Externality Policy Reform: A General-Equilibrium Analysis

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Abstract

Consider a competitive equilibrium in an economy in which an externality (whose market is missing) is the only distortion. The implications of implementing a feasible and Pareto-improving policy reform using first-best policy instruments, such as a Pigouvian tax or a direct quantity control of the externality, are well known. The aim of this paper is to study and characterize feasible and Pareto-improving policy reforms in a second-best world with an externality. We consider an extension of the Diamond and Mirrlees (1971) second-best model that incorporates an externality and a direct quantity control on the externality as an additional policy instrument. An apparently counterintuitive finding is that, starting from an initial equilibrium with no direct quantity control on the externality, it might be Pareto improving and equilibrium preserving for the regulator to mandate an *increase* in the level of a *negative* externality. In fact, it can be the case that *all* Pareto-improving and equilibrium-preserving directions of change require an increase in the negative externality. We provide intuition for these results by establishing a nexus between Guesnerie's [1977, 1995] approach to designing (tax) policy reforms and the standard Kuhn-Tucker technique for identifying the manifold of feasible Pareto-optimal states, given the policy instruments available to the government.

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1. Introductory Remarks.

It is common knowledge that, starting from a tight competitive equilibrium in an economy in which the only distortion is a single externality, a Pareto-improving move would necessarily require a reduction in the level of a negative externality or an increase in the level of a positive externality. In this paper, we seek to characterize policy reforms to combat distortions created by the presence of an externality in a more realistic economy: one in which there exist other market distortions. In particular, the second-best model we develop is one in which, apart from the presence of a negative consumption externality, there also exist distortions brought about by indirect taxation (or subsidization), which drives a wedge between consumer and producer prices and hence between the marginal rates of substitution in consumption and in production. As in the optimal taxation literature,[1] this distortion is rationalized by the inability of the government, owing to informational and administrative constraints, to implement a system of personalized lump-sum transfers. Rather, the government can implement a system of uniform lump-sum transfers.[2] To avoid the complications generated by the distribution of profits, we also assume that the government taxes away all profits of firms.[3]

Starting from an initial equilibrium in a regime where the government's policy instruments comprise only the complete system of indirect taxes, 100-percent taxation of profits, and the uniform lump-sum transfers, we study the fruitfulness of imposing quantity controls on the consumption of the externality-causing good. We focus on quantity controls rather than Pigouvian taxes because we wish to examine, in a second-best world, the effect of a comprehensive policy reform on the level of the externality. Even if we assume that the externality-generating good is not a giffen good, it is not clear in a second-best world that a comprehensive reform that includes an increase,

[1] See Ramsey [1927], Diamond and Mirrlees [1971a, 1971b], Diamond [1975], and Guesnerie [1975, 1977, 1979, 1980, 1995]).

[2] In the absence of a feasible system of personalized lump-sum transfers, the government might engage in income redistribution by taxing labor (income), one of the commodities, and redistributing some or all of the revenue equally to the consumer/workers. Or it might redistribute wealth by taxing luxuries and subsidizing necessities.

[3] Alternatively, one could make the (common) assumption of constant returns to scale (a conical production set), in which case maximal aggregate profits would be zero.

say, in a Pigouvian tax would necessarily entail a decrease in the level of the negative externality.

To sharpen our focus on the general-equilibrium interactions between externality policies and tax distortions, we consider the simplest possible externality: one in which the consumption of a particular good by one consumer imposes negative externalities on all other consumers.[4]

Our approach to the study of externality policy reforms in a general-equilibrium context draws extensively on methods and results in two important literatures that, to our knowledge, have not yet been exploited in the study of externality policies. The first is the general-equilibrium literature on tax reform, attributable primarily to Guesnerie [1977, 1995]. This research takes the view that second-best tax policies (as well as first-best policies) are typically unachievable because they require quantum leaps from the existing state of the economy. More realistic are incremental tax-policy reforms that are Pareto improving and equilibrium preserving. A theoretical counterpart of an incremental reform is a (differential) direction of change that is Pareto improving and equilibrium preserving. We adopt the same paradigm in the examination of externality policies.

The second important literature that we exploit is the research on rationing and quantity controls, attributable primarily to Tobin and Houthakker [1950-51], Neary and Roberts [1980], and Guesnerie and Roberts [1984]. The results in this literature greatly facilitate our analysis of a direct quantity control on the consumption of the externality generating commodity.

While our main objective in this research is to provide a framework for examining externality policy reform in a general-equilibrium framework, we do obtain a number of specific results, some of them surprising. Most notably, we find that, starting in a

[4] Our results would require only some minor notational changes and a system of personalized quantity controls if we were to generalize the model to allow consumption of the externality-generating commodity by multiple consumers, so long as the set of consumers can be partitioned into externality generators on the one hand and externality victims on the other. By partitioning in this way, we avoid the issues of strategic interaction that arise if some consumers are both generators and victims. The type of externality we have in mind is tobacco smoking or the playing of loud music.

(Diamond-Mirrlees) competitive equilibrium with distortional taxation and no direct externality quantity control, it might be Pareto improving and equilibrium preserving for the regulator to mandate an *increase* in the single negative externality. In fact, it can be the case that *all* Pareto-improving and equilibrium-preserving directions of change require an increase in the negative externality. While this result is at first glance counterintuitive, it is explained by the presence of other distortions in the economy, namely the wedge between consumer and producer prices, along with the feedback effects in a general-equilibrium analysis.[5]

Section 2 constructs a general-equilibrium model of policy reform with a single externality, Section 3 contains our general results, and Section 4 provides the intuition underlying the economic structure of these results. Section 5 concludes.

2. The Model.

2.1. Consumers and Producers.

We examine an economy with $n+1$ goods, the first n being the non-externality-generating goods, indexed by either k or $l = 1, \ldots, n$, and the last being the externality-generating good, indexed by ν. We assume no externalities in the production of these $n+1$ goods, so that the aggregate (economy wide) production technology is the vector sum of the individual technology sets. Denote the aggregate technology set by $Y \subset \mathbf{R}^{n+1}$ and a feasible aggregate (net) production bundle by $\langle y, y_\nu \rangle \in Y$. We assume that Y is closed and strictly convex, at least on a neighborhood of an initial equilibrium, so that, subject to a given price vector $\langle p, \rho \rangle \in \mathbf{R}^{n+1} \backslash \{0^{(n+1)}\}$,[6] the profit-maximizing

[5] A second thought may suggest that this result is not counterintuitive in that it could simply reflect the fact that a Pigouvian tax on the externality-generating commodity is set too high at the status quo and the tax is not an available instrument for reform so that the only corrective mechanism for overtaxation of the bad commodity is to mandate an increase in its consumption. We show in Section 4, however, that this is not an explanation for the counterintuitive result. In fact, this outcome can occur irrespective of whether the policymaker can implement a Pigouvian tax on the negative consumption externality.

[6] $0^{(n+1)}$ is the $(n+1)$-dimensional zero vector. Note that we do not exclude the possibility of negative prices.

aggregate production bundle is unique. In fact, we will want to maintain the existence of a continuously differentiable supply function, $\langle \eta, \eta_\nu \rangle$, defined by

$$\big\langle \eta(p,\rho), \eta_\nu(p,\rho) \big\rangle = \text{argmax}_{y,y_\nu} \big\{ p \cdot y \ + \rho \ y_\nu \mid \langle y, y_\nu \rangle \in Y \big\}, \tag{2.1}$$

at least on a neighborhood of an initial (equilibrium) production price vector.[7]

Of the $m+1$ consumers, one, indexed by h, consumes the externality-generating good as well as the n non-externality-generating goods, and the other m consumers, indexed by i or $j = 1, \ldots, m$, consume only the n non-externality-generating goods, but their welfare levels are adversely affected by this negative externality. Thus, net consumption bundles[8] are denoted by $\langle x^h, \nu^h \rangle \in \mathbf{R}^{n+1}$ and $x^i \in \mathbf{R}^n, \ i = 1, \ldots, m$, where $\nu^h \in \mathbf{R}$ is the (net) quantity of the externality-generating good consumed by consumer h. But preferences are defined over non-empty (net) consumption sets,[9] $X^h \subseteq \mathbf{R}^{n+1}$ and $X^i \subseteq \mathbf{R}^{n+1}, \ i = 1, \ldots, m$, where a generic element of X^i is $\langle x^i, \nu^h \rangle$. We assume that the preferences of consumers are represented by continuous utility functions, $U^h : X^h \to \mathbf{R}$ and $U^i : X^i \to \mathbf{R}, \ i = 1, \ldots, m$, that satisfy local nonsatiation and differentiability on a neighborhood of an initial equilibrium. In addition, we assume that U^h satisfies strict quasi-concavity on this neighborhood while each $U^i, \ i = 1, \ldots, m$, satisfies this property on an appropriate neighborhood of the n-dimensional subspace with coordinates attached to the non-externality-generating commodities.[10] We formalize our negative-externality scenario by assuming that $\partial U^i(x^i, \nu^h)/\partial \nu^h < 0, \ i = 1, \ldots, m$.

In our second-best world, consumer prices are not necessarily equal to producer prices. We denote the consumer price vector by $\langle q, \pi \rangle \in \mathbf{R}^{n+1}$, where q is the price

7 There is, of course, no aggregation problem on the supply side of a competitive economy with no production externalities (see, *e.g.*, Bliss [1975], Koopmans [1957], or Russell, Breunig, and Chiu [1998]).

8 Net of endowments, that is.

9 Endowments are implicit in our description of the economy; preferences, primitively defined over gross consumption bundles, induce an ordering on net consumption bundles, conditional on endowments, which are held constant throughout our analysis.

10 Thus, we do not preclude the existence of fundamental (global) nonconvexities. The Starrett problem of nonexistence of Arrovian competitive equilibrium is not an issue in our context.

vector for the n non-externality-generating goods and π is the price of the externality-generating good. Thus, government indirect taxes or subsidies are given by $\langle t, \tau \rangle := \langle q, \pi \rangle - \langle p, \rho \rangle \in \mathbf{R}^{n+1}$. The uniform lump-sum transfer is $R \in \mathbf{R}$.

Under the assumption that consumer h is subjected to a quantity constraint on his consumption of the externality-generating good, $\nu = \bar{\nu}$, his optimization problem and constrained indirect utility function are defined by

$$\bar{V}^h(q, \pi, R, \bar{\nu}) := \max_{x^h, \nu^h} \left\{ U^h(x^h, \nu^h) \mid q \cdot x^h + \pi\ \nu^h \leq R\ \wedge\ \nu^h = \bar{\nu} \right\}. \tag{2.2}$$

Under our regularity conditions, the solution to this optimization problem at the status quo is unique, yielding a rationed (net) demand system,

$$\begin{aligned} \overset{*}{x}{}^h &= \bar{d}^h(q, \pi, R, \bar{\nu}) \\ \overset{*}{\nu}{}^h &= \bar{d}^h_\nu(q, \pi, R, \bar{\nu}) = \bar{\nu}, \end{aligned} \tag{2.3}$$

defined on a neighborhood of initial values of the arguments. We assume, in addition, that this demand system is continuously differentiable on this neighborhood.

We now exploit the Neary and Roberts [1980] analysis of this system of rationed demands, first by relating them to the usual (unrationed) demand systems. Under our regularity conditions (most notably, strict quasi-concavity of U^h on a neighborhood), there exists a vector of virtual (support) prices that makes the unrationed demand system identical to the rationed demand system. In particular, Neary and Roberts (pp. 28–29) show that the consumer market prices differ from these virtual prices only with respect to the price of the rationed commodity. This, of course, makes sense, since the consumer is free to equate marginal rates of substitution and price ratios of unrationed goods, and it is optimal to do so. Thus, in our case, there exists a shadow price, π^h, of the externality-causing good and a level of the transfer ("income"), $\tilde{R}^h$, such that the rationed demands at prices $\langle q, \pi \rangle$ and income transfer R, are identical to the unrationed demands at prices $\langle q, \pi^h \rangle$ and income transfer $\tilde{R}^h$:

$$\begin{aligned} \bar{d}^h(q, \pi, R, \bar{\nu}) &= d^h(q, \pi^h, \tilde{R}^h) \\ \bar{\nu} = \bar{d}^h_\nu(q, \pi, R, \bar{\nu}) &= d^h_\nu(q, \pi^h, \tilde{R}^h), \end{aligned} \tag{2.4}$$

where $d^h(q, \pi^h, \tilde{R}^h)$ and $d^h_\nu(q, \pi^h, \tilde{R}^h)$ solve

$$\max_{x^h, \nu^h} \; U^h(x^h, \nu^h) \quad \text{s.t.} \quad q \cdot x^h + \pi^h \; \nu^h \leq \tilde{R}^h. \tag{2.5}$$

To interpret and provide some intuition about (2.4), consider the ration-constrained and non-ration-constrained expenditure functions dual to (2.2) and (2.5):

$$\bar{E}^h(q, \pi, \bar{\nu}, u^h) := \min_{x^h, \nu^h} \Big\{ q \cdot x^h + \pi \; \nu^h \mid U^h(x^h, \nu^h) \geq u^h \quad \wedge \quad \nu^h = \bar{\nu} \Big\} \tag{2.6}$$

and

$$E^h(q, \pi^h, u^h) := \min_{x^h, \nu^h} \Big\{ q \cdot x^h + \pi^h \; \nu^h \mid U^h(x^h, \nu^h) \geq u^h \Big\}. \tag{2.7}$$

Neary and Roberts (p. 30) establish the following intuitive relationship between these two expenditure functions:

$$\bar{E}^h(q, \pi, \bar{\nu}, u^h) = E^h(q, \pi^h, u^h) + (\pi - \pi^h) \; \bar{\nu}. \tag{2.8}$$

That is, the difference between the minimal ration-constrained cost of obtaining utility level u^h and the minimal virtual cost of obtaining this utility is given by the difference in the costs of purchasing $\bar{\nu}$ of the negative externality at the market price π and the shadow price π^h. This relationship is depicted in Figure 1 for the special case where $n = 1$ (hence q is a scalar).

