Maturity Structure and Debt Renegotiation in Sovereign Bonds

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Abstract

This paper develops a model of endogenous default with debt renegotiation for emerging economies. A small open economy faces a stochastic stream of income. The government can issue short and long term bonds and makes decision on behalf of the residents of the borrowing country. Lenders are risk-neutral and operate in a perfectly competitive financial market. Upon default, the borrowing country loses access to the financial markets and will not be able to borrow any longer. The defaulted country has to pay off the principal and interest of the restructured debt to regain access to the credit market. Debt restructuring is modeled by a Nash bargaining game. The resulted equilibrium haircut is directly related to the debt level. This feature results in a default value function that flattens out after an endogenous threshold; consequently default happens at higher debt levels compared to models without debt renegotiation. The model is calibrated to capture the default episodes in Argentina. The proposed model has several advantages over extant literature. First, the model statistics closely match the observed values. This is particularly the case for the resulted interest rate distributions for short and long term bonds, compared to previous literature. Providing a precise interest rate distribution is crucial as finding the optimal maturity structure relies on it. Furthermore, interest rates can be an indicative of financial crisis. The paper finds that endogenous debt renegotiation is an important mechanism in generating more realistic fluctuations of the interest rate. The proposed model can be used as a policy tool to predict and understand dynamics of financial crises related to debt default.

Keywords: Sovereign Default, Sovereign Bonds, Maturity Structure, Debt Renegotiation

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1 Introduction

Recurrence of financial crises and prolonged debt settlement process in emerging countries induce questions regarding the role of debt structure in sovereign default. Thus, debt maturity composition and its impact on the interest rate draw a lot of research attention. It is well-documented fact that the interest rate plays an important role in emerging economies.[1] Hence, it is crucial to investigate the reciprocal effect of debt maturity structure and the interest rate spreads faced by an emerging economy. In particular, Broner, Lorenzoni, and Schmukler (2013) show that for emerging economies short term debts are generally cheaper than long term debts. Due to an increase in dilution risks,[2] the long term borrowing is even costlier near financial crises. As a result, countries will shorten the maturity composition in the imminence of a financial crisis. This implies that while long term borrowing is expensive, the sovereign government suffers massively from rolling over the short term debt near default episodes.

This paper provides a default model with two debt instruments, short and long term bonds, and endogenous debt renegotiation. A risk averse sovereign faces a stream of income and issues bonds that will be traded in a perfectly competitive market with risk neutral lenders. Contingent on the state of the economy depending on income shocks and short and long term debts, sovereign may find it optimal to default. [3] Lenders determine the price of a bond taking into account these default probabilities. Upon default, an endogenous haircut will be applied to the short and long term bonds. Adding

[1] See Neumeyer and Perri (2005), Aguiar and Gopinath (2006)

[2] Dilution risk can be defined as the capital loss for the current long term lenders after issuance of new bonds. While both short and long term bonds are affected by the default risk, only the latter is impacted by the dilution risk.

[3] This paper does not consider the possibility of partial defaults.

the endogenous debt renegotiation feature to the model allows us to further study the dilution risk.

This paper is closely related to a set of previous studies. Arellano and Ramanarayanan (2012) proposed a similar model but with no recovery, i.e. 100% haircut. Hatchondo, Martinez, and Sosa-Padilla (2016) proposed a model with exogenous debt dilution that was constant over the states. Bi (2006) applies Yue (2010) renegotiation approach to one- and two-period bonds, representing short and long term bonds, respectively. This paper improves Bi's (2006) approach by allowing the country to issue bonds with longer maturities. in order to achieve this, we model bonds as perpetuity contracts with non-state-contingent payments that decay at a constant rate.[4] As in Macaulay (1938), different decay rates can represent bonds with different durations. This allows us to circumvent the curse of dimensionality and study the effect of bonds with longer maturities. Bi (2006) assumes the recovery rate is a function of the total outstanding debt.[5] In this paper, the state-contingent debt is a function of the income and of short and long term debts, instead of total debt. We find that expressing the recovery rate as a function of total debt is a specific case of our model when the sovereign is forced to pay the arrears in one period. We conclude that this can lead to suboptimal decision policies.

As explained in Bi (2006), interest rate and debt dilution are the main factors affecting default episodes. Most of the papers that extend Eaton and

[4] see e.g. Arellano and Ramanarayanan 2012, Hatchondo et al. 2016, and Hatchondo and Martinez 2009.

[5] As discussed in Farzad (2018a), when the sovereign issues one-period bonds and loans, the recovery rate is a function of total debt. However, in presence of bonds with different maturity, this approach relies on the presumption that upon default the total debt will be repaid in one period. The default episodes are more likely when the income realization is low and, assuming income persistence, the income is likely to remain low in the future periods. This means the country that is forced to pay back the debt in one period suffers massively from default. This process may artificially reduce the default incentives.

Gersovitz (1981) address the effect of endogenously determined interest rate. The effect of debt dilution is either discarded (Arellano 2008, Arellano and Ramanarayanan 2012) or modeled by a constant exogenous factor (Hatchondo et al. 2016). This paper models the renegotiation using Nash Bargaining game, as in Yue (2010) and Bi (2006). As in Bi (2006) and Farzad (2018a), the main focus of the proposed model is on *pari passu* debt contracts; all outstanding debts are treated equally upon default. Hence, the same haircut is applied to short and long term debts. Bi (2006) argues how lenders tend to hold short term bonds near the default to prevent the debt dilution. Hatchondo et al. (2016) show that without debt dilution the optimal maturity of debt increases by 2 years. Hence, it is crucial to take into account the effect of debt dilution in addition to the interest rate impacts.

We calibrate the model to the Argentina data. We find that the model predictions are closely related to patterns in the data. In particular, the spread dynamics match the observed data: The spread curves (difference between the interest rate of a maturity and corresponding risk free interest rate) are upwards sloping during tranquil times, but flatten out or even invert closer to financial crisis. We show that the proposed model with endogenous debt renegotiation delivers spread distribution closer to data compared to models with either no recovery rate or an exogenous one. In short, the equilibrium endogenous recovery rate allows the borrowing country to hold higher levels of debt before declaring a default. Consequently, the observed interest rates are higher than the models without debt renegotiation. At the onset of crisis, the debt dilution effect reduces the long term bond prices more than the price of the short term bonds. Hence borrowing countries shorten the maturity composition. Issuing short term bonds increases the default probability but does not affect the debt dilution risk. This results in an even larger short term spread. Studying the behavior of interest rates as done in this paper is important for two main reasons. First, high interest rates are leading indicators of financial crisis. Second, finding the optimal maturity depends of analyzing the interest dynamics for different maturities. The proposed model can be used as an important policy tool to predict and understand dynamics of financial crises related to debt default.

