Model Uncertainty and Endogenous Volatility

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Abstract

This paper identifies two channels through which the economy can generate endogenous inflation and output volatility, an empirical regularity, by introducing model uncertainty into a Lucas-type monetary model. The equilibrium path of inflation depends on agents' expectations and a vector of exogenous random variables. Following Branch and Evans (2004) agents are assumed to underparameterize their forecasting models. A Misspecification Equilibrium arises when beliefs are optimal given the misspecification and predictor proportions are based on relative forecast performance. We show that there may exist multiple Misspecification Equilibria, a subset of which are stable under least squares learning and dynamic predictor selection. The dual channels of least squares parameter updating and dynamic predictor selection combine to generate regime switching and endogenous volatility.

JEL Classifications: C53; C62; D83; D84; E40

Key Words: Lucas model, model uncertainty, adaptive learning, rational expectations, volatility.

1 Introduction

Time-varying volatility in inflation and GDP growth is an empirical regularity of the U.S. economy. This observation is often described in the applied literature as a regime shift during the 1980's which resulted in a simultaneous decline in inflation and output volatility. The 'Great Moderation', econometrically identified by Stock and Watson (2003) and McConnell and Quiros (2000), among others, is often associated with a change in the stance of monetary policy (e.g. Branch, Carlson, Evans and McGough (2004).

However, recent studies by Cogley and Sargent (2005) and Sims and Zha (2005) present evidence that drifting and regime switching inflation and output volatility is a characteristic of the post-war period. Since the Great Moderation consists of a one-time simultaneous decline in volatility, and its timing coexists with changes in Federal Reserve policy, it seems natural to seek policy explanations of this particular event. Persistently evolving inflation volatility may not always go hand in hand with changes in Federal Reserve policy. In this paper, we demonstrate that drift and regime switching in volatility may arise endogenously through model uncertainty.

Private sector expectations of future economic variables plays a key role in most monetary models (e.g. Woodford (2003)). In these self-referential models, agents' beliefs feedback positively onto the underlying stochastic process. Yet, there is no consensus among economists on how agents actually form their expectations. Many models continue to assume Rational Expectations despite the theoretical grounds to question the assumption. Instead, Evans and Honkapohja (2001) replace rational expectations with statistical learning rules. This alternative approach, it is argued, is a reasonable description of agents' actual forecasting acumen because it assumes behavior consistent with econometric practice.

Branch and Evans (2004), though, note that with computational costs and degree of freedom limitations, econometricians often underparameterize their forecasting models. It has long been known that Vector Autoregressive (VAR) Models have degrees of freedom limitations. Recent work by Chari, Kehoe, and McGrattan (2005) show that these limitations present obstacles to VAR researchers who try to uncover a model's stochastic structure from observable time-series variables. The approach in this, and our earlier, paper is to model agents as VAR econometricians who behave optimally given the restrictions imposed on them by the data. The previous paper, developed in the context of the cobweb model, derived heterogeneity as an equilibrium outcome when agents choose the dimension in which to underparameterize. This paper revisits that approach instead framing the analysis in a Lucas-type monetary model along the lines of Evans and Ramey (2003).

We confront agents with a list of underparameterized predictor functions. The economic model is self-referential in the sense that agents' expectations, a function of their underparameterization choice, depends on the underlying stochastic process which, in turn, depends on these beliefs. A *Misspecification Equilibrium* (ME) is a fixed point of this self-reinforcing process. Model uncertainty arises in the sense that agents pick the best-performing statistical model. In our framework, what constitutes the best performing model depends not only on the regressors of the model but also on the forecasting model choices of other agents. There are other approaches to model uncertainty. For instance, uncertainty about parameters (Hansen and Sargent (2005)) or about the form of the monetary policy rule. We argue, though, that econometric model uncertainty is an important component of agent behavior and this paper studies its implications.

There are two primary results in the current paper: first, when there are multiple underparameterized models from which agents must choose one, there may exist multiple stable equilibria each with distinct stochastic properties; second, when agents must adaptively learn the forecast accuracy of these models the economy will generate endogenous variation in inflation and output volatility.

This paper specifies a simple monetary model in which aggregate supply and aggregate demand depend on a vector of autoregressive exogenous disturbances and supply additionally depends on unanticipated price level changes. Motivated by the idea that cognitive and computing time constraints and degrees of freedom limitations lead agents to adopt parsimonious models, we impose that agents only incorporate a subset of these variables into their forecasting model. Following Branch and Evans (2004) we require that these expectations are optimal linear projections given the underparameterization restriction and that agents only choose best performing statistical models. Despite the bounded rationality assumption, this remains in the spirit of Muth (1961) in the sense that for each statistical model the parameters are chosen optimally. An equilibrium in beliefs and the stochastic process is a Misspecification Equilibrium. An ME extends the notion of a *Restricted Perceptions Equilibrium*, which arise in the models of Evans, Honkapohja, and Sargent (1993), Evans and Honkapohja (2001), and Sargent (1999), to settings in which agents must choose their models. We show that in the Lucas model there exist multiple ME and, moreover, the ME with homogeneous expectations are stable under least squares learning.

One implication of our theoretical model is that in a real-time dynamic version of the model agents must simultaneously estimate the parameters of their forecasting model and choose the best model based on past experience. We show that when agents use least squares to estimate the parameters of their statistical model, and base forecast performance on average mean-square forecast error of the competing models, different Misspecification Equilibria, in each of which agents coordinate on one forecasting model, can be stable.

Most interestingly, "constant gain" dynamics lead to new and distinct results. Constant gain least squares algorithms place a greater (time-invariant) weight on recent than distant observations. Constant gain, or 'perpetual learning', has been studied by Orphanides and Williams (2005) and Sargent (1999) who argue in favor of this type of estimation strategy to allow for possible structural change. In this paper we extend this idea in an important way: learning *jointly* about model parameters and model fitness. Model uncertainty arises via constant gain learning and dynamic predictor selection.

Extending constant gain learning to incorporate dynamic predictor selection, we identify two channels through which inflation and output volatility may evolve over time. The first channel is from the parameter drift induced by constant gain updating of the forecasting model parameters. Under constant gain learning, the parameters vary around their mean values, even if the economy remains at a single equilibrium. In addition, regime switching in inflation and output volatility can arise when the economy switches endogenously between high and low volatility equilibria. Thus, the second channel is through dynamic predictor selection when agents react more strongly to recent forecast errors than distant ones when assessing the fitness of a forecasting model. Through numerical simulations, we show that when there is dynamic predictor selection and parameter drift the dynamic paths of inflation and output are consistent with the empirical regularities identified by Cogley and Sargent (2005) and Sims and Zha (2005).

Our paper is related to others. Brock and Hommes (1997, 1998) study dynamic predictor selection in deterministic models which share a similar reduced form as the model in this paper. In particular, Brock and Hommes (1998) consider a case where agents place a constant weight on past forecast errors and show that complex dynamics may arise. Branch and Evans (2004) extend Brock and Hommes (1997) to a stochastic environment in which, in equilibrium, both the choice of forecasting model and the parameters of each predictor are determined simultaneously. In that paper, we show that an equilibrium may arise where agents are distributed heterogeneously across forecasting models. Moreover, under least squares learning the equilibrium may be stable under dual learning of the type described above.

The current paper departs from each of these other models in two important ways. First, Brock and Hommes focus on models with a unique steady-state while in this paper we prove the possibility of multiple equilibria. Learning and dynamic predictor selection in a model with multiple equilibria produce distinct results from Brock and Hommes (1997). Second, in our earlier paper, there was also a unique equilibrium and so the focus of the paper was on the properties of that equilibrium. With multiple equilibria, as in this paper, dual learning leads to dynamics not present in Branch and Evans (2004).

This paper proceeds as follows. Section 2 presents evidence of time-varying volatility in the U.S. economy. Section 3 presents the Lucas model with model uncertainty. Sections 4 and 5 consider the model under real-time learning. Section 6 concludes.

2 Inflation and Output Volatility in the U.S.

2.1 An Empirical Overview

In the applied literature there is widespread consensus that during the 1980's there was a decline in economic volatility. An array of econometric techniques to identify the regime shift have been employed by Bernanke and Mihov (1998), Kim and Nelson

(1999), Kim, Nelson, and Piger (2004), McConnell and Quiros (2000), Sensier and van Dijk (2004), and Stock and Watson (2003). Recently, though, Cogley and Sargent (2005) and Sims and Zha (2005) have identified repeated regime shifting economic volatility in U.S. inflation and GDP growth. While Cogley-Sargent and Sims-Zha are interested in characterizing changing monetary policy over the period they make a striking finding: during the post-war period there is persistent stochastic volatility in the economy.

Conventional macroeconomic models, however, are unable to generate persistent stochastic volatility without directly assuming the exogenous disturbances follow a Markov chain or an exogenous change in policy. In this paper, we present a model capable of generating such volatility endogenously via an adaptive learning and dynamic predictor selection process in a setting where agents may choose between competing underparameterized forecasting models. First, though, this section presents an informal accounting of the nature of stochastic volatility in the economy.

We present a series of plots, each some variant on quarterly inflation computed from the GDP deflator and quarterly GDP, which suggests the presence of stochastic volatility rigorously documented by Cogley and Sargent (2005) and Sims and Zha (2005). Our purpose in this paper is motivation and overview; we refer the reader to these other papers for formal econometric analysis. We detrend the log of real GDP using the Hodrick-Prescott filter since in the Lucas model below output is expressed as a log deviation from its trend value. Figure 1 plots inflation and (detrended) GDP for the period 1947:1-2002:2.

INSERT FIGURE 1 HERE

Inspection of Figure 1 demonstrates the 'Great Moderation' emphasized by McConnell and Quiros (2000) and Stock and Watson (2003). About 1984 there was a simultaneous decline in the volatility of inflation and GDP. This empirical feature has led to an explosion of research into monetary policy's role in bringing about the observed economic stability.[1] Broader inspection of the data, though, suggests that this was not the only simultaneous change in economic volatility. GDP appears to be slowly stabilizing throughout the sample with the exception of a period in the late 1970's. Inflation, on the other hand, seems to persistently change between high and low volatility states.

To further inspect the time-varying volatility of inflation and GDP suggested by Figure 1, Figure 2 calculates moving average estimates of the unconditional variance of inflation and GDP using a rolling window of 8 quarters. These calculations provide a rough estimate of how actual volatility changed over time. Figure 2 demonstrates,

[1]See, for example, Branch, Carlson, Evans, and McGough (2004) and the references therein.

as Figure 1 suggested, that the volatility of GDP and inflation varies over time. In particular, each series appears to move in tandem and alternate between high and low variance regimes. Sims and Zha (2005) find that 9 separate regimes fit the data best. This plot resembles the posterior mean estimates for the standard errors of the VAR innovations in Cogley and Sargent (2005) over the period 1960-2000.

INSERT FIGURE 2 HERE

Owyang (2001) presents evidence that inflation follows an ARCH process. Figure 3 plots the conditional variances from an ARCH specification for inflation and GDP to demonstrate the robustness of the finding that there is persistent stochastic volatility in both inflation and GDP. To compute the conditional variances in Figure 3 we estimated a GARCH(1,1) for an AR(4) model of inflation and GDP. This follows exactly Owyang (2001) though we also estimate a GARCH model for the volatility of GDP. Figure 3 then plots the conditional variances from the GARCH models.

INSERT FIGURE 3 HERE

Figure 3 demonstrates that the persistent and changing volatility is a hallmark of the data. In particular, the regime switching volatility is seen in both data series and not just at the time of the Great Moderation. Figure 4 plots the same conditional variance series as Figure 3, except that it focuses on the period 1955:1-2002:4. Figure 3 shows that there was particularly high volatility in 1950. In order to more clearly see how the volatility changes, even on a small scale, Figure 4 plots the more stable period.

