Predictability of Stock Returns and Asset Allocation under Structural Breaks

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Outline I

1. Introduction
2. Methodology
3. Empirical Results on Predictability under Breaks
4. Asset Allocation under Structural Breaks
   - Buy-and-hold Results
   - Asset Allocation with Rebalancing
   - Welfare Costs of ignoring Breaks
5. Extensions
6. Conclusion
Predictability of stock returns: Early papers

- Campbell (1987)
- Campbell and Shiller (1988)
- Fama and French (1988, 1989)
- Ferson and Harvey (1991)
- Goetzmann and Jorion (1993)
- Harvey (1989)
- Keim and Stambaugh (1986)
Asset Allocation under Predictability of Returns

- Ait-Sahalia and Brandt (2001)
- Ang and Bekaert (2002)
- Barberis (2000)
- Balduzzi and Lynch (1999)
- Brandt (1999), Brandt, Goyal, Santa-Clara and Stroud (2002)
- Brennan, Schwarz and Lagnado (1997), Brennan and Xia (2001)
- Detemple, Garcia and Rindisbacher (2003)
- Kandel and Stambaugh (1996)
- Xia (2001)
Key Questions for Asset Allocation under Predictability

- Parameter Estimation Errors
- Model Uncertainty
- Model Instability

Approaches for solving portfolio choice problems:

1. Plug-in estimation or calibration methods: parameters of the return process are estimated and plugged into the analytical or numerical solution of the investor’s portfolio choice

2. Method of moments approach - determine asset allocation off the Euler equation

3. Decision theory approach: integrate model estimation, portfolio selection problem typically by using a Bayesian methodology
Current Return predictability debate

- Pesaran and Timmermann (1995)
- Bossaerts and Hillion (1999)
- Lettau and Ludvigsson (2001)
- Goyal and Welch (2003)
- Cooper, Gutierrez and Marcum (2005)
- Campbell and Thompson (2005)
- Cochrane (2006)

Is the weak out-of-sample predictability due to model instability ("breaks")?
Evidence of structural breaks

- Stock and Watson (1986)
- Bai, Lumsdaine and Stock (1998)
- Ang and Bekaert (2001, 2002)
- Pastor and Stambaugh (2001)

Weakness of out-of-sample return predictability has been linked to model instability

- Schwert (2003)
- Perez-Quiros and Timmermann (2000)
- Paye and Timmermann (2005)
- Lettau and Van Nieuwerburgh (2005)
Pastor and Stambaugh (2001, p. 1207) “Finance practitioners and academics often elect to rely on more recent data ... motivated in part by concerns that the probability distribution of excess returns changes over time, experiencing shifts known as “structural breaks””

- when did the break occur? (Bai, Bai-Perron)
- How much data to use?
- inefficient only to use post-break data (Pesaran and Timmermann (2005, 2006))
- Can breaks happen in the future?
Contributions of this paper

- Develop a methodology to forecast returns and compute asset allocation in the presence of structural breaks
- Introduce a ‘meta’ distribution for the parameters across regimes and a changepoint model based on a hierarchical Hidden Markov Chain (HMC)
- Consider investment decisions under uncertainty about
  1. parameter estimates
  2. model specification
  3. number, timing (dates) and size of both past and future breaks
Empirical Findings: Predictability

Apply the method to US stock returns, using Dividend Yield or T-bill Rate as predictor variables

- Evidence of multiple breaks in return models
- Break dates coincide with major events such as changes in the Fed’s operating procedures (1979, 1982), Great Depression, World War II and the growth slowdown in early 1970s
- Structural breaks influence value and precision of parameter estimates of the return prediction model
- Return Predictability varies greatly over time
Structural breaks can have a large effect on a buy-and-hold investor’s asset allocation.

Breaks continue to have an important effect under rebalancing.