Thus, referring back to the demand systems in (2.4) and noting, from (2.8), that

$$\tilde{R}^h = R + (\pi^h - \pi) \; \bar{\nu}, \tag{2.9}$$

we see that the last equation in (2.4) implicitly defines π^h as a function of q, R, $\bar{\nu}$, and π, all of which are known values. One can then substitute this value of π^h, along with (2.9) into the first n equations in (2.4) to determine the demands for the non-rationed commodities, $\bar{d}^h(q, \pi, R, \bar{\nu})$.

Relation (2.8) yields a differential money-metric measure of the welfare effects on consumer h of a change in the quantity rationing of the externality:

$$\frac{\partial \bar{E}^h(q, \pi, \bar{\nu}, u^h)}{\partial \bar{\nu}} = \pi - \pi^h. \tag{2.10}$$

This is the differential change in income needed to just compensate consumer h for an infinitesimal change in the level of rationing of the negative externality. If the market price of the externality good exceeds (is less than) the shadow price for consumer h, a positive (negative) compensation is required to maintain his utility level when the externality level is increased (decreased).

A dual (utility) measure of the welfare change of consumer h attributable to an infinitesimal change in the rationed level of the negative externality good is $\partial \bar{V}^h(q, \pi, R, \bar{\nu})/\partial \bar{\nu}$. By comparing the first-order conditions of the ration-constrained and virtual utility maximization exercises, (2.2) and (2.5), it can be shown that

$$\frac{\partial \bar{V}^h(q, \pi, R, \bar{\nu})}{\partial \bar{\nu}} = \bar{\lambda}^h \ (\pi^h - \pi), \tag{2.11}$$

where $\bar{\lambda}^h$ is the marginal utility of money evaluated at $\langle q, \pi, R, \bar{\nu} \rangle$.

We apply the Neary/Roberts results to our specific context, one where we have as the starting point an equilibrium in a regime with no quantity rationing and we study the welfare effects of introducing externality quantity controls into the system. To this end, define the ration-unconstrained demands of consumer h in the initial situation:

$$\left\langle d^h(q, \pi, R), d^h_\nu(q, \pi, R) \right\rangle = \text{argmax}_{x^h, \nu^h} \left\{ U^h(x^h, \nu^h) \mid q \cdot x^h + \pi \ \nu^h \leq R \right\}. \tag{2.12}$$

We interpret the initial situation to be a special instance of a quantity control regime where the ration-constrained level has been set just equal to the ration-unconstrained level; *i.e.*, $\bar{\nu} = d^h_\nu(q, \pi, R)$. In that case, we also have $\bar{d}^h(q, \pi, R, \bar{\nu}) = d^h(q, \pi, R)$, and the market and consumer shadow prices are identical for all goods. In particular, $\pi^h = \pi$ and, from (2.11), we find that

$$\frac{\partial \bar{V}^h(q, \pi, R, \bar{\nu})}{\partial \bar{\nu}} = 0. \tag{2.13}$$

Thus, differentially, a forced change in the consumption of the negative externality imposes no utility loss on the consumer of this externality-producing good. The intuition for this outcome (as originally explained by Guesnerie and Roberts [1984, pp. 68–69]) is as follows: At the initial unrationed situation the indifference surface through the optimal consumption bundle is tangent to the budget hyperplane. Consequently, a forced, infinitesimally small change in consumption of the externality moves the consumer along his budget hyperplane, which is differentially equivalent to moving along the indifference surface. (In Figure 1, imagine that $\pi^h = \pi$.)

Repeated application of the envelope theorem to (2.2) yields the other components of the gradient of the indirect utility function. As the initial situation is taken as given in Section 3, to ease (and slightly abuse) the notation, we write the gradient as

$$\nabla \bar{V}^h(q, \pi, R, \bar{\nu}) = \begin{pmatrix} -\bar{\lambda}^h \; \bar{d}^h \\ -\bar{\lambda}^h \; \bar{\nu} \\ \bar{\lambda}^h \\ 0 \end{pmatrix} =: \Gamma^h. \tag{2.14}$$

Each externality victim, $i = 1, \ldots, m$, must bear the level of the negative externality passed on to her by the externality generator and choose her net consumption vector of the non-externality goods, subject to the budget constraint:

$$\bar{V}^i(q, R, \bar{\nu}) = \max_{x^i} \left\{ U^i(x^i, \bar{\nu}) \mid q \cdot x^i \leq R \right\}, \quad i = 1, \ldots, m. \tag{2.15}$$

Optimization yields the unique set of demands,

$$\overset{*}{x}^i = \bar{d}^i(q, R, \bar{\nu}), \quad i = 1, \ldots, m. \tag{2.16}$$

The quantity rationing results of Neary and Roberts [1980] prove useful here as well, providing a framework to solve for the shadow price of the externality for individual i. This is accomplished by considering a notional optimization exercise in which consumer i is free to choose the quantities of all goods, including the externality, to maximize her utility subject to a budget constraint in all $n + 1$ quantities. Under our regularity conditions (principally strict quasi-concavity and differentiability of U^i on an appropriate neighborhood), an argument analogous to that used for consumer h above identifies a unique vector of shadow prices of all $n + 1$ goods and a level of income that makes the demand system (2.16) identical to the demand system obtained from the notional optimization exercise. As in the case of consumer h, the shadow price vector differs from the market price vector faced by consumer i only with respect to the externality-generating good. Since, in her actual optimization problem, the level of the externality is not a choice variable, it is as if she faces a market price of zero for the externality good. Thus, there exists a shadow price π^i of the externality-generating good and a level of income $\tilde{R}^i$ such that the actual demands of consumer i at market prices $\langle q, 0\rangle$ and income R are identical to her notional demands at prices $\langle q, \pi^i\rangle$ and income $\tilde{R}^i$:

$$\begin{aligned} \bar{d}^i(q, R, \bar{\nu}) &= d^i(q, \pi^i, \tilde{R}^i), \quad i = 1, \ldots, m, \\ \bar{\nu} &= d^i_\nu(q, \pi^i, \tilde{R}^i), \quad i = 1, \ldots, m. \end{aligned} \tag{2.17}$$

An argument similar to that for consumer h (and also attributable to Neary and Roberts [1980, p. 30]) establishes that

$$\tilde{R}^i = R + \pi^i \bar{\nu}, \quad i = 1, \ldots, m, \tag{2.18}$$

since consumer i acts as if the market price of the negative externality were zero. Along with (2.18), the last equation in (2.17) implicitly defines π^i as a function of q, R, and $\bar{\nu}$.

In a fashion parallel to the quantity-rationing case, the differential welfare effects of a change in the level of the externality burden on consumer i can be obtained in money-metric terms as

$$\frac{\partial \bar{E}^i(q, u^i, \bar{\nu})}{\partial \bar{\nu}} = -\pi^i, \quad i = 1, \ldots, m, \tag{2.19}$$

where $\bar{E}^i$ is the expenditure function of consumer i, or, in utility terms, as

$$\frac{\partial \bar{V}^i(q, R, \bar{\nu})}{\partial \bar{\nu}} = \bar{\lambda}^i \pi^i, \quad i = 1, \ldots, m, \tag{2.20}$$

where $\bar{\lambda}^i$ is the marginal utility of income of consumer i evaluated at $\langle q, R, \bar{\nu}\rangle$. Taking the initial situation as given (and simplifying notation), the gradients of the indirect utility functions are

$$\nabla \bar{V}^i(q, R, \bar{\nu}) = \begin{pmatrix} -\bar{\lambda}^i \; \bar{d}^i \\ 0 \\ \bar{\lambda}^i \\ \bar{\lambda}^i \; \pi^i \end{pmatrix} =: \Gamma^i, \quad i = 1, \ldots, m. \tag{2.21}$$

2.2. Equilibrium-Preserving Policy Reforms.

We first extend Guesnerie's [1977, 1995] concept of a tax equilibrium to incorporate an externality quantity control:

Definition: A *tax equilibrium with a quantity control on the externality (TEQC)* is an $(m+1)$-tuple of consumption vectors, $\langle \hat{x}^h, \hat{\nu}^h \rangle \in \mathbf{R}^{n+1}$ and $\hat{x}^i \in \mathbf{R}^n$, $i = 1, \ldots, m$, an aggregate production plan, $\langle \hat{y}, \hat{y}_\nu \rangle \in \mathbf{R}^{n+1}$, a uniform lump-sum transfer, $\hat{R} \in \mathbf{R}$, two price vectors, producer prices $\langle \hat{p}, \hat{\rho} \rangle \in \mathbf{R}^{n+1} \backslash \{0^{(n+1)}\}$ and consumer prices $\langle \hat{q}, \hat{\pi} \rangle \in \mathbf{R}^{n+1}$, and a level of quantity control on the externality, $\hat{\bar{\nu}} \in \mathbf{R}$, such that the following hold:

$$
\begin{aligned}
\hat{x}^h &= \bar{d}^h(\hat{q}, \hat{\pi}, \hat{R}, \hat{\bar{\nu}}) \\
\hat{\nu}^h &= \hat{\bar{\nu}} \\
\hat{x}^i &= \bar{d}^i(\hat{q}, \hat{R}, \hat{\bar{\nu}}), \quad i = 1, \ldots, m, \\
\hat{y} &= \eta(\hat{p}, \hat{\rho}) \\
\hat{y}_\nu &= \eta_\nu(\hat{p}, \hat{\rho}) \\
\hat{x}^h + \sum_i \hat{x}^i &\leq \hat{y}
\end{aligned}
\tag{2.22}
$$

and

$$\hat{\bar{\nu}} \leq \hat{y}_\nu,$$

where $\bar{d}^h(\hat{q}, \hat{\pi}, \hat{R}, \hat{\bar{\nu}})$ and $\bar{d}^i(\hat{q}, \hat{R}, \hat{\bar{\nu}})$, $i = 1, \ldots, m$, are obtained from the consumer optimization exercises in (2.2) and (2.15) and $\langle \eta(\hat{p}, \hat{\rho}), \eta_\nu(\hat{p}, \hat{\rho}) \rangle$ is obtained from the aggregate profit-maximization condition in (2.1).[11] When the $n+1$ (weak) inequalities in (2.22) hold as equalities, we say that the TEQC is *tight*, and when any one holds as a strict inequality, we say that the TEQC is *non-tight*.

[11] Vector notation: for $\langle x, y \rangle \in \mathbf{R}^{2m}$,

$$
\begin{aligned}
x \geq y &\iff x_i \geq y_i, \; i = 1, \ldots, m, \\
x > y &\iff x_i \geq y_i, \; i = 1, \ldots, m, \; \wedge \; x \neq y, \\
x \gg y &\iff x_i > y_i, \; i = 1, \ldots, m.
\end{aligned}
$$

In the problem we examine, the status quo is an equilibrium in a regime where the only government policy instruments are the indirect taxes, the uniform lump-sum transfers, and the 100-percent tax on profits. This is a "tax equilibrium" in the tax-reform literature (Guesnerie [1977, 1995]). In the context of our problem, we can refer to this situation as a TEQC in which the quantity control on the externality is fixed at a level just equal to the unrationed Marshallian demands at the prevailing prices and lump-sum transfers. We further assume that the initial equilibrium is tight. With these qualifications, our status quo is a TEQC, as defined above, with $\hat{\bar{\nu}} = d^h_\nu(\hat{q}, \hat{\pi}, \hat{R})$, so that $\bar{d}^h(\hat{q}, \hat{\pi}, \hat{R}, \hat{\bar{\nu}}) = d^h(\hat{q}, \hat{\pi}, \hat{R})$, where d^h_ν and d^h are the unrationed Marshallian demand functions for the externality-causing and non-externality goods of consumer h, and where the (weak) inequalities in (2.22) hold as equalities:

$$\begin{aligned} \bar{d}^h(\hat{q}, \hat{\pi}, R, \hat{\bar{\nu}}) + \sum_i \bar{d}^i(\hat{q}, \hat{R}, \hat{\bar{\nu}}) &= \eta(\hat{p}, \hat{\rho}) \\ \bar{\nu} &= \eta_\nu(\hat{p}, \hat{\rho}). \end{aligned} \tag{2.23}$$

The system (2.23) contains $2n+4$ unknowns ($2(n+1)$ prices, the uniform lump-sum transfer, and the level of control on the externality) and only $n+1$ equations, implying, at first glance, $n+3$ degrees of freedom in choosing solutions. Since, however, the demand functions are homogeneous of degree zero in $\langle q, \pi, R\rangle$ and the supply functions are homogeneous of degree zero in $\langle p, \rho\rangle$, two normalizations must be adopted, reducing the number of degrees of freedom to $n+1$.

The $n+1$ degrees of freedom in solving (2.23) suggests the possibility of a neighborhood of tight TEQC around the status quo. In a similar context, Diewert [1978] has shown that, under certain regularity conditions, there does exist such a neighborhood.[12] Our interest in this paper, however, is in directions of change of government policy that are (not necessarily tight) equilibrium preserving and Pareto improving.

Definition: A *direction of change* in government policy—a *policy reform*—is a (vector-valued) derivative, denoted $\langle \dot{q}, \dot{\pi}, \dot{R}, \dot{\bar{\nu}}, \dot{p}, \dot{\rho}\rangle \in \mathbf{R}^{2n+4}$, of a differentiable, vector-valued function, $f : \mathbf{R} \to \mathbf{R}^{2n+4}$, with image $f(t) = \big\langle q(t), \pi(t), R(t), \nu(t), p(t), \rho(t)\big\rangle$.

