Threshold Integrated Moving Average Models

(Does size matter?
May be so)

Jesús Gonzalo and Oscar Martinez
U. Carlos III de Madrid

Soon at http://halweb.uc3m.es/jgonzalo

January 2004
0. Outline

- Introduction
- TIMA Models
- Assumptions
- Properties
  - Invertibility
  - Impulse Response Function
- Estimation
- Hypothesis Testing
- Application to Stock Prices (Hasbrouck (93) model on measuring price efficiency)
- Conclusions
I. Introduction

Q: Who Is the KING of the Time Series Models?

The Random Walk

\[ y_t = y_{t-1} + \epsilon_t \]

- All the Kings have problems
- In the Random Walk ALL the shocks are Permanent and/or at every \( t \) there is always a permanent shock

\[ \frac{\partial y_{t+k}}{\partial \epsilon_t} \neq 0 \text{ for } k \longrightarrow \infty \]
Solutions

• Some Standard Solutions: Permanent and Transitory decompositions
  – Uncorrelated Unobserved components (UC-0)
  – Perfect Correlated UC (Beveridge and Nelson (1981))
  – Correlated Unobserved components (UC-R)

• Some Problems:
  – Some of these decompositions may not always exist
  – In some decompositions all the shocks are permanent
  – They need identification assumptions that are not testable
  – At every $t$ there is always a permanent shock
• Some Recent Solutions
  
  – Stochastic Permanent Breaks (Engle and Smith 1999)

\[
y_t = m_t + \epsilon_t
\]
\[
m_t = m_{t-1} + q_{t-1}\epsilon_{t-1}
\]
\[
q_t = q(\epsilon_t)
\]

  * The magnitude of the persistence depends on the size of the shock
  * Problem: All the shocks are permanent


  * The magnitude of the persistence is regime-dependent
  * Problems: All the shocks are whether permanent or transitory
What are we looking for?

A type of Random Walk model with

- Two types of shocks:
  - Permanent and Transitory
- Shock’s identification assumptions that are
  - Testable
  - Coming from economic common sense

What is our proposal?

TIMA Models
II. TIMA Models

Threshold Integrated Moving Average Model (TIMA):

\[
(1 - L)y_t = x_t = \begin{cases} 
\epsilon_t - \theta_1 \epsilon_{t-1} & \text{if } |z_{t-1}| > r \\
\epsilon_t - \theta_2 \epsilon_{t-1} & \text{if } |z_{t-1}| \leq r
\end{cases}
\]

or

\[
(1 - L)y_t = x_t = \epsilon_t - \theta(z_{t-1}) \epsilon_{t-1}
\]

with

\[
\theta(z_t) = \begin{cases} 
\theta_1 & \text{if } |z_t| > r \\
\theta_2 & \text{if } |z_t| \leq r
\end{cases}
\]
Two types of TIMA models according to whether the threshold variable is observable or not:

- **Observable TIMA**
  - Example:
    
    \[ z_t = (1 - L)y_t \]

- **Unobservable TIMA**
  - Example: TIMA-shock
    
    \[ z_t = \epsilon_t \]
II. TIMA Models

Some Graphical Examples

Big Shocks are Persistent

Small Shocks are Persistent

Lag Autocorrelation

Lag Autocorrelation
II. TIMA Models

Some Graphical Examples (cont)

Big Shocks are Persistent

Small Shocks are Persistent

Same % of Permanent
Some Graphical Examples (cont)

Big Shocks are Persistent

Small Shocks are Persistent

Lag

Autocorrelation

Time Series

0 50 100 150 200

20 40 60 80 100

RW
TIMA, r=0.4
TIMA, r=1.0

Lag

Autocorrelation

0 5 10 15 20 25

0.6 0.8 1.0 1.2 1.4

RW
TIMA, r=0.4
TIMA, r=1.0

II. TIMA Models
Big Shocks are Persistent

Small Shocks are Persistent

Same % of Permanent
III. Assumptions

General Assumptions

G.0 $\varepsilon_t$ iid $(0, \sigma_\varepsilon)$, with density $\infty > f_\varepsilon (\varepsilon_t) > 0$
\hspace{1cm} $\forall \varepsilon_t$ and $\|\varepsilon_t\|_{2\gamma} < \infty$ for some $\gamma > 2$