- Parameter estimation uncertainty becomes more important since the parameters from the current regime are surrounded by larger uncertainty than the full-sample parameters.
- Possibility of future breaks affects investors’ optimal allocations even under rebalancing due to incomplete learning since they can only be detected with a lag.
Methodology

Changepoint model driven by unobserved discrete state variable
After a break, the new parameters of the return forecasting model are drawn from a meta distribution

→ we can forecast out-of-sample even in the presence of future breaks

Our approach accounts for structural breaks in return forecasting models, building on

- Chib (1998)
- Pastor and Stambaugh (2001)
- Pesaran, Pettenuzzo and Timmermann (2005)
Return Prediction Model

- **Restricted VAR:**
  \[ z_t = B' \tilde{x}_{t-1} + u_t \]

- \( z_t = (r_t, x_t)' \), \( \tilde{x}_{t-1} = (1, x_{t-1})' \)

  - \( r_t \): excess return at time \( t \)
  - \( x_{t-1} \): return predictor(s)
  - \( u_t \sim N(0, \Sigma) \)

- \( \mu_r, \mu_x \): intercepts in the return and predictor equation
- \( \beta_r, \beta_x \) coefficients on lagged predictor:

  \[ r_t = \mu_r + \beta_r x_{t-1} + \varepsilon_r t \]
  \[ x_t = \mu_x + \beta_x x_{t-1} + \varepsilon_x t \]
Break process tracked by integer-valued state variable, $S_t$. Conditional on $K$ breaks

\[ z_t = B'_1 \tilde{x}_{t-1} + u_t, \quad E[u_t u'_t] = \Sigma_1 \quad 1 \leq t \leq \tau_1 \quad s_t = 1 \]
\[ \vdots \]
\[ z_t = B'_{K+1} \tilde{x}_{t-1} + u_t, \quad E[u_t u'_t] = \Sigma_{K+1} \quad \tau_K + 1 \leq t \leq T \quad s_t = K + 1 \]

$\Upsilon_K = \{\tau_0, \ldots, \tau_K\}$ : collection of break points

Covariance matrix, $\Sigma_j$, decomposed as follows:

\[ \Sigma_j = \text{diag}(\psi_j) \times \Lambda_j \times \text{diag}(\psi_j) \]

Both volatilities and correlations can vary across regimes
Break dynamics is modeled through the transition probability matrix $P$:

\[
\tilde{P} = \begin{pmatrix}
p_{11} & p_{12} & 0 & \ldots & 0 \\
0 & p_{22} & p_{23} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & p_{KK} & p_{K,K+1} \\
0 & 0 & \ldots & 0 & p_{K+1,K+1} \\
0 & 0 & \ldots & 0 & 0 & p_{K+2,K+2} \\
\end{pmatrix}
\]

$p_{j,j}$ is assumed to be independent of $p_{i,i}$, for $j \neq i$, and is drawn from a beta distribution: $p_{j,j} \sim Beta(a, b)$.
Apply hierarchical prior setup to forecast returns out-of-sample

Location and scale parameters within each regime, \((B_j, \Sigma_j)\), are drawn from common “meta” distributions

Data from previous regimes carry information relevant for current data and for the new parameters after a future break

By using meta distributions that pool information from different regimes, historic information is used efficiently in estimating the parameters of the current regime

Special cases:

- pooled scenario (parameters are identical across regimes)
- regime-specific scenario (parameters unrelated across regimes)
To characterize the parameters of the meta distribution, we assume

\[ \text{vec}(B)_j \sim N\left( b_0, V_0 \right), j = 1, \ldots, K + 1 \]

\[ \psi_{j,i}^{-2} \sim \text{Gamma} \left( v_{0,i}, d_{0,i} \right) \]

\[ \lambda_{j,ic} \sim N\left( \mu_{\rho,ic}, \sigma^2_{\rho,ic} \right) \]

\[ b_0 \sim N\left( \mu_\beta, \Sigma_\beta \right) \]

\[ V_0^{-1} \sim W\left( v_\beta, V^{-1}_\beta \right) , \]

\[ W \left( . \right) : \text{Wishart distribution} \]

\[ \mu_\beta, \Sigma_\beta, v_\beta, V^{-1}_\beta : \text{prior hyperparameters} \]
\( \nu_{0,i} \sim \text{Exp} \left( \rho_{0,i} \right) \)

\( d_{0,i} \sim \text{Gamma} \left( c_{0,i}, d_{0,i} \right) \)

\( \rho_{0,i}, c_{0,i} \) and \( d_{0,i} \): prior hyperparameters

Hyperparameters of correlation matrix (truncated to lie on \((-1, 1)\)):

