DYNAMIC MODEL OF THE INDIVIDUAL CONSUMER

Craig B. McLaren
University Of California, Riverside

Abstract

This paper presents an alternative formulation of consumer theory that allows consumer behavior to be modeled as a dynamic process. Rather than simply predicting the optimal choices a consumer will make, this formulation provides a time dependent process by which the consumer arrives at equilibrium with the market and maintains stability with it. This formulation is built upon multivariate integral (vector) calculus and is formally analogous to the theory of electric fields in classical physics. This approach allows the consumer's Marginal Rates of Substitution (MRS) to be accepted as a theoretical given, rather than derived from hypothetical quantities such as utility or preference.

Using a basic set of axioms, a vector function giving the consumer's (observable) Marginal Values is defined from his (her) MRS. Using an additional axiom regarding the reciprocity of substitute and/or complementary goods, a scalar Use Value function is defined as the integral of the Consumer's Marginal Values using Stokes' Theorem. While functionally equivalent to utility, the consumer's Use Value is measurable and unique to constants of integration that correspond to observable quantities.

With an additional assumption that guarantees convexity of Use Value's isotimic surfaces, the formulation developed here is used to solve the traditional consumer choice problem. It is shown that, whenever the consumer holds a bundle of goods that is not his or her "optimal" one, the consumer will undergo a tatonnement–like process consisting of a series of incremental exchanges with the market until her optimal bundle is obtained.

Key Words

Dynamic Consumer Theory, Integrability, Convex Indifference Surface, Engle's Law, Antonelli Conditions, Marginal Demand, Willingness to Pay, Contingent Valuation, Vector Analysis, general equilibrium, existence, stability, tatonnement

JEL Classification Codes

B21, B41, C50, C60, D01, D11, D50

1) Introduction

This paper will express our intuitive understanding of consumer behavior in a mathematical formalism somewhat reminiscent of Newtonian Mechanics. Among other things, this will allow the consumer's actions to be modeled as dynamically evolving through time in a world that is not necessarily always in equilibrium. The notion of physical equilibrium is contained in Newton's First Law, stating that a body at rest tends to stay at rest. Such "rest" occurs when the forces on a body are balanced and hence sum to zero. This is analogous to a balancing of the incentives acting on individuals, providing no reason for any actor to change his behavior. Newton's First Law however is a special case of his Second Law, which equates the time rate of change of a body's behavior to the net (i.e. the disequilibrium of) forces acting upon it. There is no reason to expect that economic agents would not respond dynamically to a disequilibrium resulting from a sudden change in economic conditions. It is entirely reasonable to expect that a significant time period may be needed for equilibrium to be reestablished, and that the sequence of events during the transient period may determine the outcome. As Walras observed, an economy may be like the surface of a lake, over which a wind blows, always tending towards one equilibrium or another without ever reaching one.

The mathematical approach proposed here will be an alternative to the Lagrange approach common to economic analysis, much as it was an alternative approach to the Newtonian method of solving problems in physics. Each approach has a comparative advantage in addressing different kinds of problems. While the Lagrange method widened the scope of physics problems that could be solved, it did so at the cost of understanding the dynamics of how such a solution might be reached.

This paper will develop a basic model of a consumer who responds to his environment through a sequence of differentially small transactions. Each response results from the value gained in previous transactions, his or her preferences, and external conditions that may be changing. The consumer's preferences are represented by his or her Marginal Rates of Substitution (MRS) which are a function the bundle acquired through previous transactions. In essence, this model assumes that *for any given bundle of goods the consumer might hold, s/he knows how much of any one good s/he would trade for one more unit of any other good.*

While the mathematics used in the analysis may be unfamiliar, the quantitative ideas analyzed will not be, Since they will viewed from a different perspective, a deeper understanding of their intuitive meaning becomes available. By founding the model on the consumer's MRS, it will be possible to eliminate any need to reference an unobservable quantity such as utility, replacing it with Aristotle's notion of *Use Value*. Rather than the "benefit" a consumer derives from consuming a commodity, the analysis will be conducted in terms of the value the consumer *places on* the commodity, measured in terms of a standard numeraire. Care of course must be taken to insure that there is no remaining question of what Marshall referred to as the "marginal utility of money" as will be discussed shortly.

By starting with the consumer's MRS, I cannot assume a-priori that there is a utility function of which the MRS are a perfect differential. The existence of such a function will be proven from more primitive assumptions, while giving nod to the historic literature on the problem of

Integrability. Since the MRS are *a vector* of quantities, *vector analysis* is the appropriate (and most accessible) mathematical tool. From the usual assumptions of basic rationality, and the existence of a numeraire with certain properties, a vector function or *field* is defined, which indicating the *marginal values* a consumer places on goods, given the quantities of them he currently holds. Such fields such as gravity and the electric force are common in physics, rendering their analysis familiar to undergraduate students. In cost – benefit analysis, such marginal values correspond to the marginal prices a consumer would be willing to pay for additional units of the goods in question.

With no more than the currently unwritten assumption that the complementary (or substitutionary) effects between goods be mutual, I will show from *Stokes' Theorem* that the Marginal Value function must form a compete differential, and that a *Use Value* function can be defined in terms of its integral. Use Value is functionally the same as a utility though it is devoid of nebulous interpretations regarding "happiness". In terms of numeraire, the Use value function defines the value the consumer places on a finite bundle of goods, measured with respect to the value placed on some chosen reference bundle. As result, Use value can be measured cardinally, and is unique with constants of integration representing observable parameters. The Use Value function is also analogous to gravitational and electrical potential energy. The equal-potential surfaces (or curves) that are commonly used graphically to represent these functions are formally the same as indifference curves in economics.