12 See also Guesnerie [1995, p. 98].

In this construction, t can be interpreted as time. As the derivatives in the following definition are all evaluated at the initial equilibrium, we lighten the notation by eliminating the arguments in the images of demand and supply functions.

Definition: Starting from the initial tight TEQC, a policy reform, $\langle \dot{q}, \dot{\pi}, \dot{R}, \dot{\bar{\nu}}, \dot{p}, \dot{\rho} \rangle \in \mathbf{R}^{2n+4}$, is *equilibrium preserving* if

$$\nabla_q \Big[\bar{d}^h + \sum_i \bar{d}^i \Big] \; \dot{q} + \nabla_\pi \; \bar{d}^h \; \dot{\pi} + \nabla_R \Big[\bar{d}^h + \sum_i \bar{d}^i \Big] \; \dot{R} + \nabla_{\bar{\nu}} \Big[\bar{d}^h + \sum_i \bar{d}^i \Big] \; \dot{\bar{\nu}}$$

$$\leq \; \nabla_p \eta \; \dot{p} \; + \; \nabla_\rho \eta \; \dot{\rho} \tag{2.24}$$

and

$$\dot{\bar{\nu}} \; \leq \; \nabla_p \eta_\nu \; \dot{p} \; + \; \nabla_\rho \eta_\nu \; \dot{\rho}.$$

The policy reform $\langle \dot{q}, \dot{\pi}, \dot{R}, \dot{\bar{\nu}}, \dot{p}, \dot{\rho} \rangle$ is *tight* equilibrium preserving if the weak inequalities in (2.24) hold as equalities and *non-tight* equilibrium preserving if any inequality in (2.24) is a strict inequality.

Condition (2.24) can be expressed in matrix form as follows:[13]

$$\langle \dot{z}, \dot{z}_\nu \rangle := J_d \; \delta \leq J_s \; \langle \dot{p}, \dot{\rho} \rangle =: \langle \dot{y}, \dot{y}_\nu \rangle, \tag{2.25}$$

where $\langle \dot{z}, \dot{z}_\nu \rangle$ and $\langle \dot{y}, \dot{y}_\nu \rangle$ are the net changes in demand and supply, $\delta = \langle \dot{q}, \dot{\pi}, \dot{R}, \dot{\bar{\nu}} \rangle$ is the direction of change of the instruments that affect the demand side only, and

$$J_d := \begin{pmatrix} \nabla_q \big(\bar{d}^h + \sum_i \bar{d}^i \big) & \nabla_\pi \big(\bar{d}^h \big) & \nabla_R \big(\bar{d}^h + \sum_i \bar{d}^i \big) & \nabla_\nu \big(\bar{d}^h + \sum_i \bar{d}^i \big) \\ 0^{(n)} & 0 & 0 & 1 \end{pmatrix} \tag{2.26}$$

and

$$J_s := \begin{pmatrix} \nabla_p(\eta) & \nabla_\rho(\eta) \\ \nabla_p(\eta_\nu) & \nabla_\rho(\eta_\nu) \end{pmatrix} \tag{2.27}$$

are the Jacobians of the demand and supply functions.

The following assumption is fundamental to our differential comparative-static results:[14]

[13] Ordinarily, we treat vectors as abstract elements of a vector space; where matrix notation is required, we believe the reader can unambiguously infer from the context whether a vector is a column vector or a row vector.

[14] We retain Guesnerie's [1995, p. 93] name for this assumption, "local regularity 1," to draw attention to its salient role in his analysis of policy reform.

Assumption LR1: The $(n+1)\times(n+1)$ Jacobian of the aggregate supply function, J_s, is of rank n.

Under LR1, as Guesnerie [1977, 1995] has shown,[15] provided that the value of the vector of aggregate demand responses at producer prices, $\langle \hat{p}, \hat{\rho}\rangle \cdot \langle \dot{z}, \dot{z}_\nu\rangle$, is less than or equal to zero, there exists a direction of change in producer prices, $\langle \dot{p}, \dot{\rho}\rangle$, satisfying $\langle \hat{p}, \hat{\rho}\rangle \cdot \langle \dot{p}, \dot{\rho}\rangle = 0$, such that $\langle \dot{q}, \dot{\pi}, \dot{R}, \dot{\bar{\nu}}, \dot{p}, \dot{\rho}\rangle$ is equilibrium preserving. That is $\langle \dot{p}, \dot{\rho}\rangle$ results in an aggregate supply response that will at least meet the aggregate demand response to $\langle \dot{q}, \dot{\pi}, \dot{R}, \dot{\bar{\nu}}\rangle$. The fact that $\langle \dot{p}, \dot{\rho}\rangle$ satisfies $\langle \hat{p}, \hat{\rho}\rangle \cdot \langle \dot{p}, \dot{\rho}\rangle = 0$ is consistent with the normalization rule, $\|\langle p, \rho\rangle\|$ is a constant.

This result assures us that, to identify policy reforms that are equilibrium preserving, we need concentrate on directions of change only with respect to policy instruments that affect aggregate demand—namely, q, π, R, and $\bar{\nu}$—and that lead to demand changes satisfying

$$\langle \hat{p}, \hat{\rho}\rangle \cdot \langle \dot{z}, \dot{z}_\nu\rangle \leq 0. \tag{2.28}$$

Using the first identity in (2.25), along with (2.26), we can re-write this condition as

$$\langle \Phi_q, \Phi_\pi, \Phi_R, \Phi_{\bar{\nu}}\rangle \cdot \delta \leq 0, \tag{2.29}$$

where

$$\begin{aligned}
\Phi_q &:= \langle \hat{p}, \hat{\rho}\rangle \cdot \nabla_q\Big(\bar{d}^h + \sum_i \bar{d}^i\Big)\\
\Phi_\pi &:= \langle \hat{p}, \hat{\rho}\rangle \cdot \nabla_\pi\Big(\bar{d}^h\Big)\\
\Phi_R &:= \langle \hat{p}, \hat{\rho}\rangle \ \nabla_R\Big(\bar{d}^h + \sum_i \bar{d}^i\Big)\\
\Phi_{\bar{\nu}} &:= \hat{p}\ \nabla_\nu\Big(\bar{d}^h + \sum_i \bar{d}^i\Big) \ + \ \hat{\rho}.
\end{aligned} \tag{2.30}$$

In this condition, which we henceforth write as

$$\Phi \cdot \delta \leq 0, \tag{2.31}$$

[15] See also the useful discussions and interpretations by Weymark [1979] and Myles [1995].

where $\Phi = \langle \Phi_q, \Phi_\pi, \Phi_R, \Phi_{\bar{\nu}} \rangle$, Guesnerie [1977, 1995] interprets the elements of Φ as the marginal production costs of changes in the demand-side instruments, q, π, R, and $\bar{\nu}$. Thus, $\Phi \cdot \delta$ is the value at producer prices of the vector of demand responses to δ.

Starting at a TEQC, δ is tight equilibrium preserving if (2.28), and hence (2.31), holds as an equality and non-tight equilibrium preserving if it holds as an inequality. The intuition for this result is as follows. At the initial tight equilibrium, where aggregate profit is maximized on the technology set Y, the initial producer price vector $\langle \hat{p}, \hat{\rho} \rangle$ is the normal of a hyperplane supporting Y at the initial production vector $\langle y, y_\nu \rangle$ (see Figure 2). Thus, for a direction of change of consumption $\langle \dot{z}, \dot{z}_\nu \rangle$ and hence the required change in production $\langle \dot{y}, \dot{y}_\nu \rangle$ to be feasible, the angle formed by $\langle \hat{p}, \hat{p}_\nu \rangle$ and $\langle \dot{z}, \dot{z}_\nu \rangle$ must not be acute. In Figure 2, this angle is obtuse, indicating that (2.28) and (2.31) hold as strict inequalities, so that δ is non-tight equilibrium preserving. If the angle formed by $\langle \hat{p}, \hat{\rho} \rangle$ and $\langle \dot{z}, \dot{z}_\nu \rangle$ were a 90-degree angle—*i.e.*, if $\langle \dot{z}, \dot{z}_\nu \rangle$ were coincident with a portion of the supporting hyperplane—δ would be tight equilibrium preserving.

Denote the set of all equilibrium-preserving directions of change with respect to demand-side instruments by

$$Q := \left\{ \delta \in \mathbf{R}^{n+3} \mid \Phi \cdot \delta \leq 0 \right\}. \tag{2.32}$$

Although the government is one of the agents in our economy, the government's budget surplus position has not been explicitly stated in our description of the status quo, a tight TEQC. We are justified in ignoring the government's budget constraint because of a standard implication of Walras' Law: if all $n+1$ markets clear (as is the case in the TEQC) and all agents but one in the economy satisfy their budget constraints as an equality (as do our locally and globally non-satiated consumers), then so does the last agent (in this case, the government). Proof of a more general statement can be found in Guesnerie [1977, 1995]: if, in the TEQC, some commodity is in excess supply, then the government surplus is positive. Thus, $\Phi_A \cdot \delta_A < 0$ implies that, starting at a tight TEQC, the reform δ_A (differentially) generates a government surplus.

2.3. Pareto-Improving Policy Reforms.

The gradients of the indirect utility functions of the generator and the victims evaluated at the initial tight TEQC have been defined earlier as Γ^h and $\Gamma^i,\ i = 1, \ldots, m$ (see equations (2.14) and (2.21)). Thus, we have the following definition:

Definition: A direction of change with respect to $q,\ \pi,\ R$, and $\bar{\nu}$ ($\delta = \langle \dot{q}, \dot{\pi}, \dot{R}, \dot{\bar{\nu}} \rangle$) is *Pareto improving* if

$$\begin{aligned} &\Gamma^h \cdot \delta > 0 \\ &\Gamma^i \cdot \delta > 0,\ i = 1, \ldots, m. \end{aligned} \tag{2.33}$$

That is, δ is Pareto improving if it increases the utilities of all consumers.[16]

3. Results: Equilibrium-Preserving and Pareto-Improving Policy Reforms.

Let $I = \langle 1, \ldots n, n+1, n+2, \nu \rangle$ denote the coordinates of $\mathbf{R}^{n+3}$ assigned to the set of potential policy instruments (the $n+1$ consumer prices, the income transfer, and the rationed externality commodity, respectively). We will consider a restriction of the set of policy instruments to an available subset. Let A be that subset and denote by $\mathbf{R}^A$ the respective projection of $\mathbf{R}^{n+3}$ onto the space spanned by the coordinates identified in A. In general, this space would be an $|A|$-dimensional Euclidean space. Let 0_A $\left(= 0^{(|A|)} \right)$ denote the origin of $\mathbf{R}^A$ and, in an obvious notation, let δ_A, Φ_A, Γ^h_A, Γ^i_A, *etc.*, represent vectors in this space. Similarly, a particular (unspecified) component of, *e.g.*, the vectors Γ^i_A and δ_A will be denoted Γ^i_a and δ_a, respectively. Thus, for example, if the only available instruments are the income transfer and the externality control, $A = \{n+2, \nu\}$, $|A| = 2$, $\delta_A = \left\langle \dot{R}, \dot{\bar{\nu}} \right\rangle \in\ \mathbf{R}^A$, and the space $\mathbf{R}^A$ is then obtained as the projection of $\mathbf{R}^{n+3}$ onto the space spanned by the coordinates reserved for income transfers $(n+2)$ and the rationed externality commodity (ν). This is a two-dimensional Euclidean space.

[16] Weymark [1978, p. 344] makes a compelling case for this definition of Pareto improving reforms: "… a direction of price changes which is strictly Pareto-improving in the differentiable case is also strictly Pareto-improving for small finite changes … The differentiable definition can thus be viewed as an approximation to the finite version." The same cannot be said for non-strict Pareto-improving changes.

We will make extensive use of the following theorem:[17]

Theorem of the Alternative (Motzkin [1936]):[18] *Let A, C, and D be $n_1 \times m$, $n_2 \times m$, and $n_3 \times m$ matrices, where A is non-vacuous. Then there exists a vector $x \in \mathbf{R}^m$ satisfying*

$$Ax \gg 0^{(n_1)},\ Cx \geq 0^{(n_2)},\ \text{and } Dx = 0^{(n_3)} \qquad (*)$$

if and only if there do not exist vectors, $y^1 \in \mathbf{R}^{n_1}$, $y^2 \in \mathbf{R}^{n_2}$, $y^3 \in \mathbf{R}^{n_3}$, satisfying

$$A^T y^1 + C^T y^2 + D^T y^3 = 0^{(m)},\ y^1 > 0^{(n_1)},\ y^2 \geq 0^{(n_2)}, \qquad (**)$$

where A^T, C^T, D^T are the transposes of A, C, and D.

That is, there exists a solution to $(*)$ if and only if there exists no solution to $(**)$.

Let $\alpha = \langle \alpha_h, \alpha_1, \ldots, \alpha_m \rangle \in \mathbf{R}^{m+1}$ and define the polyhedral cone generated by the vectors Γ_A^h and Γ_A^i, $i = 1, \ldots, m$:

$$\Gamma_A := \left\{ \gamma \in \mathbf{R}^A \;\middle|\; \gamma = \alpha_h \Gamma_A^h + \sum_{i=1}^{m} \alpha_i \Gamma_A^i \ \wedge\ \alpha \geq 0^{(m+1)} \right\}. \tag{3.1}$$

The following results, attributable to Weymark [1979], will also be useful:

Lemma 1: There exist Pareto-improving directions of change relative to A if and only if $\Gamma_A \cap (-\Gamma_A) = \emptyset$; *i.e.*, Γ_A is pointed.