\( \mu_{\rho,i} \sim \text{N} \left( \mu_{\mu,i}, \tau_{i}^2 \right) \)

\( \sigma^{-2}_{\rho,i} \sim \text{Gamma} \left( a_{\rho,i}, b_{\rho,i} \right) \)

\( \mu_{\mu,i}, \tau_{i}^2, a_{\rho,i}, b_{\rho,i} \): prior hyperparameters

\( a \sim \text{Gamma} \left( a_0, b_0 \right) \)

\( b \sim \text{Gamma} \left( a_0, b_0 \right) \)
Impose two constraints on the parameters:

- $\beta_x < 1$
- $0 \leq \mu_x/(1 - \beta_x) \leq \bar{\mu}_x$

- $\mu_\beta = 0_{m^2}$
- $\Sigma_\beta = sc \times I_{m^2}$
- $V_\beta = \text{diag}(0.1, 10, 0.01, 0.1)$
- $c_{0,i} = 1$, $d_{0,i} = 1/\infty$ and $\rho_{0,i} = \infty$
- $\mu_{\mu,12} = 0$, $\tau_{12}^2 = \infty$, $a_{\rho,12} = 1$ and $b_{\rho,12} = 0.01$
- $a_0 = 1$ and $b_0 = 0.02$
Empirical Results

Data

- Monthly data on a portfolio of US stocks comprising firms listed on the NYSE, AMEX and NASDAQ
- Sample: 1926:12-2003:12
- Data source: CRSP
## Evidence of Breaks

### I. Excess returns - Dividend Yield

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Figure 1: Posterior probabilities of breakpoint locations for the return prediction model with seven breaks based on the dividend yield. The estimation sample is 1926:12 - 2003:12.
Evidence of seven Breaks
Break dates reasonably precisely determined
Break locations are associated with major events
  - Great Depression (1932)
  - Beginning of World War II (1940)
  - Major oil price shocks and growth slowdown (1974)
  - End of the change in the Fed’s operating procedures (1982)
  - Beginning of the bull market of the nineties (1992)
Remaining break dates (1952 and 1958) harder to interpret
### Table 2: Parameter estimates for the return ($r_t$) forecasting model with seven break points, based on the lagged dividend yield ($x_t$) as a predictor variable:

$$r_t = r_1 + r_1 x_t + r_t,$$

$$x_t = x_1 + x_1 x_t + x_t,$$

\[ r_t \sim N(0, \sigma_t^2), \]

\[ x_t \sim N(0, \sigma_t^2), \]

\[ \Pr (s_t = j | s_1 = j) = p_{jj}, \]

\[ \text{corr}(r_t; x_t) = \rho_{rj}, \]

Regimes

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<th>74-82</th>
<th>82-92</th>
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<td>0.004</td>
<td>0.002</td>
<td>0.002</td>
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<td>0.004</td>
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<td>0.001</td>
<td>0.002</td>
<td>0.002</td>
<td>0.001</td>
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<th>32-40</th>
<th>40-52</th>
<th>52-58</th>
<th>58-74</th>
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<th>82-92</th>
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<tr>
<td>mean</td>
<td>0.983</td>
<td>0.916</td>
<td>0.901</td>
<td>0.967</td>
<td>0.951</td>
<td>0.919</td>
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<td>0.006</td>
<td>0.033</td>
<td>0.035</td>
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<td>0.022</td>
<td>0.021</td>
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<tr>
<th>Regimes</th>
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<th>32-40</th>
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<tr>
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<td>0.714</td>
<td>0.443</td>
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<td>0.148</td>
<td>0.111</td>
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<td>0.013</td>
<td>0.006</td>
<td>0.016</td>
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<table>
<thead>
<tr>
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<th>74-82</th>
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<tbody>
<tr>
<td>mean</td>
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<td>-0.936</td>
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<td>0.022</td>
<td>0.023</td>
<td>0.023</td>
<td>0.020</td>
<td>0.017</td>
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<tbody>
<tr>
<td>mean</td>
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<td>0.986</td>
<td>0.990</td>
<td>0.983</td>
<td>0.992</td>
<td>0.987</td>
<td>0.988</td>
<td>N.A.</td>
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<tr>
<td>s.d.</td>
<td>0.013</td>
<td>0.010</td>
<td>0.008</td>
<td>0.013</td>
<td>0.006</td>
<td>0.010</td>
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<td>0.983</td>
<td>0.992</td>
<td>0.987</td>
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<td>N.A.</td>
</tr>
<tr>
<td>s.d.</td>
<td>0.013</td>
<td>0.010</td>
<td>0.008</td>
<td>0.013</td>
<td>0.006</td>
<td>0.010</td>
<td>0.010</td>
<td>N.A.</td>
</tr>
</tbody>
</table>