INSERT FIGURE 4 HERE

2.2 Discussion

Despite the attention given the Great Moderation, there seems to be little emphasis in the theoretical literature on accounting for the persistent stochastic volatility of the economy. Sims and Zha (2005) seek evidence in a change in the stance of monetary policy repeatedly across time. Sargent (1999) presents a theory of the rise and fall in inflation that is the result of drifting beliefs on the part of the government.[2] Orphanides and Williams (2005) account for the decline in volatility as a change in the stance of policy which pins down agents' drifting beliefs.

[2] Over a long stretch of time this would be expected to lead to periodic regime changes due to the 'escape' dynamics. For further discussion see Cho, Williams, and Sargent (2003).

In this paper, we present an alternative explanation that does not require policy changes. We present a model in which agents must choose between two alternative underparameterized models. We demonstrate the possibility of multiple coordinating equilibria with distinct stochastic properties across equilibria. We introduce model uncertainty by assuming that agents use constant gain least squares to estimate the parameters of their forecasting model. This introduces drifts into their beliefs as in Sargent (1999) and Orphanides and Williams (2005). We also augment the model to allow agents to choose their forecasting model in real time based on a geometric weighted average of recent forecast performance. In this version of the model, agents switch persistently and endogenously between forecasting models. This induces the economy to switch between high and low inflation variance equilibria. Thus, we provide two possible sources of stochastic volatility: drifting beliefs and endogenous predictor selection.

3 Model

This section extends the cobweb model with misspecification of Branch and Evans (2004) to a Lucas-type monetary model. In Branch and Evans (2004) firms choose prices based on a misspecified forecasting model of the market price. Misspecification is modeled by confronting agents with a list of underparameterized models. Agents, though, forecast optimally in the sense that they only choose the best performing statistical model. That paper establishes that, under appropriate joint conditions on the self-referential feature of the model and the exogenous disturbances, agents will be distributed heterogeneously across misspecified models.

Here we establish the existence of misspecification equilibria in a closely related Lucas-type monetary model and later sections will address dynamics of learning and predictor selection. Although the reduced form of the Lucas model is similar to the cobweb model, the slopes of the two models have opposite signs. The negative feedback of the cobweb model plays a central role in the existence of Intrinsic Heterogeneity. In the Lucas model the feedback from expectations is positive. The reinforcing aspect of expectations induces coordination by agents and raises the possibility of multiple equilibria.

3.1 Set-up

Following Evans and Ramey (1992, 2003) we assume the economy is represented by equations for aggregate supply (AS) and aggregate demand (AD):

$$\begin{aligned} AS : q_t &= \phi \left(p_t - p_t^e \right) + \beta_1 z_t \\ AD : q_t &= m_t - p_t + \beta_2 z_t + w_t \end{aligned}$$

where p_t is the log of the price level, p_t^e is the log of expected price formed in $t-1$, m_t is the log of the money supply, q_t is the deviation of the log of real GDP from trend, and η_t is an *iid* zero-mean shock. Assume that the money supply follows,

$$\begin{aligned} m_t &= p_{t-1} + \delta' z_t + u_t \\ z_t &= A z_{t-1} + \varepsilon_t \end{aligned}$$

We assume for simplicity that z_t is (2×1) and ε_t is *iid* zero-mean with positive definite covariance matrix Σ_ε. The vector z_t is also assumed to be a stationary process with the eigenvalues of A inside the unit circle. The stochastic disturbance z_t collects the serially correlated disturbances that affect aggregate supply, aggregate demand, and the money supply. The matrices β_1, β_2, δ determine which components of z affect the respective reduced form relationships via (possible) zero components.

Denoting $\pi_t = p_t - p_{t-1}$ we can write the law of motion for the economy in its expectations-augmented Phillips curve form

$$\pi_t = \frac{\phi}{1+\phi} \pi_t^e + \frac{(\delta + \beta_2 - \beta_1)'}{1+\phi} z_t + \frac{1}{1+\phi} \left(w_t + u_t \right)$$

or

$$\pi_t = \theta \pi_t^e + \gamma' z_t + \nu_t \tag{1}$$

where $\theta = \frac{\phi}{1+\phi}, \gamma' = \frac{(\delta+\beta_2-\beta_1)'}{1+\phi}, \nu_t = \frac{1}{1+\phi}\left(w_t + u_t\right)$. Note, in particular, that $0 \leq \theta < 1$. The cobweb model also takes the reduced form (1) with $\theta < 0$. This case is considered in Branch and Evans (2004).

A *rational expectations equilibrium* (REE) is a stationary sequence $\{\pi_t\}$ which is a solution to (1) given $\pi_t^e = E_{t-1}\pi_t$, where E_t is the conditional expectations operator. It is well-known that (1) has a unique REE and it is of the form

$$\pi_t = (1-\theta)^{-1} \gamma' A z_{t-1} + \gamma' \varepsilon_t + \nu_t \tag{2}$$

Output, in an REE, is white noise and does not display the time-series properties evident in the previous section. If instead agents only took one component of z into account when forecasting inflation then the reduced form weights on the components of z_t will change. Such a deviation from the REE (2) is a key insight of our model.

3.2 Model Misspecification

This paper departs from the rational expectations hypothesis (RE) and imposes that agents are boundedly rational. One popular alternative to RE is to model agents as econometricians (Evans and Honkapohja (2001)). According to this literature, agents have a correctly specified model whose parameters are estimated from a reasonable estimator. In many instances, these beliefs converge to RE. In practice, however, econometricians often misspecify their models. Professional forecasters often times restrict the number of variables and/or lags because of degree of freedom problems. Indeed, Chari, Kehoe, and McGrattan (2005) argue that econometric misspecification is central to the debate over identified Impulse Response Functions. Following Evans and Honkapohja (2001), Evans and Ramey (2003), and Branch and Evans (2004), we argue that if agents are expected to behave like econometricians then they can also be expected to misspecify their models. We impose misspecification by forcing agents to underparameterize in at least one dimension. We follow Evans and Honkapohja (2001), however, and impose that these underparameterized beliefs are optimal linear projections given the misspecification. In the next subsection we allow the model to endogenously determine the dimension in which agents underparameterize. The next Section introduces model uncertainty by replacing optimal projections with their least squares estimates.

Beliefs are formed from models that take one of the following forms

$$\pi_t^e = b^1 z_{1,t-1} \quad (3)$$

$$\pi_t^e = b^2 z_{2,t-1}. \quad (4)$$

Because z_t is a bivariate VAR(1) it is clear that (3)-(4) represents all possible underparameterized models. Informally, we view the true economic process as being driven by a high dimensional exogenous process. That agents underparameterize, or approximate their econometric models, is a reasonable description of actual forecasting behavior. The assumption that z_t is bivariate VAR(1) is, of course, made for analytical convenience. One can show the existence of Misspecification Equilibria if z_t is $n \times 1$ and follows a VAR(p). We impose that the parameters b^1, b^2 are formed as optimal linear projections of π_t on $z_{i,t}$ for $i = 1, 2$. That is, beliefs satisfy the orthogonality condition

$$E z_{i,t-1} \left(\pi_t - b^i z_{i,t-1} \right) = 0 \quad (5)$$

This condition ensures that, in an equilibrium, agents' beliefs are consistent with the actual process in the sense that their forecasting errors are undetectable within their perceived model. When this occurs we say the model is at a Restricted Perceptions Equilibrium (RPE).[3]

[3] Adam (2005b) presents experimental evidence for approximate RPE in a bivariate macro model of output and inflation.

Equilibria based on model misspecification that satisfies an orthogonality condition like (5) appear frequently in the literature. Evans and Ramey (2003) consider RPE in the Lucas-type model. Sargent (1999) focuses on the closely related self-confirming equilibria. (Hommes, Sorger, and Wagener 2002) and Branch and McGough (2005) extend (Hommes and Sorger 1998) by defining a Stochastic Consistent Expectations Equilibrium as an equilibrium in which agents have linear beliefs which are consistent with a non-linear model.

Because agents may be distributed heterogeneously across predictors, actual market beliefs for the economy are a weighted average of the individual beliefs

$$\pi_t^e = nb^1 z_{1,t-1} + (1-n)b^2 z_{2,t-1}$$

where n is the proportion of agents who use model 1.[4] Inserting these beliefs into (1) leads to

$$\pi_t = \theta\left(nb^1 z_{1,t-1} + (1-n)b^2 z_{2,t-1}\right) + \gamma' A z_{t-1} + \gamma'\varepsilon_t + \nu_t$$

Or, by combining similar terms,

$$\pi_t = \xi_1 z_{1,t-1} + \xi_2 z_{2,t-1} + \eta_t \tag{6}$$

where

$$\begin{aligned} \xi_1 &= \gamma_1 a_{11} + \gamma_2 a_{21} + \theta n b^1, \\ \xi_2 &= \gamma_1 a_{12} + \gamma_2 a_{22} + \theta(1-n) b^2, \end{aligned}$$

$\eta_t = \gamma'\varepsilon_t + \nu_t$, and a_{ij} is the ijth element of A. It follows from (5) and (6) that the optimal belief parameters are

$$\begin{aligned} b^1 &= \xi_1 + \xi_2\rho \\ b^2 &= \xi_2 + \xi_1\tilde{\rho} \end{aligned}$$

where $\rho = Ez_1 z_2 / Ez_1^2$ and $\tilde{\rho} = Ez_1 z_2 / Ez_2^2$.[5] Note that the ξ parameters are functions of b. Thus an RPE is a stationary process for π_t which satisfies (6) with parameters ξ_1, ξ_2 which solve

$$\begin{bmatrix} 1-\theta n & -\theta n\rho \\ -\theta(1-n)\tilde{\rho} & 1-\theta(1-n) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = A'\gamma \tag{7}$$

A unique RPE exists if and only if the matrix which premultiplies the parameter vector is invertible. We formalize this invertibility condition below:

Condition $\triangle$: $\triangle \neq 0$ *for all* $n \in [0,1]$, *where*

$$\triangle = 1 - \theta + \theta^2 n(1-n)(1-\rho\tilde{\rho})$$

[4]We identify model 1 as the model with the $z_{1,t}$ component and model 2 is defined symmetrically.

[5]The existence of these unconditional moments are guaranteed by the stationarity of z_t.

Remark 1 *Using the argument of Branch and Evans (2004) it can be shown that Condition* $\triangle$ *is satisfied for all* $\theta < 1$.

3.3 Misspecification Equilibrium

A Misspecification Equilibrium (ME) is an RPE which jointly determines the fraction of agents using a given model. Below we formally define the equilibrium and present results on existence of ME.

We follow Brock and Hommes (1997) in assuming the map from predictor benefits to predictor choice is a multinomial logit (MNL) map. Brock and Hommes assume that agents base their predictor decisions on recent realizations of a deterministic process. In an ME we instead assume agents base their decisions on the unconditional moments of the stochastic process. Later when we introduce learning and dynamic predictor selection the predictor choice is based on an average of past realizations.

As in Evans and Ramey (1992) we assume agents seek to minimize their forecast MSE, i.e. we assume agents maximize

$$Eu = -E\left(\pi_t - \pi_t^e\right)^2$$

This assumption is reasonable in light of the linear R.E. literature and well-known results in least-squares prediction theory. If π_t^e is conditional on full information then R.E. would minimize the expected mean-square error of one step-ahead forecasts. Thus, we preserve this structure when agents form optimal linear projections on a limited information set. The MNL approach leads to the following mapping, for each predictor $i = 1, 2$,

$$n_i = \frac{\exp\{\alpha E u_i\}}{\sum_{j=1}^{2} \exp\{\alpha E u_j\}} \tag{8}$$

Noting that $\sum_{j=1}^{2} n_j = 1$, (8) can be re-written

$$n = \frac{1}{2}\left(\tanh\left[\frac{\alpha}{2}\left(Eu_1 - Eu_2\right)\right] + 1\right) \equiv H_\alpha(Eu_1 - Eu_2)$$

where $H_\alpha : \mathbb{R} \to [0, 1]$.

The parameter α is called the 'intensity of choice'. It parameterizes agents' sensitivity to changes in forecasting success. Brock and Hommes (1997) focus on the case of large but finite α. Branch and Evans (2004) note that a drawback to finite α is it imposes that agents are not fully optimizing. In this earlier paper, it was shown that in a stochastic framework where agents underparameterize their forecasting models, heterogeneity may persist even as $\alpha \to +\infty$. This paper also emphasizes the $\alpha \to +\infty$ case so that agents behave optimally given their misspecification.