While the assumption of convexity will play much the same role her is it does in current analysis, the dynamic approach will make its intuition clearer, and its mathematical expression easier to use. Intuitively, it means no more than that there is no good or *combination of goods* for which the consumer's willingness to pay increases with his or her consumption of them. Except possibly over a limited range of consumption, it is clear that a consumer responding so to certain goods would be considered obsessed or addicted to them, an unable to function in the market in a desirable way. Goods to which an individual is likely to respond in such a way are often regarded as "vices" and banned from the marketplace. As a byproduct, the model proposed here will make it clear that convexity is not merely a convenient assumption, but a necessity that societies build institutions to maintain.

2) Some Notes on Vector Analysis

My intention is to provide an article that is of use to as wide an audience as possible. As result I have endeavored to keep the mathematics as intuitive as possible. I am quite aware that practitioners in different areas of economics may have different notions of what the term "vector" might mean. For the sake of clarity, I have used the vector concept and notation readily found in elementary physics textbooks.

Vectors are *not* to be understood as linear arrays of unrelated objects. A vector is a *type of number* having properties physically interpretable as *magnitude* and *direction*, which can be expressed equivalently with respect to any coordinate system. While some physical quantities such as mass and temperature can be expressed as *scalars* (numbers having only magnitude) quantities such as velocity and momentum must be expressed as vectors. If two vehicles collide, the force of impact will depend on their relative directions of travel as well as the magnitude of their speeds.

We will find it convenient to transform vectors between coordinate systems. I thus emphasize that it is only with respect to *a given coordinate system*, that a vector can be expressed as an array of elements. Such an array is shorthand for a vector sum of components, where each component is the scalar product of the vector in question and a basis vector of unit length that defines a coordinate axis. For the familiar three-dimensional Cartesian system with axes labeled x-y-z, the basis vectors are be denoted: $\hat{\varphi}_x, \hat{\varphi}_y, \hat{\varphi}_z$ respectively as shown in Figure 2-1.

For a given vector $\vec{A}$ its three respective components are:

$$\begin{matrix} a_x\hat{\varphi}_x & & a_x = \vec{A} \bullet \hat{\varphi}_x \\ a_y\hat{\varphi}_y & \text{where} & a_y = \vec{A} \bullet \hat{\varphi}_y \\ a_z\hat{\varphi}_z & & a_z = \vec{A} \bullet \hat{\varphi}_z \end{matrix} \qquad (2\text{-}1)$$

We can write the vector as the sum of its components i.e.:

$$\vec{A} = \quad \vec{A}_x + \vec{A}_y + \vec{A}_z = \quad a_x\hat{\varphi}_x + a_y\hat{\varphi}_y + a_z\hat{\varphi}_z \qquad (2\text{-}2)$$

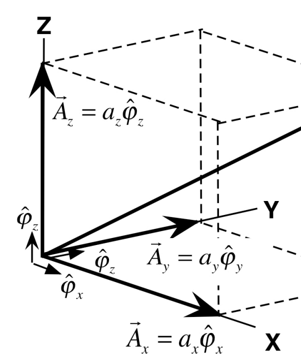

Figure 2-1. A Vector as a sum of components

Vector *functions* "map" vectors to points in space, and are commonly used to describe motion of extended, elastic bodies such as fluids. The motion of particles suspended within a fluid vary with their position within the fluid. The velocity of a particle suspended in a stream of water will be a function of its position relative to the riverbank. Particles closer to the shore will move more slowly and with a trajectory that follows curves in the riverbank, while particles near the center will move faster and in more of a straight line. Figure 2-2 illustrates of a vector field showing the velocity of exhaust gas as it escapes from an automotive tailpipe.

With respect to a given coordinate system, vector functions are expressed in component form, where the coefficients are all scalar functions of the same argument. A velocity function

$\vec{v}(x,y,z)$ used to describe the velocity of a particle suspended at a point x,y,z can be expressed in component form as $\vec{v}(x,y,z) = v_x(x,y,z)\hat{\varphi}_x + v_y(x,y,z)\hat{\varphi}_y + v_z(x,y,z)\hat{\varphi}_z$.

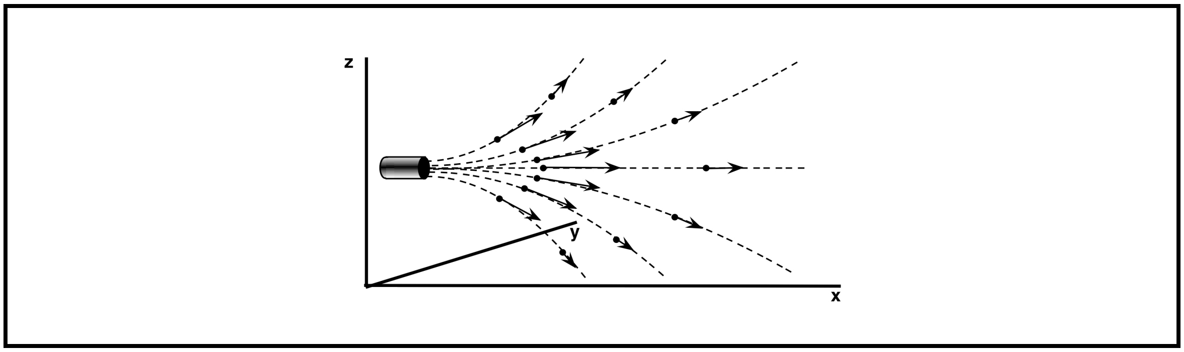

Figure 2-2. Vector field depicting the velocity of gas escaping from a pipe.

