TECHNICAL SUPPLEMENT TO NONPARAMETRIC ESTIMATION OF CONDITIONAL VALUE-AT-RISK AND EXPECTED SHORTFALL BASED ON EXTREME VALUE THEORY

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Abstract. This supplement contains statements and proofs for Lemmas 1, 2, 3 and 4 in Martins-Filho et al. (2013). In addition, there are statements for Corollary 1, Lemmas 5 and 6.

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1 Lemmas and proofs

Lemma 1. Let \( w(X_t - x; x) : \mathbb{R}^d \to \mathbb{R} \) and \( g(\varepsilon) : \mathbb{R} \to \mathbb{R} \) be measurable functions and define

\[
s(x) = \frac{1}{nh_n^d} \sum_{t=1}^{n} K\left( \frac{X_t - x}{h_n} \right) p_1 \left( \frac{X_t - x}{h_n} \right) p_2 \left( \frac{X_t - x}{h_n} \right) p_3 \left( \frac{X_t - x}{h_n} \right) w(X_t - x; x) g(\varepsilon_t)
\]

where \( K \) is a multivariate kernel given by \( K(x) = \prod_{j=1}^{d} K(x_j) \), \( h_n > 0 \) is a bandwidth, for \( i, j = 1, \ldots, d \) and \( p_1, p_2, p_3 = 0, 1 \). Assume that \( A1 \) and \( A2 \) are holding and that:

a) \( E(|g(\varepsilon)|) < \infty \) for some \( \zeta > 2 \);

b) \( w(X_t - x; x) \) satisfies a Lipschitz condition of order 1, i.e., \( |w(X_t - x; x) - w(X_t - x^k; x^k)| \leq C \|x - x^k\| \)

for some \( C > 0 \) and \( x \neq x^k \) in \( \mathbb{R}^d \) and \( |w(X_t - x; x)| < C \) for all \( x \in \mathbb{R}^d \);

c) The joint density of \( X_i \) and \( X_j \) conditional on \( \varepsilon_i \) and \( \varepsilon_j \) denoted by \( f_{X_i, X_j|\varepsilon_i, \varepsilon_j}(X_i, X_j) < C \).

Then, for an arbitrary compact set \( G \subseteq \mathbb{R}^d \), we have

\[
\sup_{x \in G} |s(x) - E(s(x))| = O_p \left( \left( \frac{\log n}{nh_n^2} \right)^{1/2} \right)
\]

provided that for \( \zeta, B > 2, \theta > 0 \), we have

\[
n^{1 - \frac{2}{\zeta} - 2\theta} h_n^d \to \infty
\]

and

\[
n^{(B+1.5)(\frac{1}{\zeta} + \theta) - \frac{B}{2} + 0.75 + \frac{d}{2}h_n^{1.75d - \frac{d}{2}(d+B)}(\log n)^{0.25 + 0.5(B-d)} \to 0.
\]

Proof. We first establish the result for the case where \( p_1 = p_2 = p_3 = 0 \). The proof follows Martins-Filho [2009] and has three steps: (1) we show that \( \sup_{x \in G} |s(x) - E(s(x))| \leq \max_{1 \leq k \leq i_n} |s_0(x^k) - E(s_0(x^k))| + 2 C \left( \frac{\log n}{nh_n^2} \right)^{1/2} \)

for a suitably defined sequence \( l_n \) and \( x^k \in G \); (2) we show that \( \sup_{x \in G} |s(x) - s^\tau(x) - E(s(x) - s^\tau(x))| = O_{as}(B_n^{1-\zeta}) \), where

\[
s^\tau(x) = \left( nh_n^d \right)^{-1} \sum_{i=1}^{n} K\left( \frac{X_t - x}{h_n} \right) w(X_t) g(\varepsilon_t) \chi_{|g(\varepsilon)| \leq B_n}
\]

with \( B_1 \leq B_2 \leq \ldots \) such that \( \sum_{i=1}^{\infty} B_i^{1-\zeta} < \infty \) for some \( \zeta > 0 \); (3) we show that for \( 0 < \Delta < \infty, B > 2 \) and \( \varepsilon_n = \left( \frac{nh_n^d}{\log n} \right)^{-1/2} \Delta, P\left( \max_{1 \leq k \leq l_n} \left| s^\tau(x^k) - E(s^\tau(x^k)) \right| \geq \varepsilon_n \right) = O(d_n) \) where

\[
d_n = n^{(B+1.5)(\frac{1}{\zeta} + \theta) - B/2 + 0.75 + \frac{d}{2}h_n^{1.75d - \frac{d}{2}(d+B)}(\log n)^{0.25 + 0.5(B-d)}}.
\]
Let $B(x_0, r) = \{ x \in \mathbb{R}^d : \| x - x_0 \| < r \}$ for $r \in \mathbb{R}^+$. $G$ compact implies that there exists $x_0 \in \mathbb{R}^d$ such that $G \subset B(x_0, r)$. Therefore, for all $x, z \in G$, $\| x - z \| < 2r$. Let $h_n > 0$ be such that $h_n \to 0$ as $n \to \infty$ where $n \in \{1, 2, 3, \cdots \}$. For any $n$, by the Heine-Borel Theorem, every infinite cover for $G$ contains a finite subcover $\left\{ B \left( x^k, C \left( \frac{n}{h_n^2 log n} \right)^{-1/2} \right) \right\}_{k=1}^{l_n}$ with $x^k \in G$ and $l_n \leq \left( \frac{n}{h_n^2 log n} \right)^{2/3}$. Step (1): For $x \in B \left( x^k, C \left( \frac{n}{h_n^2 log n} \right)^{-1/2} \right)$,

$$|s(x) - s(x^k)| = \left| \frac{1}{nh_n^d} \sum_{l=1}^{n} \left( K \left( \frac{x_l - x}{h_n} \right) w(X_t - x; x) - K \left( \frac{x_l - x^k}{h_n} \right) w(X_t - x; x) \right) \right|$$

$$\leq \frac{1}{nh_n^d} \sum_{l=1}^{n} \left| \left( K \left( \frac{x_l - x}{h_n} \right) - K \left( \frac{x_l - x}{h_n} \right) \right) \right| w(X_t - x; x)$$

$$\leq \frac{C}{h_n^d} \| x^k - x \|_E \left( \frac{log n}{nh_n^2} \right)^{1/2} \left( \frac{n}{h_n^2 log n} \right)^{-1/2} \sum_{l=1}^{n} |g(\varepsilon_l)|$$

By the measurability of $g$ and assumption A2 1) we have that $\{|g(\varepsilon_l)|\}_{l=1,2,...}$ is $\alpha$-mixing of size $-2$. By condition a) and McLeish’s LLN (White 2001, p. 49) $\frac{1}{n} \sum_{l=1}^{n} (|g(\varepsilon_l)| - E(|g(\varepsilon_l)|)) = o_p(1)$ and since $E(|g(\varepsilon_l)|) \leq C$ we have $|s(x) - s(x^k)| \leq C \left( \frac{log n}{nh_n^2} \right)^{1/2}$. Following similar arguments it is easily verified that $E(|s(x) - s(x^k)|) \leq C \left( \frac{log n}{nh_n^2} \right)^{1/2}$. Combining these two bounds, $\sup_{x \in G} |s_0(x) - E(s_0(x))| \leq \max_{1 \leq k \leq l_n} |s_0(x^k) - E(s_0(x^k))| + 2C \left( \frac{\log n}{nh_n^2} \right)^{1/2}$. Step (2): $\sup_{x \in G} |s(x) - s^*(x) - E(s(x) - s^*(x))| \leq T_1 + T_2$, where $T_1 = \sup_{x \in G} |s(x) - s^*(x)|$ and $T_2 = \sup_{x \in G} |E(s(x) - s^*(x))|$. We show that $T_1 = o_n(1)$ and $T_2 = O(B_1^{-\zeta})$ for $\zeta > 1$. Note that $T_1 = \sup_{x \in G} \left| \left( \frac{n}{h_n^d} \right)^{-1} \sum_{l=1}^{n} K \left( \frac{x_l - x}{h_n} \right) g(\varepsilon_l) w(X_t - x; x) \chi(|g(\varepsilon_l)| > B_n) \right|$. By the Borel-Cantelli Lemma for any $\epsilon > 0$ and for all $m$ satisfying $m' < m < n$ we have $P(|g(\varepsilon_m)| \leq B_n) > 1 - \epsilon$, since $\{B_t\}_{t=1,2,...}$ is an increasing sequence. By Chebyshev’s Inequality, for $t = 1, \cdots, m'$ and $\zeta > 0$, $P(|g(\varepsilon_t)| > B_n) < \frac{E(|g(\varepsilon_t)|^\zeta)}{B_n^\zeta} < \frac{C}{B_n}$ by a). Consequently, for all $\epsilon > 0$ and sufficiently large $n$, we have $P(|g(\varepsilon_t)| < B_n) > 1 - \epsilon$. Hence, for $n > \max\{N, m\}$ we have that for all $t \leq n$, $P(|g(\varepsilon_t)| < B_n) > 1 - \epsilon$ and therefore $\chi(|g(\varepsilon_t)| > B_n) = 0$ with
probability 1, which gives \( T_1 = o_n(1) \). For \( T_2 \), note that by A2 1) and A2 2)

\[
E(s(x) - s^\gamma(x)) = \frac{1}{n h_n^d} \sum_{t=1}^n \int \int K\left(\frac{X_t - x}{h_n}\right) w(X_t - x; x) g(\varepsilon_t) f_X(X_t) f(\varepsilon_t) dx_t d\varepsilon_t
\]

\[
\leq \int \int K(v) w(x + h_n v; x) f_X(x + h_n v) dv \int |g(\varepsilon)| f(\varepsilon) \chi_{\{|g(\varepsilon)| > B_n\}} d\varepsilon
\]

where \( v_l \) is the \( l^{th} \) element of \( v \)

\[
\leq C \int |g(\varepsilon)| f(\varepsilon) \chi_{\{|g(\varepsilon)| > B_n\}} d\varepsilon.
\]

The last inequality follows from A1 1), 2) and 4), condition b) and the the fact that \( w \) and \( f_X \) are continuous functions. By Hölder’s inequality, for \( \zeta > 1 \),

\[
\int \chi_{\{|g(\varepsilon)| > B_n\}} |g(\varepsilon)| f(\varepsilon) d\varepsilon \leq \left( \int |g(\varepsilon)|^\zeta f(\varepsilon) d\varepsilon \right)^{1/\zeta} \left( \int \chi_{\{|g(\varepsilon)| > B_n\}} f(\varepsilon) d\varepsilon \right)^{1 - 1/\zeta}
\]

where the first integral after the inequality is uniformly bounded by a) and by Chebyshev’s Inequality

\[
\left( \int \chi_{\{|g(\varepsilon)| > B_n\}} f(\varepsilon) d\varepsilon \right)^{1 - 1/\zeta} \leq C \left( P(|g(\varepsilon)| > B_n) \right)^{1 - 1/\zeta} \leq CB_{n}^{1 - \zeta}. \quad \text{Hence, \( T_2 = O(B_{n}^{1 - \zeta}). \) As in Step (1) and given the orders for \( T_1 \) and \( T_2 \),}

\[
\sup_{x \in V} |s(x) - E(s(x))| \leq \sup_{x \in V} |s^\gamma(x) - E(s^\gamma(x))| + \sup_{x \in V} |s(x) - s^\gamma(x) - (E(s(x)) - E(s^\gamma(x)))|
\]

\[
\leq \max_{1 \leq k \leq l^n} |s^\gamma(x) - E(s^\gamma(x))| + O \left( \left( \frac{\log n}{nh_n^d} \right)^{1/2} \right) + O(B_{n}^{1 - \zeta}) \text{ for } \zeta > 1.
\]