G.1 $|\theta_1^0 - \theta_2^0| = \delta^0 > 0$

G.2 $z_t$ is strictly stationary and $\alpha$ - mixing of size $-a$, with $a > 2/3$

G.3 $0 < \underline{p} \leq P(\{z_t < r/\mathcal{F}_{t-1}\}) \leq \overline{p} < 1$, with $r \in (0, \overline{r})$

Interpretative Assumptions

I.0 In each threshold regime the shocks are m.d.s
\hspace{1cm} $E [\varepsilon_{t+j} \mathbb{1}(|z_{t+j}| > r)/\mathcal{F}_{t-1}] = 0$ for $j \geq 0$

I.1 $z_t$ is not Granger caused in mean by $\varepsilon_t$, in the sense of $E (z_t/\varepsilon_t, \mathcal{F}_{t-1}) = E (z_t/\mathcal{F}_{t-1})$
Assumptions for Observable TIMAs

Invertibility

A.0 \( E \left( \theta^2 (z_t) / \Xi_{t-1} \right) < 1 \)

Consistency and Testing

A.1 Parameter space \( \theta^0_{(1 \times 3)} \in \Theta \) is defined by
\[
\Theta = [-1 + \delta, 1 - \delta] \times [-1 + \delta, 1 + \delta'] \times (0, \overline{r})
\]
s.t \( E \left( \theta^4 (z_t) / \Xi_{t-1} \right) \leq \overline{\lambda} < 1, \ \forall \theta \in \Theta \)
with \( \delta \) and \( \delta' > 0 \)

A.2 Conditional moment bound
\[
\max\{z', z\}
E \left( |\varepsilon_t|^{2\gamma} / z' \leq |z_t| \leq z \right) \leq \sigma^2_{\varepsilon/z} < \infty,
\]
with \( \gamma \geq 1 \), \( z' \) and \( z \) two different values of \( z_t \)

A.3 Full rank condition
\[
\min\{z', z\}
E \left( |\varepsilon_t|^2 / z' \leq |z_t| \leq z \right) \geq \sigma^2_{\varepsilon/z} > 0, \text{ and}
\]
\[
v_m \leq E \left( 1 (r < |z_t| < r + v) / \Xi_{t-1} \right) \leq M v,
\]
with \( m > 0 \), \( v > 0 \), \( M < \infty \), and \( z' \) and \( z \) two different values of \( z_t \)
Assumptions for TIMA-shock

Invertibility

A.4  \( \lambda_1 = [\partial r f^m + E (|\theta (\varepsilon_{t-1})|)] < 1, \) and \(|\theta_1| < 1; \) where \( \partial = |\theta_1 - \theta_2|, \) and \( f^m = \max_{\varepsilon} (f_{\varepsilon} (-r + e) + f_{\varepsilon} (r + e)) \)

Consistency and Testing

A.5 The parameter space \( \theta^0 \in \Theta \) is defined by \( \Theta = [-1 + \delta, 1 - \delta] \times [-1 + \delta, 1] \times (0, \overline{r}] \) s.t. \( \lambda_1^* = [\partial r f^{m,*} + \lambda_2^* (\theta_1, \theta_2, r)] < 1 \) \( \forall \theta \in \Theta, \) with \( f^{m,*} = 4 \max_{\varepsilon} f_{\varepsilon} (e), \) \( \lambda_2^* (\theta_1, \theta_2, r) = |\theta_1| (1 - \overline{p} (r)) + |\theta_2| \overline{p} (r), \) \( \sup_k P (|\varepsilon_t + k| < r) \leq \overline{p} (r), \) and \(|\theta_1| < 1 \)
III. Properties