3) Defining The Marginal Value Function

In this section I will begin building the model of the individual consumer. I begin by defining the consumer's Marginal Value function (alternately called his/her Marginal Price function) from his or her MRS, using the assumptions which I will state below. The first two assumptions entail consumer "rationality", and introduce notation. The first assumption is that the MRS "exist", I.e. that for any bundle of goods the consumer might posses, s/he knows how much of any one good s/he would exchange for one more unit of any other.

> **<u>Assumption 1: (Existence of the MRS)</u>**
> Given an economy with *n+1* commodities, x_i where $i \in \{1,2,\ldots n+1\}$. For any bundle of commodities $(x_1, x_2, \ldots x_{n+1})$, and any pair of commodities x_j and x_k within that bundle, the consumer's MRS_{j-k} (defined by Equation 3-1 below) exists with non-negative real values.

$$MRS_{i-k}(x_1, x_2, \ldots x_{n+1}) \triangleq \frac{dx_i}{dx_k} \tag{3-1}$$

Even though this assumption is quite obvious, it eliminates study of a myriad of pathological preference orderings that appear in the literature for which the MRS do not exist[1]. I argue that such orderings would represent a customer that is "confused" as to the rates at which he would

[1] See Scarf (1960) and Ingrao and Israel (1990) pp.138-40. Such pathological orderings include lexiographic or Leontief orderings, perfect compliments and the like. Scarf's examples of general equalibria which were not globally stable was based on agents for whom goods were perfect compliments. I argue that such orderings cannot represent a "rational" consumer since one would be foolish to purchase one of such goods without considering purchase of the others. Such goods are therefore sold in sets (a pair of shoes), We therefore define a set of perfect compliments as a single commodity. Elements of sets of perfect compliments may be sold separately as replacement parts. In such case though the consumer is deciding between purchasing the replacement part (and "fixing" the set that he has), or replacing the entire set. In this case the consumer does have an MRS since the "replacement part" and the "new set" are not perfect compliments.

exchange certain goods, and would thus be unable to participate in market transactions. The exception of course is the case of perfect compliments. Since a rational consumer would not consider the purchase of one perfect complement without the other(s), I argue that sets of such goods should be regarded as a single commodity.

Assumption (2), states that the exchanges a consumer is willing to make must be logically consistent:

Assumption (2) Transitivity of the MRS

For a given economy with *n+1* commodities, and for each set of commodities x_h, x_i, x_k,, where $h,i,k \in (1,2,\ldots n+1)$, the following must hold:

$$MRS_{h-k} = MRS_{h-i} \bullet MRS_{i-k} \quad (3\text{-}2)$$

Using analyses that employ a utility function, recall that for an economy with n goods, there will be only $n\text{-}1$ unique marginal rates of substitution that can be observed. By identifying one good used primary as money, we can eliminate it from the analysis by stating the MRS of the remaining goods in terms of it. This formally defining it as the standard by which value is measured. In so doing, we tacitly assume the "marginal utility of money" to be constant, a practice that has not always been accepted in the literature.

In the standard analysis, if we were to describe the marginal "benefit" the consumer derives from consuming some good x_i, we could write:

$$\frac{dU}{dx_i} = \frac{dU}{dM}\frac{dM}{dx_i} \quad (3\text{-}3)$$

where dM/dx_i represents the consumer's MRS for good x_i, in terms of numeraire M. To equate the benefit derived to the price the consumer is willing to pay, It must be arguable that $dU/dM \equiv 1$. By guaranteeing that M is generally used for money, any change in its marginal benefit to the consumer would indicate a change in the marginal benefit she derives from consumption in general. Marshall bases his argument that the marginal utility of money diminishes on the observation that a wealthier person is more likely to spend a marginal schilling on a luxury good such as a cab ride to work. This however is simply an illustration of Engle's law, which implies that the poorer man will spend his additional schilling on basic goods for which his want is less satisfied. If we regard savings and leisure as "goods" we find that any change in the consumer's behavior resulting from a change in his wealth, can be explained in terms of a change in the goods in his bundle (excluding the numeraire). When used purely as money, discussion of the constancy of the marginal utility of the numeraire provides no additional insight into the consumer's behavior.

The above argument reflects observed social behavior. If one considers economies existing even before the invention of currency, there were still goods, such as precious gems and metals, that were used for money and little else. This is likely why pretty, but otherwise useless, pieces of rock and metal are prized so highly. This reasoning is formalized in the following assumption:

Assumption (3) Existence of a standard numeraire commodity (Money)

Given an economy of $n+1$ commodities, there exists one numeraire commodity M that consumers use solely as a medium of exchange, a store of value, and/or as a unit of account.

In the following analysis, only the unit of account property will be used in practice. Using Assumptions (1) through (3), we can define the consumer's marginal price for a single good.

Definition: Marginal Value (of the i^{th} good)

For an economy with n goods $(x_1, x_2, \ldots x_n)$ and numeraire M, the consumer's *marginal value* for good x_i in term of M is a scalar function $r_i(x_1, x_2, \ldots x_n)$ of the goods the consumer holds. The marginal value the consumer places on x_i is the maximum quantity of numeraire s/he would be willing to exchange for an additional unit of it i.e.:

$$r_i(x_1, x_2, \ldots x_n) = \frac{dM}{dx_i} \tag{3-4}$$

Equation (3-4) defines a set of n functions, which contain all the information that can be observed with regard the consumer's choice behavior. Before defining the Marginal Value (vector) function, I will provide a formal definition of the commodity vector space, if for no other reason than to clarify the notation.

