# The Proportionality Principle, the Shapley Value and the Assignment of Heterogeneous Objects.

November 23, 2005

## 0.1 Introduction.

The problem fair division among agents with legitimate (but possibly unequal) claims when the total quantity of goods available is inadequate for meeting all the claims has been studied extensively in the literature.[1] When the good being divided is both homogeneous and divisible a number of appealing solutions have been proposed. Consider the case where \$8 is available with \$7 being owed to claimant A and \$3 to claimant B. Bankruptcy courts would distribute the available money in proportion to the claims giving \$5.60 (70 percent) to A and \$2.40 (30 percent) to B. Among other solutions one that has been supported by many different sets of axiomatic characterizations[2] is the Shapley Value.[3] The solution from this formula will, in general, be different from one given by the "proportional principle." [4] In this case if we divide the \$8 using the Shapley value we would give \$6 to A and \$2 to B. Notice that in this problem the preferences of the individuals are well known ("more money is better") and the good being allocated ("money") is homogeneous and easily divisible making both the proportional principle and the Shapley formula easy to apply. In this paper we develop and discuss alternative solutions to the problem of fair division when the goods being distributed are indivisible and heterogeneous and relate these solutions to the "assignment problem" for indivisible objects.

Any fair division problem involving heterogenous objects introduces the likelihood of claimants have different preferences over the objects. This raises the possibility of some allocations being unacceptable because they are inefficient. Thus, before one applies principles of equity to arrive at a just solution, as a first step one needs to identify and eliminate inefficient alloca-

[1] For excellent introductions to this literature see H. Moulin (2003) *Fair Division and Collective Welfare* and H. Peyton Young (1994) *Equity in Theory and Practice.*

[2] References.

[3] These are by no means the only solutions that have been proposed (see Moulin (2003)). For instance one could apply the "uniform rule" and divide what is available (the \$8) equally between the two individuals subject to no one getting more than her legitimate claim This would give \$5 to A and \$3 B) or one may share the shortfall of \$10 - 8 = 2 equall, inthis particular example, this would give the same solution as the Shapley value.

[4] There are by no means the only solutions that have been proposed (see Moulin (2003)). For instance one could apply the "uniform rule" and divide what is available (the \$8) equally between the two individuals subject to no one getting more than her legitimate claim or one may share the shortfall of \$10 - 8 =2 equally. In this particular example, both these approaches would give the same solution as the Shapley value.

tions. If the preferences are known then this can easily be done. However, as is often the case if preferences are private information the first task is to devise an efficient mechanism for distributing the objects under which the incentives would be such that the mechanism would be"strategy-proof": forcing individuals through their actions to truthfully "reveal" their preferences. There is a large and growing literature studying the problem of efficient and strategy-proof assignment ("the assignment problem"). In the canonical assignment model $n$ (homeless) individual's have (strict) preferences over $n$ houses. The problem is to find a one to one efficient matching of individuals to houses (given that preferences are private information). To solve this problem, two broad approaches have been adopted. Firstly, there is the market based approach inspired by the seminal work by Shapley and Scarf (1974) under which houses are allocated arbitrarily among individuals. Individuals are given one house each and are then allowed to trade with each other. Trading according to Gale's "Top Trading Cycle" algorithm leads to the unique efficient equilibrium: the unique core of this housing market.[5] An alternative approach has been to view the problem as one of mechanism design and of characterizing the class of rules which (in addition to satisfying some other desirable properties) are efficient and strategy-proof.[6] A recurring result from this mechanism design approach has been that these rules take the form of queues with individuals lining up and taking turns choosing. The first person picks her best object, the second person in the queue picks the best from what remains, and the third person picks his best from what remains after the first two have selected their houses and so on. Queuing rules are, of course, inherently inequitable and even though agents have equal claims (1 house each) the people higher up in the queue have an advantage over people lower down, in the same way that even though two theater patrons may be paying the same price for a ticket and hence have identical claims to seating, the person who is ahead in the queue can get a "better" seat. Abdulkadiroğlu and Sonmez (1998) argue that using queueing rules is equivalent to using the market based approach. There is in fact a one to one relation between queues and initial distribution of endowments. The outcome from any queuing rule leads to an element in the core for some initial distribution of houses. Conversely for any core of the housing market

[5]For other papers using this excahnge based approach, see Ma (....), Wako (...), Roth and Postlewaite (.....).

[6]For papers using this approach see ...............

there exists a queue such that the core allocation can be generated by individual choices made following the sequence described by the queue. Hence, one can conclude that the market based approach is just as inequitable as the queueing approach with the inequity being implicit in the initial distribution of resources.

Thus, while in the traditional model of fair division with homogeneous and divisible goods the focus is exclusively on the issue of fair rationing in the presence of *unequal* claims and issues of efficiency and preference revelation do not arise, the canonical assignment model deals with efficiency and preference revelation and problems associated with rationing in the presence of unequal claims do not arise. Furthermore, solutions to the assignment problem lead to queues, mechanisms that are inherently unfair. Can the models in these two areas be combined in a way that this conflict between equity and efficiency is reconciled when one is seeking a solution to the fair division problem with heterogenous and discrete objects?

The first section of our paper is devoted to extending the assignment results for the "housing market" to allow for the possibility of unequal claims and for accommodating the possibility of total claims exceeding the number of available objects. We examine both the market based approach and the queueing approach in this extended framework. We show that the assignment literature can be extended to this case and that both the market based and the queue based approach to efficiency are possible, equivalent and as in the canonical assignment model necessarily unjust. These results suggest that to achieve fairness indivisibilities need be "smoothed" out. We introduce the concept of a sharing rule where individuals are assigned fractional shares in the objects that are being distributed. These fractions can be given interpretations corresponding to two (of the three[7]) usual ways of smoothing out discreteness: randomization and rotation. In the canonical housing assignment problem one can remove the inequity of the queue by assigning spots in the queue using random draws or equivalently the initial distribution of

[7] The third method is monetization. This consists of either reducing the problem to that of fair division of a divisible homogeneous good by selling the objects and dividing the proceeds or alternatively, of devising compensation schemes where people who have an unfair advantage compensate those who are disadvantaged. Our model is applicable for problems where "monetization" would be considered inappropriate : sharing custody between divorced couples, allocating organs donated for transplantation, allocation of subsidized housing etc. (For these and many other such examples where monetization is inappropriate see Peyton Young (1994).)

houses can be made (*ex ante*) equitable by having a lottery which assigns the same probability to every possible initial distribution of houses.[8] Rotation occurs when each agent takes turns deriving utility from the possession of the object. This happens for instance when a victory trophy awarded to the team for a year is often shared among teammates: each person on the team getting possession of the trophy for a part of the year. Without confining ourself to either interpretation we describe a class of sharing rules which are based on queues in that the resulting solutions are given by convex combination of allocations from queues. These solutions to the problem include two rules one of which is analogous to Aristotelian proportional solution (A-SRule) and one that is an analogue of the Shapley value (S-SRule). (In the special case of the fair division problem of a homogenous goods, these two rules reduce to the proportional solution and the Shapley value, respectively). We argue that in problems where the individual preferences are known and all claims are treated equally, the A-SRule offers an efficient and fair solution. On the other hand, if preferences are private information and strategic considerations involving revelation of preferences are involved then the S-SRule would be the appropriate efficient and fair solution. In the case of the A-SRule the equivalence of this rule with a market based sharing method is established generalizing one of the principal results of Abdulkadiroğlu and Sonmez (1998). We use our concept of queue based sharing rules to discuss a number of other solutions that have been proposed for the fair division problem for homogeneous and divisible commodities and discuss how analogous solutions may be developed for indivisible and heterogenous goods.

