced rule, that $f_2(A)$ does exist. Let us suppose $f_2(A) = \underline{a}$. Then, by Rationality $f_1(\{\underline{a}\} = \underline{a} = f_2(\{\underline{a}\})$. As $\Re$ is element-induced in relation to F, this implies $A \sim \{\underline{a}\}$, which contradicts (PBAP).

From this point on, $\forall A \in Z$, if $\#A = 2$ and $\underline{x} \notin A$, A will be said to be a peculiar set.

Then, Steps 1 to 6 prove that,

$\forall A \in Z$ s.t. $\#A \geq 2$, if A is not a peculiar set, then $F(A) = \{\underline{a}; \overline{a}\}$; and if A is a peculiar set, then $F(A) \in \{\{\underline{a}; \overline{a}\}; \{\overline{a}; \underline{a}\}\}$. (1)

Step 7: $\forall A; B \in X$, $A \succ_{mM} B$ implies $A \succ B$:

$A \succ_{mM} B$ implies $\underline{a} P \underline{b}$ or ($\underline{a} = \underline{b}$ and $\overline{a} P \overline{b}$).

Then, $\forall A; B \in Z$, four possibilities are considered:

1. Neither A nor B are peculiar: Then $f_1(A) = \underline{a}$, $f_1(B) = \underline{b}$.

   If $\underline{a} P \underline{b}$, as $\Re$ is element-induced, then $A \succ B$.

   If ($\underline{a} = \underline{b}$ and $\overline{a} P \overline{b}$) three cases will be considered:

   – Neither A nor B are a singleton. Then, by (1), $f_2(A) = \overline{a}$, $f_2(B) = \overline{b}$. As $\Re$ is twice-iterative in relation to F, that implies $A \succ B$.

   – A is a singleton. Then ($\underline{a} = \underline{b}$ and $\overline{a} P \overline{b}$) is impossible.

   – B is a singleton (and A is not). Then, by (PBAP), $\{\overline{a}; \underline{a}\} \succ \{\underline{b}\}$, that is, $A \succ B$.

2. B is peculiar and A is not: Therefore $F(A) = \{\underline{a}; \overline{a}\}$.

   If $\underline{a} P \underline{b}$, we consider two cases: $A = \{\overline{b}\}$; and $A \neq \{\overline{b}\}$. If $A = \{\overline{b}\}$, then $A \succ B$ directly by (SRAV). If $A \neq \{\overline{b}\}$, then there exists $a^* \in A$ such that $a^* P \underline{b}$ and $a^* \neq \overline{b}$. By (1), $f_1(\{a^*; \overline{b}; \underline{b}\}) = \underline{b}$. By Rationality, $f_1(\{\overline{b}; \underline{b}\}) = \underline{b}$, and as $\Re$ is element-induced in relation to F, therefore $A \succ B$.

If $\underline{a} = \underline{b}$ and $\overline{a}P\overline{b}$, by (1) $f_1(f\overline{a};\overline{b};\underline{b}g) = \underline{b}$. By Rationality $f_1(f\overline{b};\underline{b}g) = \underline{b}$, and as $<$ is element-induced in relation to F, A Â B.

3. A is peculiar and B is not: Then $f_1(B) = \underline{b}$. If $f_1(A) = \overline{a}$, then by hypothesis $\overline{a}P\underline{b}$ and, as $<$ is element-induced, then A Â B. If $f_1(A) = \underline{a}$, then by (1), $f_2(A) = \overline{a}$. Hence, in the case ($\underline{a}P\underline{b}$), then by element-induction, A Â B. In the case ($\underline{a} = \underline{b}$ and $\overline{a}P\overline{b}$), $f_1(A) = f_1(B)$ and ($f_2(A)Pf_2(B)$ or $f_2(B)$ does not exist). Therefore A Â B

4. Both A and B are peculiar: Then four possibilities will be considered:
   - { $F(A) = f\overline{a};\underline{a}g$ and $F(B) = f\overline{b};\underline{b}g$. Then, if $\overline{a}P\overline{b}$, as $<$ is element-induced, therefore A Â B. If $\overline{a} = \overline{b}$, let us suppose $B < A$, then that would imply $\underline{b}R\underline{a}$, which would contradict the hypothesis that $\underline{a}P\underline{b}$ or ($\underline{a} = \underline{b}$ and $\overline{a}P\overline{b}$). If $\overline{b}P\overline{a}$, then, by (1), $f_1(f\overline{b};\overline{a};\underline{a}g = \underline{a}$, and, by Rationality, $f_1(f\overline{a};\underline{a}g) = \underline{a}$, reaching a contradiction.
   - { $F(A) = f\overline{a};\underline{a}g$ and $F(B) = f\underline{b};\overline{b}g$. If $\overline{a}R\overline{b}$, then $\overline{a}P\underline{b}$, and, by element-induction, A Â B. On the other hand, if $\overline{b}P\overline{a}$, then by (1), $f_1(f\overline{b};\overline{a};\underline{a}g) = \underline{a}$, which by Rationality implies $f_1(f\overline{a};\underline{a}g) = \underline{a}$, reaching a contradiction.
   - { $F(A) = f\underline{a};\overline{a}g$ and $F(B) = f\overline{b};\underline{b}g$. Then, if $\overline{a}P\overline{b}$, by (1), $f_1(f\overline{a};\overline{b};\underline{b}g) = \underline{b}$, and by Rationality, $f_1(f\overline{b};\underline{b}g) = \underline{b}$, reaching a contradiction. On the other hand, if $\overline{b}R\overline{a}$, two cases will be considered: a) $\underline{b}R\underline{a}$, which is impossible by hypothesis; and b) $\underline{a}P\underline{b}$, in which, by (1), $f_1(f\overline{b};\underline{a};\underline{b}g) = \underline{b}$ and by Rationality, $f_1(f\overline{b};\underline{b}g) = \underline{b}$, which is a contradiction.
   - { $F(A) = f\underline{a};\overline{a}g$ and $F(B) = f\underline{b};\overline{b}g$. Then, by hypothesis ($\underline{a}P\underline{b}$) or ($\underline{a} = \underline{b}$ and $\overline{a}P\overline{b}$), which by element-induction implies A Â B.