Literature Review

This paper is related to several strands of research. It builds on seminal work of Eaton and Gersovitz (1981) that developed an endogenous default model. This paper also emphasizes on the role of (endogenous) interest rate in financial crisis. This is in line with Neumeyer and Perri (2005) conclusion that interest rate spread is an important factor in explaining the business cycle fluctuations in emerging economies. As indicated by Aguiar and Gopinath (2006) models with one-period bonds are not capable of fully capturing the spread fluctuations. Arellano (2008) shows increasing the default cost in a model with one-period bonds can induce higher spreads. To generate a positive spread,even when the economy is not exposed to a high default probability, researchers added debts with longer maturities: Even in tranquil times, sovereign is exposed to a “bad” income shock over the span of a long term debt that may lead to a default episode. Taking into account this unpleasant draw, long term lenders will charge a higher interest rate. Chatterjee and Eyigungor (2008), improved Arellano’s (2008) approach by letting the sovereign to issue bonds with longer maturities. The results provide a better fit for the spread behavior in emerging economies.

Broner et al. (2013) studied the effect of debt maturity composition in emerging economies. They document that the risk premium on long term debts is relatively higher, and would increase during crisis. This shifts the debt issuance toward short term debts. To explain this fact, Arellano and Ramanarayanan (2008) and Hatchondo and Martinez (2009) developed models that include both short and long term bonds. The long term bonds are represented by perpetuity payments with a decay rate. As in Macaulay (1938), finite durations can be modeled with a proper decay rate.[6] This allows us to develop a model in which debt maturity can be changed by a choosing another decay rate.

Hatchondo et al. (2016) and Bi (2006) improved the previous works by studying the debt dilution effects. Hatchondo et al. (2016) use a constant exogenous recovery rate that is applied to defaulted debts. Bi (2006) applies the Nash Bargaining (à la Yue 2010) to find the endogenous recovery rates for one and two period bonds. Bulow and Rogoff (1989) proposed a model with Rubinstein's (1982) type of bargaining in the debt rescheduling process. As discussed by Bi (2006), not only the Nash Bargaining approach is more tractable, but it also delivers results that can be supported by Rubinstein game or other complicated game structures.[7]

Sturzenegger and Zettelmeyer (2008) show the inter-creditor equity was ex-post violated for many of the debt restructuring between 1998 and 2005. This is mainly due to higher post-exchange yields, that result in a lower NPV for higher maturity debts. Since there is not enough evidence in favor of ex-ante discrimination, Hatchondo et al. (2016), Bi (2006), and Farzad (2018a) all assume the same debt recovery rate for different debt instruments. This is

[6] Hatchondo and Martinez (2009) explain this approach is used in empirical studies as well as credit rating agencies.

[7] For instance see Bai and Zhang (2009) for the application of stochastic bargaining model.

based on the assumption of *pari passu* debts. Collective action clauses (CACs) and comparability of treatment (in Paris Club principals) can further justify applying the same haircut to short and long term bonds.[8] Due to the lack of legally-binding rules regarding the seniority of debts, we do not consider priorities in debt repayment.

The outline of the paper is as follows: Section 2 provides the theoretical model. Section 3 presents the theoretical results and propositions for the paper. Section 4 contains the quantitative analysis for the model, and finally section 5 concludes the paper.

2 Model

Time is discrete and is indexed by $t \in \{0, 1, \dots\}$. The economy faces a stream of income $\{y_1, y_2, \dots y_n\}$ that follows a Markov process with transition matrix $f(y_{t+1}|y_t)$. There is a representative agent that lives forever, with preferences represented by

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where $0 < \beta < 1$ is the discount factor. Later we assume u is a CARA utility function, hence it is continuous, strictly increasing, and strictly concave. Each period, a benevolent sovereign decides on the level of consumption and borrowings. Government can issue either short term, B_S, or long term, B_L, bonds. Short term bond is a one-period bond sold at the discount price q_S and delivers the face value next period. Long term bond, as in Arellano and Ramanarayanan (2012) and Hatchondo and Martinez (2009), is modeled by a perpetuity contract with coupon payments that decay geometrically. Figure 1 depicts the payments associated with the perpetuity contract of a bond

[8]See Das et al (2012) and Farzad (2018a)

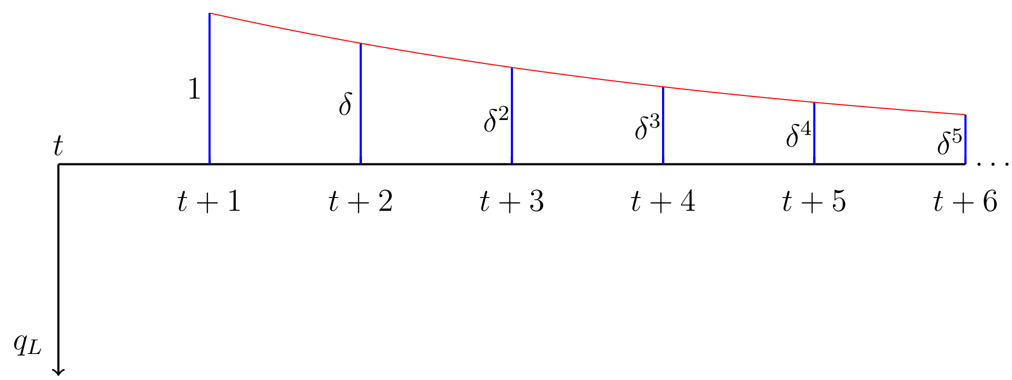


Figure 1: Long Term Bond Representation.

issued at time t with a loan face value of one. [9]