## 0.2 Efficient and Strategy-proof Assignment

Let there be $n$ individuals, $N = \{1, 2, ...., n\}$ and $m$ indivisible objects $X = \{x_1,\ x_2, ....., x_m\}$. Let $c_i$ be a positive integer representing the *number* of objects that individual $i$ is entitled to (i.e., the size of $i$'s claim) and let $\succ_i$ denote $i$'s (strict) ranking of the objects. We will use $k$ to determine the aggregate size of the claims and define a *fair division problem* as an ordered triple $\Im = \langle c; \succ; k\rangle$ where $c = (c_1, c_2, ......, c_n)$; $\sum c_i \equiv k$; and $k \geq m$. For notational convenience we define $(k - m)$ null objects: $x_{m+1,}....x_{k_i}$. We will

[8] Abdulkadiroğlu and Sonmez (1998) consider these possibility in their study of the assignment problem. Their principal result is that these two methods are equivalent. Also see Bogomolnaia and Moulin's (2001) work on random queueing.

assume that every individuals preference ordering is extended so that these null objects are all ranked at the same level and strictly below the worst “real object”.

**Example 1** Father dies leaving his library of books to his three sons. His will reads: “Of my one hundred books I leave my first son 50 my second son 30 and my third son 20. When the door of the library is opened only 80 books are found. How should the books be efficiently assigned taking into account the legitimate claims of the three individuals ?

For any fair division problem $\Im$ an *assignment function* gives us an assignment $\alpha$ assigning objects to individuals such that the number of objects assigned to any individual $i$ does not exceed his claim $c_i$. $\alpha$ is called a *matching* or *assignment* for the fair division problem. We will denote assignments using the symbols $\alpha, \beta$, ..., etc. and the objects assigned to individual $i$ under these assignments by $\alpha_i, \beta_i, ...$ , etc. For instance for Example 1 above, giving 40 books to 1, 30 to 2 and 10 to 3 (no matter which book goes to which individual) is an example of the many possible ways of assigning books to individuals. However, different individuals may like different books so looking at the many possible 40, 30, 20 matchings one could well find that some of these matchings are inefficient. Notice that this problem of inefficiency has nothing to do with the 40, 30, 10 division of claims and could for instance also arise if we gave 30 books to the first and second son and 20 to the third. What efficiency does depend on is the individuals’ preferences[9] and on which particular book is given to which individual.

### 0.2.1 The Market Based Approach

Markets are efficient and one way of using market forces to generate efficient matchings is to start with an arbitrary matching (an “initial endowment”) and allow individuals to trade with each other. A particular trading algorithm, Gales’s top trading cycle (TTC), has been used by Shapley and Scarf’s (1975) model of the “housing market” to find the unique core allocation of the market. But, their model is a special case of our fair division problem, one in which every $c_i = 1$ and $\sum c_i = k = m$. The following trading algorithm, the *myopic top trading cycle* (MTTC) describes a modification

[9] If all the individuals had identical preferences inefficient allocations would never arise as long as all the objects are assigned.

of Gale's algorithm which can be applied to our more general fair division model. This algotithm reduces to Gale's algorithm in the special case of the "housing market".

*Distribute all objects including the null objects temporarily to the individuals such that each individual i has $c_i$ objects (i.e., each individual i has less than or equal to $c_i$ "real" objects.) Let this be an initial endowment. Let each object point to the individual it is endowed to (several arrows may be pointing towards one individual) and let each individual point to her best object among the ones available. In any trading round, a MTTC forms when we have a group with the same number of individuals and objects with each individual in the group pointing to one of the objects and each object in the group pointing to one of the individuals. The objects in an MTTC are removed and each of these is assigned permanently to the individual pointing to it and is no longer available for trade in the next round. All individuals and unassigned objects remain and the process is repeated until all non-null objects are permanently assigned and only null objects remain.*

From our definition of the MTTC rule and the finiteness of individuals and objects we have the following.

**Proposition 1** *For any initial endowment the assignment from the MTTC rule exists and is unique.*

For the Shapley-Scarf "housing market" model the MTTC coincides with the TTC and for any initial endowment leads to the competitive equilibrium in the sense that it leads to the assignment of the unique allocation in the core of the economy. For any given initial endowment does the MTTC rule assign an element in the "core" of the economy in our model?

A coalition will *block* a "proposed" allocation when by redistributing their initial endowment within the group at least one of the members of the group becomes better off and no member becomes worse off as compared to the "proposed" allocation. Whether an individual becomes better off from a reallocation depends on the individuals preference on *sets of objects* (i.e., whether she considers the set of objects that she receives under a proposed allocation to be better than the set of objects that she can achieve with her coalition partners). However, the only information that we have in our Fair Division problem consists of ordinal strict preferences on objects in $X$ and not preferences on sets of objects in thepower set of $X$. There are, however, well established methods (see Bossert, Pattanaik et al Handbook........) in economics of extending the strict preference relation on $X$ to a strict preference relation on the power set of $X$, $\mathcal{P}(X)$, and we can use these extensions to fill the gap in the available information.

Consider three such preference relations on $\mathcal{P}(X)$ extending the given preference relation $\succ_i$on $X$:

A. The preference relation $\gg_i$on the power set of $\mathcal{P}(X)$ of $X$ is given by the well known *lexi-max* relation generated by $\succ_i$.

B. The preference relation $\ggg_i$ on the power set of $\mathcal{P}(X)$ of $X$ is given by the *dominance* relation defined as follows: for sets of objects $\alpha_i, \beta_i$: $\alpha_i \ggg_i \beta_i$ iff $\alpha_i \neq \beta_i$ and there exists a one to one $g$ from $\beta_i$ to $\alpha_i$ such that the object $g(x)$ is at least as good for $i$ as $x$.[10]

C. The weak preference relation $\gtrapprox_i$ on the power set of $\mathcal{P}(X)$ of $X$ is given by the *weak dominance* relation defined as follows: for sets of objects $\alpha_i, \beta_i$: $\alpha_i \gtrapprox_i \beta_i$ iff there exists an object $x \in \alpha_i \setminus \beta_i$and $y \in \beta_i \setminus \alpha_i$ such $x \succ_i y$.