Definition: Commodity Vector Space

For an economy with n goods $(x_1, x_2, \ldots x_n)$ and numeraire M, The Commodity Vector Space is the Positive orthant of the real rectilinear space $\Re^{n+}$ spanned by the mutual orthogonal basis vectors $\hat{\varphi}_i$, (where $i = 1, 2, \ldots n$). where $\hat{\varphi}_i$ represents the i^{th} commodity with all its defining attributes. The distance between any two points A and B is defined to be:

$$\left|\overline{AB}\right| = \sum_i \left|B_i - A_i\right| \tag{3-5}$$

where $\overrightarrow{AB} = \sum_i \left(B_i - A_i\right)\hat{\varphi}_i$ is the vector from A to B.

The rectilinear space differs from the familiar Euclidian space in that the path between any two points is made up of segments that are parallel to the coordinate axes. The space can be envisioned as gridded like a city street map. Paths between points are made up of segments parallel to one of the coordinate axes. The reason for defining the space in such a way is clear when one considers the magnitude of a vector $\vec{x}$ representing a bundle of goods. Intuitively, clear that the magnitude would represent the sum of the goods in the bundle. The square root of the sum of their squares would have little meaning

Definition: Marginal Value

For a consumer possessing a bundle of n goods $x_1, x_2 \ldots x_n = \vec{x}$, and marginal value $r_i(x_1, x_2, \ldots x_n) = r_i(\vec{x})$ for each commodity x_i, the consumer's marginal value function $\vec{r}(\vec{x})$ is defined by:

$$\vec{r}(\vec{x}) = r_1(\vec{x})\hat{\varphi}_1 + r_2(\vec{x})\hat{\varphi}_2 + \cdots + r_n(\vec{x})\hat{\varphi}_n \qquad (3\text{-}5)$$

4) Use Value and Integrability

As mentioned earlier, the use value the consumer places on a quantity of goods will be defined as the integral of his marginal values, taken over them as they are acquired incrementally. This of course raises the rather thorny historical issue of I*ntegrability*, which has appeared in the literature from the time it was first raised by Pareto until it was laid to rest by Samuelson and Howthakker a half century later. Before delving into the conditions that must be satisfied in order for the integral of marginal values to "exist", let us first consider what the integral of marginal values might mean economically.

Consider an economy containing two goods x_1 and x_2, and a consumer who acquires them through a series of incremental transactions. The consumer begins at time t^0 with some bundle $\vec{x}[t^0] = \vec{x}^0 = x_1^0\hat{\phi}_1 + x_2^0\hat{\phi}_2$ as shown in Figure 4-1. The consumer receives increments of wealth w in the form of a stream of income $I[t]dt = dw[t]$ with which she purchases a series of incremental bundles $d\vec{x}[t] = dx_1[t]\hat{\varphi}_1 + dx_2[t]\hat{\varphi}_2$. The ratio of goods $x_1[t]/x_2[t]$ purchased in each transaction will depend in part on their current relative market prices $\vec{p}[t] = p_1[t]\hat{\varphi}_1 + p_2[t]\hat{\varphi}_2$ which we presume are not constant. The price variations lead the consumer along a *consumption path* shown as *Path A* in Figure 4-1.

Consider the incremental purchase $d\vec{x}[t^1]$ she makes just after the sum of her prior acquisitions total some intermediate bundle $\vec{x}[t^1]$. The marginal values $\vec{r}(\vec{x}[t^1])$ she places on the goods within $d\vec{x}[t^1]$ is based on her willingness to pay for them, given her holding of $\vec{x}[t^1]$. The increment of use value $dV[t^1]$ she places on $d\vec{x}[t^1]$ is just: $r_1(\vec{x}[t^1])dx_1 + r_2(\vec{x}[t^1])dx_2 = \vec{r}(\vec{x}[t^1]) \bullet d\vec{x}$. If The consumer continues acquiring goods until she holds bundle $\vec{x}[t^*] = \vec{x}^*$, the value she will place on all goods acquired along *Path A* from $\vec{x}^0$ to $\vec{x}^*$ will be the integral:

$$\int_{\vec{x}^0 \quad Path\,A}^{\vec{x}^*} \vec{r}(\vec{x}) \bullet d\vec{x} \qquad (4\text{-}1)$$

Now let the consumer repeat the process, this time acquiring her goods with market prices $\vec{p}[t]$ varying so as to lead her along *Path B* to $\vec{x}^*$ as shown in Figure 4-1. Consider the incremental purchase she makes at time t^2. The bundle $\vec{x}[t^2]$ will generally not equal $\vec{x}[t^1]$ in the previous example, nor will prices $\vec{p}[t^2] = \vec{p}[t^1]$. As result, the mix of goods in $d\vec{x}[t^2]$ will generally be different from what she acquired at t^1, hence $dV[t^2]$ will not necessarily equal

$dV[t^1]$[2]. From the information we have so far, there is no reason to believe that the integral (Expression 4-2) indicating the value gained in the second example will equal the value given in the first.

$$\int_{\vec{x}^0}^{\vec{x}^*} {}_{Path\ B}\, \vec{r}(\vec{x}) \bullet d\vec{x} \qquad (4\text{-}2)$$