Step 8: 8A; B 2 X, A $»_{mM}$ B implies A » B:

A $»_{mM}$ B implies $\underline{a} = \underline{b}$ and $\overline{a} = \overline{b}$.

8A; B 2 Z three possibilities will be considered:

1. Neither A nor B are peculiar: Consider #A; #B ¸ 2 (Note that if one of them is a singleton then the other one must be the same set, and then, by

reflexivity, $A \sim B$). Then, by (1), $f_1(A) = \underline{a}$; $f_1(B) = \underline{b}$; $f_2(A) = \bar{a}$ and $f_2(B) = \bar{b}$. As $\succsim$ is twice-iterative, therefore $A \sim B$.

2. B is peculiar and A is not (without loss of generality). Three cases will be considered: (i): $\#A = 1$. Then, by hypothesis, $\#B = 1$, and B cannot be peculiar, reaching a contradiction. (ii): $\#A = 2$. If A is not peculiar, then $\underline{x} \in A$. Therefore, by hypothesis, $\underline{x} \in B$, which implies that B is not peculiar, reaching a contradiction. (iii): $\#A > 2$. Then, if $F(B) = \{\underline{b}, \bar{b}\}$, as $\succsim$ is twice-iterative, $A \sim B$. On the other hand, if $F(B) = \{\bar{b}, \underline{b}\}$, consider $a \in A$ s.t. $a \notin \underline{a}, \bar{a}$. Then by (1), $f_1\{\bar{b}, a, \underline{b}\} = \underline{b}$, and by Rationality $f_1(B) = \underline{b}$, which yields a contradiction.

3. Both A and B are peculiar. Then $\#A = \#B = 2$, which together with the hypothesis, implies $A = B$. By reflexivity $A \sim B$.

The results of Steps 7 and 8 together imply $\succsim = \succsim_{mM}$.

That $\succsim_{mM}$ satisfies (SRAV), (PBAP) and (SUAV) is easily proven. To prove that $\succsim_{mM}$ is twice-iterative we have to prove: a) that there exists a rational mapping F such that $\succsim_{mM}$ is element-induced in relation to $(k = 2; F)$. And b) that it is impossible to find a rational mapping F' such that $\succsim_{mM}$ is element-induced in relation to $(k = 1; F')$. To check part a), consider the corresponding part of the proof of Lemma 4. To prove b), note that $\forall x, y \in X$ such that xPy, $\{x\} \succ_{mM} \{x, y\} \succ_{mM} \{y\}$. Then, supposing that $\succsim_{mM}$ were once-iterative, if $f_1(\{x, y\}) = x$ then $\{x\} \sim \{x, y\}$, and if $f_1(\{x, y\}) = y$ then $\{y\} \sim \{x, y\}$, reaching in either case a contradiction. □

Proof of Theorem 12:

If $\succsim$ is twice-iterative, then, by definition, it is element-induced, and therefore all of the conditions of Definition 1 are satisfied.

Step 1: We will prove that $\forall A \in Z$ such that $\#A = 3$ ($A = \{\bar{a}, a, \underline{a}\}$), $f_1(A) = \bar{a}$:

By (SUAP), $\{\bar{a}, a, \underline{a}\} \succ \{a\}$, which by element-induction is only possible if

$f_1(A) \in \{\bar{a}, a\}$. Let us suppose that $f_1(A) = a$. Then, by Rationality, $f_1(\{a, \underline{a}\}) = a$. By (RAV) $\{a\} \succ \{a, \underline{a}\}$. As $\succcurlyeq$ is twice-iterative $f_2(\{a, \underline{a}\})$ exists, and $\{a\} \succ \{a, \underline{a}\}$ is only possible if $f_2(\{a, \underline{a}\}) = \underline{a}$ and $f_2(\{a\})$ also exists and is equal to a. Then $\{\bar{a}, a, \underline{a}\} \succ \{a\}$ is only possible if $f_2(\{\bar{a}, a, \underline{a}\}) = \bar{a}$. By Rationality $f_1(\{\bar{a}, a\}) = a$. By (RAV) $\{\bar{a}, a\} \succ \{\bar{a}, a, \underline{a}\}$, which by $\succcurlyeq$ being twice-iterative is impossible if $f_1(\{\bar{a}, a, \underline{a}\}) = a$ and $f_2(\{\bar{a}, a, \underline{a}\}) = \bar{a}$. Therefore, $f_1(A) = \bar{a}$.

Step 2: $\forall A \in Z$ such that $\# A = 2$, if $\exists x \in X \setminus A$ s.t. $\bar{a} P x$ then $f_1(A) = \bar{a}$ and $f_2(A) = \underline{a}$. Also, $\forall x \in X$ such that $x \neq \underline{x}, \bar{x}$, $f_1(\{x\}) = f_2(\{x\}) = x$.