Step (3): For \( \varepsilon_n = \left( \frac{nh_n^d}{\log n} \right)^{-1/2} \Delta \) with \( 0 < \Delta < \infty \), note that \( P \left( \max_{1 \leq k \leq l^n} |s^\gamma(x^k) - E(s^\gamma(x^k))| \geq \varepsilon_n \right) \leq \sum_{k=1}^n P \left( |s^\gamma(x^k) - E(s^\gamma(x^k))| \geq \varepsilon_n \right) \). Let \( s^\gamma(x^k) - E(s^\gamma(x^k)) = \frac{1}{n} \sum_{t=1}^n Z_t \) with

\[
Z_t = \frac{1}{h_n^d} K\left(\frac{X_t - x^k}{h_n}\right) w(X_t - x^k; x^k) g(\varepsilon_t) \chi_{\{|g(\varepsilon_t)| \leq B_n\}} - E \left( \frac{1}{h_n^d} K\left(\frac{X_t - x^k}{h_n}\right) w(X_t - x^k; x^k) g(\varepsilon_t) \chi_{\{|g(\varepsilon_t)| \leq B_n\}} \right).
\]

By A1 1), b) and \( |g(\varepsilon_t)| \chi_{\{|g(\varepsilon_t)| \leq B_n\}} \) \( \leq B_n \) we have that

\[
|Z_t| \leq Ch_n^{-d} B_n. \quad (7)
\]

Let \( \|Z_t\|_\infty = \inf \{ a : P(Z_t > a) = 0 \} \), then \( \sup_{1 \leq t \leq n} \|Z_t\|_\infty \leq C B_{n}^{1 - \zeta} \). Then, from Theorem 1.3 in \text{Bosq (1996)}

we have that for each \( q = 1, 2, ..., \left[ n/2 \right] \)

\[
P \left( \frac{1}{n} \sum_{t=1}^n Z_t \right) > \varepsilon_n \right) \leq 4 \exp \left( \frac{-\varepsilon_n^2 q}{8V(q)} \right) + 22 \left( 1 + \frac{4CB_{n}}{\varepsilon_n h_n^d} \right)^{1/2} \left[ n \over 2q \right].
\]
Thus, $\sigma^2(q) = \frac{2}{p^2} \sigma^2(q) + \frac{CB_n}{2l^2}$ and $\sigma^2(q) = \max_{0 \leq j \leq 2q-1} E \left( ((|jp| + 1 - jp)Z_{[jp]+1} + Z_{[jp]+2} + \cdots + Z_{j+1p}) + ((j + 1)p - (j + 1p) Z_{(j+1)p+1}) \right)^2$. We first show that $\frac{h_n^d}{p} \sigma^2(q) = O(1)$. To see this, note that

$$
\sigma^2(q) \leq \max_{0 \leq j \leq 2q-1} \left( \sum_{|jp| < i \leq (j+1)p+1} E(Z_i^2) + 2 \sum_{|jp|+1 \leq i \leq (j+1)p} \sum_{i \leq j+1p+1} |E(Z_i Z_j)| \right).
$$

Given a), b), c) and A2 we have $\sum_{|jp| < i \leq (j+1)p+1} E(Z_i^2) \leq O\left(h_n^{-d}\right)$. Since $E\left[|Z_i|^{16}\right] = O\left(h_n^{d\left(1-\frac{1}{4}\right)}\right)$ for $\delta > 2$, by Theorem 3 (1) in [Doukhan 1994] $|E(Z_i Z_j)| \leq C h_n^{\frac{d}{2}(\frac{1}{4} - 1)} \alpha(i - l)^{1 - \frac{d}{2}}$. Now, for any $l$ such that $|jp| + 1 \leq l \leq (j+1)p$ we have that $\sum_{|jp|+1 \leq i \leq (j+1)p+1} |E(Z_i Z_j)| \leq \sum_{i=1}^{p^* - 1} |E(Z_i Z_{i+l})| + \sum_{i=1}^{p^* - 1} |E(Z_i Z_{i-l})|$ where $p^* = [(j+1)p + 1] - |jp|$. Letting $d_n = h_n^\frac{d}{2} \{1 - \frac{d}{4}\}$ be a sequence of integers, we have that $d_nh_n^d \to 0$ whenever $a_1 > 1 - \frac{d}{2}$. Hence, we can write

$$
\sum_{i=1}^{p^* - 1} |E(Z_i Z_{i+l})| = \sum_{i=1}^{d_n-1} |E(Z_i Z_{i+l})| + \sum_{i=d_n}^{p^* - 1} |E(Z_i Z_{i+l})| = J_1 + J_2
$$

and easily show that $J_1 = o(h_n^{-d})$ and $J_2 = O(h_n^{-d})$. Similarly we obtain $\sum_{i=1}^{p^* - 1} |E(Z_i Z_{i-l})| = O(h_n^{-d})$.

Combining the results on the variance and covariances we have that $\frac{h_n^d}{p} \sigma^2(q) \leq C$ for $n$ sufficiently large.

Hence, we have that $ph_n^d v^2(q) \leq C + CpB_n \varepsilon_n$ and choosing $p = \left(\frac{B_n \varepsilon_n}{C}\right)^{-1}$ (with $B_n \varepsilon_n \to 0$) we have that for $n$ sufficiently large $ph_n^d v^2(q) \leq C$. Then, $4exp\left(\frac{-v^2}{a_n^2}\right) \leq 4exp\left(\frac{-v^2 h_n^d}{16\varepsilon_n}\right) \leq 4n^{-\frac{a^2}{16\varepsilon_n}}$. Now, given A2, for $B > 2$ and $n$ sufficiently large

$$
22 \left(1 + \frac{4CB_n}{\varepsilon_n h_n^d}\right)^{1/2} q_{\alpha} \left(\frac{n}{2q}\right) \leq C \left(\frac{B_n}{\varepsilon_n}\right)^{1/2} h_n^{-d/2} p^{[p]-B} h_n^{-d/2} B^{B+1.5} \varepsilon_n^{B+0.5}.
$$

Thus, $P \left( \max_{1 \leq k \leq \log n} \left| s^{(k)}(x_k) - E(s^{(k)}(x_k)) \right| \geq \varepsilon_n \right) < \left( p \frac{n}{h_n^{-d/2}} \right)^{d/2} \left(4n^{-\frac{a^2}{16\varepsilon_n}} + Cn^{-d/2} B^{B+1.5} \varepsilon_n^{B+0.5}\right)$ and if $\Delta$ is chosen such that $\frac{\Delta^2}{16C} > 1$ the first term in the sum to the right of the inequality is negligible and we have that

$$
P \left( \max_{1 \leq k \leq \log n} \left| s^{(k)}(x_k) - E(s^{(k)}(x_k)) \right| \geq \varepsilon_n \right) < CB_n^{B+1.5} (\log n)^{0.25 + 0.5(B-d)} n^{0.75 + 0.5(d-B)} h_n^{1.75d - 0.5d(d+B)}. \quad (8)
$$

Choosing $B_n \propto n^{1/\zeta + \theta}$ for $\zeta > 2, \theta > 0$ we have $B_n^{1-\zeta} < n^{-0.5 - \theta}$ and $B_n^{1-\zeta} = o(n^{-1/2})$. Furthermore, the
Since the expression on the right-hand side of the inequality converges to zero by assumption, then

\[ P \left( \max_{1 \leq k \leq l_n} \left| s^T(x_k) - E(s^T(x^*) \right) \right| \geq \varepsilon_n \) \leq Cn^{(B+1.5)(\frac{1}{\theta}+\theta)-0.5(B-d)+0.75((\log n)^{0.25+0.5(B-d)}h_n^{-1.75d-0.5d(d+B)})}

Since the expression on the right-hand side of the inequality converges to zero by assumption, then

\[ P \left( \max_{1 \leq k \leq l_n} \left| s^T(x_k) - E(s^T(x^*)) \right| \geq \varepsilon_n \) = o_p(1).

which completes the proof for the case where \( p_1 = p_2 = p_3 = 0 \). We now turn to the cases where \( p_1, p_2, p_3 \) may differ from zero. Consider first the case where only one of the exponents, say \( p_1 = 1 \) and \( p_2 = p_3 = 0 \). Provided that the bounds (5), (6), (7) continue to hold in the case where \( s(x) = \sum_{t=1}^{n} K \left( \frac{x_t-x}{h_n} \right) \left( \frac{x_t-x}{h_n} \right) w(x_t-x; x) g(\varepsilon) \) the proof follows in an analogous manner. Verification of the bounds, however, follows directly from the fact that \( K \) has compact support and is uniformly bounded. As such, whenever \( |X_t-x| \leq C, K \left( \frac{x_t-x}{h_n} \right) = 0 \). All other cases, that is when \( p_1 = p_2 = p_3 = 1 \) and \( i = j = l, i \neq j \neq l, i = j \neq l \) or \( i = l \neq j \) are treated similarly.

\[ P \left( \max_{1 \leq k \leq l_n} \left| s^T(x_k) - E(s^T(x^*)) \right| \geq \varepsilon_n \) = o_p(1).

**Lemma 2.** Assume that the kernel \( K_1 \) used to define \( \hat{m} \) satisfies assumption A1 and assumptions A2 and A3 are holding. Assume also that the bandwidth \( h_{1,n} \) used to define \( \hat{m} \) satisfies equations (3) and (4). Then, if \( E(|\varepsilon|^\zeta) < \infty, E(h^{1/2}(X_1)^\zeta) < \infty \) for some \( \zeta > 2 \) and condition c) in Lemma [2] is holding

\[ \sup_{x \in \mathbb{D}} |\hat{m}(x) - m(x)| = O_p(L_{1,n}), \tag{9} \]

where \( L_{1,n} = \left( \frac{\log n}{n h_{1,n}} \right)^{1/2} + h_{1,n} \).

**Proof.** Note that \( \hat{m}(x) - m(x) = e^T S_n^{-1}(x) c_n(x) \) where \( e^T = (1 \ 0 \ \cdots \ 0) \) is a \( 1 \times (d+1) \) vector,

\[ S_n(x) = \begin{pmatrix}
  s_0(x) & s_1(x) & \cdots & s_d(x) \\
  s_1(x) & s_{(1,1)}(x) & \cdots & s_{(1,d)}(x) \\
  \vdots & \vdots & \ddots & \vdots \\
  s_d(x) & s_{(d,1)}(x) & \cdots & s_{(d,d)}(x)
\end{pmatrix} = \begin{pmatrix}
  s_0(x) & S^T_3(x) \\
  S_1(x) & S_2(x) \\
  \vdots & \vdots \\
  S_{d-1}(x) & S_d(x)
\end{pmatrix} \]

and \( c_n(x) = \begin{pmatrix}
  c_0(x) \\
  c_1(x) \\
  \vdots \\
  c_d(x)
\end{pmatrix} \)

with \( s_0(x) = \frac{1}{n h_{1,n}} \sum_{t=1}^{n} \sum_{i=1}^{n} K \left( \frac{X_t-x}{h_{1,n}} \right) (X_t-x), s_j(x) = \frac{1}{n h_{1,n}} \sum_{i=1}^{n} K \left( \frac{X_t-x}{h_{1,n}} \right) (X_t-x), c_0(x) = \frac{1}{n h_{1,n}} \sum_{i=1}^{n} K \left( \frac{X_t-x}{h_{1,n}} \right) Y^*_t, c_j(x) = \frac{1}{n h_{1,n}} \sum_{i=1}^{n} K \left( \frac{X_t-x}{h_{1,n}} \right) (X_t-x) Y^*_t \) and \( s_{i,j}(x) = \frac{1}{n h_{1,n}} \sum_{i=1}^{n} K \left( \frac{X_t-x}{h_{1,n}} \right) (X_t-x) (X_t-x) \) for \( i, j = 1, \cdots, d \)
and $Y_t^* = Y_t - m(x) - m^{(1)}(x)(X_t - x)$. Let $G_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & h_{1n}^{-s} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_{1n}^{-s} \end{pmatrix}$ and

$$
\Sigma(x) = \begin{pmatrix} f_X(x) & 0 & \cdots & 0 \\ \frac{\mu_{K,s}}{(s-1)!} D_1^{(s-2)} f_X(x) & \frac{\mu_{K,s}}{(s-2)!} D_1^{(s-2)} f_X(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\mu_{K,s}}{(s-1)!} D_d^{(s-2)} f_X(x) & 0 & \cdots & \frac{\mu_{K,s}}{(s-2)!} D_d^{(s-2)} f_X(x) \end{pmatrix}
$$

where $D_j^{(s)} f_X(x) = D_{i_1,\ldots,i_s} f_X(x)$ where $i_1 = \cdots = i_s = j$. By partitioned inversion we obtain

$$
\Sigma^{-1}(x) = \begin{pmatrix} f_X^{-1}(x) & 0 & \cdots & 0 \\ -\frac{1}{(s-1)!} D_1^{(s-1)} f_X(x) & \frac{1}{(s-2)!} D_1^{(s-2)} f_X(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{(s-1)!} D_d^{(s-1)} f_X(x) & 0 & \cdots & \frac{1}{(s-2)!} D_d^{(s-2)} f_X(x) \end{pmatrix}
$$