Under lexi-max, the individual is mainly interested in the best object she can get. She chooses between sets of objects by comparing the best objects in the two sets, choosing the set whose best object is ranked higher; if the best object is the same in both sets then her preference between the sets is based on the second best object, if these are the same then on a comparison of the third best objects and so on. For comparisons using the dominance relation, the individual looks at whether one of the sets is clearly better in the sense of being able to associate each object in the dominated set with an object in the dominating set that is (weakly) superior to it. The requirement $\alpha_1 \neq \alpha_2$ ensures that some alternatives in the dominating set are strictly better than the corresponding alternatives in the dominated set. The weak dominance relation requires that if one removes from both sets all objects that are contained in both then among the remaining objects, something in $\alpha_i$ has to be better than something in $\beta_i$ for the set $\alpha_i$ to be considered to be at least as good as $\beta_i$. Notice that any reasonable extension of the preference relation to the power set (such as lexi-max) will necessarily declare $\alpha_i$ to be better than $\beta_i$ whenever $\alpha_i \ggg_i \beta_i$, so that $\ggg_i$ is best interpreted as a compelling "sufficient" condition and any reasonable "extension" of $\succ_i$ will have $\ggg_i$ as a sub-relation. Analogously, $\gtrapprox_i$ is best viewed as a "necessary" condition and any reasonable extension of $\succ_i$ to the power set (such as lexi-max) would be a sub-relation of $\gtrapprox_i$ .

For any given initial endowment we can define the $\gg$-core, the $\ggg$- core

[10] We need "at least as good" because the two sets may contain some identical objects.

and the $\gtrapprox$-core of the economy corresponding to whether individuals use $\gg_i$, $\ggg_i$, and $\gtrapprox_i$, respectively, to determine whether or not to block a proposed allocation in the economy. Clearly, if an allocation is blocked under $\ggg_i$ it is Pareto inferior conversely any allocation that is Pareto inferior will be blocked under $\ggg_i$. Moreover, anything that is blocked under $\ggg_i$ will also be blocked under $\gg_i$ and anything that is blocked under $\gg_i$ will be blocked under $\gtrapprox_i$ .

**Proposition 2** *For any initial endowment : (i) The $\gtrapprox$-core is a subset of the $\gg$-core which is a subset of the $\ggg$- core. (ii) The $\gg$- core is efficient and unique and is given by the assignment from the MTTC rule.*[11]

The following example illustrates Proposition 2 (ii):

**Example 2** Let there be three individuals $1, 2, 3$, three objects $x_1$, $x_2$, $x_3$ and let $c_1 = 2$, $c_2 = 1$ and $c_3 = 1$. There are two types of possible initial endowments: individual 1 has two objects and one of the other individuals has one or each has one object. Let the preferences of the three individuals be given by: $x_1 \succ_1 x_2 \succ_1 x_3$ and $x_2 \succ_i x_3 \succ_i x_1$ for $i = 2, 3$. For our illustration, we will consider the initial allocation in which 1 has an initial endowment of object $x_2$ and $x_3$ and individual 2 has $x_1$. Now consider the MTTC method. In the first round $1 \to x_1 \to 2 \to x_2 \to 1$. Thus, under the MTTC rule $x_1$ would be permanently assigned to 1 and 2 would get $x_2$. In the second round the MTTC is $1 \to x_3 \to 1$ and $x_3$ would get assigned to 1. Thus, 1 would have $\{x_1, x_3\}$ and 2 would get $\{x_2\}$. Clearly, under lexi-max, with both individuals getting their best object, the allocation would be in the $\gg$-core and the $\ggg$- core. (In this case it also belongs to the $\gtrapprox$-core.)

From the market based approach to generating efficient outcomes we now turn to examining the possibility of extending the queue based approaches to the "housing market" to our problem of fair division.

[11] Even though the other two cores are always exist, the $\gtrapprox$-core in our fair sharing economy may well be empty. To see this consider an economy with $\{x_1, x_2, , x_3, x_4\}$; $\{1, 2\}$; $c_1 = c_2 = 2$; $x_1 \succ_1 x_2 \succ_1 x_3 \succ_1 x_4$ $x_2 \succ_2 x_3 \succ_2 x_4 \succ_2 x_1$
and an initial endowment where 1 is given $x_1$ and $x_3$ and 2 is given $x_2$ and $x_4$.

### 0.2.2 The Queue Based Approach

#### Equivalence to the Market

A *queue* is a function $Q$ from $\{1, 2, ....., m, m+1, ..., k\}$ to $\{1, 2, ..., n\}$ such that for all for all $i \in \{1, 2, ..., n\}$, $|Q^{-1}(i)| \leq c_i$. The function $Q$ is interpreted as assigning turns to individuals. Each individual gets to choose an object when it is her turn to choose. Thus, $Q(1)$ picks an object, followed by $Q(2)$,followed by $Q(3), .....,$ etc. Clearly, the same individual can have multiple turns.[12] The condition $|Q^{-1}(i)| \leq c_i$ ensures that no individual gets more objects than is justified by the individuals legitimate claim. Observe that the total number of turns equals the number of objects: real and null. Thus, if individuals choose honestly then the process of sequential *honest* choice according to their turn in the $Q$ leads to an assignment of all real objects available by the $m^{th}$ turn. All individuals get the null object in turns $m+1, ..., k$. Since the turns $m+1, ..., k$ represent an artificial construct it makes sense to treat two $Q$s with the same individuals occupying these last $k-m$ spots (even though each of these individuals may be occupying a different one of these spots) as being *essentially* the same $Q$. Only $Q$s that are not *essentially the same* will be referred to as *distinct* $Q$s. Let $\mathcal{Q}$ be the set of all distinct $Qs$ . Let $\Omega$ be the set of all possible distinct allocations of non-null objects such that each $i$ gets $c_i$ or less objects. Thus, note that each element of $\Omega$ is a possible assignment and a possible initial endowment.

**Proposition 3** *Any given $Q$ yields an unique assignment. Two $Qs$ that are* ***essentially the same*** *yield the same assignment.*

**Lemma 1** *There exists a* one to one *function $\theta$ from $\Omega$ to $\mathcal{Q}$ such that if the MTTC rule together with the initial endowment $\omega$ yields the assignment $\alpha$ then the queue given by $Q = \theta(\omega)$ also assigns $\alpha$.*

**Proof.** (Sketch)Let the individuals involved in the "first round" of the MTTC be $\mathrm{G}_1$, those in the second round be $G_2$, ..., etc. Put all individuals in $\mathrm{G}_1$ higher up in the $Q$ then those in $G_2$, the individuals in $G_2$higher up in the $Q$ then those in $G_3$, ...., etc. Within the groups, $G_1$, $G_2$, ...., etc. break the ties according to whose initial endowment being traded in *that* trading cycle has the smaller index. Thus, if $\mathrm{G}_1$ involved $k_1$ individuals (*all the individuals who get their best object are in* $G_1$ and all individuals in $G_1$ get their best

[12] Thus, for instance, it is possible that $Q(3) = Q(10)$.

object) and if $G_2$ has $k_2$ individuals then the first $k_1$ spots in the $Q$ are given to the individuals involved in the $G_1$ trading cycle and the next $k_2$ spots to the individuals in the $G_2$ trading cycle. (Note that the same individual can be in both $G$'s but cannot be in the same $G$ twice). If $i_1$ and $i_2$ are in $G_1$ then which of the two has a higher spot in the first $k_1$ spots depends on what the two individuals trade in the $G_1$ cycle. If two individuals $i_1$ and $i_2$ are both in $G_1$ and $i_1$ trades his endowment $x_k$ and $i_2$ trades $x_j$ and $j < k$ then $i_2$ will be given a higher spot than $i_1$among the first $k_1$ spots.) Once all real objects have been permanently assigned through trade, individuals having unfilled claims are arbitrarily listed at the bottom of the $Q$, with all remaining claims being satisfied by the assignment of the null objects.