Table 2: Parameter estimates for the return ($r_t$) forecasting model with seven break points, based on the lagged dividend yield ($x_t$) as a predictor variable:

\[ r_t = r_j + r_j x_t^1 + r_{t-1} , \]

\[ x_t = x_j + x_j x_t^1 + x_{t-1} , \]

\[ \text{Pr} (s_t = j | s_{t-1} = j) = p_{jj} , \]

\[ \text{corr}(r_t; x_t) = r_{xj} , \]

\[ j1 + 1 \ t j. \]

The sample period is 1926:12-2003:12.
### Hyperparameters of Meta distributions

#### I Return equation

<table>
<thead>
<tr>
<th>Mean Parameters</th>
<th>mean</th>
<th>s.d.</th>
<th>95% conf interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_0(\mu_r)$</td>
<td>-0.042</td>
<td>0.038</td>
<td>-0.123 0.033</td>
</tr>
<tr>
<td>$b_0(\beta_r)$</td>
<td>1.218</td>
<td>0.508</td>
<td>0.225 2.209</td>
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</table>

#### II Dividend Yield equation

<table>
<thead>
<tr>
<th>Mean Parameters</th>
<th>mean</th>
<th>s.d.</th>
<th>95% conf interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_0(\mu_x)$</td>
<td>0.003</td>
<td>0.002</td>
<td>1.0E-04 0.008</td>
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<tr>
<td>$b_0(\beta_x)$</td>
<td>0.918</td>
<td>0.033</td>
<td>0.839 0.972</td>
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</table>

#### Correlation parameters

<table>
<thead>
<tr>
<th>$\mu_\rho$</th>
<th>mean</th>
<th>s.d.</th>
<th>95% conf interval</th>
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<tbody>
<tr>
<td></td>
<td>-0.920</td>
<td>0.042</td>
<td>-0.984 -0.831</td>
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#### Transition Probability parameters

<table>
<thead>
<tr>
<th>$a_0$</th>
<th>33.258</th>
<th>15.015</th>
<th>10.498 73.091</th>
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<tbody>
<tr>
<td>$b_0$</td>
<td>0.806</td>
<td>0.308</td>
<td>0.357 1.462</td>
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</table>
Variation in Parameter Estimates Across Regimes

- Dividend yield parameter highly persistent - mean autoregressive parameter varies from 0.90 to 0.98
- Correlation estimates for the innovations to stocks and the lagged dividend yield range from -0.96 to -0.85
- Stayer probabilities are high, mean durations ranging from 70 to 140 months
- Greater uncertainty about slope of dividend yield coefficient: parameter centered on 1.2 with a standard deviation of 0.50
- Greatest variability in parameters across regimes is associated with the effect of the dividend yield on stock returns
Evidence of seven breaks again
Break dates more dispersed than for the dividend yield model
Overlap in break dates
  - Great Depression (1934)
  - End of World War II (1947)
  - Vietnam War (1968)
  - Beginning and end of the change to the Fed’s operating procedures (1979 and 1982)
  - Beginning of the nineties’ bull market (1990)
Greatest uncertainty about slope of T-bill rate in return equation
Location of Breaks: T-bill Model

Figure 2: Posterior probabilities of breakpoint locations for the return prediction model with seven breaks based on the T-bill rate. The estimation sample is 1926:12 - 2003:12.
### Estimates (T-bill model)