Then, $\hat{m}(x) - m(x) = e^T \left( (G_n S_n(x))^{-1} - \Sigma^{-1}(x) \right) G_n c_n(x) + e^T \Sigma^{-1}(x) G_n c_n(x) = \mathcal{I}_1n(x) + \mathcal{I}_2n(x)$ By the Cauchy-Schwarz and Triangle inequalities we have

$$
|\mathcal{I}_1n(x)| \leq \left( e^T \left( (G_n S_n(x))^{-1} - \Sigma^{-1}(x) \right) e \right)^{1/2} \left( |c_0(x)| + \frac{1}{h_{1n}^s} \sum_{j=1}^d c_j(x) \right),
$$

(11)

where we note that $(G_n S_n(x))^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ with

$$
A_{11} = \left( s_0(x) - S^T_1(x) \frac{1}{h_{1n}^s} S_2(x) \right)^{-1} \frac{1}{h_{1n}^s} S_1(x) \quad \text{and}
$$

$$
A_{12} = -\frac{1}{s_0(x)} S^T_1(x) \left( \frac{1}{h_{1n}^s} S_2(x) - \frac{1}{h_{1n}^s s_0(x)} S_1(x) S_1(x)^T \right)^{-1}.
$$

(12)

By Lemma 1 we have $sup_{x \in \mathcal{U}} |s_0(x) - E(s_0(x))| = O_p \left( \left( \frac{\log n}{nh_{1n}^s} \right)^{1/2} \right)$, $sup_{x \in \mathcal{U}} |s_j(x) - E(s_j(x))| = O_p \left( \left( \frac{\log n}{nh_{1n}^s} \right)^{1/2} \right)$, and $sup_{x \in \mathcal{U}} |E(s_{(i,j)}(x)) - E(s_{(i,j)}(x))| = O_p \left( \left( \frac{\log n}{nh_{1n}^s} \right)^{1/2} \right)$. Now, given A2 3) and A1 $sup_{x \in \mathcal{U}} |E(s_0(x)) - f_X(x)| = O(h_{1n}^s)$, $sup_{x \in \mathcal{U}} \left| \frac{E(s_{(i,j)}(x))}{h_{1n}^s} - \frac{\mu_{K,s}}{(s-1)!} D_j^{(s-1)} f_X(x) \right| = O(1)$, and $sup_{x \in \mathcal{U}} \left| \frac{E(s_{(i,j)}(x))}{h_{1n}^s} - \frac{\mu_{K,s}}{(s-2)!} D_j^{(s-2)} f_X(x) \right| = O(1)$ and lastly $sup_{x \in \mathcal{U}} \left| \frac{E(s_{(i,j)}(x))}{h_{1n}^s} \right| = o(1)$ for $i \neq j$. Consequently, given that $h_{1n} \propto n^{-\frac{1}{s+1}}$ we have

$$
sup_{x \in \mathcal{U}} |s_0(x) - f_X(x)| = O_p(L_{1n})
$$

(13)
\[
\begin{align*}
\sup_{x \in \mathcal{G}} \frac{|s_j(x)|}{h_{1n}^s} - \frac{\mu_{\mathcal{C},u}}{(s-1)!} D_j^{(s-1)} f(x) &= o_p(1) \text{ which implies } \sup_{x \in \mathcal{G}} |s_j(x)| = O_p(h_{1n}^s) \\
\sup_{x \in \mathcal{G}} \frac{|s(i,i)(x)|}{h_{1n}^s} - \frac{\mu_{\mathcal{C},u}}{(s-2)!} D_j^{(s-2)} f(x) &= o_p(1) \text{ and sup } \sup_{x \in \mathcal{G}} |s(i,j)(x)| = O_p(1) \text{ which implies } \\
\sup_{x \in \mathcal{G}} |S_2(x)| &= O_p(h_{1n}^s)
\end{align*}
\]

where the absolute value and order in the last equation are taken element-wise. Now, in inequality (11) we have that

\[
e^T \left((G_n S_n(x))^{-1} - \Sigma^{-1}(x)\right) \leq \left( A_{11}(x) - \frac{1}{f(x)} \right)^2 + \sum_{j=1}^{d} A_{12j}(x)
\]

where \(A_{12j}(x)\) is the \(j^{th}\) element of \(A_{12}(x)\). Given (12), (13), (14) and (15) we have \(A_{11}(x) - \frac{1}{f(x)} = O_p(L_{1n})\), \(A_{12j}(x) = O_p(h_{1n}^s)\) which gives

\[
e^T \left((G_n S_n(x))^{-1} - \Sigma^{-1}(x)\right) \leq \left( c_0(x) + \left| \frac{1}{h_{1n}^s} \sum_{j=1}^{d} c_j(x) \right| \right).
\]

Now,

\[
Y_t^* = m(X_t) - m(x) - m^{(1)}(x)(X_t - x) + h^{1/2}(X_t) \varepsilon_t
\]

\[
= \frac{1}{2} \sum_{i_1}^{d} \sum_{i_2}^{d} D_{i_1i_2} m(x + \lambda(X_t - x))(X_{ti_1} - x_{i_1})(X_{ti_2} - x_{i_2}) + h^{1/2}(X_t) \varepsilon_t
\]

\[
= P_t(x) + h^{1/2}(X_t) \varepsilon_t
\]

and \(c_0(x) = \frac{1}{nh_{1n}} \sum_{i=1}^{n} K_1 \left( \frac{x - x_i}{h_{1n}} \right) P_t(x) + \frac{1}{nh_{1n}} \sum_{i=1}^{n} K_1 \left( \frac{x - x_i}{h_{1n}} \right) h^{1/2}(X_t) \varepsilon_t = c_{0,1}(x) + c_{0,2}(x)\). By Lemma 1 we have

\[
\frac{1}{n^{1/2}} \sup_{x \in \mathcal{G}} |c_{0,1}(x) - E(c_{0,1}(x))| = O_p \left( \frac{\log n}{nh_{1n}^{1/2}} \right).\]

Now, we observe that

\[
E(c_{0,1}(x)) = \frac{1}{h_{1n}} E \left( K_1 \left( \frac{X_t - x}{h_{1n}} \right) \left( \frac{1}{2} \sum_{i_1}^{d} \sum_{i_2}^{d} D_{i_1i_2} m(x)(X_{ti_1} - x_{i_1})(X_{ti_2} - x_{i_2}) + \cdots \right) \right)
\]

and by A2 3) and A1 \sup_{x \in \mathcal{G}} |E(c_{0,1}(x))| = O(h_{1n}^4). Hence, \sup_{x \in \mathcal{G}} |c_{0,1}(x)| = O_p(h_{1n}^4) given that \(h_{1n} \propto n^{-1/4 + \alpha}\).

Similarly, by Lemma 1 and given that \(E(c_{0,2}(x)) = 0\), \sup_{x \in \mathcal{G}} |c_{0,2}(x)| = O_p \left( \left( \frac{\log n}{nh_{1n}^{1/2}} \right)^{1/2} \right). Thus, \sup_{x \in \mathcal{G}} |c_0(x)| = O_p(L_{1n})\).

Similar arguments give

\[
\sup_{x \in \mathcal{G}} |c_j(x)| = O_p \left( h_{1n}^s + h_{1n} \left( \frac{\log n}{nh_{1n}^2} \right)^{1/2} \right).
\]
Consequently, $\sup_{x \in \mathcal{G}} I_{1n}(x) = O_p(L_{1n})$. Lastly, $I_{2n}(x) = \frac{1}{f(x)} c_0(x)$ and from the fact that $f(x)$ is uniformly bounded away from 0 and the order of $c_0(x)$ we have $\sup_{x \in \mathcal{G}} I_{2n}(x) = O_p(L_{1n})$. Therefore, $\sup_{x \in \mathcal{G}} |\hat{m}(x) - m(x)| = O_p(L_{1n})$.

Lemma 3. Assume that the kernel $K_2$ used to define $\hat{h}$ satisfies assumption A1 and assumptions A2 and A3 are holding. Assume also that the bandwidth $h_{2n}$ used to define $\hat{h}$ satisfies equations (3) and (4). Then, under the assumptions in Lemma 2 if $E(|\varepsilon_i^2 - 1|) < \infty$ and $E(h(X_i)^2) < \infty$ for some $\zeta > 2$,

$$\sup_{x \in \mathcal{G}} |\hat{h}(x) - h(x)| = O_p\left(L_{1n} + L_{2n}\right),$$

where $L_{1n} = \left(\frac{\log n}{n h_{2n}^2}\right)^{1/2} + h_{1n}^2$ and $L_{2n} = \left(\frac{\log n}{n h_{2n}^2}\right)^{1/2} + h_{2n}$.

Proof. As in Lemma 2 we write $\hat{h}(x) - h(x) = e^T \left((G_n G_n(x))^{-1} - \Sigma^{-1}(x)\right) G_n q_n(x) + e^T \Sigma^{-1}(x) G_n q_n(x) = I_{1n}(x) + I_{2n}(x)$ where

$$q_n(x) = \left(q_0(x) \quad q_1(x) \quad \cdots \quad q_d(x) \right), \quad q_0(x) = \frac{1}{n h_{2n}^2} \sum_{i=1}^n K_2 \left(\frac{x_i - x}{h_{2n}}\right) U_{i^2}, \quad q_j(x) = \frac{1}{n h_{2n}^2} \sum_{i=1}^n K_2 \left(\frac{x_i - x}{h_{2n}}\right) (x_{ij} - x_j) U_{i^2}^2, \quad U_{i^2} = \hat{U}_i - h(x) - h_1(x) \lambda_i = U_i - \hat{m}(x_i).$$

Hence, we write

$$U_{i^2} = h(X_i)(\varepsilon_i^2 - 1) + P_t^{h}(x) - 2(\hat{m}(X_i) - m(X_i))h_1^{1/2}(X_i)\varepsilon_i + (\hat{m}(X_i) - m(X_i))^2$$

where

$$P_t^{h}(x) = \frac{1}{2} \sum_{i=1}^d \sum_{i=1}^d D_{i,i} \varepsilon_i (X_{t_{i,i}} - x)(X_{t_{i,i}} - x).$$