Lemma 2: If $\Gamma_A^h \neq 0_A$ and $\Gamma_A^i \neq 0_A$, $i = 1, \ldots m$, Γ_A is pointed if and only if there exists no vector $\alpha > 0^{(m+1)}$ such that

$$\alpha_h \Gamma_A^h + \sum_i \alpha_i \Gamma_A^i = 0_A. \tag{3.2}$$

A direction of change, δ_A, is Pareto improving and equilibrium preserving relative to A if and only if

$$\Phi_A \cdot \delta_A \leq 0, \tag{3.3}$$

$$\Gamma_A^h \cdot \delta_A > 0, \tag{3.4}$$

[17] In the statement of this theorem, it is essential that we be explicit about matrix transposes, which we signify by the superscript T.

[18] See also Mangasarian [1969], Myles [1995], Blackorby [1999], and Blackorby and Brett [2000].

and

$$\Gamma^i_A \cdot \delta_A > 0, \quad i = 1, \ldots, m. \tag{3.5}$$

We turn now to our results. To this end, let $1^\nu_A = \langle 0, \ldots, 0, 1\rangle \in \mathbf{R}^A$ and define the ray $\Gamma^\nu_A = \{\gamma \in \mathbf{R}^A \mid \gamma = \kappa 1^\nu_A,\ \kappa > 0\}$.

Theorem 1: *Suppose that $\bar{\nu}$ is an available instrument and consider a tight equilibrium in which $\Phi_A \neq 0_A$ and there exists an $a \in A\backslash\{\nu\}$ such that $\Gamma^h_a \neq 0$.*

(i) There exists no Pareto-improving and equilibrium-preserving direction of change if and only if $\Phi_A \in \Gamma_A$.

(ii) There exist Pareto-improving and equilibrium-preserving directions of change and all such changes require $\dot{\bar{\nu}} > 0$ if and only if $\Phi_A \in \big(\Gamma_A - \Gamma^\nu_A\big)\backslash\big(\Gamma_A \cup (-\Gamma^\nu_A)\big) =: \xi_A$.

(iii) There exist Pareto-improving and equilibrium-preserving directions of change and all such changes require $\dot{\bar{\nu}} < 0$ if and only if

(a) $\Phi_A \in \big(\Gamma_A + \Gamma^\nu_A\big)\backslash\big(\Gamma_A \cup \Gamma^\nu_A\big) =: \Psi_A$ when $-1^\nu_A \notin \Gamma_A$

and

(b) $\Phi_A \in \Gamma^C_A$ (the complement of Γ_A) when $-1^\nu_A \in \Gamma_A$.

(iv) There exist Pareto-improving and equilibrium-preserving directions of change such that not all such directions of change require $\dot{\bar{\nu}} > 0$ and not all such changes require $\dot{\bar{\nu}} < 0$ if and only if $-1^\nu_A \notin \Gamma_A$ and $\Phi_A \in \big(\Gamma_A \cup \xi_A \cup \Psi_A\big)^C =: \chi_A$.

Proof: We first show that Pareto-improving directions of change exist when the set of instruments A includes ν and another instrument a where $\Gamma^h_a \neq 0$. Suppose not: $\mathring{K}_A = \emptyset$. By Lemma 2, the latter holds if and only if there exists an $\alpha > 0^{(m+1)}$ satisfying

$$\alpha_h \Gamma^h_A \ + \ \sum_i \alpha_i\ \Gamma^i_A = 0_A. \tag{3.6}$$

Two elements of this vector equality are

$$\alpha_h \Gamma_a^h + \sum_i \alpha_i \ \Gamma_a^i = 0$$
$$\sum_i \alpha_i \bar{\lambda}^i \pi^i = 0 \quad (\text{since } \Gamma_\nu^h = 0). \tag{3.7}$$

As $\bar{\lambda}^i > 0$, $\pi^i < 0$, and $\alpha_i \geq 0$ for all i, the second of these equalities implies that $\alpha_i = 0$ for all i, which, from the first equation, along with $\Gamma_a^h \neq 0$, implies that $\alpha_h = 0$, a contradiction. Hence $\overset{\circ}{K}_A \neq \emptyset$.

Next we prove (i). There exists no equilibrium-preserving and Pareto-improving direction of change if and only if there is no δ_A satisfying

$$-\Phi_A \cdot \delta_A \geq 0 \quad \wedge \quad \Gamma_A^h \cdot \delta_A > 0 \quad \wedge \quad \Gamma_A^i \cdot \delta_A > 0, \ i = 1, \ldots, m. \tag{3.8}$$

By Motzkin's Theorem, there exists no solution to (3.8) if and only if there exist a scalar β and a vector α satisfying

$$-\beta \ \Phi_A + \alpha_h \ \Gamma_A^h + \sum_i \alpha_i \ \Gamma_A^i = 0_A, \quad \beta \geq 0, \quad \alpha > \ 0^{(m+1)}. \tag{3.9}$$

If $\beta = 0$, $\alpha_h \ \Gamma_A^h + \sum_i \alpha_i \Gamma_A^i = 0_A$, and by Lemma 1 there exists no Pareto-improving direction of change relative to A, a contradiction. Therefore, $\beta > 0$, and

$$\Phi_A = \frac{\alpha_h}{\beta} \Gamma_A^h + \sum_i \frac{\alpha_i}{\beta} \Gamma_A^i \in \Gamma_A. \tag{3.10}$$

We use part (i) to prove part (ii). There exist Pareto-improving and equilibrium-preserving directions of change and all such changes require $\dot{\nu} > 0$ if and only if $\Phi_A \notin \Gamma_A$ and there exists no vector δ_A satisfying

$$\Gamma_A^h \cdot \delta_A > 0 \quad \wedge \quad \Gamma_A^i \cdot \delta_A > 0, \ i = 1, \ldots, m \quad \wedge \quad -1_A^\nu \cdot \delta_A \geq 0 \quad \wedge \quad -\Phi_A \cdot \delta_A \geq 0. \tag{3.11}$$

By Motzkin's Theorem, the nonexistence of a solution to the displayed inequalities is equivalent to a solution of the equality,

$$\alpha_h \Gamma_A^h + \sum_i \alpha_i \Gamma_A^i - \alpha_0 1_A^\nu - \beta \Phi_A = 0_A, \quad \alpha > 0^{(m+1)}, \quad \alpha_0 \geq 0, \quad \beta \geq 0. \tag{3.12}$$

Suppose that $\beta = 0$. Then

$$\alpha_h \Gamma_A^h + \sum_i \alpha_i \Gamma_A^i - \alpha_0 1_A^\nu = 0_A, \quad \alpha > 0^{(m+1)}, \quad \alpha_0 \geq 0, \tag{3.13}$$

and, in particular,

$$\sum_i \alpha_i \pi^i \bar{\lambda}^i - \alpha_0 = 0 \tag{3.14}$$

and

$$\alpha_h \Gamma_a^h + \sum_i \alpha_i \Gamma_a^i = 0. \tag{3.15}$$

As $\pi^i \lambda^i < 0$ for all i, $\alpha_i \geq 0$ for all i, and $\alpha_0 \geq 0$, (3.14) implies that $\alpha_i = 0$ for all i. This, in turn, along with (3.15), implies that $\alpha_h = 0$ and, hence, $\alpha = 0^{(m+1)}$, a contradiction. Therefore, $\beta > 0$. It follows from (3.12) that $\alpha_0 > 0$, for otherwise we would have $\Phi_A \in \Gamma_A$, a contradiction. Hence, in an obvious notation,

$$\Phi_A = \hat{\alpha}_h \Gamma_A^h + \sum_i \hat{\alpha}_i \Gamma_A^i + \hat{\alpha}_0(-1_A^\nu), \quad \langle \hat{\alpha}_h, \hat{\alpha}_1, \ldots, \hat{\alpha}_m \rangle > 0^{(m+1)}, \quad \hat{\alpha}_0 > 0, \tag{3.16}$$

which establishes (ii).

Next, we prove part (iii), again exploiting (i). There exist Pareto-improving and equilibrium-preserving directions of change and all such changes require $\dot{\bar{\nu}} < 0$ if and only if $\Phi_A \notin \Gamma_A$ and there exists no vector δ_A satisfying

$$\Gamma_A^h \cdot \delta_A > 0 \quad \wedge \quad \Gamma_A^i \cdot \delta_A > 0,\ i = 1, \ldots, m \quad \wedge \quad 1_A^\nu \cdot \delta_A \geq 0 \quad \wedge \quad -\Phi_A \cdot \delta_A \geq 0. \tag{3.17}$$

By Motzkin's Theorem, the nonexistence of a solution to the displayed inequalities are equivalent to a solution of the equality,

$$\alpha_h \Gamma_A^h + \sum_i \alpha_i \Gamma_A^i + \alpha_0 1_A^\nu - \beta \Phi_A = 0, \quad \alpha > 0^{(m+1)}, \quad \alpha_0 \geq 0, \quad \beta \geq 0. \tag{3.18}$$

There are four possible solutions to this equation (with $\alpha > 0^{(m+1)}$ in each case): (a) $\alpha_0 = \beta = 0$, (b) $\alpha_0 = 0$ and $\beta > 0$, (c) $\alpha_0 > 0$ and $\beta > 0$, and (d) $\alpha_0 > 0$ and $\beta = 0$. In case (a),

$$\alpha_h \Gamma_A^h + \sum_i \alpha_i \Gamma_A^i = 0, \quad \alpha > 0^{(m+1)}; \tag{3.19}$$

violating the fact that Γ is a pointed cone (Lemmas 1 and 2). Case (b) is also ruled out, since it implies that $\Phi_A \in \Gamma_A$. Case (c) yields the required result in a manner similar to (ii), a special case of the restriction in (iii). Finally, case (d) yields, in an obvious notation,

$$-1_A^{\nu} = \hat{\alpha}_h \Gamma_A^h + \sum_i \hat{\alpha}_i \Gamma_A^i \tag{3.20}$$

or $-1_A^{\nu} \in \Gamma_A$ and no restriction on Φ_A other than $\Phi_A \notin \Gamma_A$. This establishes part (iii).

Part (iv) follows in an obvious way. ■

Note that the collection of sets identified in the Theorem 1 ($\Gamma_A,\ \xi_A,\ \Psi_A$, and χ_A) is not necessarily a partition of $\mathbf{R}^A$, since, if $-1_A^{\nu} \in \Gamma_A$, cases (ii), (iiia), and (iv) are vacuous. The theorem can be re-stated in terms of partitions of $\mathbf{R}^A$ as follows:

Corollary: *Suppose that $\bar{\nu}$ is an available instrument and consider a tight equilibrium in which $\Phi_A \neq 0_A$ and there exists an $a \in A\backslash\{\nu\}$ such that $\Gamma_a^h \neq 0$.*

(1) If $-1_A^{\nu} \in \Gamma_A$, $\mathbf{R}^A$ can be partitioned into Γ_A and Γ_A^C, such that

(a) $\Phi_A \in \Gamma_A \iff \overset{\circ}{K}_A \cap Q_A = \emptyset$

and

(b) $\Phi_A \in \Gamma_A^C \iff \overset{\circ}{K}_A \cap Q_A \neq \emptyset \quad \wedge \quad \dot{\bar{\nu}} < 0 \ \ \forall\ \delta_A \in \overset{\circ}{K}_A \cap Q_A.$

(2) If $-1_A^{\nu} \notin \Gamma_A$, $\mathbf{R}^A$ can be partitioned into $\Gamma_A,\ \Psi_A,\ \xi_A$, and χ_A such that

(a) $\Phi_A \in \Gamma_A \iff \overset{\circ}{K}_A \cap Q_A = \emptyset,$

(b) $\Phi_A \in \xi_A \iff \overset{\circ}{K}_A \cap Q_A \neq \emptyset \quad \wedge \quad \dot{\bar{\nu}} > 0\ \forall\ \delta_A \in \overset{\circ}{K}_A \cap Q_A,$

(c) $\Phi_A \in \Psi_A \iff \overset{\circ}{K}_A \cap Q_A \neq \emptyset \quad \wedge \quad \dot{\bar{\nu}} < 0\ \forall\ \delta_A \in \overset{\circ}{K}_A \cap Q_A,$

and

(d) $\Phi_A \in \chi_A \iff \overset{\circ}{K}_A \cap Q_A \neq \emptyset \quad \wedge \quad \exists\ \delta_A \in \overset{\circ}{K}_A \cap Q_A$ *such that* $\dot{\bar{\nu}} \leq 0$

$\wedge \quad \exists\ \delta_A \in \overset{\circ}{K}_A \cap Q_A$ *such that* $\dot{\bar{\nu}} \geq 0.$

Theorem 1 and its Corollary are illustrated in Figure 3.[19] In each case, there is only one victim, i, and two available instruments, the externality quantity control and one other ($A = \{a, \nu\}$). Thus, $\Gamma^h_A = \langle \Gamma^h_a,\ 0 \rangle$ and $\Gamma^i_A = \langle \Gamma^i_a,\ \bar{\lambda}^i\ \pi^i \rangle$ (refer back to (2.14) and (2.21)).[20] For the purpose of illustration, we also assume that Γ^h_a is positive, so that Γ^h_A points in the positive direction of the horizontal axis in each case. Since $\bar{\lambda}^i\ \pi^i < 0$, the direction of Γ^i_A must be in the lower (open) half-space in each case; for the purpose of illustration, we assume $\Gamma^i_a > 0$ in each case (*e.g.*, that a is the lump-sum transfer).