Let us take $x \in X \setminus A$ such that $\bar{a} P x$. By Step 1 $f_1(A \cup \{x\}) = \bar{a}$. Then, by Rationality, $f_1(A) = \bar{a}$. By (RAV) $\{\bar{a}\} \succ A$. As $\succcurlyeq$ is element-induced and twice-iterative $f_2(A)$ exists, and $\{\bar{a}\} \succ A$ is only possible if $f_2(A) = \underline{a}$ and $f_2(\{\bar{a}\})$ also exists and is equal to $\bar{a}$.

Step 3: $\forall A \in Z$ such that $\# A = 3$ $(A = \{\bar{a}, a, \underline{a}\})$, $f_2(A) = \underline{a}$:

By Step 2, $f_1(\{\bar{a}, a\}) = \bar{a}$ and $f_2(\{\bar{a}, a\}) = a$. By Step 1, $f_1(A) = \bar{a}$. By (RAV), $\{\bar{a}, a\} \succ A$, which, given that $\succcurlyeq$ is twice-iterative, is only possible if $f_2(A) = \underline{a}$.

Step 4: $\forall A \in Z$ such that $\# A > 3$, $f_1(A) = \bar{a}$ and $f_2(A) = \underline{a}$:

Let us suppose that $f_1(A) = a \in A$, $a \neq \bar{a}$. Take any $A' \subset A$ s.t. $\# A' = 3$ and $\bar{a}, a \in A'$. Then, by Rationality $f_1(A') = a$, which contradicts Step 1. Therefore $f_1(A) = \bar{a}$. We know by twice-iterativeness that $f_2(A)$ exists. Let us suppose $f_2(A) = a' \in A$, $a' \neq \underline{a}$. If $a' \neq \bar{a}$, then by Rationality $f_2(\{\bar{a}, a', \underline{a}\}) = a'$, which is in contradiction with Step 3. If $a' = \bar{a}$, let us take $a \in A$ s.t. $a \neq \underline{a}, a'$. Then $f_2(A) = a'$ implies, by Rationality, $f_2(\{a', a, \underline{a}\}) = a'$, again contradicting Step 3.

In sum, from Steps 1 to 4 we can assert:

$\forall A \in Z$, $f_1(A) = \bar{a}$ and $f_2(A) = \underline{a}$ except when $A \in \{\{\underline{x}, \bar{x}\}, \{\bar{x}\}, \{\underline{x}\}\}$, in which case nothing is proved about the values of $F(A)$. (2)

Step 5: $\forall A, B \in X$, $A \succ_{Mm} B$ implies $A \succ B$:

$A \succ_{Mm} B$ implies $\bar{a} P \bar{b}$ or $(\bar{a} = \bar{b}$ and $\underline{a} P \underline{b})$. Then, $\forall A, B \in Z$, four possibilities will be considered:

1. $A, B \notin \{\{\underline{\underline{x}}, \underline{x}\}, \{\underline{x}\}, \{\underline{\underline{x}}\}\}$:
   $\overline{a} P \overline{b}$ or ($\overline{a} = \overline{b}$ and $\underline{a} P \underline{b}$) implies, by (2) and by twice iterativeness of $\succsim$, $A \succ B$
2. $A \in \{\{\underline{\underline{x}}, \underline{x}\}, \{\underline{x}\}, \{\underline{\underline{x}}\}\}$; $B \notin \{\{\underline{\underline{x}}, \underline{x}\}, \{\underline{x}\}, \{\underline{\underline{x}}\}\}$: In this case $\overline{a} P \overline{b}$ or ($\overline{a} = \overline{b}$ and $\underline{a} P \underline{b}$) is impossible.
3. $B \in \{\{\underline{\underline{x}}, \underline{x}\}, \{\underline{x}\}, \{\underline{\underline{x}}\}\}$; $A \notin \{\{\underline{\underline{x}}, \underline{x}\}, \{\underline{x}\}, \{\underline{\underline{x}}\}\}$:
   If $\overline{a} P \overline{b}$, by (2), $f_1(A) = \overline{a}$. Then, as $\succsim$ is element-induced, $A \succ B$. In this case ($\overline{a} = \overline{b}$ and $\underline{a} P \underline{b}$) is impossible.
4. $A, B \in \{\{\underline{\underline{x}}, \underline{x}\}, \{\underline{x}\}, \{\underline{\underline{x}}\}\}$:
   If $\overline{a} P \overline{b}$, then $B = \{\underline{x}\}$ and ($A = \{\underline{\underline{x}}\}$ or $A = \{\underline{\underline{x}}, \underline{x}\}$). If $A = \{\underline{\underline{x}}\}$, then $f_1(A) P f_1(B)$, and therefore $A \succ B$. If $A = \{\underline{\underline{x}}, \underline{x}\}$, then, by (PBAP), $A \succ B$. If ($\overline{a} = \overline{b}$ and $\underline{a} P \underline{b}$), then $A = \{\underline{\underline{x}}\}$ and $B = \{\underline{\underline{x}}, \underline{x}\}$. Then, by (RAV), $A \succ B$.

Step 6: $\forall A, B \in \mathcal{X}$, $A \sim_{Mm} B$ implies $A \sim B$:

$A \sim_{Mm} B$ implies $\overline{a} = \overline{b}$ and $\underline{a} = \underline{b}$.