Appealing to Macaulay's (1938) approach, one is able to use an appropriate decay rate to model a loan contract with finite duration. Macaulay defines the duration as the weighted sum of future payment dates. For instance, duration of a perpetuity with coupon payments c_n and initial price of q is

$$d = \sum_{n=1}^{\infty} n \frac{c_n (1+r)^{-n}}{q}$$

that is, each date n is weighted by the normalized (divided by the initial price $q = \sum_{n=1}^{\infty} \frac{\delta^{n-1}}{(1+r)^n}$) discounted coupon payments on that date. Applying this definition to the risk-free bonds in this paper results in

$$d = \sum_{n=1}^{\infty} n \frac{c_n (1+r)^{-n}}{q} = \frac{1}{\delta q} \sum_{n=1}^{\infty} n \delta^n (1+r)^{-n} = \frac{1}{\sum_{n=1}^{\infty} (\frac{\delta}{1+r})^n} \sum_{n=1}^{\infty} n (\frac{\delta}{1+r})^n$$

Which simplifies as

$$d = \frac{1+r}{1+r-\delta}$$

where r is the risk free interest rate. With no default, next period, the stock of long term bonds is decayed by δ; independent of the issuance date

[9]Face value of perpetuity does not affect the price.

each bond is decayed by the factor of δ. In addition, sovereign's borrowing (lending) this period will add to (subtract from) next period's stock of the bonds. Hence the law of motion for stock of long term bonds can be defined as

$$B_{L,t+1} = \delta B_{L,t} + l_t$$

where l_t is current period's issuance or repayment of long term bonds.[10] The price of new issued bonds, either short or long term, depends on the sovereign's state;

$$q_t^S(B_{S,t+1}, B_{L,t+1}, y_t), \;\; q_t^L(B_{S,t+1}, B_{L,t+1}, y_t).$$

Borrowing Country

There is a benevolent government that makes decision regarding consumption and the issuance (or repayment) of short and long term bonds. At a given state (B_S, B_L, y), a sovereign has an option to either service the debt and get the value of $W(B_S, B_L, y)$ or default and get $V^D(B_S, B_L, y)$. The value of the sovereign, $V(B_S, B_L, y)$, is the maximum of these two:

$$V(B_S, B_L, y) = \max\left\{W(B_S, B_L, y), V^D(B_S, B_L, y)\right\}$$

Contingent on honoring the debt, current consumption and new issuance (repayment) of each type of bonds will be determined according to the following problem:

$$W(B_S, B_L, y) = \max_{B'_S,\; B'_L,\; 0 \leq C} u(C) + \beta \int_Y V(B'_S, B'_L, y') f(y'|y) dy'$$

[10]For the one period bonds, δ is zero; hence, the stock of short term bonds will be the same as the amount issued (or bought) in the current period.

$$s.t. \qquad C + B_S + B_L = y + q^S(B'_S, B'_L, y)B'_S + q^L(B'_S, B'_L, y)l'_L$$

$$B'_L = \delta B_L + l'_L$$

Defaulted sovereign will be excluded from the capital market, hence only will be able to consume the current income. The arrears consist of the principal and the due interest. Contingent on the state of the economy, a haircut $1-\alpha$ will be applied to the outstanding short term and long term debt levels.

$$V^D(B_S, B_L, y) = u(y) + \beta\Big\{\int_Y W^D(\alpha(1+r)B_S, \alpha(1+r)B_L, y')f(y'|y)dy'\Big\}$$

We can assume the country has the option of permanently leaving the credit market. However with the assumptions on the utility function and the parameters used in these paper, it is never optimal to do so. Direct output loss, λ, and lacking a smoothed consumption profile make the Autarky a suboptimal choice.

$$V^{AUT}(y) = u((1-\lambda)y) + \beta\int_Y V^{AUT}(y')d\mu(y'|y)$$

Following Yue (2010), it is assumed that the sovereign will regain access to the credit market after paying the arrears in full. Until then, defaulted country has to (weakly) decrease short and long term debts, $0 \leq B'_S \leq B_S,\ \ 0 \leq B'_L \leq B_L$. Since the country is a net lender, the discount price for short and long term bonds will be the corresponding risk-free interest rate:

$$W^D(B_S, B_L, y) = \max_{0 \le B'_S \le B_S,\ 0 \le B'_L \le B_L,\ 0 \le C} u(C) + \beta \int_Y W^D(B'_S, B'_L, y') f(y'|y) dy'$$

$$C + B_S + B_L = (1 - \lambda)y + \frac{B'_S}{1+r} + \frac{l'_L}{1+r-\delta}$$

$$B'_L = \delta B_L + l'_L \tag{1}$$

When the arrears are fully paid, the sovereign will regain access to the credit market and will be able to issue new bonds:

$$W^D(0, 0, y) = V(0, 0, y)$$

Debt Renegotiation Problem

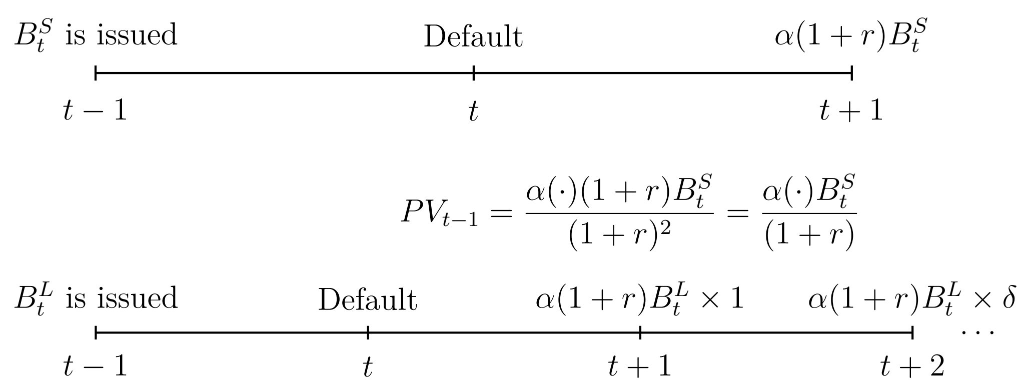


Figure 2: Present Value of Defaulted Bonds

$$PV_{t-1} = \frac{\alpha(\cdot)(1+r)B_t^L}{(1+r-\delta)(1+r)}$$

Upon default, creditors and debtors engage in a one-round renegotiation process that determines the haircuts. Following Yue (2010), the debt renegotiations will be modeled by a Nash Bargaining game; the optimal recovery rate, α, will maximizes the total surpluses of creditors and the debtor. The debtor's surplus is the difference between the default value (implying the debt renegotiations were successful and a haircut $1-\alpha$ will be applied) and the Autarky value (implying the debt renegotiation process has failed):

$$\Delta_D(B_S, B_L, y; \alpha) = V^D(B_S, B_L, y; \alpha) - V^{AUT}(y)$$

Figure 2 shows the present value of an short and long term debts issued at time $t-1$. Present values of the defaulted bonds at time t are calculated in the Figure as well. Creditors' surplus will be the present value of the recovered total debt:

$$\Delta_C(B_S, B_L, y; \alpha) = \frac{\alpha}{1+r}\Big[B_S + \frac{B_L(1+r)}{1+r-\delta}\Big]$$