The essence of Theorem 1 and its Corollary and the asymmetry between $\dot{\bar{\nu}} > 0$ and $\dot{\bar{\nu}} < 0$ are imbedded in the relationship between the the location of -1^ν_A and the sign of $\dot{\bar{\nu}}$ in the Pareto-improving directions of change (independently of the requirement of equilibrium preservation). This relationship is evoked in the following theorem:

Theorem 2: *Assume that $\bar{\nu}$ is an available instrument ($\nu \in A$) and that there exists another instrument $a \in A \backslash \{\nu\}$ such that $\Gamma^h_a \neq 0$.*[21]

(i) There exist Pareto-improving directions of change such that $\dot{\bar{\nu}} > 0$ if and only if $-1^\nu_A \notin \Gamma_A$.

(ii) There exist Pareto-improving directions of change such that $\dot{\bar{\nu}} < 0$. Moreover all such changes require $\dot{\bar{\nu}} < 0$ if and only if $-1^\nu_A \in \Gamma_A$.

Proof: (i) There exists no Pareto-improving direction of change such that $\dot{\bar{\nu}} > 0$ if and only if there exists no δ_A solving

$$\Gamma^h_A \cdot \delta_A > 0 \quad \wedge \quad \Gamma^i_A \cdot \delta_A > 0\ ,\ i = 1, \ldots, m \quad \wedge \quad 1^\nu_A \cdot \delta_A > 0. \tag{3.21}$$

[19] These diagrams are stylized. To be completely accurate, the sets of directions of change in the policy instruments might have to be restricted to a subset of $\mathbf{R}^2$ induced by a particular normalization for $\langle q, \pi, R \rangle$.

[20] The illustrations are also valid, with small changes of interpretation, if A is a superset of $\{a, \nu\}$ and changes in all instruments in the complement of $\{a, \nu\}$ relative to A are set equal to zero.

[21] Note that the maintained condition, $\Gamma^h_a \neq 0$ for some $a \in A \backslash \{\nu\}$, is necessary for the existence of *any* Pareto-improving directions of change, since, otherwise, as $\Gamma^h_\nu = 0$, we have $\Gamma^h_A = 0_A$ and $\Gamma^h_A \cdot \delta_A = 0$ for all $\delta_A \in \mathbf{R}^A$.

From Motzkin's Theorem, this holds if and only if there exists an $\langle \alpha, \alpha_0 \rangle \in \mathbf{R}^{m+2}$ satisfying

$$\alpha_h \Gamma_A^h + \sum_i \alpha_i \Gamma_A^i + \alpha_0 1_A^\nu = 0_A \quad \wedge \quad \langle \alpha, \alpha_0 \rangle > 0^{(m+2)}, \tag{3.22}$$

which is equivalent to

$$\alpha_h \Gamma_t^h + \sum_i \alpha_i \Gamma_t^i = 0 \quad \forall\, t \in A \backslash \{\nu\}, \tag{3.23}$$

$$\sum_i \alpha_i \bar{\lambda}^i \pi^i + \alpha_0 = 0, \tag{3.24}$$

and

$$\langle \alpha, \alpha_0 \rangle > 0^{(m+2)}. \tag{3.25}$$

Suppose $\alpha_0 = 0$. Then (3.24) and (3.25) imply that $\alpha_i = 0$ for all i, which in turn implies, from the first equality (using $\Gamma_a^h \neq 0$ for some $a \in A\backslash\{\nu\}$), that $\alpha_h = 0$, a contradiction of the condition that $\langle \alpha, \alpha_0 \rangle > 0^{(m+2)}$. Hence, $\alpha_0 \neq 0$. Dividing through (3.22), we obtain

$$-1_A^\nu = \frac{\alpha_h}{\alpha_0}\, \Gamma_A^h + \sum_i \frac{\alpha_i}{\alpha_0}\, \Gamma_A^i, \tag{3.26}$$

or $-1_A^\nu \in \Gamma_A$. Hence, there exists a Pareto-improving direction of change if and only if $-1_A^\nu \notin \Gamma_A$.

(ii) There exists no Pareto-improving direction of change such that $\dot{\bar{\nu}} < 0$ if and only if there exists no vector $\delta_A \in \mathbf{R}^A$ satisfying

$$\Gamma_A^h \cdot \delta_A > 0 \quad \wedge \quad \Gamma_A^i \cdot \delta_A > 0,\ i = 1, \ldots, m \quad \wedge \quad -1_A^\nu \cdot \delta_A > 0. \tag{3.27}$$

By Motzkin's Theorem, this is equivalent to the existence of a solution to

$$\alpha_h \Gamma_A^h + \sum_i \alpha_i \Gamma_A^i + \alpha_0(-1_A^\nu) = 0_A, \quad \langle \alpha, \alpha_0 \rangle\ > 0^{(m+2)}. \tag{3.28}$$

The last equation in this system is

$$\sum_i \bar{\lambda}^i \pi^i \alpha_i - \alpha_0 = 0, \tag{3.29}$$

which, with $\bar{\lambda}^i \pi^i < 0$ for all i, implies that $\alpha_0 = \alpha_i = 0$ for all i. Returning to (3.28), and remembering that $\Gamma_a^h \neq 0$ for some $a \in A\backslash\{\nu\}$, we find that $\alpha_h = 0$, contradicting

$\langle\alpha, \alpha_0\rangle > 0^{(m+2)}$. This establishes the first sentence in (ii). To prove the second part of (ii), suppose that there does not exist a solution to

$$\Gamma_A^h \cdot \delta_A > 0 \quad \wedge \quad \Gamma_A^i \cdot \delta_A > 0,\ i = 1, \ldots, m \quad \wedge \quad 1_A^\nu \cdot \delta_A \geq 0, \tag{3.30}$$

which is equivalent to the existence of a solution to

$$\alpha_h \Gamma_A^h + \sum_i \alpha_i \Gamma_A^i + \alpha_0 1_A^\nu = 0_A \quad \wedge \quad \alpha > 0^{(m+1)} \quad \wedge \quad \alpha_0 \geq 0. \tag{3.31}$$

The argument in the proof of (i) that $\alpha_0 > 0$ remains valid under the restriction in (3.31), as does the remainder of the proof that $-1_A^\nu \in \Gamma_A$.[22] ■

Theorem 2 is also illustrated in Figure 3. Panels III–V illustrate the case where $-1_A^\nu \notin \Gamma_A$ and the counterintuitive case ($\dot{\bar{\nu}} > 0$) is a possible outcome, and Panel II illustrates the case where $-1_A^\nu \in \Gamma_A$ and all Pareto-improving changes require $\dot{\bar{\nu}} < 0$.

Our last theorem identifies two sufficient conditions for the existence of Pareto-improving counterintuitive policy prescription for quantity controls on an externality:

Theorem 3: *There exist Pareto-improving directions of change such that $\dot{\bar{\nu}} > 0$ if (i) $\bar{\nu}$ and R are available instruments or (ii) $\bar{\nu}$ and q_k are available instruments and, at the initial equilibrium, the demands for the k^{th} commodity satisfy*

$$\bar{d}_k^h(q, \pi, R, \bar{\nu}) \cdot \bar{d}_k^j(q, R, \bar{\nu}) > 0 \tag{3.32}$$

for all j and

$$\bar{d}_k^i(q, R, \bar{\nu}) \cdot \bar{d}_k^j(q, R, \bar{\nu}) > 0 \tag{3.33}$$

for all i, j.

Proof: From Part (i) of Theorem 1, it suffices to show, in each case, that $-1_A^\nu \notin \Gamma_A^C$. Suppose that $n+2 \in A$ and $\nu \in A$ but $-1_A^\nu \in \Gamma_A$. From the definition of Γ_A, we know that $-1_A^\nu \in \Gamma_A$ only if

$$\sum_i \alpha_i \bar{\lambda}^i \pi^i = -1, \tag{3.34}$$

[22] In other words, the structure of the model is such that, if Pareto-improving directions of change require $\dot{\bar{\nu}} \leq 0$, they require $\dot{\bar{\nu}} < 0$, as is evident from the openness of the set $\mathring{K}_A$.

and

$$\alpha_h \bar{\lambda}^h + \sum_i \alpha_i \bar{\lambda}^i = 0, \tag{3.35}$$

for some $\alpha \geq 0^{(m+1)}$. Equation (3.35), combined with the positivity of marginal utilities of income and $\alpha \geq 0^{(m+1)}$, implies that $\alpha_h = 0$ and $\alpha_i = 0$ for all i, which, given that $\pi^i < 0$ for all i, contradicts equation (3.34).

Suppose, contrary to case (ii) that there exists a vector $\alpha > 0^{(m+1)}$ such that $-1^{\nu}_A = \alpha_h \Gamma^h_A + \sum_i \alpha_i \Gamma^i_A$, which implies that

$$\sum_i \alpha_i \bar{\lambda}^i \pi^i = -1 \tag{3.36}$$

and

$$\alpha_h \bar{\lambda}^h \bar{d}^h_k(q, \pi, R, \bar{\nu}) + \sum_i \alpha_i \bar{\lambda}^h \bar{d}^i_k(q, R, \bar{\nu}) = 0. \tag{3.37}$$

But, along with the maintained condition of part (ii), (3.37) implies that $\alpha_h = \alpha_i = 0$ for all i, contradicting (3.36). ■

4. Interpretations of the Results.

In this section, we provide some intuition for our results. Specifically we wish to understand why the counterintuitive policy (requiring the regulator to mandate an increase in the level of the negative externality) can become a possibility in the context of our second-best model (one where the policy maker cannot implement personalized lump-sum transfers) while it will never be Pareto improving and equilibrium preserving at a competitive equilibrium of a first-best world with an externality.

Our approach is to first establish and exploit a link between Guesnerie's general approach to designing policy reforms and the standard Kuhn-Tucker technique for identifying the manifold of feasible Pareto-optimal states, given the policy instruments available to the government. The link so established, along with an application of the envelope theorem, provides an economic interpretation of the conditions identified by Theorem 1 and allows us to distinguish between Guesnerie's (*strong*) Pareto-improving and equilibrium-preserving reforms and *weak* Pareto-improving and equilibrium-preserving reforms at a status quo.

Subsection 4.1 focuses on Theorem 1—in particular, on the intuition underlying the cases where Pareto-improving and equilibrium-preserving directions of change entail an increase in the negative externality, starting at a status quo with no quantity control. For good measure, we also provide a proof in Section 4.2, using our methods, of the well-known result that if first-best instruments are available to the policy makers then starting at a competitive equilibrium with no externality control, all Pareto-improving and equilibrium-preserving policy innovations require a decrease in the negative externality. The contrast between this case and the strongly counterintuitive case identified by Theorem 1 helps to explain the role of the wedge in generating the latter. Finally, in Section 4.3, we look briefly at temporary production inefficiencies along a Pareto-improving and equilibrium-preserving path with externalities.

For the sake of notational simplicity, we assume throughout this discussion that all $n+3$ demand-side instruments (as well as the producer prices) are available to the policymaker.

4.1. Explication of Theorem 1.

We associate the following Kuhn-Tucker optimization exercise with a second-best optimum:

$$\begin{aligned} U^h(u) := \max_{q,\pi,R,\bar{\nu},p,\rho} & \; V^h(q,\pi,R,\bar{\nu}) \quad \text{s.t.} \\ & V^i(q,R,\bar{\nu}) \geq u^i, \; i=1,\ldots,m, \\ & \bar{d}^h(q,\pi,R,\bar{\nu}) + \sum_i \bar{d}^i(q,R,\bar{\nu}) \leq \eta(p,\rho), \\ & \bar{\nu} \leq \eta_\nu(p,\rho), \end{aligned} \tag{KT1}$$

where u is the utility profile $\langle u^1,\ldots,u^m\rangle$ at the status quo. At a local, interior maximum, the following first-order conditions (among others) are satisfied (using the notation of Sections 2 and 3 and again suppressing the arguments of function images to ease the notation):

$$\Gamma^h - \sum_i \gamma_i \Gamma^i - \psi \cdot \left(\nabla \bar{d}^h + \sum_i \nabla \bar{d}^i \right) = 0^{(n+3)} \tag{4.1}$$

and

$$\psi \cdot \nabla \eta = 0^{(n+1)}, \tag{4.2}$$

where $\gamma = \langle \gamma_1, \ldots, \gamma_m \rangle$ is the vector of multipliers on the utility constraints in (KT1) and $\psi = \langle \psi_1, \ldots, \psi_{n+1} \rangle$ is the vector of multipliers on the resource constraints in (KT1). The latter, of course, are the shadow prices of commodities (measured in units of utility of consumer h). Invoking LR1 (that $\nabla \eta$ has rank n), we find that

$$\psi = \mu \langle p, \rho \rangle, \quad \mu > 0, \tag{4.3}$$

demonstrating that production efficiency holds at the second-best optimum.

The first two terms in each of the equalities in (4.1) represent the marginal benefits of a change in the respective instrument, measured in units of utility of consumer h (γ_i is the trade-off between utilities of consumer h and consumer i, measured along the second-best utility possiblility frontier). The last term of each equation is the marginal production cost of changing the respective instrument, again measured in units of consumer h's utility ($\mu = \psi_l / p_l$ converts the marginal production cost of changing any instrument into units of consumer h's utility). Thus, (4.1) simply says that the *net* benefit of changing any instrument is zero. Finally, since $\mu > 0$, this condition is equivalent to condition (i) in Theorem 1, the necessary and sufficient condition for the non-existence of Pareto-improving and equilibrium-preserving directions of change at a status quo (see equation (3.9)).[23]

Consider now the case where the status quo is not a local second best—that is, where there exist Pareto-improving and equilibrium-preserving directions of change. In this case, some of the $n+3$ conditions in (4.1) fail to hold. In terms of Motzkin's Theorem

[23] If $\alpha_h = 0$ in (3.9), renormalize (KT1) by maximizing the indirect utility of consumer i for whom $\alpha_i > 0$.