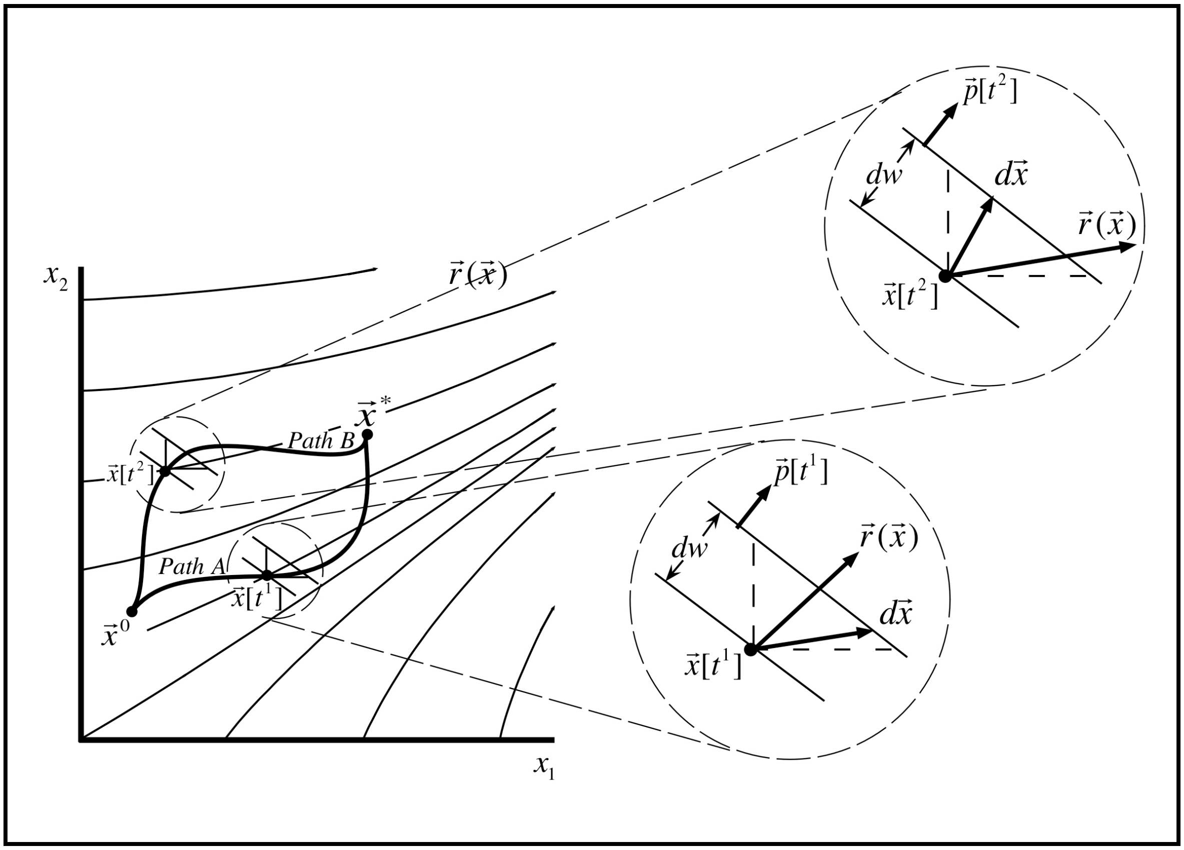

Figure 4-1

The only difference between these two exercises is that the consumer has acquired her goods in a different order[3]. It was Pareto's speculation into the possible significance of such order of consumption that initially raised the so called problem of Integrability[4]. To resolve the problem, we start by running the clock backwards for the second example. The consumer begins with $\vec{x}^*$ and *un*-acquires goods along Path B until she ends with $\vec{x}^0$. Essentially, she incrementally sells her goods back to the market at the prices she initially bought them along the path. Unless her tastes (represented by her marginal value function) have changed, we would intuitively expect that the use value she places on bundle $\vec{x}^0$ would be the same as it had been at time t^0 when she started out with it. We would therefore expect the mathematics to show no net gain or

[2] There is no reason to believe that at any time t, dv[t]=dw[t]. Prices p[t] may change fast enough to produce "corner solutions" for some incremental transactions. Hence the consumer may realize a surplus from any given transaction.
[3] Pareto (1971)
[4] Samuelson (1950)

loss in use value, were the consumer to acquire her goods along Path A, then un-acquire them along Path B, i.e. [5]:

$$\int_{\vec{x}^0}^{\vec{x}^*} {}_{Path\ A}\, \vec{r}(\vec{x}) \bullet d\vec{x} + \int_{\vec{x}^*}^{\vec{x}^0} {}_{Path\ B}\, \vec{r}(\vec{x}) \bullet d\vec{x} \quad = \oint \vec{r}(\vec{x}) \bullet d\vec{x} = 0 \tag{4-4}$$

If Equation (4-4) holds, then it is apparent that the use value $V(\vec{x}^* - \vec{x}^0)$ is truly a function of the goods themselves and not of the order by which they were acquired. This is essentially what Samuelson proved in his 1950 essay on the problem of Integrability[6]. It would be intuitively reasonable to assume such from the outset, concluding that the integral's value independent of the path taken. There is however, more compelling reasoning available. According to *Stokes' Theorem*, a central result of vector analysis, the following three statements are equivalent:

$$\oint \vec{r}(\vec{x}) \bullet d\vec{x} = 0 \quad \Leftrightarrow \quad \vec{r}(\vec{x}) = \nabla V(\vec{x}) \quad \Leftrightarrow \quad \frac{\partial r_i(\vec{x})}{\partial x_k} = \frac{\partial r_k(\vec{x})}{\partial x_i} \forall_{i,k} \tag{4-5}$$