Let $\theta \in \Theta \subset [0,1]$ be the debtor's bargaining power. The equilibrium recovery is determined in a Nash-bargaining process and maximizes the total surplus:

$$\begin{aligned} \alpha(B_S, B_L, y) = \arg\max_{\hat{\alpha}\in[0,1]} & \,[\Delta_D(B_S, B_L, y; \hat{\alpha})]^{\theta}[\Delta_C(B_S, B_L, y; \hat{\alpha})]^{(1-\theta)} \\ s.t. \qquad & \Delta_D(B_S, B_L, y; \hat{\alpha}) \geq 0, \;\; \Delta_C(B_S, B_L, y; \hat{\alpha}) \geq 0 \end{aligned} \tag{2}$$

Creditors' Problem

Creditors are assumed to be risk neutral that inelastically access to the capital at the risk-free interest rate. Hence bond prices will be the present value of the expected future payments. Assume the short and long term debt are B_S and B_L, respectively. Following Arellano and Ramanarayanan (2012), let $R(B_S, B_L)$ denote the set of output levels at which the country services the debt, and $D(B_S, B_L)$ the set of output levels at which default is optimal:

$$R(B_S, B_L) = \{y : W(B_S, B_L, y) \geq V^D(B_S, B_L, y)\}$$

$$D(B_S, B_L) = \{y : W(B_S, B_L, y) < V^D(B_S, B_L, y)\} \tag{3}$$

One-period bonds are expected to pay back the face value if the sovereign honers the debt, and the recovered value (α fraction of the face value) if the sovereign defaults. Hence, one can write the price of the one-period bond as:

$$q_{S,t} = \int_{R(b^S_{t+1}, b^L_{t+1})} \frac{f(y_{t+1}, y_t)}{(1+r)} dy_{t+1} + \int_{D(b^S_{t+1}, b^L_{t+1})} \alpha(b^S_{t+1}, b^L_{t+1}, y_{t+1}) \frac{f(y_{t+1}, y_t)}{(1+r)} dy_{t+1} \tag{4}$$

For the long term bonds, contingent on servicing the debt till n periods ahead, the present value of the period $t+n$ to creditor is $\frac{\delta^{n-1}}{(1+r)^n}$. The country may defaults in period $t+n$, contingent of servicing the debt in periods $t+1, t+2, \ldots, t+n-1$. Upon default, it is expected to receive α fraction of the the outstanding debt $\delta^{n-1}\frac{1+r}{1+r-\delta}$. Notice this paper assumes after default, the sovereign has to pay back all the outstanding debts before issuing new bonds. This implies countries cannot default twice on the same bond.

$$q_t^L = \sum_{n=1}^{\infty} \frac{\delta^{n-1}}{(1+r)^n}$$

$$\int\limits_{R(b_{t+1}^S, b_{t+1}^L)} \dots \int\limits_{R(b_{t+n}^S, b_{t+n}^L)} f(y_{t+n}, y_{t+m-1}) \dots f(y_t + 1, y_t) dy_{t+n} \dots dy_{t+1}$$

$$+ \sum_{n=1}^{\infty} \frac{\delta^{n-1}}{(1+r)^n} \int\limits_{R(b_{t+1}^S, b_{t+1}^L)} \dots \int\limits_{D(b_{t+n}^S, b_{t+n}^L)} \alpha(b_{t+n}^S, b_{t+n}^L, y_{t+n}) \frac{1+r}{1+r-\delta}$$

$$f(y_{t+n}, y_{t+n-1}) \dots f(y_t + 1, y_t) dy_{t+n} \dots dy_{t+1}$$

As in Arellano and Ramanarayanan(2012), one can write the above expression in a recursive form:

$$q_t^L = \int\limits_{R(b_{t+1}^S, b_{t+1}^L)} \frac{f(y_{t+1}, y_t)}{(1+r)} dy_{t+1} + \int\limits_{D(b_{t+1}^S, b_{t+1}^L)} \alpha(b_{t+1}^S, b_{t+1}^L, y_{t+1}) \frac{1+r}{1+r-\delta} \frac{f(y_{t+1}, y_t)}{(1+r)} dy_{t+1}$$

$$+\delta \int\limits_{R(b_{t+1}^S, b_{t+1}^L)} \Big[\int\limits_{R(b_{t+2}^S, b_{t+2}^L)} \frac{f(y_{t+2}, y_{t+1})}{(1+r)^2} dy_{t+2} \Big] f(y_{t+1}, y_t) dy_{t+1}$$

$$+\delta \int\limits_{R(b_{t+1}^S, b_{t+1}^L)} \Big[\int\limits_{D(b_{t+2}^S, b_{t+2}^L)} \frac{(1+r)\alpha(b_{t+2}^S, b_{t+2}^L, y_{t+2})}{1+r-\delta} \frac{f(y_{t+2}, y_{t+1})}{(1+r)^2} dy_{t+2} \Big] f(y_{t+1}, y_t) dy_{t+1}$$

$$+ \dots$$

since the expression in the brackets is q^L_{t+1}, the the price function reduces to:

$$q^L_t = \frac{1}{1+r}\Big\{\int_{R(b^S_{t+1},b^L_{t+1})} \Big[1+\delta q^L_{t+1}(b^S_{t+2}, b^L_{t+2}, y_{t+1})\Big] f(y_{t+1}, y_t)dy_{t+1}$$
$$+ \int_{D(b^S_{t+1},b^L_{t+1})} \frac{1+r}{1+r-\delta}\alpha(b^S_{t+1}, b^L_{t+1}, y_{t+1})f(y_{t+1}, y_t)dy_{t+1}\Big\} \tag{5}$$