One can verify that the MSE's of the predictors imply that

$$\begin{aligned} Eu_1 &= \xi_2^2 \left(\rho E z_1 z_2 - E z_2^2\right) - \sigma_\eta^2 \\ Eu_2 &= \xi_1^2 \left(\tilde{\rho} E z_1 z_2 - E z_1^2)\right) - \sigma_\eta^2 \end{aligned}$$

Define the map $F : [0,1] \to \mathbb{R}$ as

$$F(n) = Eu_1 - Eu_2 = \xi_1^2 (1 - \rho\tilde{\rho}) + \xi_2^2 \left(\rho^2 - Q\right)$$

where $Q = Ez_2^2/Ez_1^2$. If condition $\triangle$ is satisfied, $F(\cdot)$ is continuous and well-defined.

Because condition $\triangle$ is satisfied for all $\theta \in [0,1)$, there exists a well-defined mapping $T_\alpha : [0,1] \to [0,1]$ such that $T_\alpha = H_\alpha \circ F$.

Definition *A Misspecification Equilibrium (ME) is a fixed point, n^*, of T_α.*

In a Misspecification Equilibrium the forecast parameters satisfy the orthogonality condition and the predictor proportions are determined by the MNL. In equilibrium, they are, therefore, both endogenously determined.

Proposition 2 *A Misspecification Equilibrium exists.*

This result follows since $T_\alpha : [0,1] \to [0,1]$ is continuous and Brouwer's theorem ensures that a fixed point exists.[6] By developing details of the map F we are able to investigate further the set of ME.

Proposition 3 *The function $F(n)$ is monotonically increasing for all $0 \leq \theta < 1$.*

The Appendix sketches the proofs to all propositions. The precise theoretical details are presented in Branch and Evans (2004) for the cobweb model and many of the same details apply. For precise statements we refer the reader to Branch and Evans (2004).

From the equation for expected utility it can be further shown that

$$\begin{aligned} F(1) &\gtreqless 0 \text{ iff } (1-\rho\tilde{\rho})\xi_1^2(1) \gtreqless (Q-\rho^2)\xi_2^2(1) \\ F(0) &\gtreqless 0 \text{ iff } (1-\rho\tilde{\rho})\xi_1^2(0) \gtreqless (Q-\rho^2)\xi_2^2(0) \end{aligned}$$

where $Q = \frac{Ez_2^2}{Ez_1^2}$. Furthermore, from (7) we have

$$\begin{aligned} \frac{(\xi_1(1))^2}{(\xi_2(1))^2} &= \frac{\left((\gamma_1 a_{11} + \gamma_2 a_{21}) + (\gamma_1 a_{12} + \gamma_2 a_{22})\theta\rho\right)^2}{(1-\theta)^2 \left(\gamma_1 a_{12} + \gamma_2 a_{22}\right)^2} \equiv B_1 \\ \frac{(\xi_1(0))^2}{(\xi_2(0))} &= \frac{(\gamma_1 a_{11} + \gamma_2 a_{21})^2 (1-\theta)^2}{\left((\gamma_1 a_{11} + \gamma_2 a_{21})\theta\tilde{\rho} + \gamma_1 a_{12} + \gamma_2 a_{22}\right)^2} \equiv B_0 \end{aligned}$$

[6]Branch and Evans (2004) prove existence of a Misspecification Equilibrium for an n-dimensional vector z_t following a stationary VAR(p) process, provided $|\theta|$ is sufficiently small. The proof in Branch-Evans does not rely on the sign of θ.

Note that $0 < B_0 < B_1$. Recall that Q, ρ, and $\tilde{\rho}$ are determined by A and Σ_ε. The above results and Proposition 3 imply:

Lemma 4 *There are three possible cases depending on A, θ, γ and Σ_ϵ:*

1. *Condition PM: $F(0) < 0$ and $F(1) > 0$. Condition PM is satisfied when $(1 - \rho\tilde{\rho})B_0 + \rho^2 < Q < (1 - \rho\tilde{\rho})B_1 + \rho^2$.*
2. *Condition P1: $F(0) > 0$ and $F(1) > 0$. Condition P1 arises when $Q < (1 - \rho\tilde{\rho})B_0 + \rho^2$.*
3. *Condition P0: $F(0) < 0$ and $F(1) < 0$. Condition P0 arises when $Q > (1 - \rho\tilde{\rho})B_1 + \rho^2$.*

Remark: $\rho\tilde{\rho} = 1$ *is ruled out by the positive definiteness of* Σ_ε.

Below we give numerical examples of when each condition may arise.

Under Condition PM, $F(0) < 0$ and $F(1) > 0$ implies that either model is profitable so long as all agents coordinate on that model; that is, there is no incentive for agents to deviate from homogeneity. When Condition P1 or P0 holds one model always dominates the other.

Lemma 4 allows for a characterization of the set of Misspecification Equilibria for large α. Let

$$N_\alpha = \{n^* | T_\alpha(n^*) = n^*\}$$

We now present our primary existence result for large α.

Proposition 5 *Characterization of Misspecification Equilibria for large α:*

1. *Under Condition PM, as $\alpha \to \infty$, $N_\alpha \to \{0, \hat{n}, 1\}$ where $\hat{n}_1$ is s.t. $F(\hat{n}_1) = 0$.*
2. *Under Condition P0, as $\alpha \to \infty$, $N_\alpha \to \{0\}$.*
3. *Under Condition P1, as $\alpha \to \infty$, $N_\alpha \to \{1\}$.*

The remainder of the paper is primarily concerned with Case 1 in which there are multiple equilibria. It should be briefly noted that an ME does not coincide with the unique REE in (2). For all $n^* \in N_\alpha$ a comparison of (2) and the ME in (6) and (7) (for a given n^*) shows that the ME has different relative weights on the exogenous variables.[7] Interestingly, there may exist multiple ME even though there is a unique REE. The 'instability' that results from misspecification is key for generating endogenous regime change in the Lucas model.

[7] Adam (2005a) considers a New Keynesian model where agents are restricted to univariate forecasting models. In his model, though, there exist equilibria which are REE.

3.4 Further Intuition for Multiple Equilibria

The existence of multiple equilibria, and the resulting real-time learning and dynamic predictor selection dynamics, are key results. In this regard, greater intuition of when multiple equilibria arises is useful. The existence of multiple equilibria, as demonstrated by Proposition 5, depends on the asymptotic properties of the z process, the direct($\gamma' A$) and indirect (θ) effect of z on inflation. This subsection presents the intuition on the relationship between the direct and indirect effects.

For ease of exposition, assume $\rho = \tilde{\rho} = 0$. From Proposition 5 multiple ME arise when

$$B_0 < Q < B_1$$

and, it is straightforward to verify that

$$\begin{aligned} B_1 &= \frac{\left(\gamma_1 a_{11} + \gamma_2 a_{21}\right)^2}{(1-\theta)^2 \left(\gamma_1 a_{12} + \gamma_2 a_{22}\right)^2} \\ B_0 &= \frac{\left(\gamma a_{11} + \gamma_2 a_{21}\right)^2 (1-\theta)^2}{\left(\gamma_1 a_{12} + \gamma_2 a_{22}\right)^2} \end{aligned}$$

Now suppose that $\theta = 0$. Then $B_0 = B_1$ and, except for a set of measure zero, there does not exist any Q, hence any z, so that multiple equilibria exist. Instead suppose that $\theta \to 1$. Then $B_0 \to 0$ and $B_1 \to \infty$. In this instance, the entire range of uncorrelated, bivariate VAR(1) will lead to multiple equilibria.

The condition PM places restrictions on the interaction between the direct and indirect effects of the model. When there is no feedback from expectations onto the state, then it is clear that agents will always choose the predictor with the highest direct effect. When the self-referential parameter is high, then the indirect effect magnifies the direct effect of the components of z. In these instances it is possible that coordination on a particular forecasting model will produce an indirect effect sufficiently stronger than the direct effect so that this predictor dominates in expected MSE. Significantly, the existence of multiple equilibria arises from the coordinating forces of positive feedback.

3.5 Numerical Examples

We turn now to a numerical illustration. Figure 5 gives the T-maps for various values of α. The upper part of the figure shows the T-maps corresponding to (starting from $n = 0$ and moving clockwise) $\alpha = 10, \alpha = 20, \alpha = 50, \alpha = 1000$. We set

$$A = \begin{bmatrix} .5 & .001 \\ .001 & .3 \end{bmatrix}$$

$\gamma' = [.5, .75]$,

$$\Sigma_\varepsilon = \begin{bmatrix} .03 & .001 \\ .001 & .15 \end{bmatrix}$$

and $\theta = .6$. The bottom portion of the figure is the profit difference function $F(n)$.

INSERT FIGURE 5 HERE

The matrix A, Σ_ε, and θ have been chosen so that Condition PM holds. Condition PM holds under many other parameterizations as discussed above. We chose these parameters as they deliver quantitatively reasonable results in the section on real-time learning and dynamic predictor selection.

A key property of the model is that as $\alpha \to \infty$

$$H_\alpha(x) \to \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x < 0 \\ 1/2 \text{ if } x = 0 \end{cases} \tag{9}$$

and this governs the behavior of $T_\alpha = H_\alpha \circ F$. Since H_α is an increasing function and F is monotonically increasing, it follows that T_α is increasing. Under Condition PM it is clear that (9) implies existence of three fixed points for α sufficiently large. The figure illustrates this intuition.

This example makes it clear that multiple equilibria can exist in the Lucas-type monetary model. When agents underparameterize there is an incentive to coordinate on a particular forecasting model. Interestingly, though, there also exists an interior equilibrium. Below we show that this equilibrium is unstable under learning. The existence of multiple ME suggests there may be interesting learning phenomena in the model. We take up this issue in the section below.

The particular parameterization which leads to this figure produces the following asymptotic covariance matrix for z_t:

$$\Sigma_z = \begin{bmatrix} .04 & .0013 \\ .0013 & .1648 \end{bmatrix}$$

Notice that the variance of z_2 is approximately 4 times that of z_1. The effect of this can be seen in Figure 5 where the 'basin of attraction' for the $n = 0$ ME is larger than for the $n = 1$ ME. *A priori* we would expect a real-time version of this economy to spend, on average, more time near $n = 0$ than $n = 1$. This logic will be key in Section 5 below.

It should be emphasized that in other contexts there may exist a unique interior ME. Branch and Evans (2004) illustrate this case by developing the framework in the context of the cobweb model. The existence of an ME with heterogeneity– what Branch and Evans (2004) call Intrinsic Heterogeneity– exists for precisely the opposite reasoning for multiple ME in the Lucas model. In the cobweb model there is negative feedback from expectations onto the state. Under certain conditions there is an incentive for agents to deviate from the consensus model. Thus, the equilibrium forces push agents away from homogeneity. In the Lucas model the equilibrium forces, as a result of the positive feedback, push the economy towards homogeneity. These results illustrate the multiplicity of equilibrium phenomena that can arise depending on the self-referential features of a simple model.

4 Learning and Dynamic Predictor Selection

In this section we address whether the Misspecification Equilibria are attainable under real-time learning of the type emphasized in Evans and Honkapohja (2001) and dynamic predictor selection. We now substitute optimal linear projections with real-time estimates formed via recursive least squares (RLS). We also assume that agents choose their model each period based on an estimate of mean square error. The next section replaces RLS with a constant gain updating rule of the form used in Evans and Honkapohja (1993), Sargent (1999), Cho, Williams, and Sargent (2002), and Williams (2004a).