The left and middle statements confirm that the order in which the goods are acquired is irrelevant as long as $r(\vec{x})$ is a complete differential of some scalar function $V(\vec{x})$. The right hand equation is equivalent to the so-called *Antonelli conditions* for the Integrability of demand functions[7]. If any one of the statements given in Equations (4-4) can be assumed, the remaining two will follow as conclusions. It is the third statement that is intuitively the most compelling. The term $\partial r_i(\vec{x})/\partial x_k$ reflects the degree to which a consumer's holding of some good x_k impacts his willingness to acquire an additional quantity of some other good x_i. If $\partial r_i(\vec{x})/\partial x_k$ is positive, x_k complements x_i, if $\partial r_i(\vec{x})/\partial x_k$ is negative, x_k is a substitute for x_i. For the sake of clarity, we will need a more explicit definition of complementarity than what is implicit in the definition of cross price elasticity. That definition is given here.:

> **Definition: Complementarily[8]**
> For a given consumer with marginal value function $\vec{r}(\vec{x})$ holding bundle $\vec{x} = (x_1, x_2, \ldots x_n)$, the complementary effect of her possession of good x_k on the marginal value $r_i(\vec{x})$ she would pay for another good x_i is defined to be: $\partial r_i(\vec{x})/\partial x_k$.

Given this definition of complementarity, it is apparent that the Antonelli conditions state that complementary (or substitutionary) effect between goods must be mutual. Intuitively we would expect that if x_k is a substitute for x_i then the reverse must be true as well. We can now state the right hand equation of Equations 4-5 formally as an assumption.

> **Assumption (4) [Mutual Complementarily][9]**
> For a consumer with marginal value function $\vec{r}(\vec{x})$ holding bundle $\vec{x} = (x_1, x_2, \ldots x_n)$,the complementary effect of his possession of any good x_i on the marginal value $r_k(\vec{x})$ he

[5] The circle symbol on the last integral on the right side of Equation 4-1 represents integration around a closed path.
[6] Samuelson (1950)
[7] Antonelli (1971)
[8] This refers to net-complementarity.
[9] Eugen Slutsky recognized this as a testable hypothesis that must be true if demand functions were integrable See Samuelson (1950) p.357

would place on another good x_k is equal to the complementary effect of his possession of good x_k on the marginal value $r_i(\vec{x})$ he places on good x_i. Thus:

$$\frac{\partial r_i(\vec{x})}{\partial x_k} = \frac{\partial r_k(\vec{x})}{\partial x_i} \forall_{i,k} \tag{4-6}$$

If we are able to assume Equation 4-6, we know that $\vec{r}(\vec{x})$ is a complete differential of a scalar function of $V(\vec{x})$ which we can now define:

Definition: Use-Value

Given a consumer with marginal values given by $\vec{r}(\vec{x})$, for which Assumption (4) is satisfied. The Use-Value a consumer places on a bundle of goods $\vec{x}'$, measured with respect to the value she places on some other bundle $\vec{x}^0$ is defined to be:

$$V\left(\vec{x}' - \vec{x}^0\right) = \int_{\vec{x}^0}^{\vec{x}'} \vec{r}(\vec{x}) \bullet d\vec{x} \tag{4-7}$$

where integral is evaluated over any path between $\vec{x}^0$ and $\vec{x}'$.

Defining the value the consumer places on a bundle $\vec{x}'$ with respect to a reference bundle $\vec{x}^0$ has empirical advantages as that in practice, identifying a consumer who has no goods at all would be difficult to do. Measurements can thus be made with respect to a minimum, or subsistence reference bundle of the analyst's choosing.

The locus of points for which $V(\vec{x}' - \vec{x}^0)$ equals some constant is an *iso-value* (i.e. an indifference) curve or surface. With a little reasoning, it is easy to see that the shape of the indifference curves is independent of the choice of reference bundle $\vec{x}^0$. What depends on $\vec{x}^0$ is the constant value $V(\vec{x} - \vec{x}^0)$ along the curve.