For $\delta = 0$, this will reduce to the price of the short term bond, expressed in 5. With 100% haircuts, the above expression will be reduced to Arellano and Ramanarayanan (2012). Furthermore, with $\alpha = 1$ for all the states, the long term bond will turn to a risk-free bond:

$$(1+r)q_L = \int_{R(b^S_{t+1},b^L_{t+1})} 1+\delta q^L_{t+1}(b^S_{t+1}, b^L_{t+1}, y_{t+1})f(y_{t+1}, y_t)dy_{t+1}$$
$$+ \int_{D(b^S_{t+1},b^L_{t+1})} \frac{1+r}{1+r-\delta} f(y_{t+1}, y_t)dy_{t+1}$$
$$+ \int_{R(b^S_{t+1},b^L_{t+1})} \frac{1+r}{1+r-\delta} f(y_{t+1}, y_t)dy_{t+1} - \int_{R(b^S_{t+1},b^L_{t+1})} \frac{1+r}{1+r-\delta} f(y_{t+1}, y_t)dy_{t+1}$$

$$\Big(1+r-\delta \int_{R(b^S_{t+1},b^L_{t+1})} f(y_{t+1}, y_t)dy_{t+1}\Big)q_L = \Big(1+r-\delta \int_{R(b^S_{t+1},b^L_{t+1})} f(y_{t+1}, y_t)dy_{t+1}\Big)\frac{1}{1+r-\delta}$$

$$q_L = \frac{1}{1+r-\delta}$$

3 Results

Recursive Equilibrium

A recursive equilibrium for this economy consists of a set of functions defined below; for $s = (B_S, B_L, y)$:

- The country's value functions, $V(s), W(s), V^D(s), W^D(s)$, and $V(y)^{AUT}$ and policy functions of short and long term bonds, $B'_S(s), B'_L(s)$, and consumption, $C(s)$,
- Default set, $D(B_S, B_L)$, and the repayment set $R(B_S, B_L)$,
- Price functions $q_S(B'_S, B'_L, y)$ and $q_L(B'_S, B'_L, y)$ in 4 and 5, such that given the recovery rate $\alpha(s)$
    1. The default and the repayment sets are the equilibrium sets defined above,
    2. Next period's bond holdings are in agreement with the country's policy functions:

    $$b^S_{t+1} = B'_S(s), \;\; b^L_{t+1} = B'_L(s)$$

- The recovery rate function $\alpha(s)$ such that

1. Given the bond prices and the recovery rate, country solves the recursive problem
2. Given the bond prices, the value functions, the policy functions, and the default and repayment sets, recovery rate solves the debt renegotiation problem.
3. Given the recovery rate, bond prices satisfy the zero profit condition for the bondholders.

Proposition 1: $\forall \theta \in \Theta$ recursive equilibrium of the above model exist.

Lemma 1: The debt renegotiation problem is invariant to any transfer of debt that keeps $B_S + \frac{B_L(1+r)}{1+r-\delta}$ unchanged.

Proof: See Appendix

Proposition 2: The equilibrium recovery rate, $\alpha(B_S, B_L, y)$, satisfies

$$\alpha(B_S, B_L, y) = \begin{cases} 1 & (B_S + d^\delta B_L) \leq \zeta(y) \\ \frac{\zeta(y)}{(B_S + d^\delta B_L)} & (B_S + d^\delta B_L) \geq \zeta(y) \end{cases} \tag{6}$$

where $d^\delta = \frac{1+r}{1+r-\delta}$.

Proof: See Appendix

This result extends Yue (2010) to two dimensions and Farzad (2018a) to instruments with different maturities.[11] Upon default, for each level of endowment y, the value function of default is independent of $B_S + d^\delta B_L$, henceforth called the total dated debt. This simplification helps us to derive the same set of result as in Eaton and Gersovitz (1981) and Chatterjee et al. (2007), Arellano (2008), Yue (2010), and Bi (2006) all explained below.

[11]for one period debt instruments, $\delta = 0$, the relation reduces to $B_S + B_L$, as in Farzad (2018a).

Proposition 3: If default is optimal for a state (B_S^1, B_L^1, y), then it is also optimal for all (B_S^2, B_L^2, y) that $D^2 = B_S^2 + d^\delta B_L^2 \geq B_S^1 + d^\delta B_L^1 = D^1 \geq D(y)$. Hence $D(B_S^1, B_L^1) \subseteq D(B_S^2, B_L^2)$.

Corollary 1: If default is optimal for a state (B_S^1, B_L, y), then it is also optimal for all (B_S^2, B_L, y) that $B_S^2 \geq B_S^1$.

Corollary 2: If default is optimal for a state (B_S, B_L^1, y), then it is also optimal for all (B_S, B_L^2, y) that $B_L^2 \geq B_L^1$.

Corollary 3: Default probability is increasing in B_S and B_L.

Proof: See Appendix

Proposition 4: For any level of endowment the equilibrium price of short (long) term bond is decreasing in quantity demanded for short (long) term bond:

$$q_S(B_S^2, B_L, y) \leq q_S(B_S^1, B_L, y), \;\; \forall B_S^2 \geq B_S^1$$
$$q_L(B_S, B_L^2, y) \leq q_L(B_S, B_L^1, y), \;\; \forall B_L^2 \geq B_L^1$$

Proof: See Appendix

Proposition 5: Sovereign's value functions are increasing in the realized income. Hence, default incentives are higher at lower income levels.

Proof is the direct application of contraction mapping theorem.[12]

Proposition 6 below builds on Yue (2010) to shows it might be optimal for the sovereign to pay back the outstanding debt over the span of multiple periods. Let $\tilde{W^D}(D, y) = W^D(B_S, B_L, y)$ when $D = B_S + d^\delta B_L$.

Proposition 6: Consider the repayment problem 1, for a given y, if $D' = 0$ is optimum for a state described by $D = B_S + B_L$ and y, then $D' = 0$ is also optimal for all $\hat{D} < D$. Also partial payment, $D' < 0$, is optimal for all $\hat{D} > D$, if

$$\tilde{W^D}(D, y) = u((1-\lambda)y - D) + \beta \int_Y v(0, 0, y') f(y', y) dy'.$$

Proof: See Appendix

Solution Algorithm

First we need to discretize the asset and the income space. For the asset spaces, short and long term bond, we choose an upper-bound large enough that doesn't distort the optimization solution. For the income space, we use Tauchen (1986) approach to find the Markov chain approximation of the endowment.