From Lemma 2 we have $|I_{1n}(x)| \leq O_p(L_{2n}) \left(\left|q_0(x)\right| + \left|\frac{1}{h_{2n}} \sum_{j=1}^d q_j(x)\right|\right)$. Now,

$$q_0(x) = \frac{1}{n h_{2n}^2} \sum_{i=1}^n K_2 \left(\frac{x_i - x}{h_{2n}}\right) h(X_i)(\varepsilon_i^2 - 1) + \frac{1}{n h_{2n}^2} \sum_{i=1}^n K_2 \left(\frac{x_i - x}{h_{2n}}\right) \left(P_t^{h}(x)\right)$$

$$+ \frac{1}{n h_{2n}^2} \sum_{i=1}^n K_2 \left(\frac{x_i - x}{h_{2n}}\right) (\hat{m}(X_i) - m(X_i))^2 - \frac{2}{n h_{2n}^2} \sum_{i=1}^n K_2 \left(\frac{x_i - x}{h_{1n}}\right) (\hat{m}(X_i) - m(X_i))h_1^{1/2}(X_i)\varepsilon_i$$

$$= B_{1n}(x) + B_{2n}(x) + B_{3n}(x) - B_{4n}(x).$$

Given the conditions on this lemma and by the same arguments used to obtain the order of $c_{0,2}(x)$ in Lemma 2 we have that $\sup_{x \in \mathcal{G}} |B_{1n}(x)| = O_p\left(\left(\frac{\log n}{n h_{2n}^2}\right)^{1/2}\right)$. Also, by the same arguments used to obtain the order of $c_{0,1}(x)$ in Lemma 2 and provided $h_{2n} \propto n^{-\frac{1}{d+1}}$ we have that $\sup_{x \in \mathcal{G}} |B_{2n}(x)| = O_p(h_{2n}^2)$.

$$B_{4n}(x) \leq 2 \sup_{x \in \mathcal{G}} |\hat{m}(x) - m(x)| \left|\frac{1}{n h_{2n}^2} \sum_{i=1}^n K_2 \left(\frac{x_i - x}{h_{2n}}\right)\right| h_1^{1/2}(X_i)\varepsilon_i = 2 \sup_{x \in \mathcal{G}} |\hat{m}(x) - m(x)| M_n(x).$$

By Lemma 2
we have sup$_{x \in G} |M_n(x) - E(M_n(x))| = O_p \left( \left( \frac{\log n}{nh^2_n} \right)^{1/2} \right)$. Furthermore,
\[ E(M_n(x)) = E(|\xi|) \prod_{j=1}^d |K(u_j)|h^{1/2}(x+h_{2n}u) f_x(x+h_{2n}u) du \to E(|\xi|)h^{1/2}(x)f_x(x) \prod_{j=1}^d |K(u_j)| du < C \]
given A1, A2 1), A3 2) and the fact that $E(\xi^2) = 1$. Hence, sup$_{x \in G} |M_n(x)| = O_p(1)$ and from Lemma 2 we conclude that sup$_{x \in G} |B_{3n}(x)| = O_p(L_{1n}^2)$. Through similar arguments we show that sup$_{x \in G} |B_{3n}(x)| = O_p(L_{1n}^2)$ since sup$_{x \in G} \sum_{i=1}^n |K_2(\frac{x_i-x}{h_{2n}})| = O_p(1)$. Hence, sup$_{x \in G} |q_0(x)| = O_p(L_{2n} + L_{1n})$. Lastly, for the terms $q_j(x)$, we have in an analogous manner that
\[ \sup_{x \in G} |q_j(x)| = O_p \left( h_n^2 + h_{2n} \left( \frac{\log n}{nh_n^2} \right)^{1/2} \right) + h_{2n}O_p(L_{1n}) + h_{2n}O_p(L_{1n}^2). \]
Consequently, sup$_{x \in G} |T_{1n}^h(x)| = O_p(L_{2n})$. Lastly, $T_{2n}^h(x) = \frac{1}{\|f_x(x)|} q_0(x)$ and from the fact that $f_x(x)$ is uniformly bounded away from 0 and the order of $q_0(x)$ we have sup$_{x \in G} |T_{2n}^h(x)| = O_p(L_{1n} + L_{2n})$. Therefore, sup$_{x \in G} \left| \hat{h}(x) - h(x) \right| = O_p(L_{1n} + L_{2n})$. \hfill \square

**Corollary 1.** Under the assumptions of Lemma 3
\[ \sup_{x \in G} |\hat{h}^{1/2}(x) - h^{1/2}(x)| = O_p(L_{1n} + L_{2n}) \quad \text{and} \quad \sup_{x \in G} |h(x) = 0| - 1| = O_p(L_{1n} + L_{2n}), \]
where $L_{1n} = \left( \frac{\log n}{nh_n^2} \right)^{1/2} + h_n^2$ and $L_{2n} = \left( \frac{\log n}{nh_n^2} \right)^{1/2} + h_{2n}$.

**Lemma 4.** Under assumptions A1-A6 and conditions FR1 and FR2, if $\alpha \geq 1$ we have
\[ N^{1/2} \left( \frac{\hat{q}(a_n) - q(a_n)}{q(a_n)} \right) = O_p(1), \quad \text{where} \quad a_n = 1 - \frac{N}{n}. \]

**Proof.** We write
\[ \sqrt{N} \left( \frac{\hat{q}(a_n) - q(a_n)}{q(a_n)} \right) = \sqrt{N} \left( \frac{\hat{q}(a_n) - q(a_n)}{q(a_n)} \right) - \sqrt{N} \left( \frac{q(a_n) - q(a_n)}{q(a_n)} \right) = T_{1n} - T_{2n}. \]
We first show that $T_{2n}$ converges in distribution, which implies $T_{2n} = O_p(1)$. Note that
\[ P(T_{2n} \leq z) = P \left( \frac{nk_0}{\sqrt{N}} (F(y_n) - a_n) \leq -\frac{nk_0}{\sqrt{N}} (F_n(y_n) - F(y_n)) \right) \]
with $y_n = q(a_n) \left( 1 + \frac{z}{\sqrt{N}} \right)$. By the mean value theorem, $F(y_n) = a_n + f(q^*(a_n)) \frac{q(a_n) - q(a_n)}{q(a_n)}$ where $q^*(a_n) = q(a_n) \left( 1 + \lambda \frac{z}{\sqrt{N}} \right)$ for some $\lambda \in (0, 1)$. Thus,
\[ \frac{nk_0}{\sqrt{N}} (F(y_n) - a_n) = \frac{nk_0}{\sqrt{N}} f(q^*(a_n)) q(a_n) z = k_0 \frac{n(1 - F(q^*(a_n)))}{\sqrt{N}} \frac{q(a_n) f(q^*(a_n))}{1 - F(q^*(a_n))} z. \]

Since \( q^*(a_n) = q(a_n)(1 + o(1)) \) we have that \( \lim_{n \to \infty} \frac{n(1 - F(q^*(a_n)))}{N} = 1 \). In addition, given FRI and by Proposition 1.15 in [Resnick (1987)] we have \( \lim_{n \to \infty} \frac{q(a_n)f(q^*(a_n))}{n} = -\frac{1}{k_0} \), hence \( \lim_{n \to \infty} \frac{n k_0}{\sqrt{N}} (F(y_n) - a_n) = -z \).

We now show that \( \frac{n}{\sqrt{N}}(F_n(y_n) - F(y_n)) \overset{d}{\to} N(0,1) \). First, we observe that \( \frac{n}{\sqrt{N}} - \frac{\sqrt{n}}{\sqrt{1 - F(y_n)}} = o(1) \), hence we show that

\[
\frac{\sqrt{n}}{\sqrt{1 - F(y_n)}} (F_n(y_n) - F(y_n)) = \sum_{t=1}^{n} Z_{tn} \overset{d}{\to} N(0,1)
\]

where \( Z_{tn} = \frac{1}{\sqrt{n(1 - F(y_n))}} (\chi_{\{z \leq y_n\}} - E(\chi_{\{z \leq y_n\}})) \). It is readily verified that \( E(Z_{tn}) = 0 \) and \( V(Z_{tn}) = n^{-1}F(y_n) \). Hence, given that \( \sum_{t=1}^{n} E(|Z_{tn}|^3) \leq 2(n(1 - F(y_n)))^{-1/2} = o(1) \) we have by Liapounov’s CLT that

\[
\frac{n}{\sqrt{N}}(F_n(y_n) - F(y_n)) \overset{d}{\to} N(0,1). \]

Hence, \( T_{2n} \overset{d}{\to} N(0, k_0^2) \). We now show that \( T_{1n} = O_p(1) \) by establishing that \( T_{1n} \) converges in distribution. As above,

\[
P(T_{1n} \leq z) = P\left( \frac{n k_0}{\sqrt{N}} (F(y_n) - a_n) = -\frac{n k_0}{\sqrt{N}} (\bar{F}(y_n) - F(y_n)) \right)
\]

and we establish that \( \frac{n}{\sqrt{N}}(\bar{F}(y_n) - F(y_n)) \overset{d}{\to} N(0,1) \). We start by noting that for some \( \lambda_t \in (0,1) \)

\[
\bar{F}(y_n) = \int_{-\infty}^{y_n} \frac{1}{nh_{3n}} \sum_{t=1}^{n} K_3 \left( \frac{y - \varepsilon_t}{h_{3n}} \right) dy - \int_{-\infty}^{y_n} \frac{1}{nh_{3n}^2} \sum_{t=1}^{n} K_3^{(1)} \left( \frac{y - \varepsilon_t}{h_{3n}} \right) dy (\varepsilon_t - \varepsilon_t)
\]

\[
+ \frac{1}{2} \int_{-\infty}^{y_n} \frac{1}{nh_{3n}} \sum_{t=1}^{n} K_3^{(2)} \left( \frac{y - \varepsilon_t^*}{h_{3n}} \right) dy (\varepsilon_t - \varepsilon_t)^2 = Q_{1n} - Q_{2n} + Q_{3n}.
\]

where \( \varepsilon_t^* = \lambda_t \varepsilon_t + (1 - \lambda_t) \varepsilon_t \). Therefore, \( \frac{n}{\sqrt{N}}(\bar{F}(y_n) - F(y_n)) = \frac{n}{\sqrt{N}} (Q_{1n} - F(y_n) - Q_{2n} + Q_{3n}) \). From equation (17) in the proof of Theorem 1 we have

\[
Q_{3n} \leq \frac{1}{2 nh_{3n}^2} \sum_{t=1}^{n} \left| K_3^{(1)} \left( \frac{y_n - \varepsilon_t^*}{h_{3n}} \right) \right| \left| O_p(L_{1n}) + O_p(L_{2n}) \right|^2
\]

\[
\leq \frac{1}{2 nh_{3n}^2} \sum_{t=1}^{n} \left| K_3^{(1)} \left( \frac{y_n - \varepsilon_t^*}{h_{3n}} \right) \right| \left| O_p(L_{1n}) + O_p(L_{2n}) \right|^2
\]

\[
= O_p \left( \frac{L_{1n}^2}{h_{3n}} \right) Q_{31n} + O_p \left( \frac{L_{1n}^2 + L_{2n}^2}{h_{3n}} \right) Q_{32n}
\]