INVERTIBILITY

- We focus on \((1 - L)y_t = x_t\)
- We use the general invertibility definition introduced by Granger and Andersen (1978) improved by Hallin (1980)

**Definition** (Granger and Andersen): The process \(x_t = g(x_{t-1}, \varepsilon_{t-1}, \ldots, x_{t-p}, \varepsilon_{t-p}) + \varepsilon_t\) will be invertible if

\[
\lim_{t \to \infty} E \left( e_t^2 \right) = 0
\]

with

\[e_t = \varepsilon_t - \hat{\varepsilon}_t = \varepsilon_t - (x_t - g(x_{t-1}, \hat{\varepsilon}_{t-1}, \ldots, x_{t-p}, \hat{\varepsilon}_{t-p}))\]

**RESULTS**

**Theorem 1** (Observable TIMA) Given G.0 and A.0 the process \(\{x_t\}\) is invertible.

**Theorem 2** (TIMA-shock) Given G.0 and A.4 the process \(\{x_t\}\) is invertible.
Persistence

Persistence: It is the effect of a shock, $\varepsilon_t$, at time $t$, in the future sample path of the series, $\{y_{t+k}\}_{k=0}^{\infty}$.

Permanent shock: If this effect in $y_{t+k}$ does not vanish when $k \to \infty$.

To analyze this property we use the General Impulse Response Function (Koop, Pesaran and Potter(1996), and Potter (2000)) defined as:

$$GI (k, \varepsilon_t, w_{t-1}) =$$

$$E \left[ y_{t+k} / \varepsilon_t, w_{t-1} \right] - E \left[ y_{t+k} / w_{t-1} \right],$$

$$k = 0, 1, ...,$$

with $w_{t-1}$ a particular history of $\mathcal{S}_{t-1}$. 
Some Results

Under the assumptions \textbf{I.0} or \textbf{I.1}, the TIMA model (1) has the following GI:

\[ GI(k > 0, \varepsilon_t, w_{t-1}) = [(1 - \theta(z_t))] \varepsilon_t \]

Case: \(|\theta_1| < 1, \theta_2 = 1\)

\[ GI(k > 0, \varepsilon_t, w_{t-1}) = \begin{cases} 
(1 - \theta_1) \varepsilon_t & \text{if } |z_t| > r \\
0 & \text{if } |z_t| \leq r 
\end{cases} \]

Case: \(\theta_1 = 0, \theta_2 = 1\)

\[ GI(k > 0, \varepsilon_t, w_{t-1}) = \begin{cases} 
\varepsilon_t & \text{if } |z_t| > r \\
0 & \text{if } |z_t| \leq r 
\end{cases} \]
Some Simulations

TIMA Model:

$$(1 - 0.5L)(1 - L)y_t = \begin{cases} 
\epsilon_t - 0.5\epsilon_{t-1} & \text{if } |\epsilon_{t-1}| > 0.6 \\
\epsilon_t - \epsilon_{t-1} & \text{if } |\epsilon_{t-1}| \leq 0.6 
\end{cases}$$

TAR Model:

$$y_t = \begin{cases} 
y_{t-1} + \epsilon_t & \text{if } |\epsilon_{t-1}| > 0.6 \\
0.5y_{t-1} + \epsilon_t & \text{if } |\epsilon_{t-1}| \leq 0.6 
\end{cases}$$

IMA Model:

$$y_t = y_{t-1} + \epsilon_t - 0.2\epsilon_{t-1}$$
GI \left( k, \varepsilon_t, w_{t-1} \right) / \varepsilon_t

TIMA

\begin{align*}
\text{Density} & \quad \text{GI of big shock} \\
0.85 & \quad 0.90 \quad 0.95 \quad 1.00 \quad 1.05 \quad 1.10 \quad 1.15 \\
\end{align*}

\begin{align*}
\text{Density} & \quad \text{GI of small shock} \\
-0.5 & \quad 0.0 \quad 0.5 \quad 1.0 \quad 1.5 \\
\end{align*}