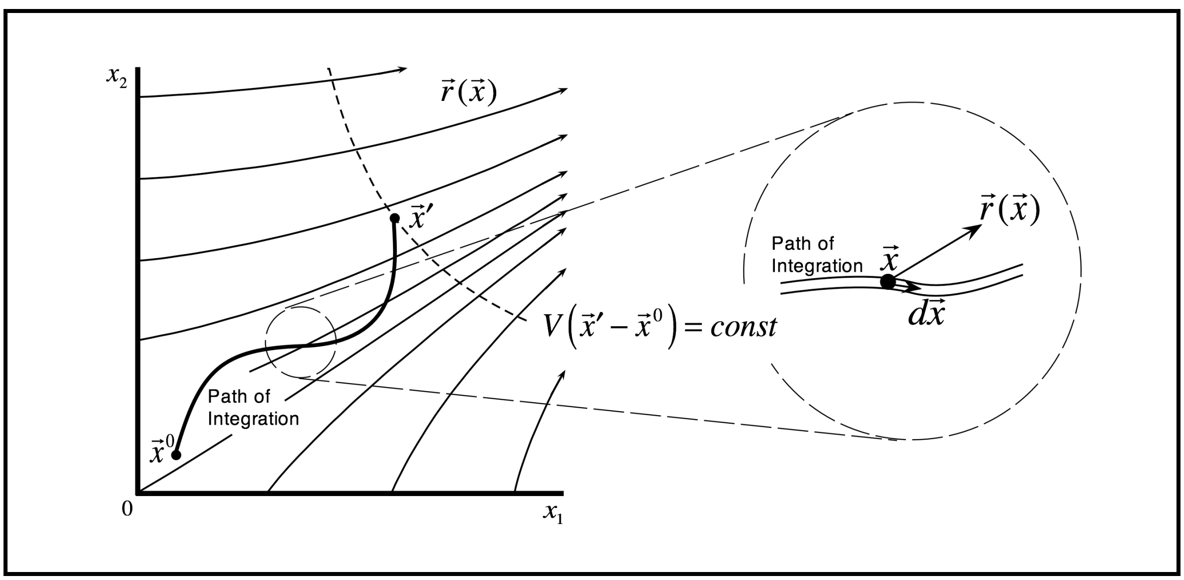

Figure 4-2

Depending on the analysis, it may be convenient to represent the consumer's characteristics with a network diagram showing both his marginal values and his indifference curves as given in Figure 4-3

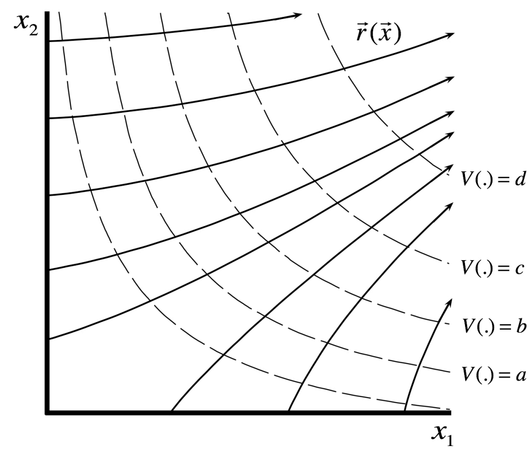

Figure 4-3

5) Convexity and the Assumption of Non Addiction

The assumption presented in this section will guarantee that the indifference curves of the use value function be convex to the origin. The assumption will be expressed in terms of the consumer's marginal values, and its interpretation explored from a psychological and social perspective to see it if it is justified. As we will see, such an assumption is merely an extension of Jevons' law of diminishing marginal utility to include *linear combinations* of goods. The role the assumption plays in the workings of markets though may explain why most cultures restrict the presence of goods for which consumers may become addicted, i.e. goods for which the more they consume, the greater effort they will expend to acquire more.

To insure a unique solution to the consumer problem we require that for every possible set of positive prices, the budget plane must make contact with one of the consumer's indifference surfaces at exactly one point[10]. This condition will be met if the indifference surfaces of the consumer's use value function are *convex*. Geometrically, convexity is assured if, for every pair of points A and B on an indifference surface, the chord joining them lies entirely interior to the surface. Such surfaces of course appear as rounded, thought not necessarily symmetric bowls, with their bottoms oriented towards the origin.

Figure 5-1 illustrates how this description of convexity can be translated into the language of vectors. Consider a convex surface, a slice of which is represented as the curve joining points A and B. For the sake of generality, we allow the surfaces' radii of curvature to be different in different directions as well as non-constant as one moves from point to point on the surface.

[10] If we allow the surfaces to be "quasi" convex (to have flat spots), the plane will touch the surface over the entire flat region, assuming the plane is oriented parallel to the flat region.

Additionally we allow the surface to twist as one proceeds from A to B. Vectors $\vec{N}_A$ and $\vec{N}_B$ are of arbitrary positive magnitude, and are normal to the surface at points A and B respectively. (To illustrate the twist, $\vec{N}_A$ is shown pointing out of the plane of the drawing and towards the viewer, while $\vec{N}_B$ points out of the plane and away from the viewer.) From Figure 5-1 it is apparent that if we project $\vec{N}_A$ and $\vec{N}_B$ onto the chord $\overrightarrow{AB}$, the projections would point towards each other. A projection of their vector difference $\vec{N}_B - \vec{N}_A$ onto the chord would thus point in the opposite direction as the vector $\overrightarrow{AB}$. The surface in Figure 5.1 will be convex as long as:

$$\left(\vec{N}_B - \vec{N}_A\right) \bullet \overrightarrow{AB} < 0 \tag{5-1}$$

where $\overrightarrow{AB}$ is a vector from A to B.