We replace the equilibrium stochastic process (6) with one which has time-varying beliefs and predictor proportions. Below we provide details on how the key relationships are altered. This section briefly discusses the stability of the equilibrium under recursive least squares. We model least squares learning as in Branch and Evans (2004). Agents have a RLS updating rule with which they form estimates of the belief parameters b_t^1, b_t^2. They also estimate the MSE of each predictor by constructing a moving average of past forecast errors with equal weight given to all time periods. Given estimates for the belief parameters and predictor fitness, agents choose their forecasting model according to the MNL map in real time.[8]

We now assume the equilibrium stochastic process is given by

$$\pi_t = \xi_1(b_{t-1}^1, n_{1,t-1})z_{1,t-1} + \xi_2(b_{t-1}^2, n_{1,t-1})z_{2,t-1} + \eta_t.$$

Agents use a recursive least squares updating rule,

$$\begin{aligned} b_t^1 &= b_{t-1}^1 + \kappa_t R_{1,t}^{-1} z_{1,t-1}\left(\pi_t - b_{t-1}^1 z_{1,t-1}\right) \\ b_t^2 &= b_{t-1}^2 + \kappa_t R_{2,t}^{-1} z_{2,t-1}\left(\pi_t - b_{t-1}^2 z_{2,t-1}\right) \end{aligned}$$

[8] A point made in Branch and Evans (2004) is that stability of a steady-state depends on how more recent forecast errors are weighted in the moving average calculation. In particular, as the most recent error is weighted more heavily then instability will result as in Brock and Hommes (1997, 1998).

where

$$\begin{aligned} R_{1,t} &= R_{1,t-1} + \kappa_t \left(z_{1,t-1}^2 - R_{1,t-1} \right) \\ R_{2,t} &= R_{2,t-1} + \kappa_t \left(z_{2,t-1}^2 - R_{2,t-1} \right) \end{aligned}$$

We consider two possible cases for the gain sequence κ_t: under decreasing gain, $\kappa_t = t^{-1}$ so that $\kappa_t \to 0$; under constant gain, $\kappa_t = \kappa \in (0, 1)$.

We also assume agents recursively update mean-square forecast error according to

$$MSE_{j,t} = MSE_{j,t-1} + \lambda_t \left((\pi_t - \pi_{j,t}^e)^2 - MSE_{j,t-1} \right), \ j = 1, 2.$$

We again consider two possible cases for the gain sequence λ_t: under decreasing gain, $\lambda_t = t^{-1}$ so that $\lambda_t \to 0$; under constant gain, $\lambda_t = \lambda \in (0, 1)$.

We first look at the case of decreasing gain for both κ_t, λ_t. We then turn in the next Section to our main emphasis of constant gain updating.

4.1 Stability under decreasing gain

In this subsection we study whether the sequence of estimates b_t^1, b_t^2 and predictor proportions $n_{1,t}$ converge to a Misspecification Equilibrium.[9] Our aim is to use simulations to ascertain which equilibrium is stable under real-time learning and dynamic predictor selection. Establishing analytical convergence is beyond the scope of this paper.

We continue with the parameterization in the previous section which yielded multiple ME. We set

$$A = \begin{bmatrix} .5 & .001 \\ .001 & .3 \end{bmatrix}, \Sigma_\varepsilon = \begin{bmatrix} .03 & .001 \\ .001 & .15 \end{bmatrix}$$

and $\gamma' = [.5, .75]$. We also set $\theta = .6$ and $\alpha = 1000$.[10] We simulate the model for 5,000 time periods. The initial value of the VAR is equal to a realization of its white noise shock, i.e., $z_0 = \varepsilon_0$. The initial value $n_{1,0}$ is drawn from a uniform distribution on $[0, 1]$ and $b_{j,0}$, $j = 1, 2$ is drawn from a uniform distribution on $[0, 2]$. The initial estimated variances are set $R_{1,0} = R_{2,0} = 1$.

Figure 6 illustrates the results of two representative simulations. The top panel plots the simulated proportions n_t against time. Recall that for the chosen parameters

[9] Because the analysis is numerical we are being deliberately vague in what sense these sequences converge.

[10] Similar results were obtained for other parameter settings. In particular, the speed of convergence is sensitive to larger values of θ and α.

there exist three equilibria. The plot demonstrates that only the equilibria with homogeneous expectations are stable under learning and dynamic predictor selection. The dynamics quickly converge to either $n = 0$ or $n = 1$. The bottom panel plots the reduced form equilibrium parameters b^1_{t-1}, b^2_{t-1}. In each panel there are two horizontal lines which correspond to the parameter values in either the $n = 0$ or $n = 1$ ME. In the b^1 panel the top horizontal line corresponds to the $n = 1$ equilibrium and in the b^2 panel the top horizontal line is for the $n = 0$ equilibrium. As seen in the top panel, these parameters converge to their ME values. Which equilibrium the dynamics converge to depends on the basins of attraction. As we emphasize in the next section, these basins are sensitive to the parameterization of the z_t process. Thus, we conclude that ME with $n \in \{0, 1\}$ are locally stable under learning and dynamic predictor selection.

INSERT FIGURE 6 HERE

The intuition for this stability is as follows. The multiple equilibria results from an incentive for agents to coordinate on a single model. These coordinating forces yield the interior equilibrium with heterogeneity unstable under learning. Suppose the dynamics begin in a neighborhood of the interior equilibrium. Because the profit function is monotonically increasing, as more agents mass onto a particular model then more agents will also want to use that model. The dynamics are repelled from the neighborhood of the interior steady-state and towards one of the other ME. To which ME the dynamics converge depends on the basin of attraction in which the initial conditions lie.

This result is, again, distinct from the result in Branch and Evans (2004). In that paper, there is a unique Misspecification Equilibrium which is stable under learning. In this paper we have multiple equilibria on the boundary of the unit interval which are locally stable under learning. This distinction leads to interesting dynamics when agents update with a constant gain learning algorithm.

5 Real-time Learning with Constant Gain

It has been suggested by Sargent (1999), among others, that agents concerned with structural change should use a constant gain version of RLS to generate parameter estimates. A constant gain algorithm involves a time-invariant gain which places a high relative weight on recent versus distant outcomes. If agents are concerned about structural change then a constant gain algorithm will better pick up a change in parameters. It has also been argued by Orphanides and Williams (2005) that constant gain learning is more reasonable than RLS learning because the learning rule itself is stationary where it is time-dependent in RLS.

In the Lucas model with misspecification we showed that there may exist multiple equilibria. Moreover, a subset of these equilibria are stable under learning with a decreasing gain algorithm such as RLS. In these equilibria there is an incentive for agents to coordinate on the same forecasting model. If a large enough proportion of agents suddenly switch forecasting models then the economy will switch from one stable ME to another. Agents concerned with this possibility should use a constant gain algorithm instead of a decreasing gain to account for possible regime change.

There has been an explosion in research which adopts constant gain learning rules. Examples include Bullard and Cho (2002), Cho and Kasa (2003), Cho, Williams, and Sargent (2002), Evans and Honkapohja (1993, 2001), Evans and Ramey (2003), Kasa (2004), Orphanides and Williams (2005), Sargent (1999) and Williams (2004a,b). In many of these models constant gain learning can lead to abrupt changes or 'escapes' in the dynamics. For example, models with multiple equilibria such as Evans and Honkapohja (1993, 2001) occasional shocks can lead agents to believe the economy has shifted to a new equilibrium. The result of these beliefs is a self-confirming shift to the new equilibrium. Unlike sunspot equilibria, these shifts are driven entirely by agents' recursive parameter estimates. In Sargent (1999), Cho and Kasa (2003), Cho, Williams, and Sargent (2002), Bullard and Cho (2002), and McGough (2004) occasional large shocks may lead to temporary deviations from the equilibrium uniquely stable under RLS.

The same logic underlies the use of constant gain RLS for parameter estimation carries over to the estimate of the relative fitness of the two forecast rules. Agents who are concerned about structural change, including shifts taking the form of occasional regime changes, would want to allow for the possibility that the better performing forecast rule may shift over time. In order to remain alert to such shifts, agents would weight recent forecast errors more heavily than past forecast errors when computing the average mean square error of each rule. This is equivalent to a constant gain estimate of the average mean square error and leads to dynamic predictor selection following a stochastic process.

In this section we examine the implications of constant gain learning and dynamic predictor selection in the Lucas model with multiple misspecification equilibria. Note, though, that we expect to find distinct dynamics from the studies listed above. This is because in each misspecification equilibrium the mean inflation rate, hence mean output, is the same. Instead the variance of inflation differs across equilibria. We show that endogenous inflation and output volatility arise through two channels: (1) the drift in beliefs from parameter learning with a constant gain RLS; (2) dynamic predictor selection with a geometric average of past squared forecast errors.

Our results show that this combination can generate the observed volatility presented in Section 2. In Branch and Evans (2004) the joint learning of parameters and dynamic predictor selection was presented as a novel extension of Evans and

Honkapohja (2001) and Brock and Hommes (1997), but that model possesses a unique equilibrium and the focus was on heterogeneity and stability. Here the focus is on endogenous volatility resulting from the dual learning process in a set-up with multiple equilibria.

5.1 Joint learning with constant gain algorithms

Because with constant gains $\kappa_t = \kappa > 0, \lambda_t = \lambda > 0$ the dynamics will not converge to a Misspecification Equilibrium. We note that because the ME with $n \in \{0, 1\}$ are stable under decreasing gain learning we anticipate that the dynamics will spend a considerable portion of their time in a neighborhood of the stable ME's.

Under constant gain, $MSE_{j,t}$ estimates the MSE as an average of past squared forecast errors with weights declining geometrically at rate $1-\lambda$. Similarly, constant gain least squares aims to minimize a weighted sum of squared errors where the weight declines geometrically at rate $1-\kappa$. In choosing κ, λ there is a trade-off in tracking structural change versus filtering noise. How strongly $n_{j,t}$ and $b_{j,t}$ react to these shocks then depends on λ, κ, the 'intensity of choice' parameter in the MNL mapping α, and the relative size of the basins of attraction of the two stable steady-state ME. Switching as a result of changes in relative MSE is the second source of endogenous volatility. Suitable choices of λ, κ, α determine the degree to which the model exhibits parameter drift and/or endogenous switching between basins of attraction.

As a means of illustrating the intuition we first present a simulation from a parameterization designed to yield striking results. We first let the asymptotic moments of the z process differ markedly. Set

$$A = \begin{bmatrix} .5 & .001 \\ .001 & .3 \end{bmatrix}, \Sigma_\varepsilon = \begin{bmatrix} .2 & .1 \\ .1 & 3.2 \end{bmatrix}$$

$\gamma' = [.5, .5]$, $\theta = .95$, and $\alpha = 1000$. We set $\kappa = .15$ and $\lambda = .35$. With this parameterization the asymptotic covariance matrix for z is

$$\Sigma_z = \begin{bmatrix} .2668 & .1190 \\ .1190 & 3.5166 \end{bmatrix}$$

Figures 7-8 illustrate typical trajectories when $\alpha = 1000, 10$ respectively. The figure illustrates a number of switches between equilibria during the period 1500-2000. Notice that in this plot the system spends most of its time at the $n = 0$ ME. Moreover, when the dynamics switch to the $n = 1$ ME it is for relatively short periods. This is because the basin of attraction for $n = 0$ is relatively large, and it takes a greater accumulation of shocks to place the economy in the $n = 1$ basin. This parameterization was designed to make the volatility differences dramatic. In so doing we set the variance of inflation at the $n = 0$ ME much greater than at the $n = 1$ ME. Because the variance of z_2 is much greater than z_1 and z_1, z_2 are weakly correlated, the basin of attraction for the 'lower' ME is larger. It is only when a very large proportion of agents use the z_1 forecasting model is it in the best interests of all agents to use that model. Notice also that these dynamics exist for both large and small values of α.

INSERT FIGURES 7-8 HERE

Once a switch takes place there is considerable differences in inflation volatility. Notice that during the periods of frequent switches between ME– so that, on average, more time is spent at the $n = 1$ ME– the inflation volatility switches between a high rate and a low rate. In the $n = 1$ ME there is no positive feedback from z_2 through expectations onto the inflation rate. Thus, in the $n = 1$ ME a larger relative weight is placed on z_1 which is a random variable with a lower asymptotic variance. Hence, we see much lower inflation variances.