Using Taylor’s Theorem we can write for some \( \lambda \in (0,1) \) and \( \varepsilon_t^{**} = \lambda_t \varepsilon_t + (1 - \lambda_t) \varepsilon_t \) that

\[
Q_{32n} \leq \frac{1}{nh_{3n}^2} \sum_{t=1}^{n} \left| K_3^{(1)} \left( \frac{y_n - \varepsilon_t}{h_{3n}} \right) \right|^2 + \frac{1}{nh_{3n}^2} \sum_{t=1}^{n} \left| K_3^{(2)} \left( \frac{y_n - \varepsilon_t^{**}}{h_{3n}} \right) \right| (\varepsilon_t^{**} - \varepsilon_t) \varepsilon_t^{**}
\]

\[
= Q_{321n} + Q_{322n}.
\]
Note that

\[ E(Q_{31n}) = y_n(1 - F(y_n)) \int |K_3^{(1)}(\psi)| \left( \frac{y_n - h_{3n}\psi}{1 - F(y_n - h_{3n}\psi)} \right) \frac{1 - F(y_n - h_{3n}\psi)}{1 - F(y_n)} \frac{y_n - h_{3n}\psi}{y_n} d\psi \]

and since by part a) of Proposition 1.15 in [Resnick 1987]

\[ \frac{(y_n - h_{3n}\psi)f(y_n - h_{3n}\psi)}{1 - F(y_n - h_{3n}\psi)} \to -1/k_0, \quad \frac{1 - F(y_n - h_{3n}\psi)}{1 - F(y_n)} \to 1 \]

and \( \frac{(y_n - h_{3n}\psi)}{y_n} \to 1 \) we have that \( E(Q_{31n}) = y_n(1 - F(y_n)) \int |K_3^{(1)}(\psi)|d\psi(1 + o(1)) \). By part c) of the same Proposition 1.15, \( 1 - F(x) = C\exp(-g(x)) \) where \( g(x) = \int_{z_0}^x t^{-1}\alpha(t)dt \) and \( \alpha(t) \to \alpha \) as \( t \to \infty \) for some \( z_0 \) and all \( x > z_0 \). Hence, \( g(x) \approx a\log \frac{x}{z_0} \) and \( 1 - F(x) \approx C\exp\left(-a\log \frac{x}{z_0}\right) \). Consequently, \( \lim_{x \to \infty} x(1 - F(x)) = C\lim_{x \to \infty} \frac{x}{\exp(g(x))} = C \lim_{x \to \infty} \frac{x}{(x/z_0)^{a\alpha}} = C \) for \( \alpha \geq 1 \). Thus, we conclude that \( y_n(1 - F(y_n)) = O(1) \) and \( Q_{321n} = O_p(1) \). For \( Q_{322n} \), note that \( |\varepsilon_i^e - \varepsilon_i| \leq \lambda_t (O_p(L_{1n}) + O_p(L_{1n} + L_{2n})|\varepsilon_i|) \) and since \( \lambda_t \in (0, 1) \) we have

\[ Q_{322n} \leq O_p(L_{1n}) \frac{1}{h_{3n}^2} \sum_{i=1}^n K_3^{(2)} \left( \frac{y_n - \varepsilon_i^e}{h_{3n}} \right) \frac{n}{\sqrt{nh_{3n}}} + \frac{O_p(L_{1n} + L_{2n})}{h_{3n}^2} \sum_{i=1}^n K_3^{(2)} \left( \frac{y_n - \varepsilon_i^e}{h_{3n}} \right) \frac{n}{\sqrt{nh_{3n}}} \frac{|\varepsilon_i|^3}{\sqrt{nh_{3n}}} \]

Given that \( |K_3^{(2)}(x)| \leq C, E(|\varepsilon_i|^3) < \infty \) and the fact that \( O_p((L_{1n} + L_{2n})h_{3n}^{-2}) = o_p(1) \) given the orders of \( h_{1n}, h_{2n} \) and \( h_{3n} \) we conclude that \( Q_{32n} = O_p(1) \). Following similar arguments we have that \( Q_{31n} = O_p(1) \) and consequently \( Q_{3n} = O_p \left( \frac{L_{1n}^2}{h_{3n}^2} \right) + O_p \left( \frac{(L_{1n} + L_{2n})^2}{h_{3n}^2} \right) \). Thus,

\[ \frac{n^{1/2}}{(1 - F(y_n))^{1/2}} Q_{3n} = \frac{n}{\sqrt{Nh_{3n}}} C \left( \log n + \frac{h_{1n}^2 + h_{2n}^2}{nh_{3n}^2} \right) = o_p(1) \quad (20) \]

for \( i = 1, 2 \) given the orders of \( h_{1n}, h_{2n}, h_{3n} \) and \( N \). We now turn to the study of \( Q_{2n} \). Using equation (16) in the proof of Theorem 1 we write

\[ Q_{2n} = \frac{1}{nh_{3n}^2} \sum_{i=1}^n K_3^{(1)} \left( \frac{y - \varepsilon_i}{h_{3n}} \right) dy \left( (m(X_i) - \hat{m}(X_i))(h^{-1/2}(X_i) - h^{-1/2}(X_i)) \chi_{(h(X_i), > 0)} \right) \]

\[ + h^{-1/2}(X_i)(m(X_i) - \hat{m}(X_i))( \chi_{(h(X_i), > 0)} - 1) + h^{-1/2}(X_i)(m(X_i) - \hat{m}(X_i)) \]

\[ + \left( \frac{h^{1/2}(X_i)}{h^{1/2}(X_i)} - 1 \right) \chi_{(h(X_i), > 0)} \varepsilon_t + \left( \chi_{(h(X_i), > 0)} - 1 \right) \varepsilon_t \]

\[ = \sum_{j=1}^5 Q_{2jn} \]
We investigate the order of each $Q_{2jn}$ separately. First, we note that from Lemmas 3, 4 and Corollary 1

$$Q_{21n} \leq \frac{1}{n^3} O_p(L_{1n})O_p(L_{1n} + L_{2n}) \frac{1}{n} \frac{1}{h_{3n}} \sum_{t=1}^{n} \left| \int_{-\infty}^{\gamma_n} K_3(1) \left( \frac{y - \xi_t}{h_{3n}} \right) dy \right|$$

(22)

with $\frac{1}{n^3} O_p(L_{1n})O_p(L_{1n} + L_{2n}) \leq \frac{1}{n^3} O_p(L_{2n}^2 + L_{2n}^2) = o_p\left( \frac{(1 - F(y_n))^{1/2}}{n^{1/2}} \right)$. Furthermore,

$$E \left( \frac{1}{n} \frac{1}{h_{3n}} \sum_{t=1}^{n} \int_{-\infty}^{\gamma_n} K_3(1) \left( \frac{y - \xi_t}{h_{3n}} \right) dy \right) = \frac{1}{h_{3n}} \left( K_3(1) \left( \frac{y - \xi_t}{h_{3n}} \right) \right) = \int |K_3(\psi)| |f(y_n - h_{3n}\psi)| d\psi$$

$$= o(1)$$

since $f(y_n + h_{3n}\psi) \to 0$ as $n \to \infty$ and $|K_3(\psi)| < C$. Hence, $\frac{1}{n} \frac{1}{h_{3n}} \sum_{t=1}^{n} \int_{-\infty}^{\gamma_n} K_3(1) \left( \frac{y - \xi_t}{h_{3n}} \right) dy = o_p(1)$ and

$$\frac{n^{1/2}}{(1 - F(y_n))^{1/2}} Q_{21n} = o_p(1).$$

(23)

Since,

$$Q_{22n} \leq \frac{1}{n^2} \sum_{t=1}^{n} \int_{-\infty}^{\gamma_n} K_3(1) \left( \frac{y - \xi_t}{h_{3n}} \right) dy \sup_{\text{sup}_{x \in G} h^{-1/2}(x) |\sup_{x \in G} m(x) - \hat{m}(x)| \sup_{x \in G} |X(h(x) > 0)|} - 1,$$

we immediately conclude by Lemmas 3, 4, Corollary 1, the fact that $h(x)$ is bounded away from zero and

$$\frac{1}{n^2} \sum_{t=1}^{n} \int_{-\infty}^{\gamma_n} K_3(1) \left( \frac{y - \xi_t}{h_{3n}} \right) dy = o_p(1)$$

that

$$\frac{n^{1/2}}{(1 - F(y_n))^{1/2}} Q_{22n} = o_p(1).$$

(24)

From Theorem 2 in Martins-Filho and Yao (2009) we write

$$\hat{m}(X_i) - m(X_i) = \frac{1}{nh_{1n}^d fX(X_i)} K_1 \left( \frac{X_i - X_i}{h_{1n}} \right) Y_i^*(1 + o_p(1))$$

(25)

uniformly on $G$, where $Y_i^*$ is as defined in Lemma 2. Therefore,

$$Q_{23n} = -\frac{1}{n} \frac{1}{h_{3n}} \sum_{t=1}^{n} \int_{-\infty}^{\gamma_n} K_3(1) \left( \frac{y - \xi_t}{h_{3n}} \right) dy \sup_{x \in G} h^{-1/2}(x) \frac{1}{nh_{1n}^d fX(X_i)} \sum_{t=1}^{n} K_1 \left( \frac{X_i - X_i}{h_{1n}} \right) Y_i^*(1 + o_p(1))$$

$$= -(1 + o_p(1)) \frac{1}{n} \frac{1}{h_{3n}} \sum_{t=1}^{n} \int_{-\infty}^{\gamma_n} K_3(1) \left( \frac{y - \xi_t}{h_{3n}} \right) \left( \frac{X_i - X_i}{h_{1n}} \right) \left( h_{1n}^{-1/2}(X_i) \right) \left( h_{1n}^{-1/2}(X_i) \xi_i \right)$$

$$+ \frac{1}{2} h^{-1/2}(X_i)(X_i - X_i)^T m^{(2)}(Z_{ii})(X_i - X_i)$$

where $Z_{ii} = \lambda X_i + (1 - \lambda) X_i$. Now, let

$$\psi_n(w_i, w_i) = \frac{1}{h_{1n}^d h_{3n} fX(X_i)} K_3 \left( \frac{y_n - \xi_t}{h_{3n}} \right) K_1 \left( \frac{X_i - X_i}{h_{1n}} \right)$$

$$\times \left( \frac{h_{1n}^{-1/2}(X_i) \xi_i + \frac{1}{2} h_{1n}^{-1/2}(X_i)(X_i - X_i)^T m^{(2)}(Z_{ii})(X_i - X_i) \right)$$

$$12$$
for \( w_t = (X_t \varepsilon_t) \) and define \( \phi_n(w_t, w_i) = \psi_n(w_t, w_i) + \psi_n(w_i, w_t) \). Then, we write \( Q_{23n} = -(1 + o_p(1)) \frac{1}{n^2} \sum_{t=1}^{n} \sum_{i=1}^{n} \phi_n(w_t, w_i) \) and

\[
\frac{1}{2n^2} \sum_{t=1}^{n} \sum_{i=1}^{n} \phi_n(w_t, w_i) = \frac{1}{2n^2} \sum_{t=1}^{n} \sum_{i=1}^{n} \phi_n(w_t, w_i) + \frac{1}{2n^2} \sum_{i=1}^{n} \sum_{t \neq i}^{n} \phi_n(w_t, w_i) \\
= \frac{1}{2n^2} \sum_{t=1}^{n} \sum_{i=1}^{n} \phi_n(w_t, w_i) + \frac{1}{n^2} \sum_{t=1}^{n} \sum_{t < i}^{n} \phi_n(w_t, w_i).
\]

Now,

\[
\frac{1}{n^2} \sum_{t=1}^{n} \sum_{i=1}^{n} \phi_n(w_t, w_i) = \frac{1}{n^2} \sum_{t=1}^{n} \sum_{i=1}^{n} \Phi_n(w_t, w_i) + \frac{1}{n^2} \sum_{t=1}^{n} \sum_{t < i}^{n} (\theta_{ni} + \theta_{nt} - \theta_n) \\
= Q_{231n} + Q_{232n}
\]

where \( \theta_{ni} = \int \phi_n(w_t, w_i) dP(w_t), \theta_n = \int \phi_n(w_t, w_i) dP(w_i) dP(w_t), \Phi_n(w_t, w_i) = \phi_n(w_t, w_i) - \theta_{ni} - \theta_{nt} + \theta_n \)
and \( P(w_t) \) be the probability measure associated with the vector \( w_t \). Then, \( \Phi_n(w_t, w_i) \) is a symmetric function, and for fixed \( w_i, E(\Phi_n(w_t, w_i)) = 0 \). By part (ii) of Lemma A.2 in [Gaı̈, 2007], for \( \delta > 0 \),

\[
E \left( \frac{1}{n^2} \sum_{t=1}^{n} \sum_{t < i}^{n} \Phi_n(w_t, w_i) \right)^2 \leq \frac{C}{n^2} M^{1/(1+\delta)} \text{ where}
\]

\[
M = \max_{1 \leq t \leq n} \left\{ E \left( |\Phi_n(w_1, w_i)|^{2(1+\delta)} \right), \int \int |\Phi_n(w_1, w_i)|^{2(1+\delta)} dP(w_1) dP(w_i) \right\}.
\]