The solution algorithm is as follows:

1. Start with an arbitrary initial value for the recovery rate, α. For simplicity we can start with $\alpha \equiv 0.50$.

[12] See Stockey, Lucas, and Prescott (1989)

2. Choose arbitrary values for bond prices, q_S and q_B. One can assign the risk-free prices as the initial values.

3. Solve the sovereign's problem and derive the policy functions, repayment and default sets.

4. Update the prices according to 4 and 5 . Find the difference between the initial prices and the updated ones. Solve the sovereign's problem in step 3 until the updated prices are the close enough to the the ones calculated before.

5. Find the recovery rate that solves the Nash bargaining game in 2. Find the difference between the initial value and the updated one, and start over from step 3 as long as this difference is above the preset tolerance.

4 Quantitative Analysis

Parametrization

Table 1 shows the selected parameters for the quantitative analysis. As mentioned before, preferences are modeled with a CARA utility function. Following the literature, the coefficient of risk aversion, σ, is set to two:

$$u(C) = \frac{C^{1-\sigma}}{1-\sigma}$$

.

Endowment process is estimated using Argentina's GDP using data from Ministry of Finance (MECON). The quarterly data (real, seasonally adjusted) starts from the first quarter of 1980 till the default episode of 2001, the last quarter of 2001. As in Arellano (2008), this paper assumes a log-normal AR(1) process for the GDP:

$$\log(y_t) = \rho \log(y_{t-1}) + \varepsilon_t, \quad E[\varepsilon] = 0 \text{ and } E[\varepsilon^2] = \eta^2$$

The estimated persistence, ρ and error standard deviation, η, are $\rho = 0.95$ and $\eta = 0.02$. Then Markov chain with 21 discrete endowment state is constructed Using Tauchen (1986).

The annual risk-free interest rate is set to $r = 4\%$, which is the average 1-year yield of the US bonds in that period. Decay rate, $\delta = 0.936$ is selected to represent 10-year default-free duration. Output loss during default is $\lambda = 2\%$, as suggested by Sturzenegger (2002).

Time preference, β, and borrower's bargaining power, θ are calibrated to match the model moments to the observed data. Time preference is calibrated to match the default frequency in data. According to Reinhart, Rogoff, and Savastano (2003) Argentina experienced four default episodes from 1824 to 1999. Including the 2001 default results in an annual (quarterly) default frequency equals to 2.8% (0.7%). According to Benjamin and Wright (2009), the average debt recovery rate in the 2001 default was 37%. In order to match this moment, the debtor' bargaining power, θ is calibrated to $\theta = 0.83$.

Simulation Results

Figure 3 plots the price functions for short term and long term bonds as a function of choice of short term and long term debts, respectively. To illustrate the role of endowment, the prices are depicted for high and low

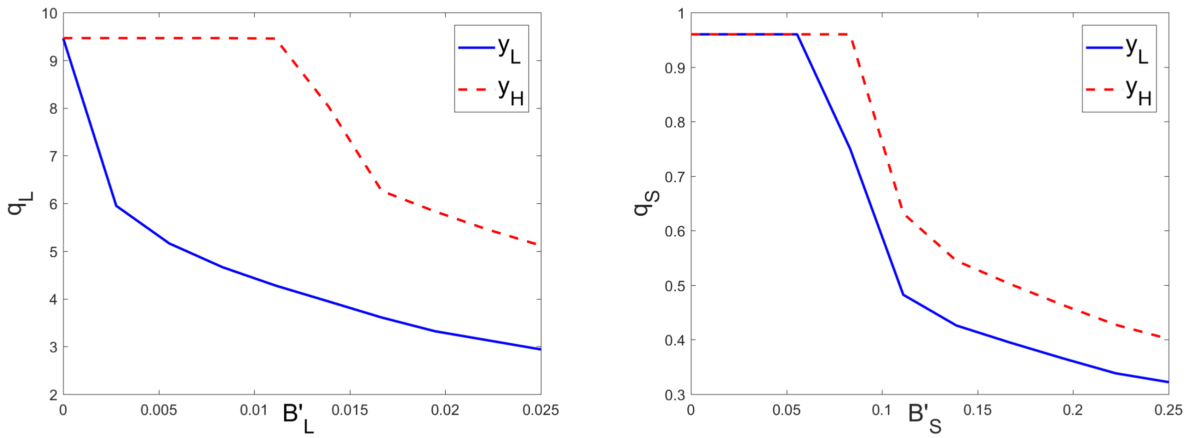
Parameter	Value	Target
one-year risk free rate	$r = 4\%$	U.S. annual interest rate
Risk Aversion	$\sigma = 2$	Literature
Decay factor	$\delta = 0.936$	10-year Default-free duration
Output loss	$\lambda = 2\%$	Sturzenegger (2002)
Stochastic structure	$\rho = 0.95$, η=0.025	Argentina output (1980Q1: 2001Q4)

Calibration	Value	Target statistics
Borrower's discount factor	$\beta = 0.94$	2.8% Default Probability
Borrower's bargaining power	$\theta = 0.83$	37% Ave. Debt Recovery Rate

Table 1: Parameter Values

values of income, y_H and y_L. As stated by Proposition 4, the price functions are decreasing functions in their corresponding choice variables; which easily can be seen in the Figure. Price functions clearly show the term premium for the low levels of income; for the low levels of short term debt holding, the price is the same as the risk-free bond, while the price of the long term bond is below the risk-free price even for small stock of long term holdings. Only for high levels of income and low levels of long term debt, the term premium disappears.

Table 2 summarizes the model statistics. As shown in the table, mean spread for short and long term bonds are 4.32% and 6.21%, respectively. These values are close to the corresponding values in Argentina data. Nevertheless, the short term spreads are higher than the observed values in the data, while the long term spread is below the observed data. The ratio of trade balance (TB) standard deviation to output standard deviation in the model closely follows the data. The model overestimates consumption standard deviation, which may suggest using a more sophisticated output loss function. The average debt to GDP in the model is 0.36, which explains more than 80% of the debt to GDP in data. This is an improvement compared to the previous



(a) Long Term Bond Price

(b) Short Term Bond Price

Figure 3: Bond Price Functions. Panel (a) shows the price of the long term bond as a function of next period's stock of long term bonds for two income levels y_H and y_L. Panel (b) shows the corresponding values for the short term bond.

models; Arellano and Ramanarayanan (2012) could only explain only half of the debt to GDP ratio for Brazil. The model captures the short term debt to the total debt; 8% in the model versus 11% in the data. Finally the average default duration in the model matches the data. Allowing the defaulted country to pay back the arrears in more than one period, provides a more realistic model. The last two rows of Table 2 show the calibrated parameters.