(equation (3.9)), every pair, $\alpha := \langle \alpha_h, \alpha_1, \ldots, \alpha_m \rangle > 0^{(m+1)}$ and $\beta \geq 0$ violates at least one of the $n+3$ conditions,

$$\alpha_h \Gamma^h + \sum_i \alpha_i \Gamma^i - \beta \Phi = 0^{(n+3)}. \tag{4.4}$$

Thus, for an appropriately chosen $\alpha > 0^{(m+1)}$ and $\beta \geq 0$, we partition the set of coordinates of $\mathbf{R}^{n+3}$ into B and C such that, at the status quo (and in an obvious notation analogous to that introduced to identify available instruments at the beginning of Section 3),

$$\alpha_h \Gamma^h_b + \sum_i \alpha_i \Gamma^i_b - \beta \Phi_b = 0 \quad \forall\, b \in B. \tag{4.5}$$

and

$$\alpha_h \Gamma^h_c + \sum_i \alpha_i \Gamma^i_c - \beta \Phi_c \neq 0 \quad \forall\, c \in C. \tag{4.6}$$

By Motzkin's Theorem, (4.5) is equivalent to the non-existence of directions of change with respect to instruments corresponding to the elements of B, namely δ_B, such that $\Gamma^h_B \cdot \delta_B > 0$, $\Gamma^i_B \cdot \delta_B > 0$ for all i, and $\Phi_B \cdot \delta_B \leq 0$; *i.e.*, the status quo is a local second best with respect to the instruments in B.

Now let $|B| = t$ and denote the vector of instrument values corresponding to the elements of B by $\tilde{b} = \langle \tilde{b}_1, \ldots, \tilde{b}_t \rangle$. Similarly, $\tilde{c} = \langle \tilde{c}_1, \ldots, \tilde{c}_{n+3-t} \rangle$ is the vector of instrument values corresponding to the elements of C. We now show that the status quo satisfying (4.5) and (4.6) is a solution to the following Kuhn-Tucker problem:

$$\begin{aligned} \tilde{U}^h\Big(u, \tilde{c}\Big) := \max_{\tilde{b},p,\rho} \, & V^h(\tilde{b}, \tilde{c}) \quad \text{s.t.} \\ & V^i(\tilde{b}, \tilde{c}) \geq u^i, \; i = 1, \ldots, m, \\ & \bar{d}^h(\tilde{b}, \tilde{c}) + \sum_i \bar{d}^i(\tilde{b}, \tilde{c}) \leq \eta(p, \rho), \\ & \bar{\nu} \leq \eta_\nu(p, \rho). \end{aligned} \tag{$KT2$}$$

The first-order conditions for a local interior optimum include (4.2), but (4.1) is replaced by

$$\Gamma^h_B + \sum_i \gamma_i \Gamma^i_B - \mu \Phi_B = 0_B. \tag{4.7}$$

As $\mu > 0$, a comparison of conditions (4.5) and (4.7) indicates that the status quo is a solution to $KT2$ above. The conditions in (4.5) thus have the interpretation that, at the status quo, the marginal net benefit of changing any instrument in B is zero. On the other hand, the marginal net benefit of changing any instrument in C,

$$\Gamma^h_{\tilde{c}_r} + \sum_i \gamma_i \Gamma^i_{\tilde{c}_r} - \mu \Phi_{\tilde{c}_r}, \tag{4.8}$$

need not be zero.

Theorem 1 provides conditions where, at the status quo, *strict* Pareto-improving and equilibrium-preserving directions of change exist. The Kuhn-Tucker approach, on the other hand, indicates the existence of *weak* Pareto-improving and equilibrium-preserving reforms at the status quo. Let $\langle \overset{*}{b}(u,\tilde{c}),\ \overset{*}{p}(u,\tilde{c}),\ \overset{*}{\rho}(u,\tilde{c})\rangle$ be the vector of optimal values of $\tilde{b}$, p, and ρ corresponding to $KT2$. A direction of change in instruments in C, δ_c, implies a direction of change of the instruments in B: $\langle \delta_B, \dot{p}, \dot{\rho}\rangle = \langle \nabla_{\tilde{c}} \overset{*}{b}(u,\tilde{c}),\ \nabla_{\tilde{c}} \overset{*}{p}(u,\tilde{c}),\ \nabla_{\tilde{c}} \overset{*}{\rho}(u,\tilde{c})\rangle \cdot \delta_C$. From the envelope theorem it follows that, starting at the status quo (a solution to $KT2$), implementing a reform, $\langle \delta_B, \dot{p}, \dot{\rho}, \delta_C\rangle$, leads to a new state where the welfare of h is improved, the welfare of the victims is unchanged, and the resource constraints continue to hold if and only if

$$\left[\Gamma^h_C + \sum_i \gamma_i \Gamma^i_C - \mu \Phi_C\right] \cdot \delta_C > 0. \tag{4.9}$$

We now provide some characteristics of directions of change that are *strict* Pareto improving and equilibrium preserving.

Result 1: Suppose that there exist $\langle \alpha_h, \alpha_1, \ldots, \alpha_m\rangle > 0^{m+1}$ and $\beta \geq 0$ such that (4.5) and (4.6) hold at the status quo. Let $\langle \delta_1 \ldots, \delta_{n+3}\rangle$ be Pareto-improving and equilibrium-preserving. Then

$$\sum_{a=1}^{n+3} \left[\alpha_h \Gamma^h_a + \sum_i \alpha_i \Gamma^i_a - \beta \Phi_a\right] \delta_a > 0; \tag{4.10}$$

(*i.e.*, the sum of changes in marginal net benefits is strictly positive.)

Proof: Recall that δ is Pareto-improving and equilibrium-preserving if and only if (2.31) and (2.33) hold, which, under the above parameter restrictions, is equivalent to

$$\begin{aligned} -\beta \sum_{a=1}^{n+3} \Phi_a \delta_a \geq 0, \\ \alpha_h \sum_{a=1}^{n+3} \Gamma_a^h \delta_a \geq 0, \\ \alpha_i \sum_{a=1}^{n+3} \Gamma_a^i \delta_a \geq 0, \quad i = 1, \ldots, m, \\ \alpha_h \sum_{a=1}^{n+3} \Gamma_a^h \delta_a + \sum_{a=1}^{n+3} \alpha_i \, \Gamma_a^i \delta_a > 0. \end{aligned} \tag{4.11}$$

Summation and rearrangement yields (4.10). ■

Result 2: Assume that (4.5) and (4.6) hold at the status quo and that δ is a Pareto-improving and equilibrium-preserving direction of change. Then there exists a $c \in C$ such that $\delta_c \neq 0$; *i.e.*, a Pareto-improving and equilibrium-preserving direction of change involves a non-zero direction of change in some instrument for which marginal net benefits are not equal to zero at the status quo.

Proof: Suppose $\delta_c = 0$ for all $c \in C$. Since δ is Pareto-improving and equilibrium-preserving, (2.31) and (2.33) imply that $\delta_B \neq 0_B$ and δ_B is Pareto-improving and equilibrium-preserving. But this contradicts the fact that there does not exist a Pareto-improving and equilibrium-preserving direction of change in δ_B. ■

Result 3: Assume that (4.5) and (4.6) hold at the status quo and that δ is a Pareto-improving and equilibrium-preserving direction of change. Then there exists a $c \in C$ such that $\left[\alpha_h \Gamma_c^h + \sum_i \alpha_i \Gamma_c^i - \beta \Phi_c\right] \delta_c > 0$; *i.e.*, there exists a direction of change in some instrument in C that leads to an increase in net benefits.

Proof: Result 1 and the fact that δ is a Pareto-improving and equilibrium-preserving direction of change imply that

$$\left[\alpha_h \Gamma_B^h + \sum_i \alpha_i \Gamma_B^i - \beta \Phi_B\right] \delta_B + \left[\alpha_h \Gamma_C^h + \sum_i \alpha_i \Gamma_C^i - \beta \Phi_C\right] \delta_C > 0, \tag{4.12}$$

which, from (4.5) and (4.6), implies that

$$\left[\alpha_h \Gamma_C^h + \sum_i \alpha_i \Gamma_C^i - \beta \Phi_C\right] \delta_C > 0, \tag{4.13}$$

yielding the result. ■

Cases (ii) and (iii)(a) of Theorem 1 correspond to status quos where (assuming $A = I$) $B = \{1, \ldots, n, n+1, n+2\}$ and $C = \{\nu\}$. In particular, in case (ii), (4.6) holds as

$$\alpha_0 = \alpha_h \Gamma_\nu^h + \sum_i \alpha_i \Gamma_\nu^i - \beta \Phi_\nu > 0, \tag{4.14}$$

and in case (iii)(a), (4.6) holds as

$$-\alpha_0 = \alpha_h \Gamma_\nu^h + \sum_i \alpha_i \Gamma_\nu^i - \beta \Phi_\nu < 0. \tag{4.15}$$

Thus, the strong counterintuitive case (ii) corresponds to a status quo where the marginal net benefits of changing any instrument other than $\bar{\nu}$ is zero, while the marginal net benefits of changing $\bar{\nu}$ is positive. Given that $\Gamma_\nu^h = 0$, $\Gamma_\nu^i = \bar{\lambda}^i \pi^i < 0$ for all i, $\alpha > 0^{(m+1)}$, and $\beta \geq 0$, (4.14) holds if and only if $\Phi_\nu < 0$ and $|\beta \Phi_\nu| > |\sum_i \alpha_i \Gamma_\nu^i|$; *i.e.*, the marginal net gains from a change in $\bar{\nu}$ accrue entirely from the reduction in production costs that it entails at the status quo.[24] Since $\dot{\bar{\nu}} > 0$ itself directly increases production cost, this reduction must be generated by secondary effects of the change in $\bar{\nu}$ on the demands for other commodities of all agents. Hence both weak and strict Pareto-improving and equilibrium-preserving directions of change involve $\dot{\bar{\nu}} > 0$. The latter follows from result 3 above and the fact that ν is the only instrument in C.

We show next why the direct and secondary effects of $\dot{\bar{\nu}} > 0$ can result in a reduction of production costs starting at a tight TEQC while this is not true in a first-best world.

[24] Note that this explication makes it clear that the strong counterintuitive case does not arise trivially from the possibility of the Pigouvian tax on the externality-generating commodity being set too high at the status quo, so that the obvious prescription for the introductory quantity control is to mandate an increase in $\bar{\nu}$. Clearly, this result and its explication would go through even if no Pigouvian taxation were implemented at the status quo (*i.e.*, even if $\pi = \rho$ at the status quo. Note the contrast to the well-known case of overproduction of the negative externality-generating commodity in a competitive equilibrium with no Pigouvian tax.

4.2. Other Policy Regimes.

To understand the role played by the other distortions in our system in generating the counterintuitive results of Theorem 3, we now consider a policy regime in which the government can implement all of the first-best instruments: a common set of prices $\langle p, \rho \rangle$ faced by consumers and producers alike, a set of personalized lump-sum transfers $\langle R^h, R^1, \ldots, R^m \rangle$, and a direct quantity control $\bar{\nu}$ on the consumption of the externality-generating commodity. We refer to the feasible states in this policy regime as the set of competitive equilibria with a quantity control (CEQC).

We first show that, starting at a status quo that is a tight CEQC, there exists no Pareto-improving and equilibrium-preserving direction of change with respect to the policy instruments $\langle p, \rho, R^h, R^1, \ldots, R^m \rangle$. Let B be the set of coordinates of $\mathbf{R}^{n+m+2}$ reserved for these instruments and denote a direction of change as $\delta_B = \langle \dot{p}, \dot{\rho}, \dot{R}^h, \dot{R}^1, \ldots, \dot{R}^m \rangle$. Let

$$V^h\big(p, \rho, R^h, \bar{\nu}\big) = U^h\Big(\bar{d}^h\big(p, \rho, R^h, \bar{\nu}\big), \bar{\nu}\Big) \tag{4.16}$$

and

$$V^i\big(p, R^i, \bar{\nu}\big) = U^i\Big(\bar{d}^i\big(p, R^i, \bar{\nu}\big), \bar{\nu}\Big), \quad i = 1, \ldots, m. \tag{4.17}$$

Analogously to the modeling in Section 2, take the initial situation (the status quo) as given and denote the derivatives of these indirect utility functions with respect to all $n + m + 2$ of the instruments in B by

$$\Gamma_B^h = \begin{pmatrix} \bar{\lambda}^h p \ \nabla_p \bar{d}^h \\ \bar{\lambda}^h p \ \nabla_\rho \bar{d}^h \\ \bar{\lambda}^h p \ \nabla_{R^h} \bar{d}^h \\ 0^{(m)} \end{pmatrix}$$

$$\Gamma_B^i = \begin{pmatrix} \bar{\lambda}^i p \ \nabla_p \bar{d}^i \\ 0 \\ \vdots \\ 0 \\ \bar{\lambda}^i p \ \nabla_{R^i} \bar{d}^i \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad i = 1, \ldots, m, \tag{4.18}$$

where we have exploited the first-order conditions for the consumers' utility maximization exercises. Similarly, denote the $(n+1)\times(n+m+2)$ matrix of derivatives of excess demands with respect to the $n+m+2$ instruments by $E_B = D_B - S_B$, where

$$D_B = \begin{pmatrix} \nabla_{\langle p,\rho\rangle}\left(\bar{d}_1^h + \sum_i \bar{d}_1^i\right) & \nabla_{R^h}\bar{d}_1^h & \nabla_{R^1}\bar{d}_1^1 \cdots & \nabla_{R^m}\bar{d}_1^m \\ \vdots & \vdots & \vdots & \cdots \\ \nabla_{\langle p,\rho\rangle}\left(\bar{d}_n^h + \sum_i \bar{d}_n^i\right) & \nabla_{R^h}\bar{d}_n^h & \nabla_{R^1}\bar{d}_n^1 \cdots & \nabla_{R^m}\bar{d}_n^m \\ 0^{(n+1)} & 0 & 0 & 0 \end{pmatrix} \tag{4.19}$$

and

$$S_B = \left[\nabla_{\langle p,\rho\rangle}\eta(p,\rho)\ \mathbf{0}\right] \tag{4.20}$$

(where $\mathbf{0}$ is an $(n+1)\times(m+1)$ matrix of zeros).