By the \( c_r \) and Cauchy-Schwartz inequalities we have that

\[
E \left( \Phi_n(w_1, w_i) \right)^{2(1+\delta)} \leq C \left( E \left( |\phi_n(w_1, w_i)|^{2(1+\delta)} \right) + \int \int |\phi_n(w_1, w_i)|^{2(1+\delta)} dP(w_1) dP(w_i) \right)
\]

and

\[
\int \int |\Phi_n(w_1, w_i)|^{2(1+\delta)} dP(w_1) dP(w_i) \leq C \left( \int \int |\phi_n(w_1, w_i)|^{2(1+\delta)} dP(w_1) dP(w_i) + E|\phi_n(w_1, w_i)|^{2(1+\delta)} \right).
\]
Hence, we investigate the order of $E[\phi_n(w_t, w_i)]^{2(1+\delta)}$ for $t > i$. Note that,

$$E[\phi_n(w_t, w_i)]^{2(1+\delta)} \leq CE[\psi_n(w_t, w_i)]^{2(1+\delta)}$$

$$\leq Ch_{3n}^{-(1+2\delta)}h_{1n}^{-d(1+\delta)} \frac{1}{h_{3n}h_{1n}^{4}}E \left( \frac{1}{f_X(X_i)} \right)^{2(1+\delta)} K_3 h_{1n}^{2(1+\delta)} \left( \frac{y_n - \varepsilon_t}{h_{3n}} \right)$$

$$\times K_1^{2(1+\delta)} \left( \frac{X_t - X_i}{h_{1n}} \right) \left( \frac{h^{1/2}(X_i)}{h^{1/2}(X)} \right)^{2(1+\delta)}$$

$$+ \left( \frac{1}{2} h^{-1/2}(X_i)(X_t - X_i)^T m(2)(Z_{it})(X_t - X_i) \right)^{2(1+\delta)}$$

$$= Ch_{3n}^{-(1+2\delta)}h_{1n}^{-d(1+\delta)}(Q_{2311} + Q_{2312}).$$

Since $t > i$ and $\varepsilon_t$ is independent of $X_t$ we have

$$\frac{y_n}{1 - F(y_n)} Q_{2311} < CE(\varepsilon_t^{2(1+\delta)}) \frac{y_n}{1 - F(y_n)} \int \int K_3^{2(1+\delta)}(\psi_1)f(y_n - h_{3n}\psi_1) \left( \frac{h^{1/2}(X_i)}{h^{1/2}(X_t - h_{1n}\psi_2)} \right)^{2(1+\delta)}$$

$$\times K_1^{2(1+\delta)}(\psi_2)f_{X, X_i|\varepsilon_t=y_n-h_{3n}\psi_1}(X_t - h_{1n}\psi_2, X_i)d\psi_1d\psi_2dX_t$$

$$< CE(\varepsilon_t^{2(1+\delta)}) \int \int K_3^{2(1+\delta)}(\psi_1)f(y_n - h_{3n}\psi_1) \frac{y_n - h_{3n}\psi_1}{1 - F(y_n - h_{3n}\psi_1)}$$

$$\times \frac{1 - F(y_n - h_{3n}\psi_1)}{1 - F(y_n)} K_1^{2(1+\delta)}(\psi_2) \left( \frac{h^{1/2}(X_i)}{h^{1/2}(X_t - h_{1n}\psi_2)} \right)^{2(1+\delta)} \frac{y_n}{y_n - h_{3n}\psi_1}$$

$$\times \sup_{\varepsilon_t} f_{X, X_i|\varepsilon_t}(X_t - h_{1n}\psi_2, X_i)d\psi_1d\psi_2dX_t.$$

Consequently, $Q_{2311} = O \left( \frac{1 - F(y_n)}{y_n} \right)$ by A5 1) and 3). Now,

$$\frac{y_n}{1 - F(y_n)} Q_{2312} < C \frac{y_n}{1 - F(y_n)} \int \int K_3^{2(1+\delta)}(\psi_1)f(y_n - h_{3n}\psi_1)h_{1n}^{4(1+\delta)}$$

$$\times \int \int \sup_{\varepsilon_t} K_1^{2(1+\delta)}(\psi_2)(\psi_2^T\psi_2)^{2(1+\delta)}f_{X, X_i|\varepsilon_t}(X_t - h_{1n}\psi_2, X_t)d\psi_1d\psi_2dX_t$$

$$= O(h_{1n}^{4(1+\delta)})$$

since $h^{1/2}(X), f_X(x) > 0$, and every element of $m(2)$ is uniformly bounded. Hence, $Q_{2312} = O \left( \frac{1 - F(y_n)}{y_n} h_{1n}^{4(1+\delta)} \right)$ and $E[|\phi_n(w_t, w_i)|^{2(1+\delta)}] = O \left( h_{3n}^{-(1+2\delta)}h_{1n}^{-d(1+2\delta)} \frac{1 - F(y_n)}{y_n} \right)$. Now, for $t > i$

$$\int \int |\phi_n(w_t, w_i)|^{2(1+\delta)}dP(w_t)dP(w_i) \leq \int \int |\psi_n(w_t, w_i)|^{2(1+\delta)}dP(w_t)dP(w_i)$$
\[ \leq C h_{3n}^{-(1+2\delta)} h_{1n}^{-d(1+\delta)} \left( \frac{1}{h_{3n}h_{1n}^{3(1+\delta)}} \int \int f_X(X_i) \right)^{2(1+\delta)} \]
\[ \times K_3^{2(1+\delta)} \left( \frac{y_n - \varepsilon_i}{h_{3n}} \right) K_1^{2(1+\delta)} \left( \frac{1}{h_{1n}} \right) \left( \frac{h_{1/2}(X_i)}{h_{1/2}(X_i)} \right)^{2(1+\delta)} \]
\[ + \left( \frac{1}{2} h_{1/2}(X_i)(X_i - X_i)^T m(x)(X_i - X_i) \right)^{2(1+\delta)} \]
\[ \times dP(w_i) dP(w_i) = C h_{3n}^{-(1+2\delta)} h_{1n}^{-d(1+2\delta)} (Q_{2313} + Q_{2314}) \]

Since \( X_i \) and \( \varepsilon_i \) are independent \( dP(w_i) = f_X(X_i)f(\varepsilon_i)dX_id\varepsilon_i \) and

\[ Q_{2313} = \frac{1}{h_{3n}} \int \int K_3^{2(1+\delta)} \left( \frac{y_n - \varepsilon_i}{h_{3n}} \right) f(\varepsilon_i)d\varepsilon_i E(\varepsilon_i)^{2(1+\delta)} \]
\[ \times \left( \frac{1}{h_{1n}} \right) \left( \frac{h_{1/2}(X_i)}{h_{1/2}(X_i)} \right) f_X(X_i)f_X(X_i)dX_idX_i. \]

Note that \( E(\varepsilon_i)^{2(1+\delta)} < C \) and \( \int K_3^{2(1+\delta)} \left( \frac{y_n - \varepsilon_i}{h_{3n}} \right) f(\varepsilon_i)d\varepsilon_i = O \left( \frac{1-F(y_n)}{y_n} \right). \) The remaining integral can be written as

\[ \int \int \left( \frac{h_{1/2}(X_i)}{h_{1/2}(X_i)} \right)^{2(1+\delta)} \frac{1}{f_X(X_i - h_{1n})} K_1^{2(1+\delta)} (\psi) f_X(X_i - h_{1n})f_X(X_i)d\psi dX_i \]
\[ \to \int K_1^{2(1+\delta)} (\psi)d\psi \int f_X^{-2(1+\delta)}(X_i)dX_i < C \]

since \( \sup_{x \in V} f_X(x) \) and \( h(x) > C > 0. \) Consequently, \( Q_{2313} = O \left( \frac{1-F(y_n)}{y_n} \right). \) Given that all elements of \( m(2)(x) \) are uniformly bounded above, similar arguments give \( Q_{2314} = O \left( \frac{1-F(y_n)}{y_n} h_{1n}^{4(1+\delta)} \right). \) Combining the orders of \( Q_{2313}, \) \( Q_{2314} \) we have \( \int \int |\phi_n(w_i,w_i)|^{2+\delta}dP(w_i)dP(w_i) = O \left( \frac{1-F(y_n)}{y_n} h_{3n}^{-(1+2\delta)} h_{1n}^{-d(1+2\delta)} \right). \)

Hence, we can write \( E(Q_{2313}^2) = O \left( n^{-2} \left( \frac{1-F(y_n)}{y_n} \right)^{(1+\delta)} h_{3n}^{-(1+2\delta)}(1+\delta) h_{1n}^{-(d(1+2\delta))} \right) \) and consequently
\[ Q_{2311} = O_p \left( n^{-1} \left( \frac{1-F(y_n)}{y_n} \right)^{1/2(1+\delta)} h_{3n}^{-(1+2\delta)/2(1+\delta)} h_{1n}^{-d(1+2\delta)/2(1+\delta)} \right). \]

Thus,
\[ \frac{n^{1/2}}{(1 - F(y_n))^{1/2}} Q_{2313} = O_p \left( \left( nh_{3n}^{1+2\delta} h_{1n}^{d(1+2\delta)} (n(1 - F(y_n))) \right)^{1/2(1+\delta)} \right). \]

Since \( y_n \to \infty \) and \( n(1 - F(y_n)) \to \infty \) we have that \( \frac{n^{1/2}}{(1 - F(y_n))^{1/2}} Q_{2313} = o_p(1) \) provided \( nh_{3n}^{1+2\delta} h_{1n}^{d(1+2\delta)} \to \infty, \)

which is verified given that \( h_{1n} \propto n^{-\psi/(2(1+\delta))} \) and \( h_{3n} \propto n^{-\psi/(2(1+\delta)) + \delta}. \) We now consider \( Q_{2321} = \frac{1}{n} \sum_{t=1}^{n} \sum_{t<i} (\theta_{ni} + \)
\[ Q_{23n} = \frac{1}{2n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} (\theta_{ni} + \theta_{nj} - \theta_n) = \frac{n - 1}{n^2} \sum_{t=1}^{n} \theta_{nt} - \frac{1}{2n^2} \frac{n^2 - n}{2} \theta_n \]

\[ = \frac{n - 1}{n^2} \sum_{t=1}^{n} \int \phi_n(w_t, w_i)dP(w_i) + O(1)\theta_n. \]

Let \( \theta_n = I_{1n} + I_{2n} \) where

\[ I_{1n} = \frac{2}{h_{3n}h_{1n}^d} \int \int \frac{1}{f_x(x_i)} K_3 \left( \frac{y_n - \varepsilon_i}{h_{3n}} \right) K_1 \left( \frac{x_t - x_i}{h_{1n}} \right) K_1 \left( \frac{x_t - X_i}{h_{1n}} \right) \varepsilon_i dP(w_i) dP(w_t) \]

\[ I_{2n} = \frac{1}{h_{3n}h_{1n}^d} \int \int \frac{1}{h^{1/2}(x_i)/f_x(x_i)} K_3 \left( \frac{y_n - \varepsilon_i}{h_{3n}} \right) K_3 \left( \frac{x_t - X_i}{h_{1n}} \right) (X_{it} - X_i)^T m^{(2)}(Z_{it})(X_t - X_i) \times dP(w_i)dP(w_t). \]

We note that \( E(I_{1n}) = 0 \) since \( E(\varepsilon_i) = 0 \) and

\[ E(I_{2n}) = \frac{2}{h_{3n}} E \left( K_3 \left( \frac{y_n - \varepsilon_i}{h_{3n}} \right) \right) \frac{1}{h_{1n}^d} \int \int K_1 \left( \frac{x_t - x_i}{h_{1n}} \right) K_1 \left( \frac{x_t - X_i}{h_{1n}} \right) (x_t - X_i)^T m^{(2)}(Z_{it})(x_t - X_i) \times f_x(x_i) dX_i dX_t = I_{21n} I_{22n}. \]

Using Proposition 1.15 in Resnick (1987) we have \( \frac{\theta_n}{-F(y_n)} I_{21n} \to -2k_0 \). Furthermore, given that \( h(x) > C > 0 \) and \( f_x \) have \( s \) bounded partial derivatives \( I_{22n} = O(h_{1n}^s) \). Hence, \( \theta_n = O \left( h_{1n}^{s} \frac{1-F(y_n)}{y_n} \right) \).