If one follows the construction above then the same initial endowment cannot lead to the two essentially distinct $Q$s since for the this to happen, the "$G$" structure must be the same and within each $G$ the same object has to be traded. In other words each individual must have had exactly the same initial endowment. ■

**Example 3** Consider Example 2 with the initial endowments and preference as described. Consider the $Q$ in which the first three turns are given to individuals $1, 2$ and then to 1, again, and 3 choose last. If the individuals choose honestly according to this $Q$, in the $\gg$-core allocation generated 1 will get $\{\{x_1, x_3\}$, 2 will get $\{x_2\}$ and 3 the empty object. This $Q = \theta(\omega)$ for the initial endowment $\omega$ where 1 has an initial endowment of object $x_2$ and $x_3$ and individual 2 has $x_1$.

**Lemma 2** *The cardinality of the set $\Omega$ (i.e., the set of all possible initial endowments) is equal the cardinality set of $\mathcal{Q}$, the set of all distinct $Q$s.*[13]

**Proof.** Lemma 1 implies that $|\mathcal{Q}| \geq |\Omega|$. To show that the converse is also true observe that if all individuals are have the *same* preferences over objects and individuals choose honestly according to distinct $Q$s, the assignments will be distinct and that each of these assignments can be interpreted as distinct

[13]Enumerating using the rules of permutations and combinations, we can actually see that the number of distinct initial endowments and the number of essentially distinct $Q$s is:

$$\frac{k!}{c_1!c_2!...c_n!(k-m)!}$$

initial endowment. This shows that there are at least as many distinct initial endowments as distinct $Q$s (i.e., $|\Omega| \geq |\mathcal{Q}|$) . ■

The following result establishes the equivalence of the market based method of MTTC and the $Q$ based method of assigning the objects:

**Proposition 4** *There exists a* one to one *and* onto *function $\theta$ from $\Omega$ to $\mathcal{Q}$ such that if the MTTC rule together with the initial endowment $\omega$ yields the assignment $\alpha$ then the queue given by $Q = \theta(\omega)$ also assigns $\alpha$.*

**Proof.** Consider any initial endowment $\omega$ and using Proposition 2 let $\omega^*$ be the unique $\gg$-core assigned by the MTTC rule. By lemma 1 using the one-to -one function $\theta$ an "unique" $Q$ is associated with the $\omega$ with yields the same assignment, $\omega^*$. Consider the set of all initial endowments $\Omega(\omega^*)$ which assign the same $\gg$-core, $\omega^*$, using the MTTC rule and the set of $Q$s: $\mathcal{Q}(\omega^*)$ of distinct $Q$s such that $Q = \theta(\omega)$ for all $\omega \in \Omega(\omega^*)$. By Lemma 1 the number of $\omega$s in $\Omega(\omega^*)$ is exactly equal to the number of $Q$s in $\mathcal{Q}(\omega^*)$. Thus, all we need to complete our proof is to show to that $\theta$ is onto and hence there does not exist a distinct $Q$ such that $Q$ assigns $\omega^*$ and $Q \notin \mathcal{Q}(\omega^*)$. If one assumed to the contrary that such a $Q$ existed this would imply (using lemma 1) that the number of distinct $Q$s, $|\mathcal{Q}|$, is strictly greater than the number of distinct initial endowments, $|\Omega|$, violating Lemma 2. ■

### Serial Dictatorship and Strategy-proofness

One of the difficulties that we would face if we tried to use either the MTTC or the associated $Q$s to generate efficient assignments is that *unlike in the case of the canonical housing assignment problem,* these rules are in general not strategy-proof and individuals will have an incentive to misrepresent their preferences to achieve a better outcome for themselves. To see this consider the following example:

**Example 4** Consider Example2 with the initial endowments and preference as described. The particular core allocation generated by the $Q$ in which individuals $1, 2, 1$(second turn) and 3 choose honestly in that sequence. If so, 1 will get $\{x_1, x_3\}$, 2 will get $\{x_2\}$ and 3 the empty object. If using the same $Q$, 1 lied and picked $x_2$ as his first pick , she would end up with an allocation $\{x_1, x_2\}$ instead of $\{x_1, x_3\}$. Since, $\{x_1, x_2\} \ggg_1 \{x_1, x_3\}$, it would clearly be in her interest to manipulate the system. An analogous misrevelation of preferences of course would result if the MTTC assignment is used and in the first round 1 (falsely) pointed to $x_2$ (instead of $x_1$).

********To characterize a set of *assignment rules* that are strategy-proof we will define some additional properties and a special kind of $Q$.

In the what follows we us the standard notation of using $\succ$ or $(\succ_i)$ to denote a preference profile and $(\succ_{-i}, \succ'_i)$ to denote a profile which only differs from $\succ$ in its $i^{th}$ component.

**Strategy-Proofness.** An assignment rule $f$ is $\gg$-**Strategy-Proof** iff for any $\succ, \succ'$ such that $\succ' = (\succ_{-i}, \succ'_i)$ if $\alpha = f(\langle c; \succ; k\rangle)$ and $\alpha' = f(\langle c; \succ'; k\rangle)$ then $\alpha \gg \alpha'$. Replacing $\gg$ with $\ggg$ (respectively, $\gtrapprox$) defines $\ggg$-**Strategy-Proof** (resp., $\gtrapprox$-**Strategy-Proof** ).

**Neutrality.** An assignment rule $f$ is **Neutral** iff for any permutation $\pi$ of the set of objects, if $\succ'$ can be obtained from $\succ$ by permuting the objects using $\pi$ and if $\alpha = f(\langle c; \succ; k\rangle)$ and $\alpha' = f(\langle c; \succ'; k\rangle)$ then $\alpha' = \pi(\alpha)$.

**Efficiency.** An assignment rule $f$ is **Efficient** iff for any $\alpha = f(\langle c; \succ; k\rangle)$ there does not exist an assignment $\beta \neq \alpha$ such that for all $i$, $\beta_i \neq \alpha_i$ implies $\beta_i \ggg_i \alpha_i$.

**Non-Bossiness.** An assignment rule $f$ is **Non-Bossy** iff for all $\succ, \succ'$ such that $\succ' = (\succ_{-i}, \succ'_i)$ if $\alpha = f(\langle c; \succ; k\rangle)$ and $\alpha' = f(\langle c; \succ'; k\rangle)$, $\alpha_i = \alpha'_i$ implies for all $j \in N : \alpha_j = \alpha'_j$ .

**Separable.** An assignment rule $f$ is **Separable** iff all fair division problems $\langle c; \succ; k\rangle$, $\langle c; \succ'; k\rangle$ if $\alpha = f(\langle c; \succ; k\rangle)$ and $\alpha' = f(\langle c; \succ'; k\rangle)$ then for all $i$, $|\alpha_i| = |\alpha'_i|$.