These simulations suggest an interpretation to the empirical regularity discussed in the beginning of the paper. In the Lucas model with model underparameterization there may exist multiple equilibria where agents ignore some relevant information when forecasting inflation. If there are significant differences between the information they incorporate and ignore, then the expectational feedback will make the inflation variances differ across these equilibria. To explore this hypothesis further we parameterize the model so that unconditional variances have plausible magnitudes. We also seek to isolate the contributions of parameter learning and mean-square error learning to the endogenous volatility.

We now set the parameters as in Section 3.4 and Section 4, which we reproduce here for convenience:

$$A = \begin{bmatrix} .5 & .001 \\ .001 & .3 \end{bmatrix}, \Sigma_\varepsilon = \begin{bmatrix} .03 & .001 \\ .001 & .15 \end{bmatrix}$$

with $\theta = .6$, $\gamma' = (.5, .75)$ $\kappa = .01$, $\lambda = .04$. We simulate the model, first with a transient period of length 15000, and then for 5000 periods in which we report the results in the figures below. To isolate the effect of parameter drift versus dual learning our strategy is as follows. We first present results where we fix the proportion of agents n to one of its ME values, but allow agents to update their parameters with constant gain least squares. This is analogous to the approach pursued, for example, by Orphanides and Williams (2005) in a full-information setting. We then present simulations with dual learning.

Figure 9 presents the results from a typical simulation. There are 5 panels in the figure. Beginning from the northwest and moving clockwise they are: predictor proportion n, belief parameters b_t^1, b_t^2, time t estimated unconditional variance of output and price respectively. The unconditional variances are computed as moving averages with window length 200 of the variance of the simulated time series. We set $n = 0$, though similar results obtain if we instead set $n = 1$. The horizontal lines in the figure are the ME values.

INSERT FIGURE 9 HERE

Figure 9 shows that some of the endogenous volatility can be attributed to parameter drift. With a constant gain in the least-squares algorithm agents are sensitive to structural change. This is why in the two panels on the right hand side of the figure there is considerable parameter drift. This parameter drift manifests itself in the reduced-form parameters of the model and induces some endogenous volatility. However, it does not generate the type of regime-shifting volatility that was documented in Section 2 and elsewhere in the literature.

Figure 10 now puts both elements together to illustrate that dual learning can account for endogenous volatility. Figure 10 demonstrates combining parameter drift and dynamic predictor selection induces a stochastic process for inflation and output with volatility which both drifts and switches between high and low volatility regimes.

INSERT FIGURE 10 HERE

5.2 Further discussion

As a means of further discussion, an overview is helpful. We take a business cycle model where only unexpected shocks matter for real output fluctuations. We assume bounded rationality but preserve the spirit of Muth's hypothesis and find that there exist multiple equilibria in a model with a unique REE. Moreover, these multiple equilibria arise because the self-referential feature of the Lucas model provides an incentive for agents to forecast with the same model. Each equilibrium can be characterized by the forecasting model which generates it, and each predictor produces distinct forecasts. For practical purposes, the important theoretical implication of the multiple equilibria result is that the self-referential property alters the effect the exogenous stochastic processes have on inflation and output; the positive feedback from expectations onto inflation reinforces the effect of exogenous disturbances. As agents switch forecasting models, the underlying equilibrium stochastic process changes. This theoretical finding is the basis for the learning and predictor selection dynamics in this Section.

The model in this paper is an extension of the learning literature and Brock-Hommes' A.R.E.D.. In the current paper beliefs and the choice of forecasting model are jointly determined. In contrast to our earlier paper set in a cobweb model – whose primary distinction is a negative feedback from beliefs onto the state – we find multiple equilibria. This insight suggested, and our results confirm, that a dynamic version of the model can lead to new and important results.

Previous work by Orphanides and Williams (2005) and Sargent (1999) highlight the role 'perpetual learning' might play in the Great Moderation. But, as has been argued elsewhere the actual U.S. experience has been regime shifting and drifting volatility. The results of this section suggest a new avenue for exploring how an economy might endogenously generate shifting inflation and output volatility.

In particular, we identify two channels. Parameter learning with a constant gain version of least squares produces drifting volatility, but does not generate regime shifting volatility (Figure 9). However, the inclusion of constant gain dynamic predictor selection, in which agents estimate a geometric average of past squared forecast errors for each competing model, can lead to distinct shifts in inflation and output volatility (Figure 10). As with constant gain parameter updating, the use of constant gain in estimates of predictor fitness can be interpreted as a way of providing robustness to structural change.

With dual constant gain learning, shocks can occasionally lead agents to switch forecast models. This, via the feedback of expectations onto the state, produces a regime switch in inflation and output volatility that can have varying durations. Evidence presented in Cogley and Sargent (2005) and Sims and Zha (2005) suggest that drifting and regime switching volatility are important elements of the empirical record. The simulation results in this Section demonstrate that a simple self-referential economic model in which agents choose between competing parsimonious predictors provide a possible explanation for this finding.

It is important to emphasize how natural the assumptions are that generate these results. We model agents as econometricians, in effect, taking the motivation of the learning literature seriously. Because of computational limitations and degrees of freedom problems agents are forced to underparameterize by omitting at least one variable and/or lag from their forecasting model. Although the agents are boundedly rational, they are 'in the spirit' of Muth's original hypothesis since agents only select best-performing statistical models. In the real-time dynamic version of the model we again assume agents behave as econometricians by recursively updating parameter and goodness of fit estimates in light of new data and remaining vigilant against structural change.

6 Conclusion

This paper considered a simple Lucas-type monetary model where inflation is driven by an exogenous process and expectations of current inflation. We introduce model uncertainty and underparameterization to the model. We assume that agents choose the best performing statistical models from a list of misspecified forecasting functions. When agents' predictor choices are endogenous to the model, there exists an equilibrium in the stochastic process, agents' beliefs, and the proportion of agents using a given model. Moreover, there may exist multiple Misspecification Equilibria each with distinct stochastic properties. Numerical simulations show that a subset of these equilibria are stable under least squares learning. If agents adopt dual learning with constant gains, then the system can endogenously switch between equilibria producing time-varying inflation and output volatility.

There is empirical evidence of time-varying inflation and GDP volatility that is consistent with the equilibrium and real-time learning properties of our model. Importantly, we identify two channels through which the economy may generate endogenously drifting and regime-switching economic volatility. The first channel is through drifting parameter estimates that arise from an adaptive learning rule alert to possible structural change. Drifting parameter estimates imply mean forecasts consistent with their equilibrium values, but with occasional departures which induces economic volatility that does not exist in a long-run equilibrium. The second channel is through dynamic predictor selection. Analogously, predictor selection rules that remain alert to possible structural change can lead agents to switch forecast rules in response to occasional large shocks. Such shocks can induce switching between equilibria and produce persistent swings in inflation and output volatility.

These results are significant because they extend Branch and Evans (2004) by showing that underparameterization and expectational feedback are important elements of the economic process. Strikingly, we are able to obtain these results in a simple Lucas-model that has a unique rational expectations equilibrium. The results of this paper show that there are potentially important implications to models with dual learning of parameter estimates and dynamic forecasting model selection.

A Appendix

Proof of Proposition 3. The proof of the proposition follows Lemma 5 in Branch and Evans (2004). Here we briefly summarize the argument and amend it as necessary. We can rewrite (7) as

$$S(n_1)\xi = A'\gamma,$$

where $\xi' = (\xi_1, \xi_2)$ and $S(n_1)$ is the indicated 2×2 matrix. We seek to sign $dF/dn_1 = (dF/d\xi)'(d\xi/dn_1)$. Following Branch and Evans (2004) it can be verified that

$$\begin{aligned} dF/dn_1 &= 2\theta\xi' K(n_1)\xi, \text{ where} \\ K &= \begin{pmatrix} 1-\rho\tilde{\rho} & 0 \\ 0 & \rho^2 - Q \end{pmatrix} S^{-1} \begin{pmatrix} 1 & \rho \\ -\tilde{\rho} & -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{(r^2-1)\left(-1+(1+n(r^2-1))\theta\right)}{(1-\theta)+(n-1)n(r^2-1)\theta^2} & \frac{\sqrt{Q}r(r^2-1)(\theta-1)}{(1-\theta)+(n-1)n(r^2-1)\theta^2} \\ \frac{\sqrt{Q}r(r^2-1)(\theta-1)}{(1-\theta)+(n-1)n(r^2-1)\theta^2} & \frac{-Q(r^2-1)\left(1-r^2\theta+n(r^2-1)\theta\right)}{(1-\theta)+(n-1)n(r^2-1)\theta^2} \end{pmatrix} \end{aligned}$$

Here $r^2 = \rho\tilde{\rho}$ with $0 \leq r^2 < 1$. Notice that K is symmetric. It is easily seen that the necessary and sufficient condition for monotonicity that K is positive semidefinite is satisfied. ■

Proof of Proposition 5. Our proof again follows Branch and Evans (2004).

1. From our earlier paper it was established that for each α the map T_α has a fixed point denoted $n^*(\alpha)$, and, moreover, $\exists\{\alpha(s)\}_s$ $s.t.$ $\alpha(s) \to \infty \Rightarrow n(\alpha(s)) \to \bar{n}$ for some $\bar{n}$ which is a fixed point to the map $lim_{\alpha(s)\to\infty} T_{\alpha(s)}$. The proposition claims that $\bar{n} \in \{0, \hat{n}, 1\}$ where $F(\hat{n}) = 0$. That $\hat{n}$ is a fixed point was proven in Proposition 8 of Branch and Evans (2004). Following the arguments for Conditions P0 and P1 in that proposition, it is clear that $F' > 0$ implies $\bar{n} \in \{0, 1\}$ is a fixed point.

2. Parts 3 and 4 follows from the proof to part (1.).

■

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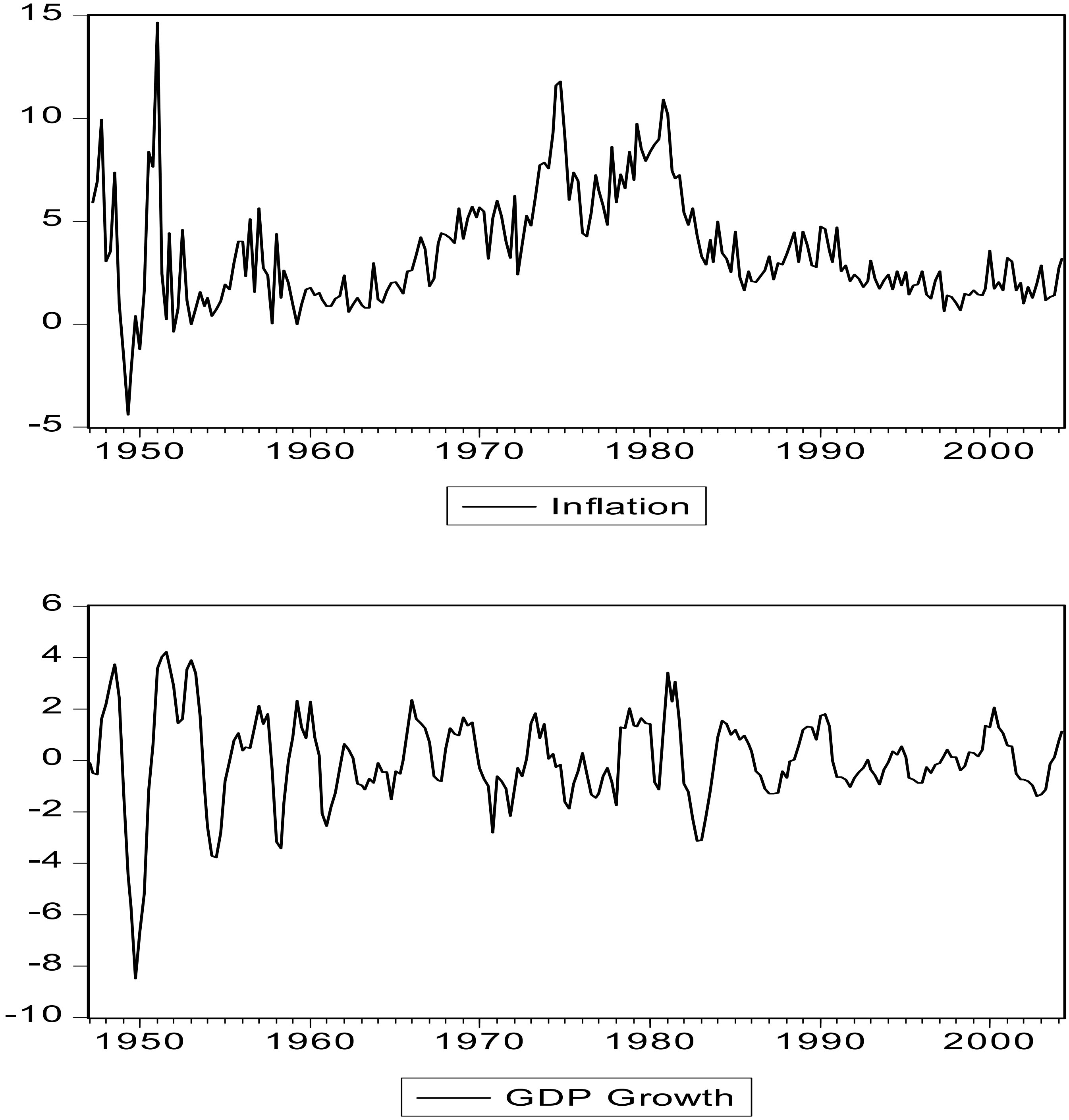

Figure 1. Log of inflation and detrended log GDP, 1947:1-2004:2.