Again, three possibilities will be considered:

1. $A, B \notin \{\{\underline{\underline{x}}, \underline{x}\}, \{\underline{x}\}, \{\underline{\underline{x}}\}\}$:
   By (2) $f_1(A) = \overline{a}$; $f_2(A) = \underline{a}$; $f_1(B) = \overline{b}$; $f_2(B) = \underline{b}$. As $\succsim$ is element-induced and twice-iterative, $A \sim B$.
2. $A \in \{\{\underline{\underline{x}}, \underline{x}\}, \{\underline{x}\}, \{\underline{\underline{x}}\}\}$; $B \notin \{\{\underline{\underline{x}}, \underline{x}\}, \{\underline{x}\}, \{\underline{\underline{x}}\}\}$ (without loss of generality):
   This case is impossible given that $\overline{a} = \overline{b}$.
3. $A, B \in \{\{\underline{\underline{x}}, \underline{x}\}, \{\underline{x}\}, \{\underline{\underline{x}}\}\}$: Then $\overline{a} = \overline{b}$ and $\underline{a} = \underline{b}$ implies $A = B$, and by reflexivity, $A \sim B$.

The results of Steps 5 and 6 together imply $\succsim = \succsim_{Mm}$.

That $\succsim_{Mm}$ satisfies (RAV), (PBAP) and (SUAP) is easily proven. To prove that $\succsim_{Mm}$ is twice-iterative, see the corresponding part in the proof of $\succsim_{mM}$.

Proof of Theorem 13:

The following proof is made provided that X contains at least three elements. The case $\#X = 1$ is degenerate, and in the case $\#X = 2$, if $\succsim$ satisfies (SRAV) and if it is an n-times iterative rule in relation to an Eliminative F, then directly $\succsim = \succsim_{ImM}$.

If $\succsim$ is n-times iterative, it is element-induced by definition and therefore all of the conditions of Definition 1 are satisfied.

Step 1: We will prove that, under the properties of $\succsim$, $\forall A \in Z$ such that $\# A = 3$ $(A = \{\overline{a}; a; \underline{a}\})$, $f_1(A) = \underline{a}$:

Let us suppose that $f_1(A) \neq \underline{a}$. By (SUAV) $\{a\} \succ A$. As $\succsim$ is element-induced $\{a\} \succ A$ is only possible if $f_1(A) = a$. Hence, by n-times iterativeness $f_2(A)$ exists and by Elimination in Uncertain Prospects, $f_2(A) \in \{\overline{a}; \underline{a}\}$. $f_2(A) = \overline{a}$ is impossible because $\{a\} \succ A$ and $\succsim$ is n-time iterative. Therefore $f_2(A) = \underline{a}$. Then, as $\succsim$ is element induced, $f_2(\{a\}) = a$. At this stage the proof is similar to the proof of Step 1 in Theorem 11, for which (PBAP) is used, but by Lemma 10 (PBAP) is satisfied by $\succsim$.

Step 2: $\forall A \in Z$ such that $\# A = 2$ and $\underline{x} \in A$, $f_1(A) = \underline{a}$ $(= \underline{x})$.

As $\# X \geq 3$, $\exists x \in X \backslash A$ such that $xP\underline{a}$. By Step 1 $f_1(\{A \cup \{x\}) = \underline{a}$. Then, by Rationality, $f_1(A) = \underline{a}$.

Step 3: $\forall A \in Z$ such that $\# A > 3$, $f_1(A) = \underline{a}$: See the proof of Step 4 of Theorem 11.

Step 4: $\forall A \in Z$ such that $\# A = 3$ $(A = \{\overline{a}; a; \underline{a}\})$, $f_2(A) = \overline{a}$:

By Step 1 $f_1(A) = \underline{a}$. By n-times iterativeness $f_2(A)$ exists. Let us suppose $f_2(A) \neq \overline{a}$. Then, by Elimination in Uncertain Prospects, $f_2(A) = a$. By (SRICH) $A \succ \{\overline{a}; \underline{a}\}$. If $f_1(\{\overline{a}; \underline{a}\}) = \underline{a}$, then by n-times iterativeness $f_2(\{\overline{a}; \underline{a}\})$ exists, and by Elimination in Uncertain Prospects, $f_2(\{\overline{a}; \underline{a}\}) = \overline{a}$. Then, as $\succsim$ is element-induced, $\{\overline{a}; \underline{a}\} \succ A$, reaching a contradiction. If $f_1(\{\overline{a}; \underline{a}\}) = \overline{a}$, then as $\succsim$ is element-induced, $\{\overline{a}; \underline{a}\} \succ A$, again reaching a contradiction.
Therefore, $f_2(A) = \overline{a}$.

Step 5: $\forall A \in Z$ such that $\# A > 3$, $f_2(A) = \overline{a}$.

By n-times iterativeness $f_2(A)$ exists. By Elimination in Uncertain Prospects and Step 3, $f_2(A) \neq \underline{a}$. Let us suppose $f_2(A) = a'$ s.t. $a' \neq \overline{a}; \underline{a}$. By Step 3

$f_1(A) = \underline{a}$. By Step 1 $f_1(\{\overline{a}, a', \underline{a}\}) = \underline{a}$. By Step 4 $f_2(\{\overline{a}, a', \underline{a}\}) = \overline{a}$. Therefore, by Rationality, $f_2(A) \neq a'$, which turns into a contradiction.

Step 6: $\forall A \in Z$ such that $\#A = 2$, $f_1(A) = \underline{a}$ and $f_2(A) = \overline{a}$:

If $\underline{x} \in A$, $f_1(A) = \underline{a}$ by Step 2. By n-times iterativeness $f_2(A)$ exists, and by Elimination in Uncertain Prospects $f_2(A) = \overline{a}$.