Strategy-proofness is the requirement that the assignment rule be such that individuals have no incentive to misreveal their preferences in order to get a better assignment of objects where "better" can be represented by three different extensions of the individuals' preference relation on the set of objects $X$ to the power set of $X$. Neutrality ensures that all objects are treated symmetrically under the assignment rule and Efficiency represents the requirement that no "Pareto" improvement is possible. The condition of Non-Bossiness is a form of requirement first introduced by Sonnenschein and Satterthwaite (1981) and has since been used in the vast majority of papers dealing with mechanism design. This condition requires that an individual by signalling a different preference cannot affect the outcome for some other individual without making his own outcome different. The final condition of separability expresses the requirement that while which objects an individual gets might depend on his preference, the *number* of objects assigned depends on claims and the total available and is independent of the preference profile.

The relationship shown in the proposition below between the different kinds of strategy-proofness follows immediately from their definition and the relationship between $\gg$, $\ggg$ and $\gtrapprox$ .

**Proposition 5** ***$\gtrapprox$-Strategy-Proofness$\Rightarrow\gg$-Strategy-Proofness$\Rightarrow\ggg$-Strategy-Proofness.***

To characterize the set of assignment functions satisfying $\gg$-Strategy-Proofness, neutrality, efficiency and non-bossiness we introduce a special type of $Q$.

**Serial Dictatorship.** An assignment rule is **serially dictatorial** iff the assignment is from a $Q$ which satisfies the following property: If for any two integers $p$ , $q \in \{1, 2, ..., m\}$, $Q(p) = i$ and $Q(q) = i$ then for all integers $s$, $q < s < p$, $Q(s) = i$. A **Serial Dictatorship** is **unrestricted** if for any for any two integers $p$ , $q \in \{1, 2, ., m, .., k\}$, $Q(p) = i$ and $Q(q) = i$ then for all integers $s$, $q < s < p$, $Q(s) = i$.

In other words a serial dictatorship is a $Q$ such that no *other* individual occupies spots between two non-trivial spots occupied by the same individual and an unrestricted serial dictatorship is one where no other individual occupies spots between *any* two spots occupied by the same individual.

**Lemma 3** *Let $f$ be an assignment rule satisfying Efficiency, Neutrality, Non-bossiness, Separability and $\gg$-Strategy-Proofness and let all individuals have identical orderings $x_1 \succ_i x_2.... \succ_i x_m$. Then, for integers $j$ and $h$ such that $j < h < m$ if $x_j$ and $x_h$ are assigned by $f$ to individual $i$ the for all integers $p$ between $j$ and $h$, $x_p$ is also assigned by $f$ to individual $i$.*

The following is a version of a result by Svensson (1999) extended from a canonical assignment problem to a fair division problem.

**Theorem 1** *An assignment rule satisfies Efficiency, Neutrality, Separability, Non-bossiness and $\gg$-Strategy-Proofness iff it is a serial dictatorship.*

**Remark 1** In fact it is easy to check that a serial dictatorship is $\gtrapprox$-Strategy-Proof. This implies that for all reasonable extensions of preferences to the power set, individuals will have no incentive to manipulate the system by signalling false preferences. Moreover, this implies that using Proposition 5 we get the following:

**Corollary 1** *An assignment rule satisfies Efficiency, Neutrality, Separability, Non-bossiness and $\gtrapprox$-Strategy-Proofness iff it is a serial dictatorship..*

**Remark 2** The theorem implies that some but not all initial distribution of resources combined with the MTTC would yield assignment rules that are "strategy-proof." It is interesting to note that for these initial distributions (ones that using Proposition 4 correspond to serial dictatorships) the MTTC gives us a very strong equilibrium of the exchange market: an assignment that is not only in the $\gg$-core as with all $Q$s but one which is in the $\gtrapprox$-core of the economy. This would imply that for all reasonable extensions of preferences on $X$ to preferences on the power set of $X$, the assignment arrived at through MTTC exchange would not be blocked for these distinguished initial endowments.***********

## 0.3 Sharing and Fair Division.

Consider the classic contested garment problem from the Talmud (..... references.....):

**Example 5** Two hold a garment; one claims it all and the other claims half of it. What is an equitable division of the garment?

The proportional solution would divide the garment between the two individuals in the proportions 2/3 and 1/3 (i.e., in proportion to their claims). The Talmudic solution (in this case the Shapley Value) would give 3/4 to the individual claiming all of it and 1/4 to the person claiming 1/2. (The justification being that the person claiming 1/2 has already conceded 1/2 and that it is the half in dispute (the contested 1/2) that should be divided between the two individuals). Now, no matter which of these two ways of dividing one favors, by division clearly what is *not* being suggested is that the garment be cut up into these proportions. This would make the garment worthless to both individuals. So what does division or sharing mean in this context? In the presence of indivisibilities there are two non-pecuniary[14] sharing methods that are often used in the real world to smooth out the problem posed by indivisibility. Most frequently the method used is rotation (custody of a child when the parents divorce, soldiers given a break by being rotated in and out of a war theater, etc.). Somewhat less frequently randomization is used (for instance, lotteries are sometimes used to award government subsidized housing in India and for distributing over subscribed IPOs). Wherever the

[14] Sharing not involving monetary transfers.

costs associated with sharing by rotation are small, this method will be the preferred since using this method can lead to being both *ex ante* and *ex post* fairness where as randomization is only an *ex ante* fair sharing method. However, there are cases where rotation is unduly burdensome (sharing houses (other than vacation houses which are often time-shared) or sharing organs available for transplantation) and randomization would be preferred if other methods of assigning priority of one claim over another is not available. The appropriateness of the method of sharing depends clearly depends on the nature of the object itself. Thus, without giving the sharing process either of these two specific interpretations (rotation/probability) we can ask the question what is the appropriate "share" of each object that an individual should get on the basis of his claim and the claims of others and the preference profile so that there is no inequity.

One can think of an assignment as a $k \times k$ boolean matrix $B = (b_{ij})$; $b_{ij} \in \{0, 1\}$; $\sum_i b_{ij} = 1$ and $\sum_j b_{ij} = 1$. The rows will represent the agents *unit* claims with the first $c_1$ rows representing 1's claims for $c_1$ objects, the second $c_2$ rows, 2's claims..., etc. Each row has a single one with all the other entries being zero. The columns represent the objects, with the last $k - m$ columns representing null objects. Each column, also, has just a single one (the rest of the entries being zeros) indicating exactly how and to meet which claim, the object has been assigned.

Thus, we can view an *assignment rules* discussed in the last section as functions that for any sharing problem $\langle c; \succ; k \rangle$ specifies a unique $k \times k$ boolean matrix, $B$.

A *sharing matrix* is a $k \times k$ bi-stochastic matrix $D = (d_{ij})$, $1 \geq d_{ij} \geq 0$, $\sum_j d_{ij} = \sum_i d_{ij} = 1$ (i.e., with non-negative entries each less than or equal to one with each row and column adding up to 1). Here, if $\underline{h}_i = \sum_{j=1}^{i-1} c_j$ then $d_{(\underline{h}_{ii}+1)j}, d_{(\underline{h}_i+2)j}......, d_{c_ij}$ represents the fraction of good $j$ that $i$ receives to meet each of his $c_i$ unit claims.