#### Regimes

<table>
<thead>
<tr>
<th></th>
<th>Full sample</th>
<th>27-34</th>
<th>34-47</th>
<th>47-52</th>
<th>52-68</th>
<th>68-79</th>
<th>79-82</th>
<th>82-90</th>
<th>90-03</th>
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<tr>
<td>$\mu_r$ mean</td>
<td>0.009</td>
<td>-0.001</td>
<td>0.007</td>
<td>0.016</td>
<td>0.027</td>
<td>0.037</td>
<td>0.092</td>
<td>0.021</td>
<td>0.009</td>
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<tr>
<td>$\mu_r$ s.d.</td>
<td>0.003</td>
<td>0.018</td>
<td>0.005</td>
<td>0.011</td>
<td>0.007</td>
<td>0.019</td>
<td>0.034</td>
<td>0.024</td>
<td>0.010</td>
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<tr>
<td>$\beta_r$ mean</td>
<td>-1.402</td>
<td>-0.439</td>
<td>-3.343</td>
<td>-4.353</td>
<td>-8.215</td>
<td>-8.203</td>
<td>-9.976</td>
<td>-2.452</td>
<td>-0.942</td>
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<tr>
<td>$\beta_r$ s.d.</td>
<td>0.731</td>
<td>6.886</td>
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<td>10.542</td>
<td>3.015</td>
<td>4.074</td>
<td>3.656</td>
<td>4.086</td>
<td>2.724</td>
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<tr>
<td>$\sigma_r$ mean</td>
<td>0.055</td>
<td>0.109</td>
<td>0.059</td>
<td>0.036</td>
<td>0.033</td>
<td>0.047</td>
<td>0.047</td>
<td>0.049</td>
<td>0.044</td>
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<tr>
<td>$\sigma_r$ s.d.</td>
<td>0.001</td>
<td>0.008</td>
<td>0.003</td>
<td>0.004</td>
<td>0.002</td>
<td>0.003</td>
<td>0.006</td>
<td>0.004</td>
<td>0.002</td>
</tr>
<tr>
<td>$\mu_x \times 100$ mean</td>
<td>0.004</td>
<td>0.006</td>
<td>0.001</td>
<td>0.017</td>
<td>0.011</td>
<td>0.032</td>
<td>0.132</td>
<td>0.049</td>
<td>0.006</td>
</tr>
<tr>
<td>$\mu_x \times 100$ s.d.</td>
<td>0.002</td>
<td>0.004</td>
<td>0.001</td>
<td>0.006</td>
<td>0.005</td>
<td>0.014</td>
<td>0.069</td>
<td>0.021</td>
<td>0.004</td>
</tr>
<tr>
<td>$\beta_x$ mean</td>
<td>0.986</td>
<td>0.958</td>
<td>0.924</td>
<td>0.844</td>
<td>0.959</td>
<td>0.937</td>
<td>0.832</td>
<td>0.914</td>
<td>0.974</td>
</tr>
<tr>
<td>$\beta_x$ s.d.</td>
<td>0.005</td>
<td>0.021</td>
<td>0.027</td>
<td>0.065</td>
<td>0.020</td>
<td>0.030</td>
<td>0.075</td>
<td>0.037</td>
<td>0.012</td>
</tr>
</tbody>
</table>

Table 4: Parameter estimates for the return ($r_t$) forecasting model with seven break points, based on the lagged T-bill rate ($x_t$) as a predictor variable:

\[
 r_t = r_j + r_j x_t + r_t, \\
 x_t = x_j + x_j x_t + x_t, \\
 Pr (s_t = j | s_{t-1} = j) = p_{jj}, \\
 corr(r_t; x_t) = r_{xj}
\]

The sample period is 1926:12-2003:12.
### Table 4: Parameter estimates for the return ($r_t$) forecasting model with seven break points, based on the lagged T-bill rate ($x_{t-1}$) as a predictor variable:

\[ r_t = r_j + r_j x_{t-1} + x_t, \]

\[ x_{t+1} = x_j + x_j x_t + x_t, \]

\[ \sigma_x \times 100 \]