By Motzkin's Theorem, there does not exist a δ_B such that

$$\Gamma_B^h \cdot \delta_B > 0 \quad \wedge \quad \Gamma_B^i \cdot \delta_B > 0,\ i = 1,\ldots,m \quad \wedge \quad E_B \cdot \delta_B \leq 0^{(n+1)} \tag{4.21}$$

if and only if there exists $\alpha = \langle \alpha_h, \alpha_1, \ldots, \alpha_m\rangle > 0^{(m+1)}$ and $\beta := \langle \beta_1 \cdots, \beta_n, \beta_\nu\rangle \geq 0^{(n+1)}$ such that

$$\alpha_h \Gamma_B^h + \sum_i \alpha_i \Gamma_B^i - \beta \cdot E_B = 0^{(n+m+2)}. \tag{4.22}$$

Exploiting the homogeneity of η, we find that the following specifications of α and β satisfy (4.22):

$$\begin{aligned} \alpha_h &= \mu/\bar{\lambda}^h, \\ \alpha_i &= \mu/\bar{\lambda}^i, \quad i = 1,\ldots,m, \\ \beta_k &= \mu p_k, \quad k = 1,\ldots,n, \\ \beta_\nu &= \mu\rho, \end{aligned} \tag{4.23}$$

where $\mu > 0$.

Next we show that the status quo offers a solution to the following Kuhn-Tucker problem:

$$\begin{aligned} \hat{U}(u,\bar{\nu}) := \max_{p,\rho,\mathcal{R}} & V^h(p,\rho,R^h,\bar{\nu}) \quad \text{s.t.}, \\ & V^i(p,R^i,\bar{\nu}) \geq u^i,\ i = 1,\ldots,m, \\ & \bar{d}^h(p,\rho,R^h,\bar{\nu}) + \sum_i \bar{d}^i(p,R^i,\bar{\nu}) \leq \eta(p,\rho), \\ & \bar{\nu} \leq \eta_\nu(p,\rho), \end{aligned} \tag{KT3}$$

where $\mathcal{R} = \langle R^h, R^1, \ldots, R^m \rangle$. The first-order conditions include the following:

$$\Gamma^h_B + \sum_i \gamma_i \cdot \Gamma^i_B - \psi \; E_B = 0^{(n+m+2)}. \tag{4.24}$$

where $\gamma_i, \; i = 1, \ldots, m, \; \psi_l, \; l = 1, \ldots, n$, and ψ_ν are the appropriate multipliers. A comparison of (4.22) and (4.24) makes it clear that the status quo is a solution to (KT3) where $\bar{\nu}$ is set equal to its status quo value; $u^i = U^i(x^i, \bar{\nu}), \; i = 1, \ldots, m, h$; and $x^i, i = 1, \ldots, m$, are set equal to their status quo values. Note that in this characterization, $\psi_l = \beta_l/\alpha_h = \mu p_l/\alpha_h, \; l = 1, \ldots, n$, and $\psi_\nu = \beta_\nu/\alpha_h = \mu\rho_\nu/\alpha_h$ at the status quo, indicating that the status quo is a production-efficient solution to (KT3).

Using (4.23), (4.22) can be re-written as

$$\alpha_h \Gamma^h_B + \sum_i \alpha_i \Gamma^i_B - \mu \langle p, \rho \rangle \cdot \big[D_B - S_B \big] = 0^{(n+m+2)}, \tag{4.25}$$

or, using the fact that $\langle p, \rho \rangle S_B = 0^{(n+m+2)}$ and recalling that $\langle p, \rho \rangle D_B = \Phi_B$ (the production cost of changing the instruments in B),

$$\alpha_h \Gamma^h_B + \sum_i \alpha_i \Gamma^i_B - \mu \Phi_B = 0^{(n+m+2)}. \tag{4.26}$$

Thus, at the status quo, the marginal net benefits of changing the instruments in B are all zero. Consider, however, the marginal net benefit of changing $\bar{\nu}$, MNB_ν. Applying the envelope theorem, this is obtained by differentiating the Lagrangian of (KT3), $L(p, \rho, \mathcal{R}, \bar{\nu})$, with respect to $\bar{\nu}$ and evaluating the Lagrangian multipliers at the status quo using (4.23):

$$\frac{\partial L(p, \rho, \mathcal{R}, \bar{\nu})}{\partial \bar{\nu}} = \frac{\partial V^h(p, \rho, R^h, \bar{\nu})}{\partial \bar{\nu}} + \sum_i \frac{\lambda^h}{\lambda^i} \frac{\partial V^i(p, \rho, R^i, \bar{\nu})}{\partial \bar{\nu}} - \mu \Phi_\nu. \tag{4.27}$$

Employing the same arguments that led to (2.11) and (2.20) in Section 2, we can re-write (4.27) as

$$\frac{\partial L(p, \rho, \mathcal{R}, \bar{\nu})}{\partial \bar{\nu}} = \bar{\lambda}^h (\pi^h - \rho) + \sum_i \lambda^h \pi^i - \bar{\lambda}_h \Phi_\nu. \tag{4.28}$$

A competitive equilibrium (CE) is a special CEQC where the quantity control on the externality has been fixed at its unconstrained Marshallian demand evaluated at the prevailing prices, in which case $\pi^h = \rho$. Thus, in a CE, the marginal net benefit of changing $\bar{\nu}$ is

$$MNB_\nu = \sum_i \lambda^h \pi^i - \bar{\lambda}_h \Phi_\nu. \tag{4.29}$$

We can now contrast a CE with a tax equilibrium (TE), which is a special TEQC where the quantity control on the externality has been fixed at its unconstrained demand). We saw in section 4.1 above that, at a TE corresponding to the strong counterintuitive case identified by Theorem 3, condition (4.14) holds. This condition means that, at the status quo, the marginal net benefit from changing $\bar{\nu}$ is positive. The marginal net gains from changing $\bar{\nu}$ come from the fact that such a change brings about (via the changes in the consumer demands that it induces) a reduction in marginal production cost that more than offsets the induced loss in welfare of the victims. In contrast, at a CE in the absence of a wedge between producer and consumer prices and in which personalized lump-sum transfers are feasible, the marginal production cost of changing $\bar{\nu}$ is

$$\Phi_\nu = \langle p,\ \rho \rangle \begin{pmatrix} \partial\big(\bar{d}^h + \sum_i \bar{d}^i\big)/\partial \bar{\nu} \\ 1 \end{pmatrix} = 0, \tag{4.30}$$

the last identity following from differentiation of the consumer budget constraints and aggregation across all consumers. (Given the absence of a wedge between producer and consumer prices and the possibility of personalized lump-sum transfers, the marginal production cost of a change in $\bar{\nu}$ is also the marginal aggregate consumer expenditure attributable to a change in $\bar{\nu}$. A change in $\bar{\nu}$, holding prices and income fixed, just moves each consumer along his or her budget hyperplane.) Hence, the last term in (4.29) vanishes, and

$$MNB_\nu = \bar{\lambda}^h \sum_i \pi^i. \tag{4.31}$$

Since we are dealing with a negative externality, in a CE we have

$$MNB_\nu = \bar{\lambda}^h \sum_i \pi^i < 0. \tag{4.32}$$

In other words, we obtain, using these methods the well-known prescription that, at a CE, all Pareto-improving and equilibrium-preserving directions of change require a decrease in the externality-generating commodity. The strong counterintuitive case in Theorem 1 thus has to be attributed to the wedge that exists between producer and consumer prices in our second-best model, which prevents the marginal production cost of changing $\bar{\nu}$ from being equal to zero (and, of course, the wedge itself is attributable to the infeasibility of personalized lump-sum transfers).

4.3. Temporary Production Inefficiency.

The tax reform literature (see, especially, Guesnerie [1977, 1995], Smith [1983], and Myles [1995]) has focused on the possibility of temporary production inefficiency: when Pareto-improving and equilibrium-preserving reforms exist but all such reforms are necessarily *non-strict* equilibrium preserving. Assuming that Pareto-improving reforms exist and $\Phi_A \neq 0$, this phenomenon occurs if and only if $\Phi_A \in -\Gamma_A$ (Guesnerie [1977, 1995]). If, however, the lump-sum transfer is an available instrument and the demand system satisfies "positivity of marginal (production) cost of (uniform) income transfer" (Guesnerie's [1995, p. 95] assumption LR2), $\Phi_R = \langle \hat{p}, \hat{\rho} \rangle \; \nabla_R \Big(\bar{d}^h + \sum_i \bar{d}^i \Big) > 0$, temporary production inefficiency cannot occur (Smith [1983] and Guesnerie [1995]). Thus, temporary inefficiency cannot occur if normality of demands dominates inferiority in the precise sense of assumption LR2. A sufficient condition, of course, is normality of demand for all commodities.

Guesnerie's [1995, p.150] proof of this proposition goes straight through when an externality quantity control is an available instrument. In each case the intuition is obvious: if any reform leads to production inefficiency and hence to a surplus in the government budget, the policymaker can use the surplus to increase the uniform transfer so that the resulting change in demands moves the economy (differentially) back to the frontier. If, however, an externality exists but the quantity control on the externality-generating commodity is not an available instrument, the Smith/Guesnerie proof does not go through. Intuitively, this is because the increase in R changes the demand for the externality-generating commodity by consumer h, which in turn has undetermined effects on the demands for all other commodities by the victims, possibly undermining the effects of the increase in R.

It turns out that temporary production inefficiency is ruled out in the strong counterintuitive case, even though the increase in $\bar{\nu}$ itself generates a surplus. To see this, note that temporary inefficiency,

$$\Phi_A = \tilde{\alpha}_h\big(- \Gamma^h_A\big) + \sum_i \tilde{\alpha}_i\big(- \Gamma^i_A\big), \quad \langle \tilde{\alpha}_h, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_m \rangle > 0^{(m+1)}, \tag{4.33}$$

implies that

$$\Phi_\nu = - \sum_i \tilde{\alpha}_i \bar{\lambda}^i \pi^i > 0. \tag{4.34}$$

But recall from the discussion of equation (4.14) that the counterintuitive case requires that $\Phi_\nu < 0$. The intuition for this result can also be found in the discussion of equation (4.14). The strong counterintuitive case requires that the mandated increase in $\bar{\nu}$ generate a government surplus ($\Phi_\nu < 0$), which can then be used to increase the welfare of the externality victims, so long as an appropriate instrument exists to effect this increase. The types of instruments that would suffice are indicated in Theorem 3. The necessity of temporary inefficiency, on the other hand, requires that the increase in $\bar{\nu}$ generate a government deficit ($\Phi_\nu > 0$), which is incompatible with the strong counterintuitive case.

5. Conclusion.

In this paper, we have imbedded a consumption-externality quantity control in a standard tax-reform framework. In the spirit of Guesnerie's [1977, 1995] analysis of the relationship between Pareto-improving and equilibrium-preserving directions of change of available policy instruments, we have analyzed the introduction of externality quantity controls in a second-best economy. An apparently counterintuitive result emerges: starting at an initial tight equilibrium with no quantity controls, it is possible that the Pareto-improving and equilibrium-preserving directions of change, if they exist, entail a government-mandated *increase* in the quantity consumed of a commodity that generates a negative externality. Indeed, it is the case that, under some preference profiles and cost structures, all Pareto-improving and equilibrium-preserving reforms *require* such an increase.

In the specific context of our model, the reason for our counterintuitive results, as explained in the intuitive interpretations in Section 4, is the existence of distortions elsewhere in the economy, reflected in the wedge between consumer and producer prices, which is in turn necessitated by the infeasibility of personalized lump-sum transfers. This second-best aspect of our model makes possible the existence of status quos in which marginal net gains can be achieved only by changing the level of the externality. This explains the strong counterintuitive case, where all Pareto-improving and equilibrium-preserving directions of change require an increase in the level of the negative externality (the marginal net gains from increasing the level of the externality being positive at such status quos). In some status quos, however, the welfare effects of changes in other (available) policy instruments can offset (can be offset by) the welfare losses (gains) from changing the quantity control on the negative externality. This explains the weaker counterintuitive case in which there exist Pareto-improving and equilibrium-preserving policies involving an increase in the level of the negative externality.