A *sharing function (*SF*)* is a function that for any sharing problem $\langle c; \succ; k \rangle$ specifies an unique sharing matrix.

A *basis* of a sharing function $f$ is a set of *assignment rules* $f_1, f_2, ....., f_k$ such $f$ is a convex combination of the assignment rules with each rule getting a positive weight (i.e., there exists $0 < \theta_j \leq 1$, $j = 1, 2, ..., k$, $\sum \theta_j = 1$ such that $f = \sum_{j=1}^{k} \theta_j f_j$). Thus, a sharing rule tells us how, given the claims, preferences and objects available, how each good is divided and distributed to meet each claim.

A *sharing rule (SR)* is a function $\gamma$ that for any sharing problem $\langle c; \succ; k\rangle$ gives us a $n \times m$ matrix $A = (a_{ij})$. The number $a_{ij}$ is interpreted as the share of good $j$ that individual $i$ receives.

A sharing rule $\gamma$ is derived from a sharing function $f$ (we will say $f. \Rightarrow \gamma$) , if $f(\langle c; \succ; k\rangle) = (d_{ij})$, $\gamma(\langle c; \succ; k\rangle) = (a_{ij})$ and $a_{ij}$ is obtained by adding the entries in column $j$ in all the rows corresponding to the claims of any individual $i$. (i.e., $a_{ij} = \sum_{p=\underline{h}_i}^{c_i} d_{pj}$ where $\underline{h}_i = \sum_{j=1}^{i-1} c_j$). If $f$ is based on the assignment rules $f_1, f_2, ....., f_k$ we will say the sharing rule $\gamma$ is based on $f_1, f_2, ....., f_k$ .

We will examine sharing rules that are based on the types of $Q$'s that we have discussed in the last section. Our results, so far, have shown that while one can use queueing rules for efficient assignment, these rules are necessarily unfair. There are two types of inequities associated with these rules that we will address: *positional inequity* (the inequity caused by one person being allowed to choose before another person and hence one being able to get a "better" alternative than the other) and *rationing inequity* (the possible unfairness in the sharing of the shortage between claims and objects available when, for instance, we have a few individuals to having large claims which are fully met while many individuals with small claims receive nothing).[15]

## 0.3.1 Positional Inequity and the Proportional Rule

We first turn to *positional inequity*. This problem that would occur with any queueing rule even if there were enough objects to satisfy all the claims. people choosing early have an unfair advantage over those who choose later. One way out is to use a set of assignment rules as a basis of a sharing rule. The disadvantage that one individual may have in one $Q$ in the basis can be cancelled out by giving that individual an advantageous position in another $Q$ in the basis. since the sharing rule is derived from a convex combinations of the assignment rules in the basis, the final shares may then be deemed to be fair.

While defining weights and a basis defines an unique sharing rule, different basis may lead to the same sharing rule. Since each basis and weights can correspond to a different kind of normative justifications, what this means is that often the same rule can be justified in a number of different ways.

[15] Indeed the separability of the last section represents the requirement that these two issues: of "efficiency" and of "rationing" can be (partly) separated out, the latter depending *only* on the claims profile and on the number of objects available.

Our next result exploits this property of sharing rules and provides three different justifications for a sharing rule that, we will argue, is an analogue of the proportional rule.

A fundamental notion of justice is anonymity or "equal treatment of equals", a property that should be satisfied by all fair division rules. In most models, this is interpreted as all individuals being treated symmetrically and is defined as the requirement that when individual preferences are permuted the outcome should be permuted. But, when individuals have different claims this notion of treating equals equally is clearly inappropriate.. One possible interpretation of "equal treatment" is that the sharing rules should treat all *claims* equally (i.e., "equal treatment of equal claims").

To see how this could work, first consider the fact that in the real world very often all claims do not have the same ("equal" ) status. There are often categories of "preferred debt".[16] When an estate is being divided the will itself may specify whose claims get priority. To distinguish the claims that the individuals hold consider the following notation. Let $c_{i1}, c_{i2}......c_{ic_{ii}}$ be individual $i$'s unit claims for $i = 1, 2, ....n$. Now, treating claims *unequally* consists of a prioritizing of claims and can be represented by a linear ordering of the claims. One can think of generating an assignment rule under which each individual exercises their claims in sequence according to the priority ranking of these claims. In other words, in this case what we have is a queue on the set of unit claims. We can use this queue to select the owner of the claim to take his turn in selecting an object. One way of treating all claims *equally* is to have a sharing rule derived from a sharing function given by the equally weighted convex combination of assignments generated by all possible prioritizations of claims. Let us call this sharing rule the Aristotelian Proportional Sharing Rule (A-SRule), $\gamma_A$.

**Theorem 2** *Let $\mathcal{Q}$ be the set allocation rules generated by essentially distinct queues and let $\mathcal{R}$ be the set of assignment rules generated by the MTTC for all possible distinct initial endowments and let $f_{\mathcal{Q}} = \frac{1}{|\mathcal{Q}|}\sum_{h\in\mathcal{Q}} f_h$ and let $f_{\mathcal{R}} = \frac{1}{|\mathcal{R}|}\sum_{h\in\mathcal{R}} f_h$. If $\gamma_{\mathcal{Q}}$ and $\gamma_{\mathcal{R}}$ are such that $f_{\mathcal{Q}} \Rightarrow \gamma_{\mathcal{Q}}$ and $f_{\mathcal{R}} \Rightarrow \gamma_{\mathcal{R}}$ then $\gamma_A = \gamma_{\mathcal{Q}} = \gamma_{\mathcal{R}}$.*

**Proof.** $\gamma_{\mathcal{Q}} = \gamma_{\mathcal{R}}$.follows immediately from Proposition 4.[17] It tells us that

[16] Note that sometimes some claims do have priority : shares , preference shares, bonds, secured debt, for instance.

[17] This extends a result in Abdulkadiroğlu and Sonmez (1998). Clearly, an even stronger

tells us that attaching an equal weights to rules based on every possible distinct $Q$ results in the same shares of alternatives for individuals[18] as attaching equal weights to each possible initial endowment and assigning the weighted average of the $\gg$-cores.

Notice that a linear order on unit claims generates a $Q$ in a natural way by identifying the individuals who own each unit claim. However, different prioritizations may lead to the same $Q$. (If $c_1 \geq 2$, a prioritization in which all of individual 1's unit claims occupy the first $c_1$ priorities leads to the same $Q$ as one in which the priorities of the first two unit claims are interchanged). To see that $\gamma_A = \gamma_Q$ observe that the fraction of the $Q$'s in which individual $i$ gets the $p^{th}$ pick for $p \in \{1, 2, ..., m\}$ is the same as the fraction of the priority rankings of unit claims in which $i$ exercises the $p^{th}$ pick and is given by $\frac{c_i}{k}$.[19] ■

The next example illustrates Theorem 2.