TAR

\begin{align*}
\text{Density} & \quad \text{GI of big shock} \\
0.0 & \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1.0 \quad 1.2 \\
\end{align*}

\begin{align*}
\text{Density} & \quad \text{GI of small shock} \\
-15 & \quad -10 \quad -5 \quad 0 \quad 5 \quad 10 \\
\end{align*}

Random Walk

\begin{align*}
\text{Density} & \quad \text{GI of big shock} \\
0.85 & \quad 0.90 \quad 0.95 \quad 1.00 \quad 1.05 \quad 1.10 \quad 1.15 \\
\end{align*}

\begin{align*}
\text{Density} & \quad \text{GI of small shock} \\
0.5 & \quad 1.0 \quad 1.5 \quad 2.0 \\
\end{align*}
Mean \( GI (k, \varepsilon_t, w_{t-1}) / \varepsilon_t \)
Method: Least Squares

Objective Function:

\[ Q_T(\theta) = \sum_{t=1}^{T} e_t^2(\theta) \]

with \( e_t = \theta(z_{t-1})e_{t-1} + x_t \)

Results for Observable TIMAs

**Theorem 3** Under assumptions G.0 – G.4 and A.1 – A.3,

\( \hat{\theta}_{i,T} = \theta_i^0 + O_p\left(T^{-1/2}\right) \) and \( \hat{r}_T = r_0 + O_p\left(T^{-1}\right) \).

Results for TIMA-Shock

**Theorem 4** Under assumptions G.0 – G.1 and A.5,

\( \hat{\theta}_{i,T} = \theta_i^0 + O_p\left(T^{-1}\right) \) and \( \hat{r}_T = r_0 + O_p\left(T^{-1}\right) \).
V. Inference

Results for Observable TIMA

Theorem 5 (r known) Under assumptions $G.0 - G.4$, $A.1 - A.3$, with $r$ known, and $H(\theta^0)$ being a positive matrix,

$$T^{1/2} \left( \hat{\theta} - \theta^0 \right) \xrightarrow{d} N \left( 0, 4H^{-1}(\theta^0) \Omega H^{-1}(\theta^0) \right).$$

Theorem 6 (r unknown) Under assumptions $G.0 - G.4$ and $A.1 - A.3$ with $r$ estimated by LS, and $H(\theta^0)$ being a positive matrix,

$$T^{1/2} \left( \hat{\theta}(\hat{r}) - \theta^0 \right) \xrightarrow{d} N \left( 0, 4H^{-1}(\theta^0) \Omega H^{-1}(\theta^0) \right).$$

Results for TIMA Shock

Theorem 7 Under assumptions $G.0 - G.4$ and $A.5$, $z_t = \tilde{e}_t$, with $r$ estimated by LS and $H(\theta^0)$ being a positive matrix,

$$T^{1/2} \left( \hat{\theta}(\tilde{e}) - \theta^0 \right) \xrightarrow{d} N \left( 0, 4H^{-1}(\theta^0) \Omega H^{-1}(\theta^0) \right),$$

where $\hat{\theta}(\tilde{e})$ is a second step estimator of $\theta^0$. 
Testing Strategy

Step 1: Test the Threshold Hypothesis \( \theta_1 = \theta_2 \)

- Obtain the Test Statistic, with \( R = (1, -1) \),

\[
W_T (r) = \left( R \hat{\theta}_T (r) \right)' \left[ RVar \left( \hat{\theta}_T (r) \right) R' \right]^{-1} \left( R \hat{\theta}_T (r) \right)
\]

\[
W_T = \sup_r W_T (r) \implies \text{Something}
\]

- Obtain the \( p - values \) by Bootstrap methods

Step 2: Transitory Hypothesis \( \theta_j = 1 \) for some \( j = 1, 2 \)