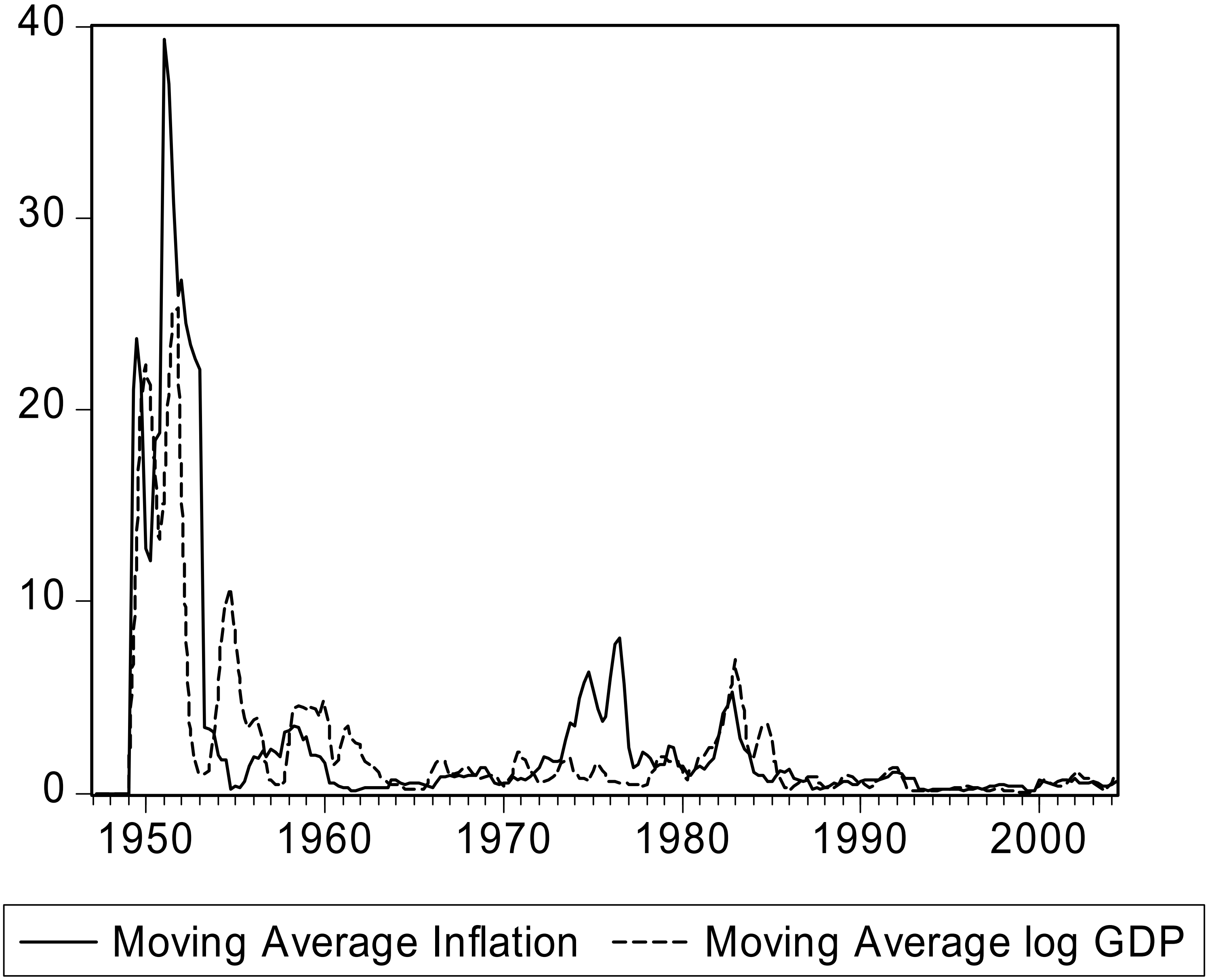

Figure 2. Moving Averages (with window length of 8 quarters) of unconditional variance of inflation and detrended log GDP.

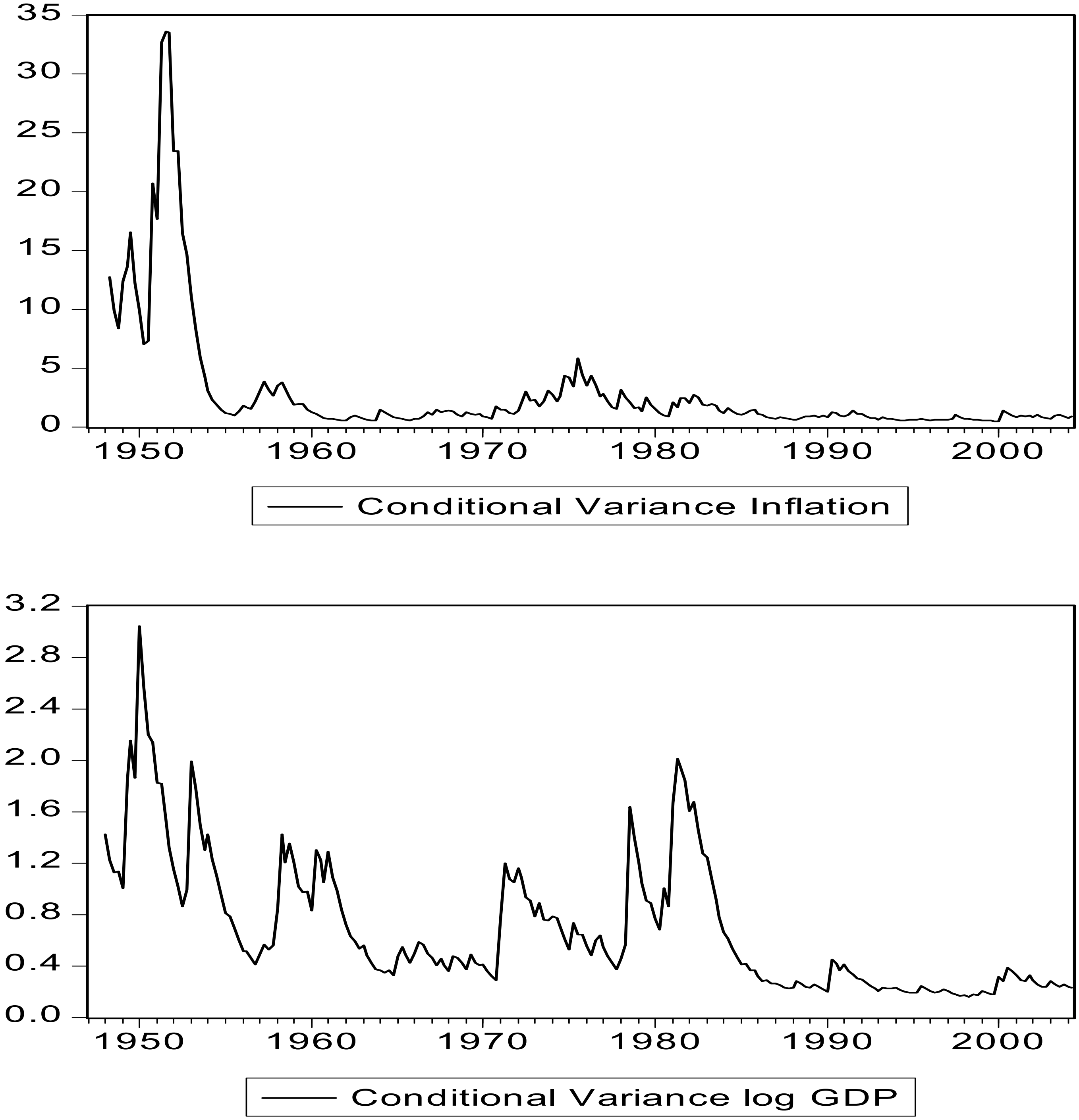

Figure 3. Conditional Variances from a GARCH(1,1) model of an AR(4) process for Inflation and log GDP. Sample: 1947:1-2004:2.

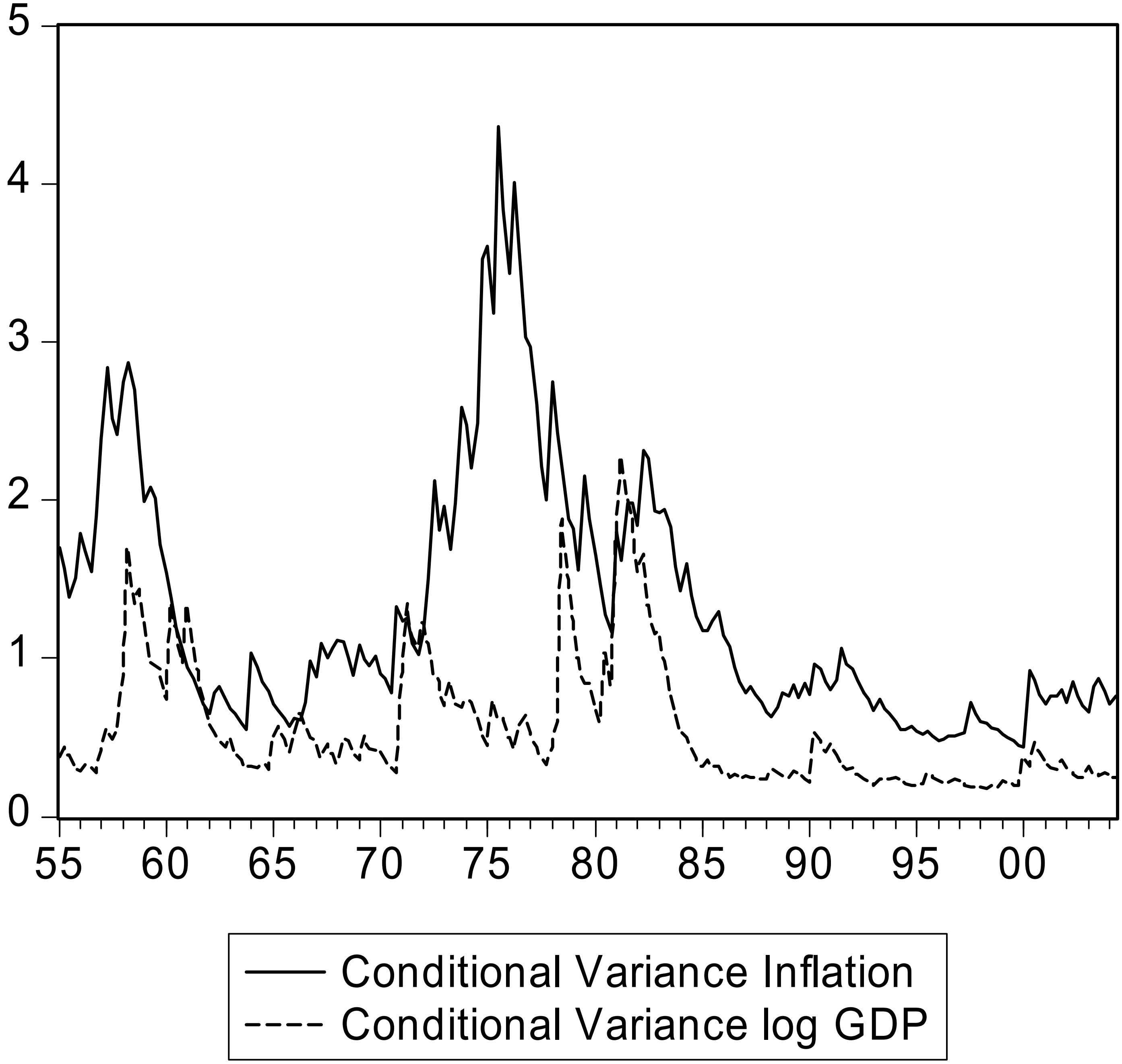

Figure 4. Conditional Variances from a GARCH(1,1) model of an AR(4) process for Inflation and log GDP. Sample: 1955:1-2004:2.