**Example 6** Consider the contested garment problem described in Example 5. The objects are 2 halves let us call them $h_1$and $h_2$. Individual 1 claims two of the halves, let us call his unit claims $c_{11} = \frac{1}{2}$ and $c_{12} = \frac{1}{2}$. Individual 2 claims any one of the two halves, thus, say, $c_{21} = \frac{1}{2}$. We have discussed three ways of proceeding: (i) Attach equal weights to assignment rules that arise out of all possible initial endowments together with "trade". In this case there would be no trade and the three possible assignments are 1 gets both halves, 1 gets $h_1$ and 2 gets $h_2$ and 1 gets $h_2$ and 2 gets $h_1$. The equally weighted average is $(2/3, 1/3)$. (ii) Attach equal weights to all the $Qs$ and choose a equally weighted average of the outcomes from all possible $Q$s. There are three $Q$'s possible: $(1, 2, 1)$; $(2, 1, 1)$ and $(1, 1, 2)$. Giving their outcomes equal weight would also give the $(2/3, 1/3)$ sharing of the garment. (iii) Ranking the claims and assigning according to choices based on the claims, attaching equal weights to every ranking. The six possible rankings $(c_{11}, c_{21}, c_{12})$, $(c_{21}, c_{12}, c_{11})$, $(c_{12}, c_{11,}c_{21})$, $(c_{12}, c_{21}, c_{11})$, $(c_{21}, c_{11,}c_{12})$ and $(c_{11}, c_{12,}c_{21})$. Once again attaching equal weights to the assignments arising out of these six rankings of the claims would lead to a $(2/3, 1/3)$ sharing of the garment.

To see the close relationship between the A-SRule and the proportional

statement is possible..........statement....

[18] The same probability distribution on alternatives, under the randomized sharing interpretation.

[19] $\frac{\frac{c_i(k-1)!}{c_1!c_2!...c_n!(k-m)!}}{\frac{k!}{c_1!c_2!...c_n!(k-m)!}} = \frac{c_i(k-1)!}{k!} = \frac{c_i}{k}$

rule for the homogeneous and divisible goods, consider, particularly, the case where all individuals have identical preferences over the objects. In this case, the share of every good that an individual receives is the same and is proportional to the individuals' claims. (The share of individual $i$ is given by $\frac{c_i}{k}$.) Thus, Theorem 2 can be seen as providing three different justifications (using the core with "equality" of initial endowments, "equality" of claims and equal treatment of all possible $Q$'s) for using this rule.

### 0.3.2 Positional Inequity and Strategy-proofness

We know from our analysis in the last section that all $Q$s are not "strategy-proof".[20] Hence, the A-SRule would be useful only when the preferences of the individuals are either known or can easily be deduced. When preferences are private information, to avoid manipulation, what if we based our sharing rule on the set $\mathcal{S}$ of all possible serial dictatorships? Given our Theorem 1 this would certainly be a reasonable thing to do. Let us define the Shapley type sharing rule (S-SRule) as a sharing rule given by derived from $\gamma_s = \frac{1}{|\mathcal{S}|}\sum_{h\in\mathcal{S}} f_h$. Can this rule be justified by an equity principle?

For any every division problem $\langle c;\succ;k\rangle$, any just sharing rule should surely satisfy the following very mild equity condition.

**Weak Anonymity.** A sharing rule $\gamma$ is **weakly anonymous** iff for any fair division problem $\langle c;\succ;k\rangle$ if all individual preferences, $\succ_i$, and claims, $c_i$, are identical then every object should be equally shared among the individuals.

Recalling that in an unrestricted serial dictatorship there is no external restriction on the *extent*[21] of his claims that an individual can exercise when his turn comes, (*other than that* imposed by $c_i$ and the actual availability of objects) we have the following:

**Proposition 6** *Let $\tau$ be a sharing rule based a set of* unrestricted *serial dictatorships, $\mathcal{S}$. Then, $\tau$ is weakly anonymous iff it is the S-SRule.*

**Example 7** Consider the contested garment problem described in Example 5 . There are two possible serial dictatorships $(1,2)$ and $(2,1)$. In the first

[20] Neither are the other two types of assignment rules that we have suggested as forming the basis of the A-SRule.

[21] Thus, in Example 2 the serial rule (123) is unrestricted if 1 is allowed to pick two objects. We could of course create a restricted serial dictatorship by still allowing 1 to choose first but restricting the number of the objects he can choose to 1.

case the garment is assigned $(1, 0)$ in the second case $(1/2, 1/2)$. The equally weighted average being the shares $(3/4, 1/4)$.

**Example 8** Consider the problem in Example 2 . A sharing matrix for the S-SRule in this case will be given by:

$$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \end{matrix}$$

indicating that individual 1 gets good $x_1$ while good $x_2$ and good $x_3$ are shared equally among all the three individuals. Contrast this with the A-SRule in which while 1 still gets good $x_1$; $x_3$ is shared equally among all the three individuals, but, in this case, 1 gets just $1/6$, with 2 and 3 getting a $5/12$ share each of $x_2$.

In the case of the S-SRule, if all the individuals have identical preference, each individual has the same share of every object, this share being given by the Shapley Value calculated on the basis of the claims. Thus, the share of the top ranked object is the Shapley value of 1 unit being distributed among individuals with claims $c_1, c_2, ..., c_n$; for the second best alternative it is the Shapley Value for 2 units being distributed to the individuals with the claims $c_1, c_2, ..., c_n$. The proportions of the worst (real) object are given by the Shapley value where $m$ units of are being distributed to individuals with the claims $c_1, c_2, ..., c_n$.

### 0.3.3 Rationing Inequity.

Neither the S-SRule nor the A-SRule addresses the question of "fair rationing" which may arise in some problems. Consider the case of the unrestricted serial dictatorship. If one of the individuals has large claim, say a claim ofof all $m$ objects then any one picking after him will get anull object. It can argued that such an *assignment rule i*s unfair and sharing rules should not include such rules in their basis. Both the A-SRule and the S-SRule treat small and large claims "proportionally" and include such assignment rules in their basis. One can think of instances when one may wish to treat large and small claims differently by using sharing rules based on assignment rules in which large claims are restricted. For instance when a bank fails it may be in societies' interest to protect small depositors by paying them in full and dividing the shortfall among larger (and presumably richer) depositors who are more able to bear the loss. On the other hand when all the depositors are large and the losses small, sharing losses equally rather might be thought of being the appropriate norm. Application of these norms of equity in rationing give us versions of the uniform sharing rules and the uniform loss sharing rules (see Moulin (2003)). Both the A-SRule and the S-SRule can be modified using these principles. In each case the process involves changing the claim shares $c$ to claim shares $\overline{c}$ using an equitable rationing norm and then applying the A-SRule (or the S-SRule) to the fair division problem $\langle \overline{c}; \succ; k\rangle$ (rather than to $\langle c; \succ; k\rangle$ as is done in the usual case).