If $\underline{x} \notin A$, then let us suppose that $f_1(A) = \overline{a}$. By Step 1 $f_1(\{\overline{a}, \underline{a}, \underline{x}\}) = \underline{x}$. By Step 4 $f_2(\{\overline{a}, \underline{a}, \underline{x}\}) = \overline{a}$.

By Alternate Iteration Independence $f_3(\{\overline{a}, \underline{a}, \underline{x}\}) = f_1(\{\overline{a}, \underline{a}, \underline{x}\} \setminus f_1(\{\overline{a}, \underline{a}, \underline{x}\})) = f_1(\{\overline{a}, \underline{a}\}) = \overline{a}$, which is in contradiction with Elimination in Uncertain Prospects. Therefore $f_1(A) = \underline{a}$, and by Elimination in Uncertain Prospects $f_2(A) = \overline{a}$.

In sum, Steps 1 to 6 prove that:

$$\forall A \in Z \text{ such that } \#A \geq 2,\ f_1(A) = \underline{a} \text{ and } f_2(A) = \overline{a} \qquad (3)$$

Step 7: $\forall A \in Z$ such that $\#A = m \geq 2$, $\forall l \in N$ s.t. $l \leq k$, let us denote by A_l and A^l the subsets of A consisting respectively of the l-worst elements and l-best elements of A according to P. Then, $\forall i \in N$, $i \leq m$,

$$f_i(A) = \begin{cases} \min(A \setminus A_{(i-1)/2}) & \text{if } i \text{ is odd and } m \geq (i-1) \\ \max(A \setminus A^{(i-2)/2}) & \text{if } i \text{ is even and } m \geq (i-1) \\ \text{does not exist} & \text{if } m < (i-1) \end{cases}$$

The proof is direct applying Alternate Iteration Independence, Elimination in Uncertain Prospects, and (3).

Step 8: $\forall A, B \in X$, $A \succ_{lmM} B$ implies $A \succ B$:

$\forall A \in Z$, let $A_0 = A$ and

$$n_A = \begin{cases} \#A/2 & \text{if } \#A \text{ is even} \\ (\#A - 1)/2 & \text{if } \#A \text{ is odd} \end{cases}$$

If $n_A > 0$, let, for all $t = 1, \ldots, n_A$, $A_t = A_{t-1} \setminus \{\underline{a_{t-1}}, \overline{a_{t-1}}\}$. For all $A, B \in Z$, let $n_{AB} = \min(\{n_A, n_B\})$.

Then, $A \succ_{lmM} B$ implies that $\exists t \in \{0, \ldots, n_{AB}\}$ such that $(A_s \sim_{mM} B_s \forall s$

t) and ($A_t \succ_{mM} B_t$ or $B_t = \emptyset$)

Now, four cases are considered:

– $\#A, \#B > 1$:

By Step 7, $f_i(A) = \begin{cases} \min(A \setminus A_{(i-1)/2}) & \text{if } i \text{ is odd and } m \geq (i-1) \\ \max(A \setminus A^{(i-2)/2}) & \text{if } i \text{ is even and } m \geq (i-1) \\ \text{does not exist} & \text{if } m < (i-1) \end{cases}$.

And analogously, we know the values of any $f_i(B)$. Then, by hypothesis and Step 7, $\exists l \leq n$ such that $\forall i \in N$, $i < l$, $f_i(A) = f_i(B)$ and $[f_l(A) R f_l(B)$ or ($f_l(A)$ exists and $f_l(B)$ does not exist)]. As $\succeq$ is element-induced and n-times iterative, then $A \succ B$.

– $\#A = 1$ and $\#B > 1$: By definition of element-induced rule, $\forall A \in Z$, $f_1(A) \in A$. Therefore $f_1(A) = \underline{a}$. By (3) $f_1(B) = \underline{b}$ and $f_2(B) = \overline{b}$. $A \succ_{lmM} B$ implies $\underline{a} R \underline{b}$. If $\underline{a} P \underline{b}$, then, as $\succeq$ is element-induced, $A \succ B$. If $\underline{a} = \underline{b}$, then $A \succ_{lmM} B$ implies $\overline{a} R \overline{b}$. It is only possible that $\underline{a} = \overline{a}$ if $\overline{b} = \underline{b}$, that is, if $\#B = 1$, which is a contradiction.

– $\#A > 1$ and $\#B = 1$. By (3) $f_1(A) = \underline{a}$ and $f_2(A) = \overline{a}$. Also, by the definition of element-induced rule, $f_1(B) = \underline{b}$, and $f_2(B)$, if it exists, is equal to $\underline{b}$. On the other hand $A \succ_{lmM} B$ implies $\underline{a} R \underline{b}$. If $\underline{a} P \underline{b}$, then since $\succeq$ is element-induced, $A \succ B$. If $\underline{a} = \underline{b}$, then as $\#B = 1$, $\overline{a} P \overline{b}$. Since $\succeq$ is element-induced and n-times iterative, hence $A \succ B$.

– $\#A = 1$, $\#B = 1$. Then $A \succ_{lmM} B$ implies $\underline{a} P \underline{b}$, that is, $f_1(A) P f_1(B)$. Then, by element induction, $A \succ B$.

Step 9: $\forall A, B \in X$, $A \sim_{lmM} B$ implies $A \sim B$:

By definition of $\succeq_{lmM}$, $A \sim_{lmM} B$ implies $A = B$. Then, by reflexivity of $\succeq$, $A \sim B$.