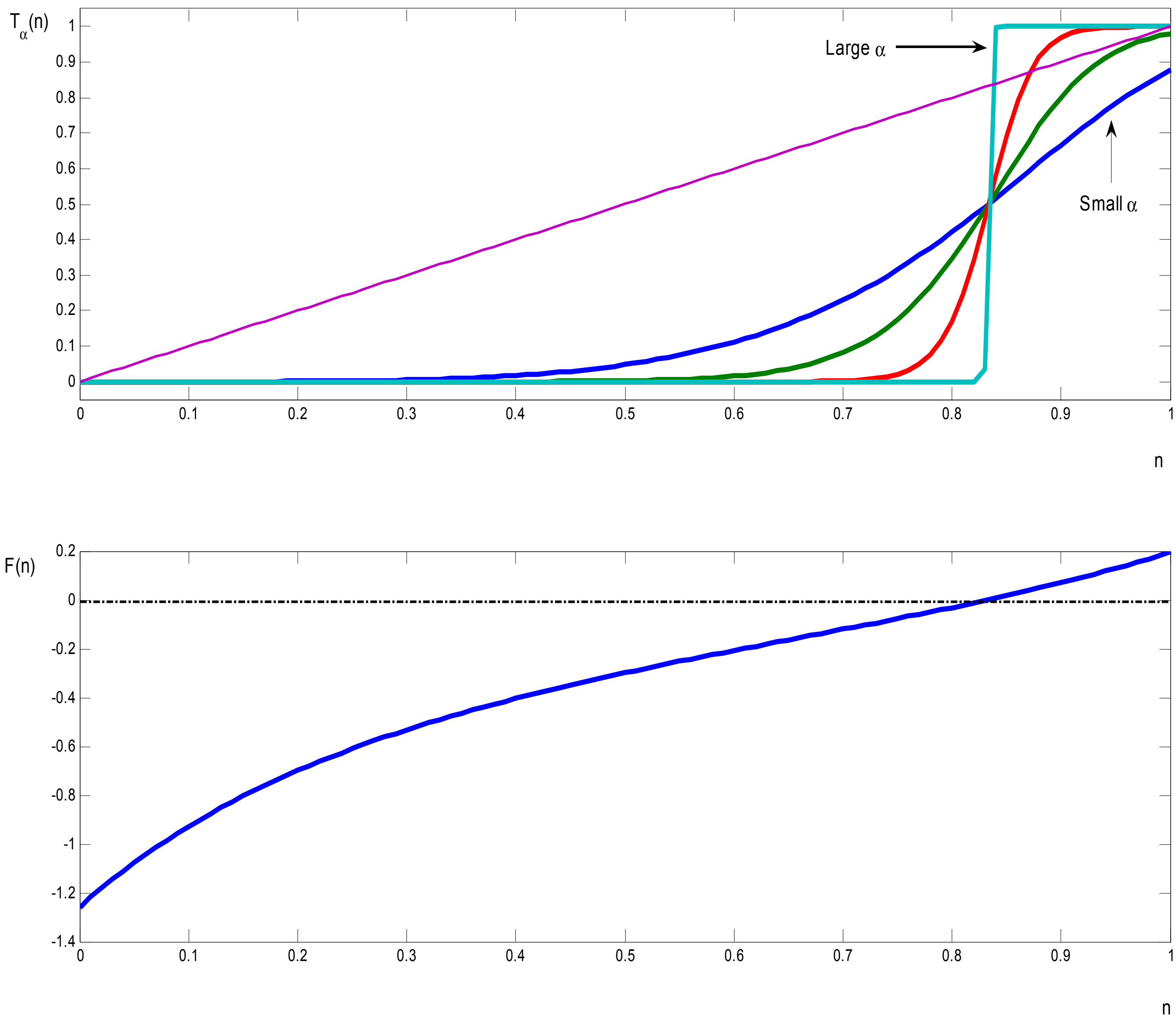

Figure 5. T-map for various values of α and θ=*.60*.

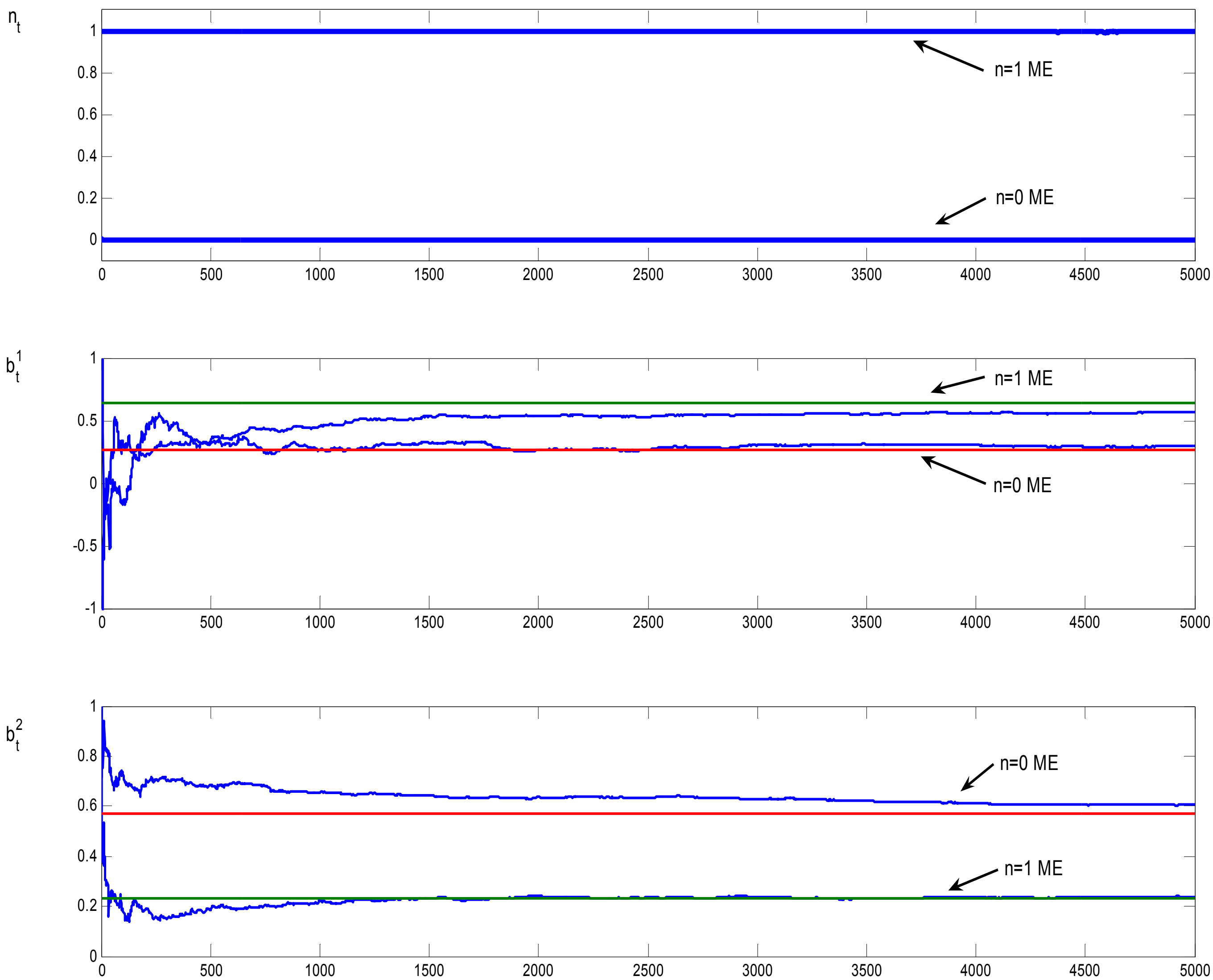

Figure 6. Two RLS learning and dynamic predictor selection trajectories converging to ME. Note: horizontal lines correspond to equilibrium parameter values.

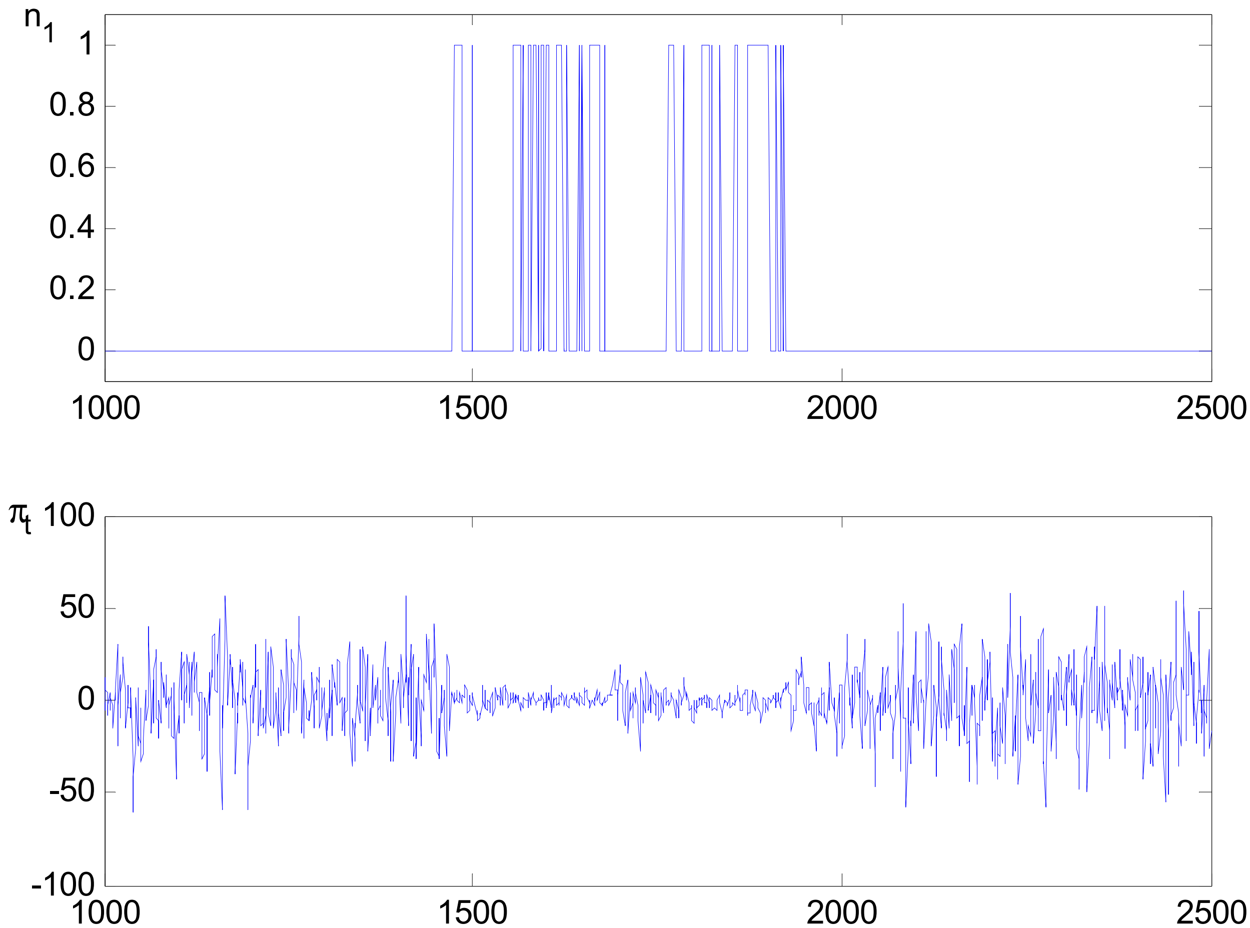

Figure 7. Constant gain learning with θ=.95, κ=.15, λ=.35, α=100.