- Obtain the Test Statistic, with \( R = (0, 1) \),

\[
W_T (\hat{\tau}) = \left( R \hat{\theta}_T (\hat{\tau}) - 1 \right)' \left[ RVar \left( \hat{\theta}_T (\hat{\tau}) \right) R' \right]^{-1} \left( R \hat{\theta}_T (\hat{\tau}) - 1 \right)
\]

\[
W_T \overset{d}{\to} \chi^2 (1)
\]
Alternative Method for Testing the Threshold Hypothesis

- Adjust an IMA model to \( y_t \):
  \[
  (1 - L)y_t = \hat{\varepsilon}_t - \hat{\theta}\hat{\varepsilon}_{t-1}
  \]

- Regress \( \hat{\varepsilon}_t \) on \( \hat{\varepsilon}_{t-1} \), and obtain \( R^2_R \)

- Regress \( \hat{\varepsilon}_t \) on \( \hat{\varepsilon}_{t-1} \), \( 1(|z_{t-1}| < r)\hat{\varepsilon}_{t-1} \), and obtain \( R^2_{UR}(r) \)

- Construct \( W_T(r) = KF_T(r) \), where

\[
F_T(r) = \frac{(R^2_{UR}(r)-R^2_R(r))(T-K)}{(1-R^2_{UR}(r))K},
\]

and obtain the \( p-value \) of \( \sup_r(W_T(r)) \) by bootstrap methods
Extra Issues

• More Dynamics

Always in the AR part

• Number of Regimes


• Inference on the Threshold Parameter $r$

"Subsampling Inference in Threshold Autoregressive Models" (Gonzalo and Wolf) (2003), (http://halweb.uc3m.es/jgonzalo/)
A Simulation Exercise

- Case A (Observable TIMA): $z_t \ iid$

For Size, $DGP_1$:

$$y_t = y_{t-1} + \varepsilon_t,$$

For Power, $DGP_2$:

$$(1 - L)y_t = \begin{cases} 
\varepsilon_t & \text{if } |z_{t-1}| > 0.3 \\
\varepsilon_t - \varepsilon_{t-1} & \text{if } |z_{t-1}| \leq 0.3 
\end{cases}$$

- Case B (TIMA-Shock): $z_t = \varepsilon_t$

For Size, $DGP_1$:

$$y_t = y_{t-1} + \varepsilon_t$$

For Power, $DGP_2$:

$$(1 - L)y_t = \begin{cases} 
\varepsilon_t & \text{if } |\varepsilon_{t-1}| > 0.3 \\
\varepsilon_t - \varepsilon_{t-1} & \text{if } |\varepsilon_{t-1}| \leq 0.3 
\end{cases}$$
## Results for Observable TIMA

### $H_0$: $\theta_1 = \theta_2 = 0$ - No Threshold

<table>
<thead>
<tr>
<th></th>
<th>Sup</th>
<th>Mean</th>
<th>Exp</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T=200$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha = 5%$</td>
<td>0.045</td>
<td>0.035</td>
<td>0.035</td>
</tr>
<tr>
<td>$\alpha = 10%$</td>
<td>0.105</td>
<td>0.095</td>
<td>0.105</td>
</tr>
<tr>
<td>$T=400$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha = 5%$</td>
<td>0.050</td>
<td>0.035</td>
<td>0.040</td>
</tr>
<tr>
<td>$\alpha = 10%$</td>
<td>0.090</td>
<td>0.070</td>
<td>0.090</td>
</tr>
</tbody>
</table>

### $H_a$: $\theta_1 = 0, \theta_2 = 1, r = 0.3$

<table>
<thead>
<tr>
<th></th>
<th>Sup</th>
<th>Mean</th>
<th>Exp</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T=200$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha = 5%$</td>
<td>0.96</td>
<td>0.86</td>
<td>0.94</td>
</tr>
<tr>
<td>$\alpha = 10%$</td>
<td>0.98</td>
<td>0.91</td>
<td>0.97</td>
</tr>
</tbody>
</table>