The results of Steps 8 and 9 together imply $\succeq = \succeq_{lmM}$.

That $\succeq_{lmM}$ satisfies (SRAV), (SUAV) and (SRICH) is easily proven. To prove that there exists a pair (k; F) such that k = n, that F is rational, Alternate Iteration Independent and Eliminative, and that $<_{Im\,M}$ is element-induced in relation to (k; F), see the corresponding part of the proof in Lemma 4. Moreover, to prove that $<_{Im\,M}$ is n-times iterative, note that $<_{Im\,M}$ is a linear ordering, and therefore Lemma 5 applies. ¤

Proof of Theorem 14: The case X = 1 is degenerate and if X = 2, then by directly applying (RAV) and the fact that < is an n-times iterative rule in relation to an Eliminative F, we reach $<=<_{LMm}$. Hence, the following proof is made provided that X contains at least 3 elements.

If < is n-times iterative, then it is also element-induced by de¯nition, and therefore all of the conditions of De¯nition 1 are satis¯ed.

Step 1: We will prove that, under the properties of <, then 8A 2 Z such that #A = 3 (A = f$\overline{a}$; a; $\underline{a}$g), $f_1(A) = \overline{a}$:

By (SUAP) f$\overline{a}$; a; $\underline{a}$g Â fag. Since < is element-induced f$\overline{a}$; a; $\underline{a}$g Â fag is only possible if $f_1(A) = \overline{a}$ or $f_1(A) = a$. Let us suppose $f_1(A) = a$. By Rationality, f_1(fa; $\underline{a}$g) = a, and by n-times iterativeness and Elimination in Uncertain Prospects f_2(fa; $\underline{a}$g) exists and is equal to $\underline{a}$. By (RAV) fag Â fa; $\underline{a}$g. Therefore, as < is element-induced, f_1(fag) = f_2(fag) = a. Again, by (SUAP), f$\overline{a}$; a; $\underline{a}$g Â fag. By n-times iterativeness $f_2(A)$ exists. If $f_1(A) = a$, by Elimination in Uncertain Prospects f$\overline{a}$; a; $\underline{a}$g Â fag is only possible if $f_2(A) = \overline{a}$. Also, by Rationality, f_1(f$\overline{a}$; ag) = a. In sum, we have, by Elimination in Uncertain Prospects and n-times iterativeness, the following values of F: f_1(f$\overline{a}$; ag) = a; f_2(f$\overline{a}$; ag) = $\overline{a}$; f_3(f$\overline{a}$; ag) does not exist; $f_1(A) = a$; $f_2(A) = \overline{a}$; $f_3(A) = \underline{a}$. If < is n-times iterative, then A Â f$\overline{a}$; ag, which results in a contradiction with (RAV). Hence, 8A 2 Z such that #A = 3, $f_1(A) = \overline{a}$.

Step 2: 8A 2 Z such that #A = 2, if 9x 2 XnA s.t. $\overline{a}$Px (that is, A 6= f$\underline{x}$; $\underline{x}$g), then $f_1(A) = \overline{a}$ and $f_2(A) = \underline{a}$.

Let us take x 2 XnA such that $\overline{a}$Px. By Step 1 f_1(A [fxg) = $\overline{a}$. Then, by Rationality, $f_1(A) = \overline{a}$. By n-times iterativeness and Elimination in Uncertain

Prospects $f_2(A)$ exists and is equal to $\underline{a}$.

Step 3: $\forall A \in Z$ such that $\#A = 3$ $(A = \{\overline{a}; a; \underline{a}\})$, $f_2(A) = \underline{a}$ and $f_3(A) = a$:

By Step 2, $f_1(\{\overline{a}; a\}) = \overline{a}$ and $f_2(\{\overline{a}; a\}) = a$. By Step 1, $f_1(A) = \overline{a}$. By n-times iterativeness, we know that $f_2(A)$ and $f_3(A)$ exist. Let us suppose that $f_2(A) = a$. Then, by Elimination in Uncertain Prospects, $f_3(A) = \underline{a}$ and $f_3(\{\overline{a}; a\})$ does not exist. Hence, by element-induction, $A \succ \{\overline{a}; a\}$, which contradicts (RAV)

Step 4: $\forall A \in Z$ such that $\#A > 3$, $f_1(A) = \overline{a}$ and $f_2(A) = \underline{a}$: See Step 4 in the proof of Theorem 12.

Step 5: $f_1(\{\underline{x}; \underline{x}\}) = \underline{x}$ and $f_2(\{\underline{x}; \underline{x}\}) = \underline{x}$:

Let $a \in X$, $a \neq \underline{x}; \underline{x}$. By Step 3 $f_3(\{a; \underline{x}; \underline{x}\}) = \underline{x}$, which by Alternate Iteration Independence is the same element as $f_1(\{a; \underline{x}; \underline{x}\} \setminus (f_1(\{a; \underline{x}; \underline{x}\}))$. By Step 1 $f_1(\{a; \underline{x}; \underline{x}\}) = a$. Therefore $f_1(\{\underline{x}; \underline{x}\}) = \underline{x}$. By Elimination in Uncertain Prospects, $f_2(\{\underline{x}; \underline{x}\}) = \underline{x}$.