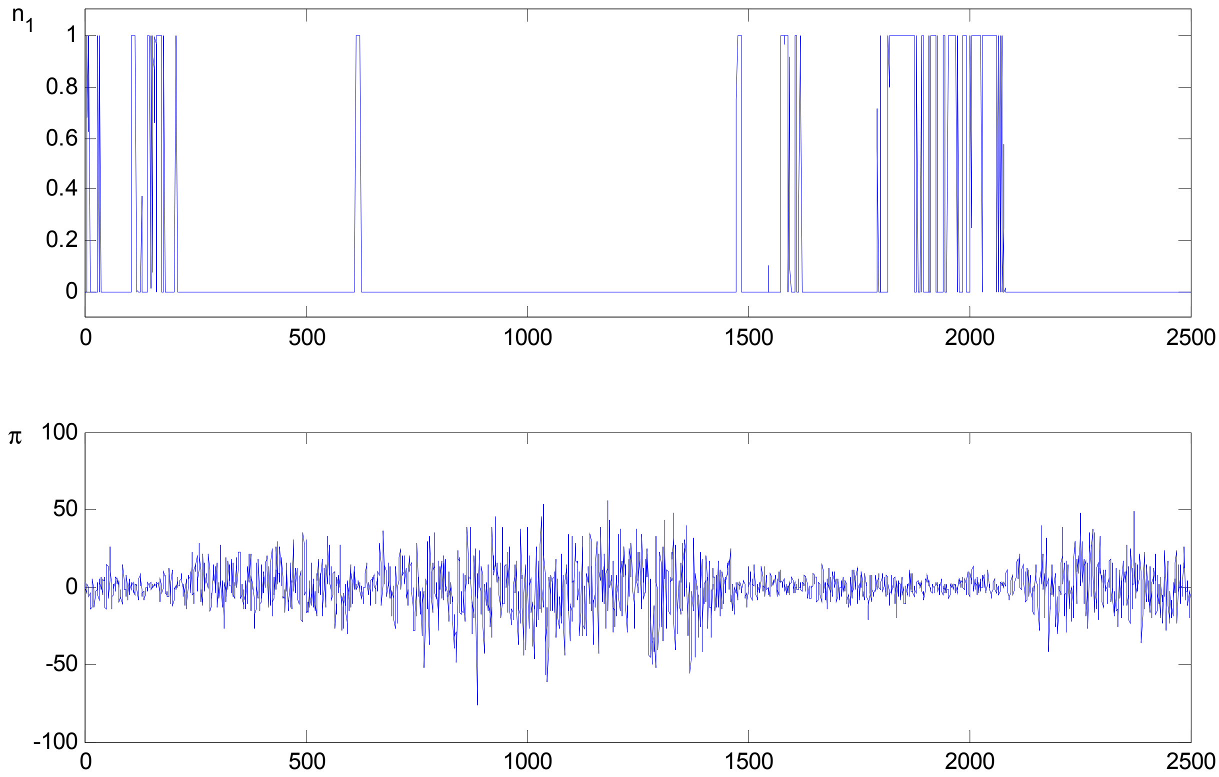

Figure 8. Constant gain learning with θ=.95, κ=.15, λ=.35, α=10.

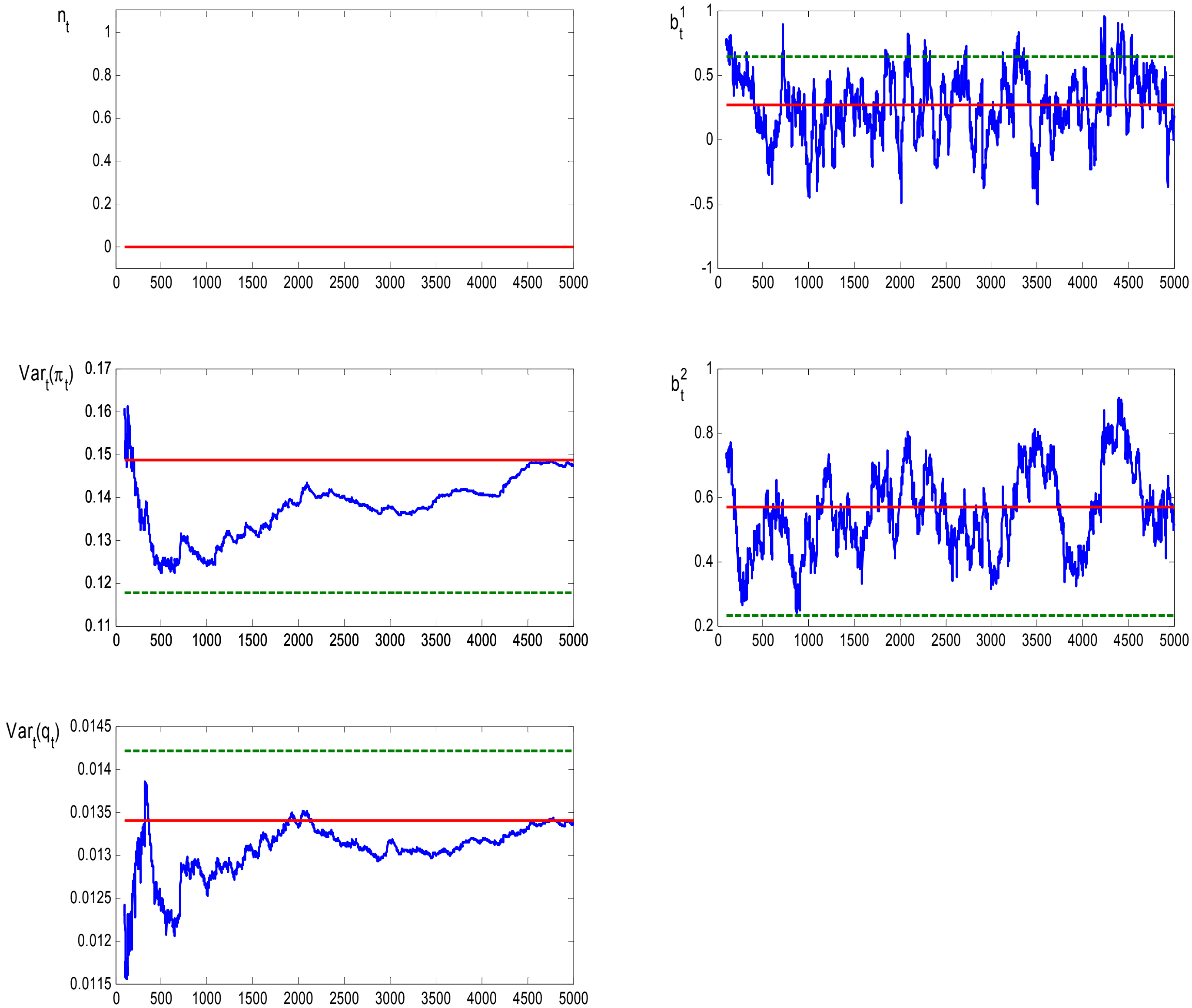

Figure 9. Parameter learning and no dynamic predictor selection with *n=0*. Solid line is *n=0 ME* and dashed line represents *n=1 ME*.

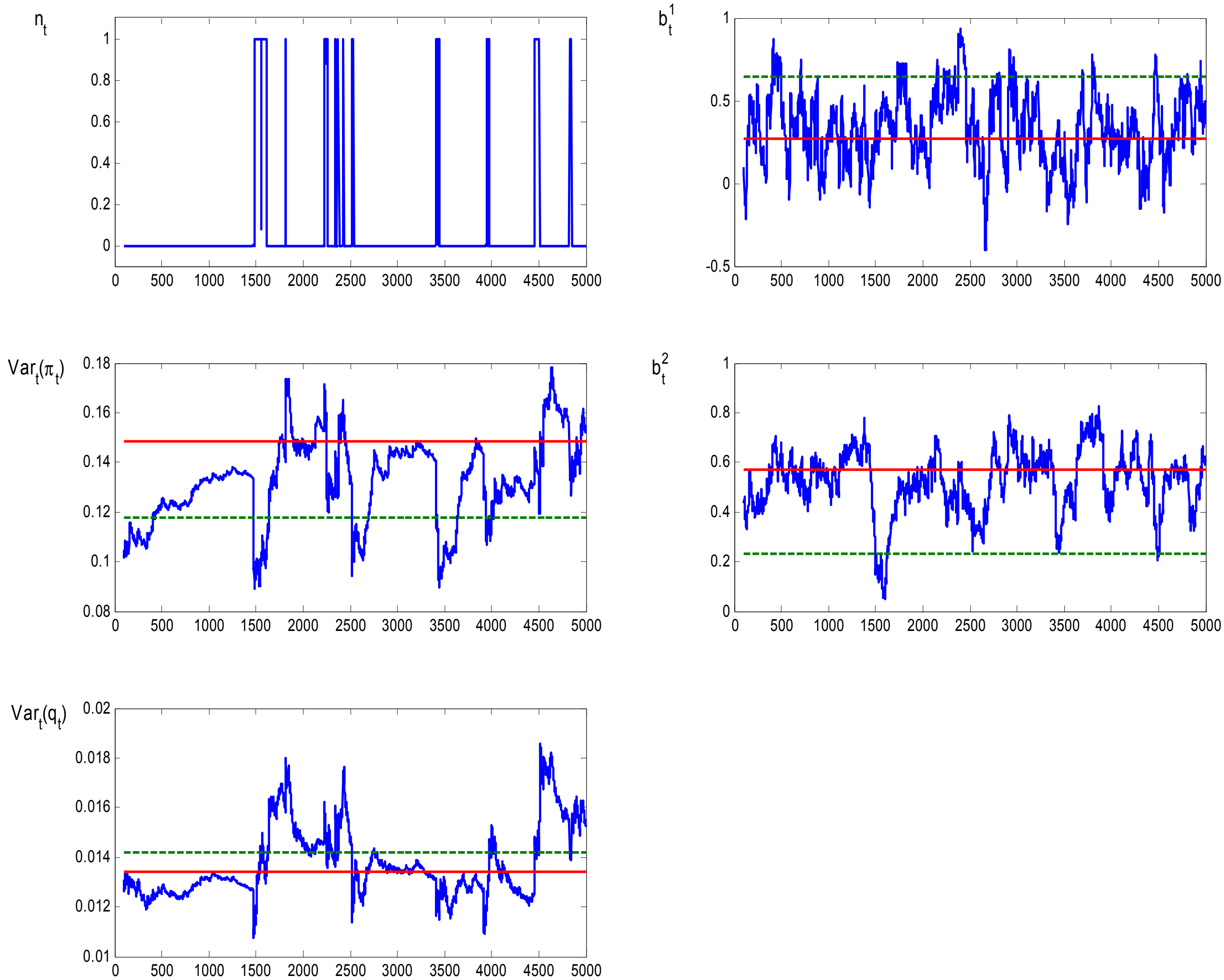

Figure 10. Parameter learning and dynamic predictor selection.