For the uniform principle, the transformation of $c$ to $\overline{c}^U$ is carried out so as to divide the *number* of objects as equally as possible without exceeding any ones legitimate claim. One can think of the process as follows: Let $\widetilde{c}_i = 1$ for all $n$. If $\sum \widetilde{c}_i = n \geqq m$, the process stops and everyone's claim has a value of one ($\overline{c}_i = 1$ for all $n$). If not, for all individuals such that $c_j = 1$, set $\widetilde{c}_j = \overline{c}_j = 1$. For the remaining individuals let $\widetilde{c}_i = 2$. If $\sum \widetilde{c}_i \geqq m$, then process stops and everyone's claim has equal to $\min\{c_i, 2\}$. If not, $\widetilde{c}_i = \min\{c_i, 3\}$ and the process is carried on until for some $q$, $\sum_i \min\{c_i, q\} \geqq m$. Thus, letting $\phi(q) = \sum_i \min\{c_i, q\}$ where $q$ is a positive integer and $q^U$ be a value of $q$ such that $\phi(q^U) \geqq m$ and $\phi(q^U - 1) < m$. Then, $\overline{c}_i^U = \min\{c_i, q^U\}$

The uniform loss principle tries to share the shortage as equally as possible with the restriction that no ones claim becomes negative. The process is analogous to the uniform principle. First, we reduce every ones claim by 1 that is $\widetilde{c}_i = c_i - 1$. If $\sum \widetilde{c}_i \leq m$, then $\overline{c}_i = c_i$; if not consider $\widetilde{c}_i = \max\{c_i - 2; 0\}$ and check to see if $\sum \widetilde{c}_i \leq m$. If this is so then $\overline{c}_i = \min\{c_i - 1, 0\}$. If not we go to the next step, where claims are reduced "one more unit." More precisely, let $\xi(q) = \sum_i \max\{c_i - q, 0\}$ and define $q^u$ as the value of $q$ such that $\xi(q^u + 1) \leq m$ and $\xi(q^u) > m$. Then, $\overline{c}_i^u = \max\{c_i - q^u, 0\}$.

Now, we can define the uniform proportional (A$^U$-SRule), the uniform loss proportional rule (A$^u$-SRule), the uniform Shapley Type (S$^U$-SRule) and the uniform loss Shapley Type (S$^u$-SRule), by changing the claim shares $c$ to claim shares $\overline{c}$ using an equitable rationing norm and then applying the A-SRule (or the S-SRule) to the fair division problem $\langle \overline{c}; \succ; k\rangle$ to obtain uniform and uniform loss versions of these rules

****** ....We need to analyze these rules

**Proposition 7** *(i) If the claim of every individual is large relative to the total* *of objects available in that $\overline{c}_i^U = \min\{c_i, q^U\} = q^U$ for all $i$, then $A^U$-SRule $= S^U$-SRule and gives every individual equal shares in every good. (ii) If the shortage of goods is small in that $\sum_i (c_i - 1) < m$, then $A^U$-SRule=A-SRule and the $S^U$-SRule= S-SRule.*

....*****

### 0.3.4 $Q$ Based Sharing Rules on the Full Domain

While almost all matching models use the preference framework that we have adopted in this paper requiring individuals to have strict preferences on objects, there have been two papers (Svensson ,....) showing that queuing methods can be extended to the full domain of preferences allowing for individual indifference between objects. While the model itself becomes notationally much more complex and parallel results become technically difficult to derive, a description of the extended queueing rules themselves are quite transparent and can be easily described. In this case, the queuing method works as follows: the first person in the queue instead of picking a single objects indicates a group of her best objects. The second person in the queue selects her best object(s) subject to the first person getting one of her best, the third person indicates what her best would be subject to the first and second receiving one of their selections and so on. Clearly, the A-SRule and S-SRule that we have described can easily be extended along these lines. We have claimed that the A-SRule and the S-SRule are natural extensions of the Proportional Rule and the Shapley Value from the homogeneous case. To reinforce this argument notice that our extensions of these rules to the full domain of preference allowing for individual indifference among object allows us to have a nested model in which the standard fair division problem with homogeneous goods is a special case. We started the paper by discussing the fair division problem in which \$8 was available for A and B with claims \$7 and \$3. We can interpret this problem in our framework as there being 8 identical objects $c_1 = 7$ $c_2 = 3$ and both individuals having identical preferences under which all objects are equally desirable. S-SRule would work as follows. There are two serial dictatorships $(A, B)$ and $(B, A)$. Applying the queuing methods as extended to the full domain, we can see the money would be divided $(7, 1)$ for the serial dictatorship $(A, B)$ and $(5, 3)$ under the serial dictatorship $(B, A)$. The equally weighted average of these two cases would give us the $(6, 2)$ Shapley division. Arguing similarly, using all possible rankings of the 10 claims ($\{c_{11}, c_{12}, ..., c_{17}, c_{21,}c_{22}, c_{33}\}$) it is possible to demonstrate that applying the A-SRule will give us the proportional division of \$5.60 ( 70 percent) to A and \$2.40 (30 percent). Under uniform division the claims of the individuals would be $\overline{c}_1 = 5$, $\overline{c}_2 = 3$ and under uniform loss division, $\overline{c}_1 = 6$ and $\overline{c}_2 = 2$. The usual interpretation in the literature of the "uniform" rules would correspond $(5, 3)$ and the $(6, 2)$ division of the \$8 a result that we would get with both the uniform versions of both the A-SRule and the S-SRule applied to this case.

*In general, in any model dividing multiple units of a "homogeneous" good our extensions of the A-SRule and S-SRule will give us the proportional and Shapley Value methods of sharing. Notice in particular for this result, divisibility plays no role except in the interpretation of the division. If the goods are divisible then the shares represent physical division, otherwise, they can be interpreted as we have been doing in terms of rotation or in terms of lotteries. If the good is sufficiently divisible, the uniform (resp. the uniform loss) versions of the A-SRule and the S-SRule will coincide and give the same result as we get for the uniform and uniform loss rules in the traditional fair division literature for homogeneous and divisible goods. For the case of homogeneous goods, since preference revelation is not an issue, the proportional solution is preferred since unlike the Shapley rule, it treats all claims equally.*

## 0.4 Conclusion.

In this paper we extend existing solutions of matching models for efficiently assigning single objects to every individual to a model in which each individual is matched to possibly many objects. Both the older market based approach and the newer queue based approach are explored. Two main results in the literature one relating the equilibria ("cores") of markets to queueing rules ( Reference....) and a characterization result for serial dictatorships (reference....) are established for our extended framework. We demonstrate a close relationship between standard sharing solutions, such as the proportional solution and the Shapley Value, to solutions based on queueing methods. We show that versions of these sharing rule may be viewed as convex combinations of particular queueing rules. In the special case when the goods being shared are homogeneous, the standard proportional solution and the Shapley solution emerge as a special cases of our extended versions of these sharing rules. This shows that the underlying norms of these rules do not depend on homogeneity, divisibility or cardinality. We can conclude that the principles underlying the proportional and Shapley solution can be applied to a much wider class of sharing problems involving the sharing of indivisible and heterogenous objects as long as we interpret "division" appropriately as sharing either through rotation or randomization. The proportional sharing method is a more just solution as it treats all claims equally but it may not be implementable if preferences are private information. In this case the Shapley type solution is preferable. In presence of a shortage, both rules can be modified to make rationing more equitable by modifying these rules using principles such as the uniform principle or the uniform loss principle.