<table>
<thead>
<tr>
<th></th>
<th>0.040</th>
<th>0.033</th>
<th>0.004</th>
<th>0.011</th>
<th>0.027</th>
<th>0.045</th>
<th>0.129</th>
<th>0.047</th>
<th>0.025</th>
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</thead>
<tbody>
<tr>
<td><strong>mean</strong></td>
<td>0.005</td>
<td>0.067</td>
<td>0.003</td>
<td>0.170</td>
<td>-0.011</td>
<td>-0.189</td>
<td>-0.311</td>
<td>0.302</td>
<td>0.067</td>
</tr>
<tr>
<td><strong>s.d.</strong></td>
<td>0.032</td>
<td>0.026</td>
<td>0.026</td>
<td>0.046</td>
<td>0.033</td>
<td>0.034</td>
<td>0.033</td>
<td>0.037</td>
<td>0.030</td>
</tr>
<tr>
<td>( \rho_{rx} )</td>
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<td></td>
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<tr>
<td><strong>mean</strong></td>
<td>0.985</td>
<td>0.991</td>
<td>0.981</td>
<td>0.992</td>
<td>0.989</td>
<td>0.976</td>
<td>0.987</td>
<td>N.A.</td>
<td></td>
</tr>
<tr>
<td><strong>s.d.</strong></td>
<td>0.012</td>
<td>0.007</td>
<td>0.015</td>
<td>0.006</td>
<td>0.009</td>
<td>0.019</td>
<td>0.010</td>
<td>N.A.</td>
<td></td>
</tr>
</tbody>
</table>
Assume power utility over terminal wealth:

\[ u(W_{T+h}) = \frac{W_{T+h}^{1-\gamma}}{1-\gamma}, \quad \gamma > 0. \]

\( T \rightarrow T + h \): Holding period  
\( \gamma \): Coefficient of relative risk aversion  
\( W_{T+h} \): Terminal wealth \((W_T = 1)\)

\[ W_{T+h} = (1 - \omega) \exp(r_f h) + \omega \exp(r_f h + r_{T+1} + \ldots + r_{T+h}) \]
Buy-and-hold investor solves the following program

$$\max_{\omega} E_T \left( \frac{((1 - \omega) \exp(r_f h) + \omega \exp(r_f h + R_{T+h}))^{1-\gamma}}{1 - \gamma} \right)$$

subject to the no short-sales constraints $0 \leq \omega < 1$
No Breaks, no Estimation Uncertainty

\[ p(R_{T+h}|\hat{\Theta}, S_{T+h} = 1, Z_T) : \text{predictive return distribution ignoring parameter estimation uncertainty and breaks} \]

\[ \Theta = (\mu, \beta_0, \Sigma) : \text{VAR parameters} \]
\[ \hat{\Theta} : \text{Parameter estimate} \]
\[ Z_T : \text{Information set} \]

Investor maximizes

\[
\max_{\omega} \int u(W_{T+h}) p(R_{T+h}|\hat{\Theta}, S_{T+h} = 1, Z_T) dR_{T+h}
\]
Integrating over the posterior distribution, $\pi(\Theta|S_{T+h} = 1, Z_T)$, leads to the predictive distribution of returns conditioned only on the observed sample and the assumption of no breaks:

$$p(R_{T+h}|S_{T+h} = 1, Z_T) = \int p(R_{T+h}|\Theta, S_{T+h} = 1, Z_T) \times \pi(\Theta|S_{T+h} = 1, Z_T) d\Theta.$$ 

Investor solves the asset allocation problem

$$\max_\omega \int u(W_{T+h})p(R_{T+h}|S_{T+h} = 1, Z_T)dR_{T+h}.$$
Condition on $K$ historic breaks but no new break in $[T + 1, T + h]$:

$$
p(R_{T+h}|S_{T+h} = S_T = K + 1, Z_T) = \int p( R_{T+h}|\Theta_{K+1}, S_{T+h} = K + 1, S_T = K + 1, Z_T) \times \pi( \Theta_{K+1}|H, p, S_T, Z_T) \, d\Theta_{K+1}
$$

$$\Theta_{K+1} = (\text{vec}(B)_{K+1}, \psi_{K+1}, \Lambda_{K+1}) : \text{parameters from state } K + 1$$

$H$ = hyperparameters of the meta distribution =

$$\left( b_0, V_0, v_{0,1}, d_{0,1}, \ldots, v_{0,m}, d_{0,m}, \mu_{\rho,12}, \sigma_{\rho,12}^2, \ldots, \mu_{\rho,m-1m}, \sigma_{\rho,m-1m}^2, a, b \right)$$