Number of Monte Carlo replications, $N=200$
Number of Bootstrap replications, $B=500$
## Results for TIMA Shock

<table>
<thead>
<tr>
<th></th>
<th>H₀: ( \theta_1 = \theta_2 = 0 )</th>
<th>Sup</th>
<th>Mean</th>
<th>Exp</th>
</tr>
</thead>
<tbody>
<tr>
<td>T=200</td>
<td>( \alpha = 5% )</td>
<td>0.035</td>
<td>0.07</td>
<td>0.045</td>
</tr>
<tr>
<td></td>
<td>( \alpha = 10% )</td>
<td>0.075</td>
<td>0.105</td>
<td>0.100</td>
</tr>
<tr>
<td>T=400</td>
<td>( \alpha = 5% )</td>
<td>0.065</td>
<td>0.075</td>
<td>0.06</td>
</tr>
<tr>
<td></td>
<td>( \alpha = 10% )</td>
<td>0.115</td>
<td>0.12</td>
<td>0.115</td>
</tr>
</tbody>
</table>

Power Table for TIMA Shock Model under construction !!!!

Number of Monte Carlo replications, N=200
Number of Bootstrap replications, B=500
Application: Stock Prices

We apply the **TIMA Shock** for measuring the deviations between actual transaction prices and implicit efficient prices, following a method developed by Hasbrouck (1993).

We decompose the transaction price into the sum of two components:

\[ p_t = m_t + s_t \]

**Efficient price:** \( m_t = m_{t-1} + w_t \)

**Pricing error:** \( s_t = \alpha w_t + \eta_t \)

The pricing error can come from diverse microstructure effects and it has been described by Lo and MacKinlay (1988), Frama and French (1988) and Poterba and Summers (1988) among others.
The proposed measure of the deviation is the dispersion of the transitory components, $\sigma_s$.

Unfortunately without assumptions about $s_t$ this measure, $\sigma_s$, is not identified.

The standard identification conditions are:

- $\alpha = 0, \eta_t \neq 0 \Rightarrow (\text{UC}-0)$
- $\alpha > 0, \eta_t = 0 \Rightarrow (\text{B-N})$

We propose the following model:

$$m_t = m_{t-1} + 1(|w_t| > r)w_t$$

$$s_t = 1(|w_t| \leq r)w_t$$

From it is straightforward to obtain the following representation:

$$(1 - L)p_t = \begin{cases} 
    w_t & \text{if } |w_{t-1}| > r \\
    w_t - w_{t-1} & \text{if } |w_{t-1}| \leq r 
\end{cases}$$
We test this model with S&P500 daily series.
Results

$H_0$: Linear model is REJECTED (p-value=0.061)

$H_0$: Unit root in the MA part of the small shocks regime is NOT REJECTED (p-value=0.46)

**CONCLUSION:** TIMA-shock model is NOT rejected. Therefore SIZE matters.

Estimated Model under linearity:

$$p_t - p_{t-1} = \hat{\varepsilon}_t - 0.017\hat{\varepsilon}_{t-1}$$

Estimated Model under TIMA-shock structure:

$$p_t - p_{t-1} = \hat{\varepsilon}_t - 1(|\hat{\varepsilon}_{t-1}| < 0.0043)\hat{\varepsilon}_{t-1}$$

Proportion of Transitory Shocks: 0.27%.

<table>
<thead>
<tr>
<th>CASE</th>
<th>$\alpha = 0$</th>
<th>$\eta_t = 0$</th>
<th>TIMA</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$(UC = 0,$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$Watson(1986))$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\hat{\sigma}_s \times 100$</td>
<td>0.181</td>
<td>0.024</td>
<td>0.126</td>
</tr>
</tbody>
</table>
If you want to test whether SIZE matters or not for permanent and transitory issues, TIMA is the PERFECT model to use.