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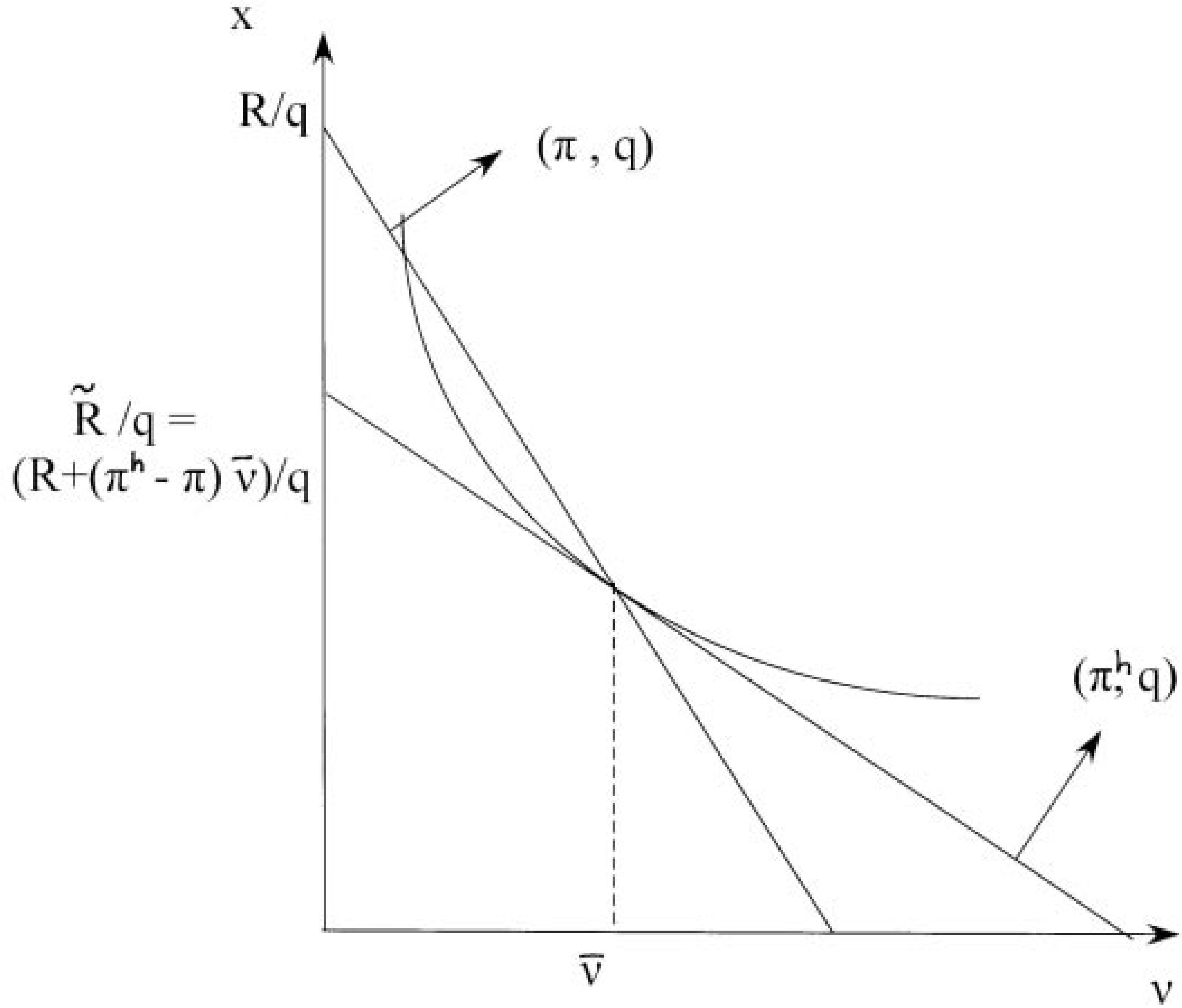


Figure 1.

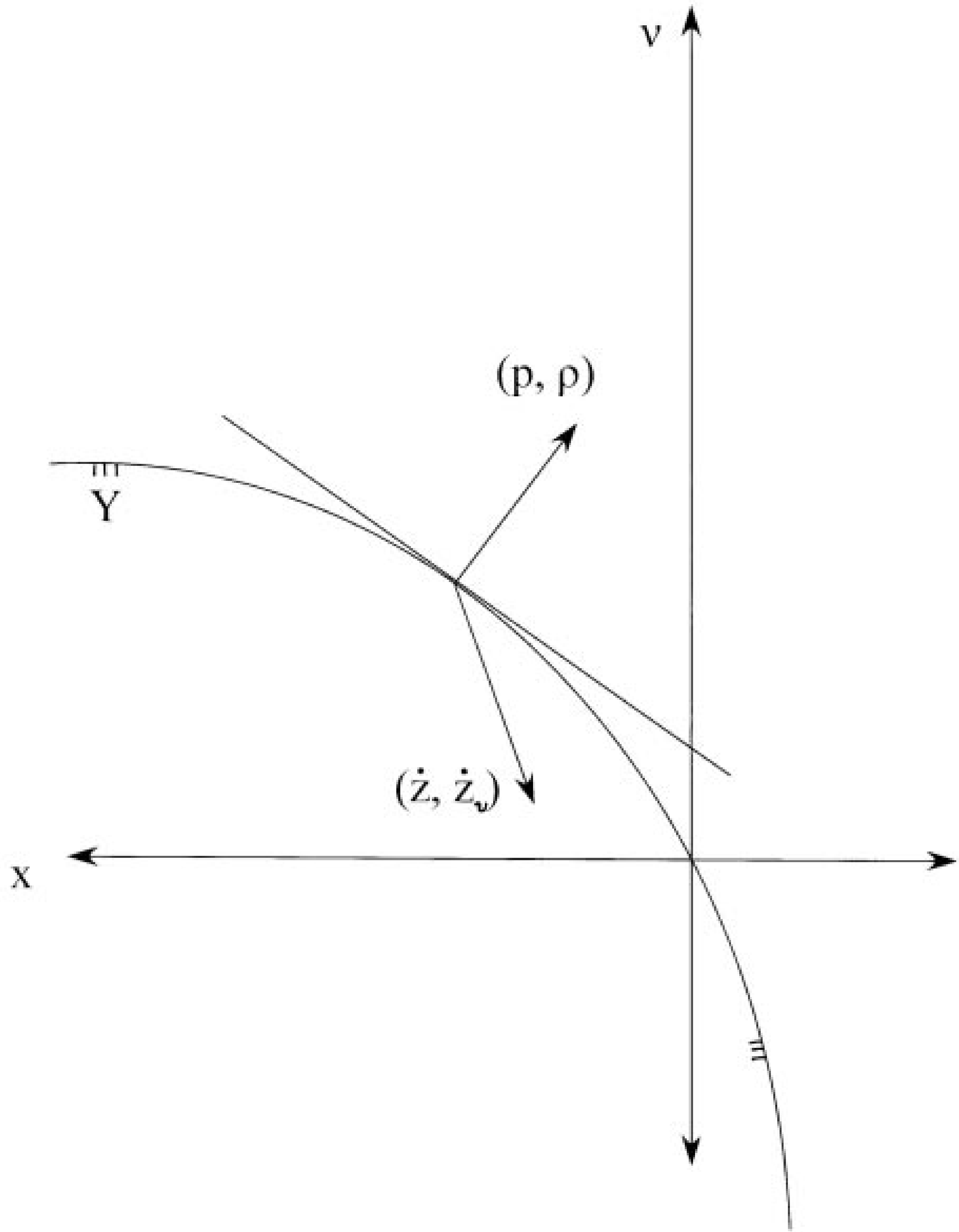
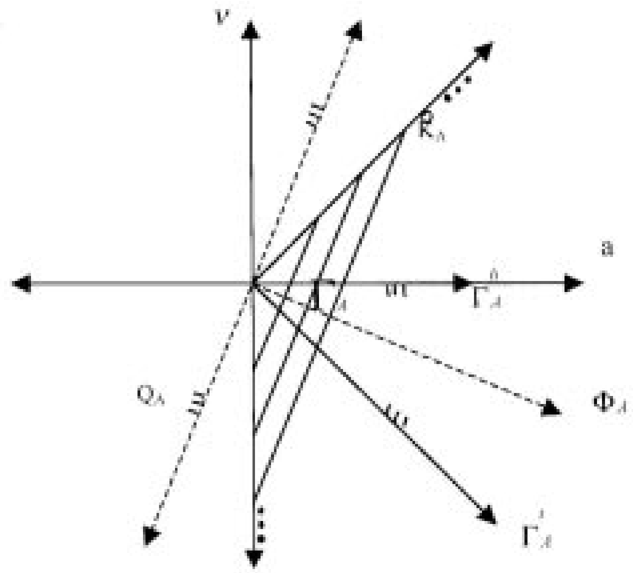


Figure 2.

I. Theorem 1, Case (i).



II. Corollary, Case 1b.

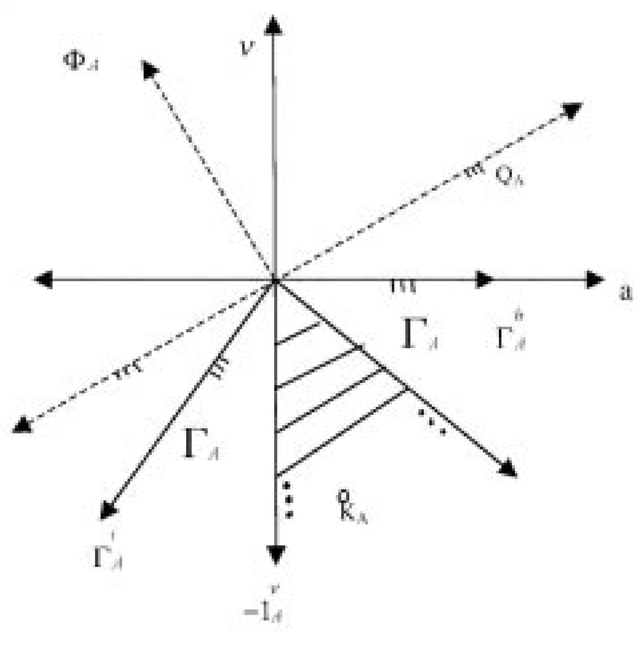
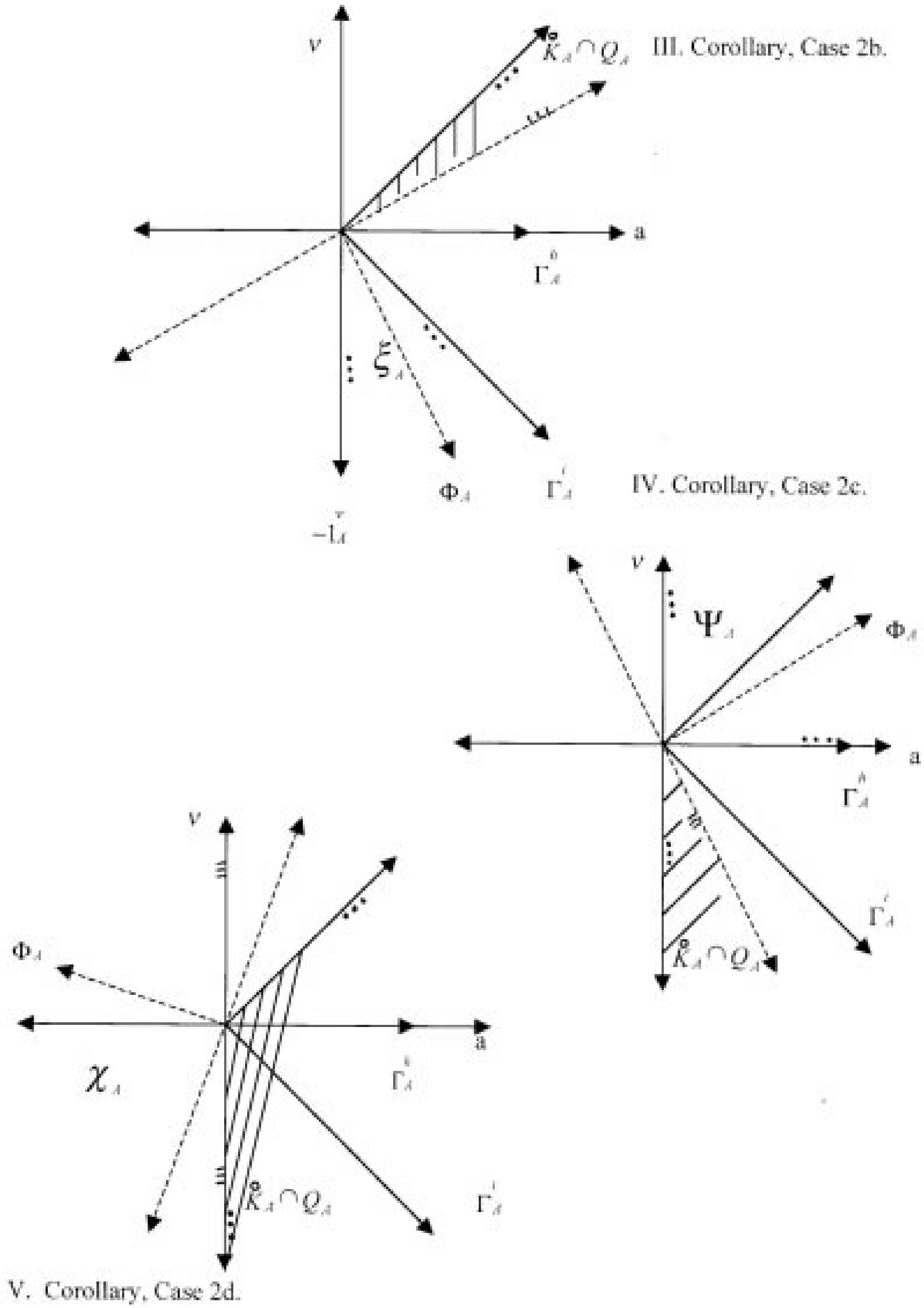


Figure 3.



III. Corollary, Case 2b.
v
$\mathring{K}_A \cap Q_A$
a
Γ_A^h
ξ_A
Φ_A
Γ_A^l
-1_A^v
IV. Corollary, Case 2c.
v
Ψ_A
Φ_A
a
Γ_A^h
Γ_A^l
$\mathring{K}_A \cap Q_A$
v
Φ_A
a
χ_A
Γ_A^h
$\mathring{K}_A \cap Q_A$
Γ_A^l
V. Corollary, Case 2d.

Figure 3.