Investor solves the portfolio problem

$$\max_\omega \int u(W_{T+h})p(R_{T+h}|S_{T+h} = S_T = K + 1, Z_T) \, dR_{T+h}$$
Past and Future Breaks

Condition on up to $n_b$ breaks over $[T, T + h]$. Let $j$ track the date where a break occurs. Probabilities of zero, one, up to $n_b$ breaks:

$$p(S_{T+h} = K + 1|S_T = K + 1, Z_T) = p^h_{K+1,K+1}$$

$$p(S_{T+h} = K + 2|S_T = K + 1, Z_T) = \sum_{j_1=1}^{h} (1 - p_{K+1,K+1}) \sum_{j_2=j_1+1}^{h} p^{j_1-1}_{K+1,K+1} (1 - p_{K+1,K+1})$$

$$\times p^{j_2-j_1-1}_{K+2,K+2} (1 - p_{K+2,K+2})$$

$$\vdots$$

$$p(S_{T+h} = K + n_b + 1|S_T = K + 1, Z_T) = \sum_{j_1=1}^{h-n_b+1} \ldots$$

$$\sum_{j_{n_b} = j_{n_b-1} + 1}^{h} \left( \prod_{j=1}^{n_b} p_{K+j,K+j} (1 - p_{K+j,K+j}) \right).$$
Use T-bill rate or Dividend Yield as predictor variable?

- Avramov (2002)
- Cremers (2002)

BMA over predictor variables, $X$, and number of breaks, $\bar{K}_x$:

$$p(R_{T+h}|Z_T) = \sum_{x=1}^{\bar{X}} \sum_{k=0}^{\bar{K}_x} p_x(R_{T+h}|S_T = k_x + 1, Z_T) p(M_{k_x}|Z_T).$$
Empirical Asset Allocation Results

- Mean-reverting component in returns means that the risk of stock returns grow slower than in the IID model (Barberis, 2000)
- Parameter estimation uncertainty generally reduces a risk averse investor’s demand for stocks: New information leading investors to revise downward their belief about mean stock returns is similar to a permanent negative dividend shock

Interaction between parameter estimation uncertainty and structural breaks:
- Breaks mean that bad draws of the parameters of the return model will eventually cease to affect returns following future breaks
- Breaks lower the precision of current parameter estimates and increase the importance of parameter estimation uncertainty
- Which effect dominates depends on the variability in the parameters across regimes and on the average duration of the regimes
Figure 4: Optimal Asset Allocation as a function of the investment horizon for a buy-and-hold investor with power utility over terminal wealth, $U(W_T + h) = \frac{1}{1+\gamma} W_T + h$, where $h$ is the forecast horizon and $\gamma$ is the coefficient of relative risk aversion. The panels show allocations to stocks under the assumption that the current regime parameters and the dividend yield are set at the values from the regime prevailing during 1958-1974. The dotted line shows allocations starting from the end of the 1952-1974 regime. The dashed line shows the full-sample allocation ignoring breaks and parameter estimation uncertainty. The dashed/dotted line shows allocations based on full-sample parameter values (no breaks) but accounting for parameter uncertainty. Finally, the solid line shows allocations under the model which accounts for both past and future breaks.
Initial value of (persistent) predictor variable matters a lot

Allocation to stocks increases in the horizon if the initial value of the dividend yield is very low and breaks are ignored.

If past breaks are accounted for but future breaks are ignored, the asset allocation can be flat or increasing in the horizon.

If both past and future breaks are modeled, we see a non-monotonic or sometimes strongly declining allocation to stocks, the longer the investment horizon.

Parameter instability has a larger effect on a buy-and-hold investor’s optimal asset allocation than parameter estimation uncertainty.
Asset Allocation under Predictability from T-bill Rate

Figure 5: Optimal Asset Allocation as a function of the investment horizon for a buy-and-hold investor with power utility over terminal wealth, \( U(W_t + h) = \frac{1}{1 + \gamma} W_t^\gamma \), where \( h \) is the forecast horizon and \( \gamma \) is the coefficient of relative risk aversion. The panels show allocations to stocks under the assumption that the T-bill rate is set at its value at the end of the sample, \( T_B = 0.83\% \). The dotted line shows allocations starting from the regime at the end of the sample (2003:12). The dashed line shows the full-sample allocation ignoring breaks and parameter estimation uncertainty. The dashed/dotted line shows allocations based on full-sample parameter values (no breaks) but accounting for parameter uncertainty. Finally, the solid line shows allocations under the model that accounts for both past and future breaks.
Rebalancing under Predictability from the Yield