In sum, Steps 1 to 5 prove that:

$$\forall A \in Z \text{ such that } \#A \geq 2,\ f_1(A) = \overline{a} \text{ and } f_2(A) = \underline{a} \qquad (4)$$

Step 6: $\forall A \in Z$ such that $\#A = m \geq 2$, $\forall l \in N$ such that $l \leq m$, let us denote by A_l and A^l the subsets of A consisting respectively on the l-worst elements and l-best elements of A according to P. Then $\forall i \in N$, $i \leq m$,

$$f_i(A) = \begin{cases} \max(A \setminus A^{(i-1)/2}) & \text{if } i \text{ is odd and } m \geq (i-1) \\ \min(A \setminus A_{(i-2)/2}) & \text{if } i \text{ is even and } m \geq (i-1) \\ \text{does not exist} & \text{if } m < (i-1) \end{cases}$$

The proof is direct applying Alternate Iteration Independence, Elimination in Uncertain Prospects, and (4).

Step 7: $\forall A; B \in X$, $A \succ_{LMm} B$ implies $A \succ B$, and $A \sim_{LMm} B$ implies $A \sim B$:

The proof is analogous to that of Steps 8 and 9 in Theorem 13, for which (PBAP) is necessary, but by Lemma 10, (PBAP) is satisfied by $\succcurlyeq$.

That $\succcurlyeq_{LMm}$ satisfies (RAV) and (SUAP) is easily proven. To prove that there exists a pair $(k; F)$ such that $k = n$, that F is rational, Alternate Iteration Independent and Eliminative, and that $\succcurlyeq_{lmM}$ is element-induced in relation to $(k; F)$, see the corresponding part of the proof in Lemma 4. Moreover, to prove that $\succcurlyeq_{LMm}$ is n-times iterative note that $\succcurlyeq_{LMm}$ is a linear ordering, and therefore Lemma 5 applies. □

The following examples establish the independence of the conditions used in Theorems 11, 12, 13 and 14, provided that $\#X \geq 2$. The case $\#X = 1$ is clearly degenerate. For all of the examples provided below, we will assume that sets are ordered according to R from their best to their worst element.

1. $\succcurlyeq_{mM}$
   - $\succcurlyeq_{lm}$ satisfies (SRAV), (SUAV) and (PBAP) but is not twice-iterative.
   - Let $X = \{x; y; z\}$, and let $\succcurlyeq$ be the ordering over X induced the following F: for all $A \in Z$ s.t. $A \neq \{x; y\}$, $F(A) = \{\underline{a}; \overline{a}\}$, and $F(\{x; y\}) = \{x; x\}$. Then $\succcurlyeq$ is a twice-iterative rule that satisfies (SUAV) and (PBAP), but not (SRAV).
   - $\succcurlyeq_{Mm}$ is a twice-iterative rule that satisfies (SRAV) and (PBAP), but not (SUAV).
   - Let $X = \{x; y; z\}$, and let $\succcurlyeq$ be the ordering over X induced by the following F: for all $A \in Z$ such that $A \neq \{x; y\}$, $F(A) = \{\underline{a}; \overline{a}\}$, and $F(\{x; y\}) = \{y; y\}$. Then $\succcurlyeq$ is a twice-iterative rule that satisfies (SUAV) and (SRAV), but not (PBAP).
2. $\succcurlyeq_{Mm}$
   - Let $X = \{x; y; z\}$, and let $\{x\} \succ \{x; y\} \sim \{x; z\} \succ \{x; y; z\} \succ \{y\} \succ \{y; z\} \succ \{z\}$. Then $\succcurlyeq$ satisfies (RAV), (SUAP) and (PBAP), but is not element-induced. Therefore it cannot be twice-iterative.
   - Let $X = \{x; y; z\}$, and let $\succcurlyeq$ be the ordering over X induced by F such that, $\forall A \in X$, $A \neq \{x; y\}$, $F(A) = \{\overline{a}; \underline{a}\}$, and $F(\{x; y\}) = \{x; x\}$. Then

$\Re$ is a twice-iterative rule that satisfies (SUAP) and (PBAP), but not (RAV).

- $\Re_{mM}$ is a twice-iterative rule that satisfies (RAV) and (PBAP), but not (SUAP).
- Let $X = \{x, y, z\}$, and let $\Re$ be the ordering over X induced by the following F: $\forall A \in X$, $A \neq \{y, z\}$, $F(A) = \{\overline{a}, \underline{a}\}$, and $F(\{y, z\}) = \{z, z\}$. Then $\Re$ is a twice-iterative rule that satisfies (SUAP) and (RAV), but not (PBAP).

3. $\Re_{ImM}$

- Let $X = \{x, y\}$, and let $\Re$ be the ordering over X induced by F such that, $\forall A \in Z$, $F(A) = \underline{a}$. Then $\Re$ is element-induced in relation to an Alternate Iteration Independent and Eliminative mapping F, and it satisfies (SRAV), (SRICH) and (SUAV), but it is not n-times iterative.
- Let $X = \{x, y, z\}$, and let $\{x\} \succ \{x, y\} \succ \{y\} \succ \{x, y, z\} \succ \{y, z\} \succ \{x, z\} \succ \{z\}$. Then $\Re$ satisfies (SRAV), (SUAV) and (SRICH). It is possible to find a rational and Eliminative mapping F with which $\Re$ is element-induced, (and as $\Re$ is a linear ordering then it is n-times iterative). But it is impossible to find an Alternate Iteration Independent, Eliminative and rational F in relation to which $\Re$ is element-induced.
- Let $X = \{x, y, z\}$, and let $\Re$ be the ordering over X induced by the following F: $F(X) = \{z, y, x\}$; $F(\{x, y\}) = \{x, y, y\}$; $F(\{x, z\}) = \{z, z, x\}$; $F(\{y, z\}) = \{z, y, y\}$; $F(\{x\}) = \{x, x\}$; $F(\{y\}) = \{y, y\}$; and $F(\{z\}) = \{z, z\}$. Then $\Re$ satisfies (SRAV), (SUAV) and (SRICH), and it is an n-times iterative rule in relation to a mapping F which is Alternate Iteration Indepedent, but F is not Eliminative.
- Let $X = \{x, y\}$ and let $\Re$ be the ordering over X induced by: $F(\{x, y\}) = \{x, y\}$; $F(\{x\}) = x$; and $F(\{y\}) = y$. Then $\Re$ is an n-times iterative rule in relation to a mapping F which is Alternate Iteration Independent and Eliminative. Also, $\Re$ satisfies (SUAV) and (SRICH), but it does not