Now,

\[ \frac{n-1}{n} \sum_{t=1}^{n} \int \phi_n(w_t, w_i)dP(w_i) = (1 + O(n^{-1})) \frac{1}{n} \sum_{t=1}^{n} \left( \int \psi_n(w_t, w_i)dP(w_i) \right) \]

\[ + \left( \int \psi_n(w_t, w_i)dP(w_i) \right) \]

\[ = (1 + O(n^{-1})) \frac{1}{n} \sum_{t=1}^{n} (I_{11t} + I_{12t} + I_{13t}) \]

where,

\[ I_{11t} = \frac{1}{h_{3n}} E \left( K_3 \left( \frac{y_n - \varepsilon_i}{h_{3n}} \right) \right) \frac{1}{h_{1n}^d} \int K_1 \left( \frac{x_t - x_i}{h_{1n}} \right) K_1 \left( \frac{x_t - X_i}{h_{1n}} \right) dX_i h^{1/2}(X_t) \varepsilon_t \]

\[ I_{12t} = \frac{1}{h_{3n}} E \left( K_3 \left( \frac{y_n - \varepsilon_i}{h_{3n}} \right) \right) \frac{1}{h_{1n}^d} \int K_1 \left( \frac{x_t - x_i}{h_{1n}} \right) K_1 \left( \frac{x_t - X_i}{h_{1n}} \right) \frac{1}{2 h^{1/2}(X_i)} \]

\[ \times (X_t - X_i)^T m^{(2)}(Z_{it})(X_t - X_i) dX_i \]

\[ I_{13t} = \frac{1}{h_{3n}} E \left( K_3 \left( \frac{y_n - \varepsilon_i}{h_{3n}} \right) \right) \frac{1}{f_x(x_i)} \frac{1}{h_{1n}^d} \int K_1 \left( \frac{x_t - x_i}{h_{1n}} \right) K_1 \left( \frac{x_t - X_i}{h_{1n}} \right) \frac{1}{2 h^{1/2}(X_i)} \]

\[ \times (X_t - X_i)^T m^{(2)}(Z_{it})(X_t - X_i) f_x(x_i) dX_i. \]
From the order of $I_{21n}$ and the fact that $(X_t^T \varepsilon_t)$ is strictly stationary, $X_t$ independent of $\varepsilon_t$, we have for $t \neq i$ that $\frac{1}{n} \sum_{t=1}^{n} I_{21t} = O_p \left( n^{-1/2} \frac{1-F(y_n)}{y_n} \right)$. Now,

$$\frac{y_n}{1-F(y_n)} \frac{1}{n} \sum_{t=1}^{n} I_{12t} = \frac{y_n}{1-F(y_n)} E \left( K_3 \left( \frac{y_n - \varepsilon_i}{h_{3n}} \right) \right) \times \frac{1}{n} \sum_{t=1}^{n} \int \frac{h_{1n}^2}{2h^{1/2}(X_i - h_{1n} \phi_t)} K_1(\phi_t) \phi_t^T m^{(2)}(X_i + \lambda h_{1n} \phi_t) \phi_t d\phi_t$$

$$= O(1) \frac{1}{n} \sum_{t=1}^{n} \gamma_{nt}$$

and $E \left( \left( \frac{1}{n} \sum_{t=1}^{n} \gamma_{nt} \right)^2 \right) = \frac{1}{n^2} \sum_{t=1}^{n} E(\gamma_{nt}^2) + \frac{1}{n} \sum_{t \neq i} E(\gamma_{nt} \gamma_{ni})$. From previous arguments we immediately conclude that $E(\gamma_{nt}^2) = O(h_{1n}^2)$. For the second term, note that $\frac{1}{n^2} \sum_{t=1}^{n} \sum_{t \neq i} E(\gamma_{nt} \gamma_{ni}) \leq \frac{1}{n} \sum_{t=1}^{n} \sup_{t \neq i} \sum_{t=1}^{n} |E(\gamma_{nt} \gamma_{ni})|$, 

Letting $J_n = \sum_{i=1}^{n} |E(\gamma_{nt} \gamma_{ni})|$, we observe that for $\delta_1 > 2$, by Theorem 3(1) in Doukhan (1994) we have

$$|E(\gamma_{nt} \gamma_{ni})| \leq S(8E(|\gamma_{nt}|^{\delta_1})^{2/\delta_1} \alpha(i)^{1-2/\delta_1}$$

(27)

where $E(|\gamma_{nt}|^{\delta_1}) = O(h_{1n}^\delta_1)$. Hence, by assumption A2 1) $J_n \leq O(h_{1n}^2) \sum_{i=1}^{n} \alpha(i)^{1-2/\delta_1} = O(h_{1n}^2)$. Similarly,

$$\sum_{i=1}^{n} |E(\gamma_{nt} \gamma_{ni})| = O(h_{1n}^2)$$

and consequently $\frac{1}{n^2} \sum_{t=1}^{n} \sum_{t \neq i} E(\gamma_{nt} \gamma_{ni}) \leq O(n^{-1} h_{1n}^2)$. As a result, we write

$$E \left( \left( \frac{1}{n} \sum_{t=1}^{n} \gamma_{nt} \right)^2 \right) = O(n^{-1} h_{1n}^2)$$

and $\frac{y_n}{1-F(y_n)} \frac{1}{n} \sum_{t=1}^{n} I_{12t} = O_p \left( n^{-1/2} h_{1n}^2 \right)$. Now, let $\frac{1}{n} \sum_{t=1}^{n} I_{13t} = O(1) \frac{1}{n} \sum_{t=1}^{n} \eta_{nt}$ where $\eta_{nt} = \frac{1}{n} \sum_{t=1}^{n} E(\gamma_{nt})$ and $\frac{1}{n} \sum_{t=1}^{n} \gamma_{nt} = O \left( n^{-1} h_{3n} h_{1n}^2 \frac{1-F(y_n)}{y_n} \right)$. Furthermore, for $\delta_1 > 2$ we have

$$\sum_{i=1}^{n} E(|\eta_{nt} \eta_{ni} + 1|) \leq 8 \sum_{i=1}^{n} \left( E(|\eta_{nt}|^{\delta_1})^{2/\delta_1} \alpha(i)^{1-2/\delta_1} = O \left( h_{3n}^2(1-\delta_1) h_{1n}^{2a} (1-F(y_n))^{2/\delta_1} \right) \right)$$

as $\delta_1 > 2$ and $y_n \rightarrow \infty$. Thus, provided that $\frac{h_{1n}^2}{h_{3n}^2(1-F(y_n))} \rightarrow 0$, which is verified when $h_{1n} \propto n^{1/2(2s+d)}$, $h_{3n} \propto n^{-s/(2s+d)+\delta}$, we have $\sum_{i=1}^{n} E(|\eta_{nt} \eta_{ni} + 1|) = o(1-F(y_n))$. Thus, $E \left( \left( \frac{1}{n} \sum_{t=1}^{n} \eta_{nt} \right)^2 \right) = O \left( n^{-1} h_{3n} h_{1n}^{2a} (1-F(y_n)) \right)$.
\[ o(n^{-1}(1 - F(y_n))). \] Hence, \( \frac{1}{n} \sum_{t=1}^{n} T_{13t} = o_p \left( \frac{(1 - F(y_n))^{1/2}}{n^{1/2}} \right) \) and
\[
\frac{n - 1}{n^2} \sum_{t=1}^{n} \int \phi_n(w_i, w_t) dP(w_t) = O_p \left( \frac{n^{-1/2} 1 - F(y_n)}{y_n} \right) + O_p \left( \frac{n^{-1/2} h_{1n}^s 1 - F(y_n)}{y_n} \right)
+ o_p \left( (1 - F(y_n))^{1/2} \right) = o_p \left( \frac{(1 - F(y_n))^{1/2}}{n^{1/2}} \right).
\]
Consequently, \( Q_{232n} = O \left( h_{1n}^s \frac{1 - F(y_n)}{y_n} \right) + o_p \left( (1 - F(y_n))^{1/2} \right) \) and
\[
Q_{23n} = O_p \left( h_{1n}^s \frac{1 - F(y_n)}{y_n} \right) + o_p \left( (1 - F(y_n))^{1/2} \right).
\]
Now, whenever \( i = t \) we have \( Q_{23n} = -\frac{1}{nh_{3n}^2} K_i(0) \frac{1}{n} \sum_{t=1}^{n} \nu_{nt} \) where \( \nu_{nt} = \frac{1}{n} \sum_{i=1}^{n} K_3 \left( \frac{y_n - \varepsilon_t}{h_{3n}} \right) \frac{\varepsilon_t}{f(x_t)} \). \( E(\nu_{nt}^2) \) is \( O(h_{3n}^{-1}) \) since \( y_n(1 - F(y_n)) = O(1) \) if \( \alpha > 1 \). Also, for \( \delta_1 > 2 \) we have \( \sum_{i=1}^{n} |E(\nu_{nt} \nu_{nt+p})| \leq 8(E(\nu_{nt}^2))^{2/\delta_1} \alpha(i)^{1-2/\delta_1} \) where \( E(\nu_{nt}^2) = O(h_{3n}^{-2}) \) if \( E(\varepsilon^2) < \infty \). Hence, \( \sum_{i=1}^{n} |E(\nu_{nt} \nu_{nt+p})| = O(h_{3n}^{-2}) \) and consequently \( E \left( \left( \frac{1}{n} \sum_{t=1}^{n} \nu_{nt} \right)^2 \right) = O(n h_{3n}^{-2}) \). Thus, we have that \( Q_{23n} = O_p \left( n^{-3/2} h_{1n}^{-1} h_{3n}^{-1} \right) \) and \( Q_{23n} = o_p \left( (1 - F(y_n))^{1/2} \right) \) since \( n(1 - F(y_n)) \to \infty \) and \( n^{-1/2} h_{1n}^{-1} h_{3n}^{-1} = o(1) \). Overall, we have \( Q_{23n} = o_p \left( (1 - F(y_n))^{1/2} \right) + O_p \left( h_{1n}^s (1 - F(y_n)) \right) \) and consequently
\[
\frac{n^{1/2}}{(1 - F(y_n))^{1/2}} Q_{23n} = o_p(1). \tag{28}
\]
We now consider \( Q_{24n} \) which can be written as
\[
Q_{24n} = \frac{1}{nh_{3n}} \sum_{t=1}^{n} K_3 \left( \frac{y_n - \varepsilon_t}{h_{3n}} \right) \left( \frac{h_{1/2}(X_t)}{h_{1/2}(X_t) - 1} \right) \left( \chi_{\{h(X_t) > 0\}} - 1 \right) \varepsilon_t
+ \frac{1}{nh_{3n}} \sum_{t=1}^{n} K_3 \left( \frac{y_n - \varepsilon_t}{h_{3n}} \right) \left( \frac{h_{1/2}(X_t)}{h_{1/2}(X_t) - 1} \right) \varepsilon_t = Q_{241n} + Q_{242n}.
\]
By Lemmas 3.4 and Corollary 1 we have \( \sup_{x \in \mathcal{X}} \left| \chi_{\{h(X_t) > 0\}} - 1 \right| = O_p(L_{1n} + L_{2n}) \) and \( \sup_{x \in \mathcal{X}} \left| h_{1/2}(X_t) \right| = O_p(L_{1n} + L_{2n}). \) Thus, \( Q_{241n} \leq O_p(L_{1n}^2 + L_{2n}^2) \frac{1}{nh_{3n}} \sum_{t=1}^{n} K_3 \left( \frac{y_n - \varepsilon_t}{h_{3n}} \right) ||\varepsilon_t|| \). Since \( 1 - F(y_n) = o(1) \), by Proposition 1.15 in Resnick (1987) we have \( \frac{1}{nh_{3n}} \sum_{t=1}^{n} K_3 \left( \frac{y_n - \varepsilon_t}{h_{3n}} \right) ||\varepsilon_t|| = o_p(1) \). Hence, \( Q_{241n} = o_p(L_{1n}^2 + L_{2n}^2) \) and
\[
\frac{n^{1/2}}{(1 - F(y_n))^{1/2}} Q_{241n} = o_p(1). \] Now, we can write \( Q_{242n} \) as
\[
Q_{242n} = -\frac{1}{2nh_{3n}} \sum_{t=1}^{n} K_3 \left( \frac{y_n - \varepsilon_t}{h_{3n}} \right) \left( \frac{1}{h(X_t)} (\hat{h}(X_t) - h(X_t)) \right) + o_p \left( \frac{(1 - F(y_n))^{1/2}}{n^{1/2}} \right),
\]
hence it suffices to study the order of the first term. Given that \( \hat{h}(X_t) \) is a local linear estimator, and under
where the last inequality follows from Lemma 3 and Corollary 1. Given that