Figure 9: Optimal Asset Allocation as a function of the investment horizon for an investor who optimally rebalances every 12 months and has power utility over terminal wealth, 

\[ U(W_T + h) = \frac{1}{(1+h)^\gamma} \]

where \( h \) is the forecast horizon and \( \gamma \) is the coefficient of relative risk aversion. The two panels show allocations to stocks under the assumption that the dividend yield is set at the average during the period 1958-1974, \( Yld_{1958-1974} = 3.1\% \). The dotted line shows allocations starting from the end of the 1958-1974 regime. The dashed line shows the full-sample allocation ignoring breaks and parameter estimation uncertainty. The dashed/dotted line shows allocations based on full-sample parameter values (no breaks) but accounting for parameter uncertainty. Finally, the solid line shows allocations under the composite model which accounts for past and future breaks.
Rebalancing under Predictability from the T-bill rate

Figure 10: Optimal Asset Allocation as a function of the investment horizon for an investor who optimally rebalances every 12 months and has power utility over terminal wealth, $U(W_T + h) = \frac{1}{\gamma} W^\gamma$, where $h$ is the forecast horizon and $\gamma$ is the coefficient of relative risk aversion. The two panels show allocations to stocks under the assumption that the T-bill rate is set at the average during the period 1952-1968, $T_{bi} = 2.68\%$. The dotted line shows allocations starting from the end of the 1952-1968 regime. The dashed line shows the full-sample allocation ignoring breaks and parameter estimation uncertainty. The dashed/dotted line shows allocations based on full-sample parameter values (no breaks) but accounting for parameter uncertainty. Finally, the solid line shows allocations under the model which accounts for both past and future breaks.
Rebalancing Results

- Allocation to stocks goes from being decreasing under no rebalancing to being marginally increasing under rebalancing.
- Rebalancing provides an efficient way for investors to adjust their asset allocation in case a future adverse shock hits the parameters of the stock return equation.
- Optimal asset allocations with and without breaks continue to be very different due to differences in the parameter estimates.
- Breaks have a very significant effect on asset allocations even under rebalancing.
Welfare Costs of ignoring Breaks

Compute certainty equivalent return (CER) for investor who ignores breaks versus an investor who accounts for breaks

- For the dividend yield model, the CER lies between 0.1 and zero percent
- The CER grows to 1.5 percent at short horizons under predictability from the T-bill rate
CER under rebalancing and Predictability

Figure 12: Certainty equivalence returns (in annualized percentage terms) under rebalancing as a function of the investment horizon for different levels of risk aversion. For each panel, the solid line shows the difference in certainty equivalence returns between a model that allows for past and future breaks and a model that ignores breaks. The dashed/dotted line shows the difference in certainty equivalence returns between a model that allows for past and future breaks and a model based on the last regime that considers past breaks but ignores future breaks.

\[ \gamma = 5, \ Yld_T = 3.1\% \]

\[ \gamma = 5, \ Tbi_T = 2.68\% \]
Our model allows for breaks to the covariance matrix of returns and is capable of accounting for heteroskedasticity in returns insofar as this coincides with the identified regimes.

The mean value of the standard deviation of returns varies significantly from a level around 10% around the Great Depression to a level near 3-4% in the middle of the sample.
Time-variations in Volatility

Figure 13: Standard deviations of the predictive distribution of excess returns when the predictor variable is the dividend yield (top panel) or the T-bill rate (bottom panel) under a model with seven breaks.
Conclusion

Our analysis accounts for

1. model uncertainty
2. parameter uncertainty
3. uncertainty about the number and size of historical breaks
4. uncertainty about future (out-of-sample) breaks

Empirical results

- Parameters of standard forecasting models appear to be highly unstable and subject to multiple shifts
- Many of the breaks coincide with important historical events
- Once such breaks are accounted for, the possibility of future breaks has a large impact on the optimal asset allocation