satisfy (SRAV).

{ $<_{LMm}$ is an n-times iterative rule in relation to certain mapping F which is Alternate Iteration Independent and Eliminative. Furthermore, $<$ satis¯es (SRAV) and (SRICH), but it does not satisfy (SUAV).

{ Let X = fx; y; zg, and let $<$ be the ordering over X induced by F such that, F(X) = fz; y; xg; F(fx; yg) = fx; yg; F(fx; zg) = fz; xg; F(fy; zg) = fz; yg; F(fxg) = fx; xg; F(fyg) = fy; yg; and F(fzg) = fz; zg. Then $<$ is n-times iterative in relation to an Alternate Iteration Independent and Eliminative mapping F. Also, $<$ satis¯es (SRAV) and (SUAV), but it does not satisfy (SRICH).

4. $<_{LMm}$

{ Let X = fx; yg, and let $<$ be the ordering over X induced by the following F: 8A 2 Z, F(A) = <u>a</u>. Then $<$ is element-induced in relation to an Alternate Iteration Independent and Eliminative mapping F, and it satis¯es (RAV) and (SUAP), but it is not n-times iterative.

{ Let X = fx; y; zg, and let $<$ be the ordering over X induced by the following F: F(X) = fx; z; yg; F(fx; yg) = fx; yg; F(fx; zg) = fx; zg; F(fy; zg) = fz; yg; F(fxg) = fx; xg; F(fyg) = fy; yg; and F(fz; zg) = fzg. Then $<$ satis¯es (RAV) and (SUAP), and it is an n-times iterative rule in relation to an Eliminative mapping F, but F is not Alternate Iteration Independent.

{ Let X = fx; y; zg, and let $<$ be the ordering over X induced by the following F: F(X) = fx; z; yg, F(fx; yg) = fx; y; yg, F(fx; zg) = fx; z; zg, F(fy; zg) = fy; z; zg, F(fxg) = fx; xg, F(fyg) = fy; yg, and F(fzg) = fz; zg. Then $<$ satis¯es (RAV) and (SUAP), and it is an n-times iterative rule in relation to F, which is Alternate Iteration Independent, but not Eliminative.

{ Let X = fx; y; zg, and let $<$ be the ordering over X induced by the following F: F(X) = fx; z; yg, F(fx; yg) = fx; yg, F(fx; zg) = fx; zg,

$F(\{y,z\}) = \{y,z\}$, $F(\{x\}) = \{x\}$, $F(\{y\}) = \{y\}$, and $F(\{z\}) = \{z\}$. Then $\succcurlyeq$ is an n-times iterative rule in relation to an Alternate Iteration Independent and Eliminative mapping F. Also, $\succcurlyeq$ satisfies (SUAP), but it does not satisfy (RAV).

– $\succcurlyeq_{ImM}$ is an n-times iterative rule in relation to certain F which is Alternate Iteration Independent, and Eliminative. Also, $\succcurlyeq_{ImM}$ satisfies (RAV), but it does not satisfy (SUAP).

7 Final Remarks

Unlike other works in the field, in the previous sections the problem of choice under complete uncertainty has been approached at three analytical levels. At the first level, the model concentrates on element-induced evaluation processes. Two different kinds of arguments support this assumption. The first one is merely based on the confirmation that almost all work in the field so far has converged to this type of rules. Secondly, deliberative arguments, supported by experimental evidence, lead to the idea that element-induced processings provide a fair and flexible equilibrium between two important factors in the context of uncertainty: computational costs, and the desire to choose accurately.

At the second level, an adaptation of the classical principle of revealed preference has been applied to the mental process of deciding which outcome(s) is (are) representative(s) or focal(s) within a particular action, that is, what we have called "evaluation processes." The result is a class of rules where individual attitudes towards uncertainty do not play yet any role . This aspect has been introduced at the third analytical level by means of a few simple axioms, allowing us to characterize some criteria of the literature as particular cases of element-induced rational rules.

Some plausible rules, such as median-based rules or second-best based rules (see Nitzan and Pattanaik [29], Sen [36], Baigent and Gaertner [3] or Gaertner and Xu [17, 18]) are element-induced (and clearly seek some kind of procedural rationality) but they fail our assumption of Rationality. This suggests that such assumption is susceptible to adaptations and modifications and leaves open questions for further investigation.

Finally, some other non-element-induced processings have been quoted in the previous sections, such as the satisficing rule, the majority of confirming dimensions rule, and others. Some of these have been well studied from an experimental psychology point of view, but little has been studied regarding theoretical and axiomatic formalization.

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