$$A$$

a case of

Due to the similarity of the arguments, we omit the details. However, it is worth pointing out that in the assumptions A1-A3 we can write

$$O$$

Following arguments that are similar to those used in the study of $$Q_{23n}$$, we show that

$$n^{1/2} \left(1 - F(y_n)\right)^{1/2} Q_{24n} = o_p(1).$$  \tag{30}$$

Due to the similarity of the arguments, we omit the details. However, it is worth pointing out that in the case of $$Q_{24n}$$, three dimensional U-Statistics appear due to the structure of (29). As such, we must use part (i) of Lemma A.2 in Gao (2007). This in turn requires assumptions A5 2) and 4).

Now, let $$A_t = \{ \omega : |\chi_h(X_t) - 1| = 0 \}$$ and $$B_t = \{ \omega : h(X_t) < \hat{h}(X_t) \leq \delta \}$$ for some $$\delta > 0$$. Clearly, $$A_t^c \subseteq B_t^c$$ so that $$\chi A_t^c \leq \chi B_t^c \leq h(X_t) - \hat{h}(X_t)$$. Thus,

$$Q_{25n} = \frac{1}{nh_{3n}} \sum_{t=1}^{n} K_3 \left( \frac{y_n - \varepsilon_t}{h_{3n}} \right) \left| \chi_h(X_t) - 1 \right| \varepsilon_t A_t^c$$

$$\leq \frac{1}{nh_{3n}} \sum_{t=1}^{n} K_3 \left( \frac{y_n - \varepsilon_t}{h_{3n}} \right) \left| \chi_h(X_t) - 1 \right| \left| \varepsilon_t \right| \frac{h(X_t) - \hat{h}(X_t)}{\delta}$$

$$\leq O_p(L_{1n}^2 + L_{2n}^2) \frac{1}{nh_{3n}} \sum_{t=1}^{n} K_3 \left( \frac{y_n - \varepsilon_t}{h_{3n}} \right) \left| \varepsilon_t \right|$$

where the last inequality follows from Lemma 3 and Corollary 1. Given that $$\frac{1}{n h_{3n}} \sum_{t=1}^{n} K_3 \left( \frac{y_n - \varepsilon_t}{h_{3n}} \right) \left| \varepsilon_t \right| = O_p(1)$$ we have

$$n^{1/2} \left(1 - F(y_n)\right)^{1/2} Q_{25n} = o_p(1).$$  \tag{31}$$

Combining the orders obtained in equations (23), (24), (28), (30) and (31) we have

$$n^{1/2} \left(1 - F(y_n)\right)^{1/2} Q_{2n} = o_p(1).$$  \tag{32}$$

Lastly, we show that $$\frac{n}{\sqrt{N}} (Q_{1n} - F(y_n)) \overset{d}{\rightarrow} N(0, 1)$$. First, we put $$q_{1n} = \frac{1}{h_{3n}} \int_{-\infty}^{y_n} K_3 \left( \frac{y - \varepsilon_t}{h_{3n}} \right) dy$$ and write

$$\frac{n}{\sqrt{N}} (Q_{1n} - F(y_n)) = \sum_{t=1}^{n} \frac{1}{\sqrt{n(1 - F(y_n))}} (q_{1n} - E(q_{1n})) + \sum_{t=1}^{n} \frac{1}{\sqrt{n(1 - F(y_n))}} (E(q_{1n}) - F(y_n))$$

$$= I_{1n} + I_{2n}.$$
Clearly, \( E\left(\frac{1}{\sqrt{n(1-F(y_n))}} (q_{1n} - E(q_{1n}))\right) = 0 \) and \( V\left(\frac{1}{\sqrt{n(1-F(y_n))}} (q_{1n} - E(q_{1n}))\right) = \frac{s_n^2}{n(1-F(y_n))} \) where
\[
s_n^2 = \int \frac{1}{h_{3n}} b\left(\frac{y_n - u}{h_{3n}}\right) F(u)du - \left( \int \frac{1}{h_{3n}} K_1 \left(\frac{y_n - u}{h_{3n}}\right) F(u)du \right)^2
\]
and \( b(x) = 2K_3(x) \int_x^\infty K_3(y)dy \). Define \( s^2 = F(y_n)(1-F(y_n)) \) and write \( \frac{s_n^2}{1-F(y_n)} = \frac{s_n^2}{1-F(y_n)} + F(y_n) \).

Since, \( \frac{s_n^2-s^2}{1-F(y_n)} = o(h_{3n}) \) and \( F(y_n) \rightarrow 1 \) as \( n \rightarrow \infty \) we have \( \frac{s_n^2}{1-F(y_n)} \rightarrow 1 \). By Liapounov’s CLT, \( I_{1n} \xrightarrow{d} N(0,1) \) provided that \( nE(|Z_{in}|^3) \rightarrow 0 \) as \( n \rightarrow \infty \), where
\[
Z_{in} = \frac{1}{\sqrt{n(1-F(y_n))}} \left( \frac{1}{h_{3n}} \int_{-\infty}^{y_n} K_3 \left(\frac{y - \varepsilon_i}{h_{2n}}\right) dy - E \left( \frac{1}{h_{3n}} \int_{-\infty}^{y_n} K_3 \left(\frac{y - \varepsilon_i}{h_{3n}}\right) dy \right) \right).
\]

The condition is verified by noting that
\[
\left| \left( \frac{1}{h_{3n}} \int_{-\infty}^{y_n} K_3 \left(\frac{y - \varepsilon_i}{h_{2n}}\right) dy - E \left( \frac{1}{h_{3n}} \int_{-\infty}^{y_n} K_3 \left(\frac{y - \varepsilon_i}{h_{3n}}\right) dy \right) \right) \right| \leq 2
\]
since \( \frac{1}{h_{3n}} \int_{-\infty}^{y_n} K_3 \left(\frac{y - \varepsilon_i}{h_{2n}}\right) dy \leq 1 \). Consequently, \( |Z_{in}| \leq \frac{2}{\sqrt{n(1-F(y_n))}} \) and
\[
n E(|Z_{in}|^3) \leq \frac{2n}{\sqrt{n(1-F(y_n))}} \frac{s_n^2}{n(1-F(y_n))} \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

Integrating by parts we have
\[
|E(q_{1n}) - F(y_n)| = \left| \int (-h_{3n})^j \psi K_3(\psi) f(y_n) + \sum_{j=1}^{m_1-1} \frac{(-h_{3n})^j}{(j+1)!} \frac{d}{dy_n^j} f(y_n) \right|
\]
\[
+ \left( \frac{(-h_{3n})^m}{m!} \right) \int \psi f(y_n)dy_n
\]
where \( y_n^* = \lambda(y_n-h_{3n}\psi)+(1-\lambda)y_n \) for some \( \lambda \in (0,1) \). Since \( K_3 \) is an \( m_1^{th} \)-order kernel and \( \left| \frac{d}{dy_n^j} f(y_n) \right| < C \),

we have that \( |E(q_{1n}) - F(y_n)| \leq C \frac{h_{3n}^{m_1+1}}{(m_1+1)!} \int \psi f(y_n)dy_n \) \( = O(h_{3n}^{m_1+1}) \). Hence, \( I_{2n} = O \left( \frac{n}{\sqrt{n}} h_{3n}^{m_1+1} \right) = o(1) \), given the orders of \( h_{3n} \) and \( N \), and
\[
\frac{n}{\sqrt{N}} (Q_{1n} - F(y_n)) \xrightarrow{d} N(0,1).
\]

Equations (20), (32) and (33) show that \( \frac{n}{\sqrt{N}} \left( \hat{f}(y_n) - F(y_n) \right) \xrightarrow{d} N(0,1) \), and by consequence \( T_{1n} = O_P(1) \) which completes the proof. \( \square \)

**Lemma 5.** Let \( a_n = 1 - \frac{N}{n} \) and for \( i = 1, \cdots, N \) define \( Z_i = \varepsilon_i - q_i(a_n) \) whenever \( \varepsilon_i > q_i(a_n) \) and for \( i = 1, \cdots, N_1 \) define \( Z_i' = \varepsilon_i - q(a_n) \) whenever \( \varepsilon_i > q(a_n) \). If \( \Delta_\sigma = \frac{1}{N} \sum_{i=1}^{N} \frac{\partial}{\partial \sigma} \log g(Z_i; \sigma_N, k_0) \sigma_N - \sigma_N \)

20
\[
\frac{1}{N} \sum_{i=1}^{N} \frac{\partial}{\partial \sigma} \log g(Z_i; \sigma_N, k_0) \sigma_N \text{ and } \Delta_k = \frac{1}{N} \sum_{i=1}^{N} \frac{\partial}{\partial k} \log g(Z_i; \sigma_N, k_0) - \frac{1}{N} \sum_{i=1}^{N} \frac{\partial}{\partial k} \log g(Z_i'; \sigma_N, k_0), \text{ then}
\]
\[
N^{1/2} \Delta_\sigma = b_1 \sqrt{N} q_n(\alpha_n) - q(\alpha_n) + O_p(1) \text{ and } N^{1/2} \Delta_k = b_2 \sqrt{N} q_n(\alpha_n) - q(\alpha_n) + O_p(1), \text{ where } b_1 = -\frac{\alpha(1+\alpha)}{2+\alpha}, \text{ and } b_2 = \left(-\frac{\alpha^2(1+\alpha)}{2+\alpha} + \frac{\alpha^3}{1+\alpha}\right).
\]

**Proof.** The proof is identical to that of Lemma 3 in Martins-Filho et al. (2014) by substituting their \( U_{(n-N)} \) with \( \varepsilon_{(n-N)} \).

**Lemma 6.** \( E \left( \log \left( 1 - \frac{k_0 Z'_i}{\sigma_N} \right) \left( 1 - \frac{k_0 Z'_i}{\sigma_N} \right)^{-1} \left( k_0 Z'_i \right)^{-1} \right) = -\frac{1}{\alpha} + \frac{\alpha}{(1+\alpha)^2} + O(\varepsilon_{(n-N)}) \)

**Proof.** The proof is identical to that of Lemma 4 in Martins-Filho et al. (2014) by substituting their \( U_{(n-N)} \) with \( \varepsilon_{(n-N)} \).

**References**