Spread Curves

To provide a clearer picture regarding the spread behaviors, we provide the spread distribution for different model specifications. In particular, the model in this paper is compared to a model with a zero recovery rate, á la Arellano and Ramanarayanan (2012), and with a constant exogenous recovery rate, á la Hatchondo et al. (2016). The corresponding distributions are reported in Table 3.

As the table suggests, the model provides spread distributions that match

Model Statistics

	Model	Data
$Mean\ s_S(\%)$	4.32	4.22
$Mean\ s_L(\%)$	6.21	6.29
$SD(TB)/SD(y)$	0.47	0.48
$SD(C)/SD(y)$	1.11	1.03
$(q_S B_S + q_L B_L)/y$	0.36	0.43
$(q_S B_S)/(q_S B_S + q_L B_L)$	0.08	0.11
Default Duration (years)	4.40	4.50
Default Frequency	2.80%	2.80%
Ave. Recovery Rate	38.35%	37%

Table 2: Model Statistics

Spread Curves

	s_S			s_L		
	Overall Mean	$< 50th$	$\geq 50th$	Overall Mean	$< 50th$	$\geq 50th$
Data	4.22	1.12	7.32	6.29	4.65	7.93
Model	4.32	1.09	7.56	6.21	4.38	8.04
$\alpha = 0\%$	3.15	0.03	6.27	3.15	2.24	4.06
$\alpha = 37\%$	3.81	0.68	5.86	5.63	4.20	7.06

Table 3: Effect of Endogenous Debt Renegotiation

the data. To put this into perspective, two sets of spread distributions are provided: first spread distributions for a model with no recovery rate ($\alpha = 0\%$) and also for a model with a constant recovery rate ($\alpha = 37\%$), as suggested by Benjamin and Wright (2009). Although the average short (long) term bond spread in the model is higher (lower) than data, the overall distribution of spreads match the realized spreads.

Discussion

In this section, an intuitive argument is presented to explain the behavior of the interest rate spreads in the models with and without debt renegotiation. First notice the model with endogenous renegotiation results in larger spread means. In addition, according to Table 2, our model delivers higher debt to GDP ratio. The association between these two can be explained by recalling that the high spreads occur at the high levels of debt. In other words, since in our model the default is optimal at higher levels of debt, the observed spreads (spreads before the default episodes) are larger than the models without debt renegotiation.

Figure 4 provides a schematic depiction of debt renegotiation effect. The optimal recovery rate in 6, makes the default value function to decrease at the slope of one, which flattens out after some endogenous level of debt (which is a function of income, $\zeta(y)$). However for the exogenous recovery rate model, the default value the value function decreases at a milder slope compared to endogenous recovery.[13] As a result, for comparatively the same debt servicing value, W, the default is optimal at higher levels of debt for the endogenous recovery rate model. This in turn will result in higher spreads in the model compared to previous model specifications.

5 Conclusion

This paper develops a default model with endogenous debt renegotiation for a small open economy. The equilibrium recovery rate in the model allows the borrowing country to hold higher levels of debt. Consequently near the defaults, the resulted spreads are higher compared to the model specifications

[13]The default value decreases at a constant rate $\alpha < 1$.

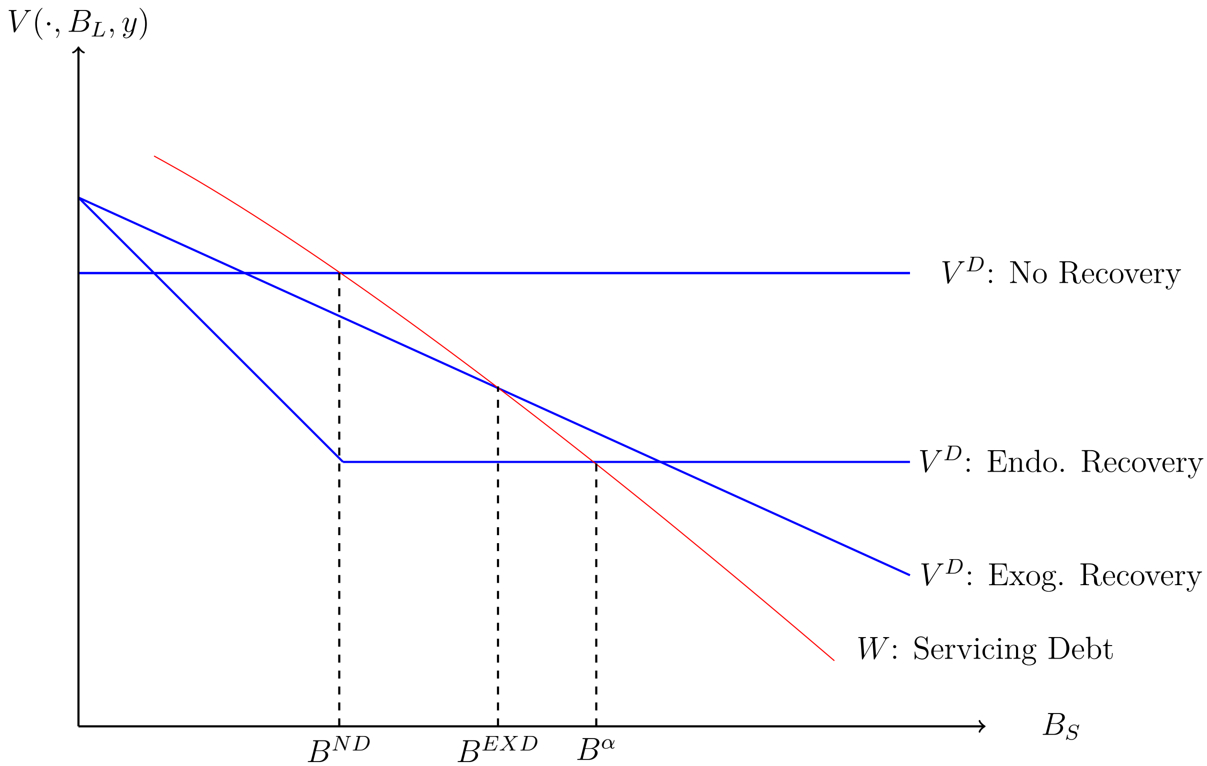


Figure 4: Schematic View of Value Functions.

without the endogenous restructuring of the debt.

A better understanding of the interest rate behavior will help the countries to select a maturity structure that minimizes the outflow of resources at the onset of financial crisis. One immediate extension of this problem is to include the possibility of partial defaults; i.e. adding bonds with different maturities to a partial default model (Arellano, Mateos-Planas, and Rios-Rull, 2013). This paper assumes the same haircut for different maturities. Any deviation from this assumption affects the optimal maturity structure. That is, whether the borrowing country be treated differently after default, affects the ex-ante decisions regarding the debt levels. This further sheds light on the liquidity-insurance trade-off between short and long term debts.

Appendix

Proof of Proposition 1: The proof follows Yue (2010). We can show in the similar manner that following problems have a fixed point:

1. The bond price functions, given the default and repayment sets, the policy functions of the borrowing country, and the recovery rate,
2. The debt renegotiation problem, given the bond prices,
3. The borrowing country value functions, given the bond prices and recover rate.

Proof of Lemma 1:

$$W^D(B_S, B_L, y) = \max_{0 \leq B'_S \leq B_S,\ 0 \leq B'_L \leq B_L,\ 0 \leq C} u(C) + \beta \int_Y W^D(B'_S, B'_L, y') f(y'|y) dy'$$

$$C + B_S + B_L = (1-\lambda)y + \frac{B'_S}{1+r} + \frac{B'_L - \delta B_L}{1+r-\delta}$$

$$C = (1-\lambda)y + \Big[\frac{B'_S}{1+r} - B_S\Big] + \Big[\frac{B'_L}{1+r-\delta} - \frac{B_L(1+r)}{1+r-\delta}\Big]$$

Letting $X = \frac{B_L(1+r)}{1+r-\delta}$, we can rewrite the above budget constraint as:

$$C = y + \Big[\frac{B'_S}{1+r} - B_S\Big] + \Big[\frac{X'_L}{1+r} - X_L\Big]$$

$$C + B_S + X_L = (1-\lambda)y + \Big[\frac{B'_S}{1+r} + \frac{X'_L}{1+r}\Big]$$

Since $X'_L \leq X_L$ implies $B'_L \leq B_L$, 1 is invariant to any transfer of debt as long as $B_S + \frac{B_L(1+r)}{1+r-\delta}$ is fixed. Since V^D solely depends on W^D, debtor's surplus is invariant to a transformation shown above. The present values show creditors' surplus is invariant to such transformations as well.

Proof of Proposition 2[14]:

Appealing to Lemma 1, W^D and consequently V^D is a function of $\alpha(B_S + d^\delta B_L)$. Hence $\Delta_D = V^D - V^{Aut}$ is a function of $\alpha(B_S + d^\delta B_L)$. Δ_C in 2 is also a function $\alpha(B_S + d^\delta B_L)$. This means the solution to 2 can be written as $\alpha(B_S + d^\delta B_L) = \zeta(y)$. Since $\alpha \in [0, 1]$, $\alpha = 1,\ \forall(B_S + d^\delta B_L) \leq \zeta(y)$ and $\frac{\zeta(y)}{(B_S+d^\delta B_L)},\ \forall(B_S + d^\delta B_L) \geq \zeta(y)$.

Proof of Proposition 3:

The proof follows Eaton and Gersovitz (1981) and Chatterjee et al. (2007), Arellano(2008), and Yue(2010). Since the value of default is independent of total dated debt level, for any debt level above $D^1 =\leq D^2$, $W(B_S^2, B_L^2, y) \leq W(B_S^1, B_L^1, y) \leq V^D(B^1, L^1, y) = V^D(B^2, L^2, y)$. Proof of the corollaries is by contradiction.

Proof of Proposition 4:

First notice by increasing short term bonds $R(B'_S, B'_L)$ and $D(B'_S, B'_L)$ in 3 will (weakly) shrink and expand, respectively. Also α (weakly) decreases according to proposition 2. Hence, the price of short term bond is (weakly) decreasing.

[14] Extension of Yue (2010).

For the long term bonds, we can appeal to the contraction mapping characteristics.[15] Using the fact that by increasing long term bonds $R(B'_S, B'_L)$ and $D(B'_S, B'_L)$ in 3 will (weakly) shrink and expand, respectively, one can show that the price of long term bond is a decreasing function in B_L.

Proof of Proposition 6:

The proposition is a direct consequence of properties of policy functions; if $D_1 > D_2$, then $D'_1 > D'_2$.

Notice $\tilde{W^D}(D, y)$ is decreasing and concave in D; if $D_1 > D_2$, then $\tilde{W^D}(D_1, y) < \tilde{W^D}(D_2, y)$ and $\frac{\partial}{\partial D}\tilde{W^D}(D_1, y') < \frac{\partial}{\partial D}\tilde{W^D}(D_2, y') < 0$.

By contradiction, let $D'_1 < D'_2$ for $D_1 > D_2$. The optimization problem requires:

$$\frac{1}{1+r}u'(y + \frac{D'_1}{1+r} - D_1) = -\beta \int_Y \frac{\partial}{\partial D}\tilde{W^D}(D'_1, y')f(y', y)dy'$$

$$\frac{1}{1+r}u'(y + \frac{D'_2}{1+r} - D_2) = -\beta \int_Y \frac{\partial}{\partial D}\tilde{W^D}(D'_2, y')f(y', y)dy'$$

However due to concavity of u,

$$u'(y + \frac{D'_1}{1+r} - D_1) \geq u'(y + \frac{D'_2}{1+r} - D_2)$$

Which means,

$$-\beta \int_Y \Big[\frac{\partial}{\partial D}\tilde{W^D}(D'_1, y') - \frac{\partial}{\partial D}\tilde{W^D}(D'_2, y')\Big] f(y', y)dy' \geq 0$$

Which is a contraction.

[15] See Stokey, Lucas, and Prescott (1989)

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