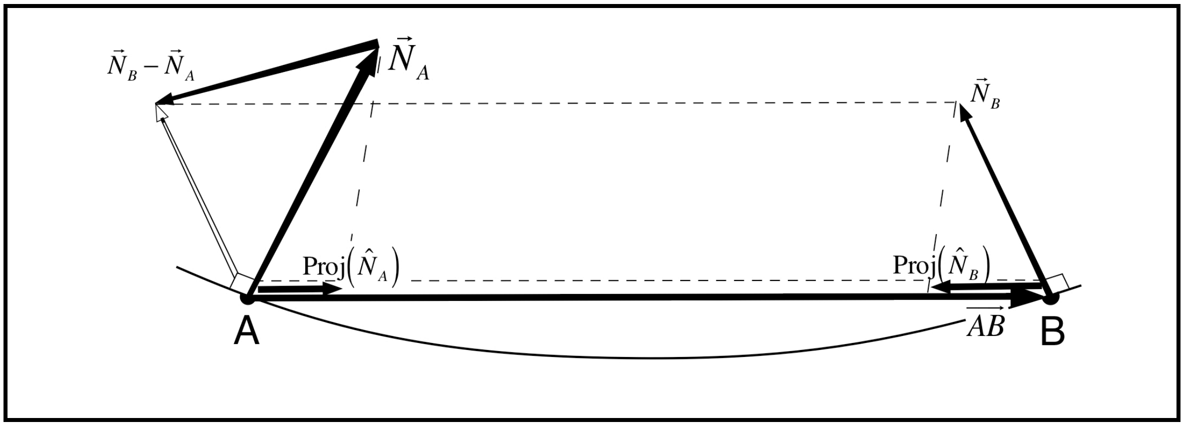

Figure 5-1

For our purposes, indifference surfaces are of less interest than the marginal value functions that generate them. Since $\vec{r}(\vec{x})$ is the gradient of $V(\vec{x})$ we know that for every point $\vec{x}$, $\vec{r}(\vec{x})$ is normal to the indifference surface of $V(\vec{x})$ passing through it. I define the vector $\overrightarrow{AB}$ as a displacement in commodity space $\Delta\vec{x}$. By replacing points A and B with $\vec{x}$ and $\vec{x}+\Delta\vec{x}$, the normal vectors at these points become $\vec{r}(\vec{x})$ and $\vec{r}(\vec{x}+\Delta\vec{x})$ respectively. Equation 5-1 becomes:

$$\left(\vec{r}(\vec{x}+\Delta\vec{x}) - \vec{r}(\vec{x})\right) \bullet \Delta\vec{x} < 0 \tag{5-2}$$

If we consider $\Delta\vec{x}$ to be a small increment of a single good x_i Equation 5-2 reduces two the familiar law of diminishing marginal utility, stated in marginal value form[11].

$$\frac{r_i(\vec{x}+\Delta x_i) - r_i(\vec{x})}{\Delta x_i}\left(\Delta x_i\right)^2 < 0 \quad \Leftrightarrow \quad \frac{\partial r_i(\vec{x})}{\partial x_i} = \frac{\partial^2 V(\vec{x})}{\partial x_i^2} < 0 \tag{5-3}$$

[11] Rather than saying that the benefit derived reduces with consumption, we say that it is the consumer's willingness to pay that reduces with consumption.

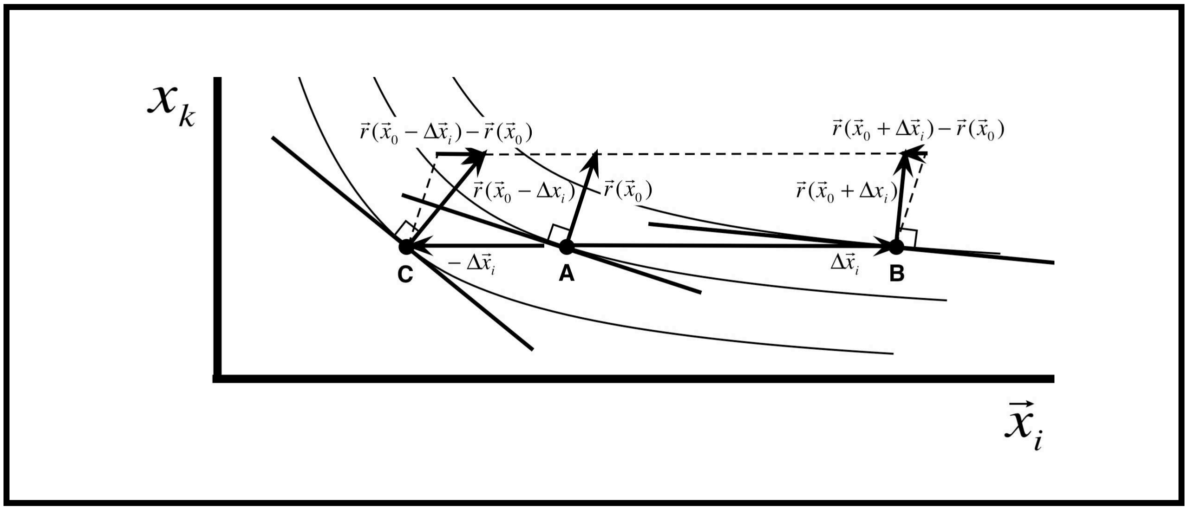

Figure 5-2

The more general result is obtained by allowing $\Delta\vec{x}$ to represent any linear combination of goods. I begin by expanding the dot product of Equation (5-2) according to its definition.

$$\sum_i \left[r_i\left(\vec{x} + \Delta\vec{x}\right) - r_i\left(\vec{x}\right) \right] \Delta x_i < 0 \tag{5-4}$$

Since the argument of each r_i is a function of all goods x_i, and each displacement Δx_i is assumed to be small, we can apply the mean value theorem[12] to each term in the square brackets obtaining:

$$r_i\left(\vec{x} + \Delta\vec{x}\right) - r_i\left(\vec{x}\right) = \sum_k \frac{\partial r_i\left(\vec{x} + \theta\Delta\vec{x}'\right)}{\partial x_k} \Delta x_k \qquad 0 < \theta < 1 \tag{5-5}$$

Since $\theta\Delta\vec{x}'$ represents a very small displacement from $\vec{x}$, we can ignore it and substitute Equation (5-5) into Equation (5-4):

$$\sum_i \sum_k \frac{\partial r_i\left(\vec{x}\right)}{\partial x_k} \Delta x_k \Delta x_i < 0 \tag{5-6}$$

Equation (5-6) is the quadratic form often used to describe convex indifference surfaces, written in terms of marginal values. Equation (5-6) can be in matrix or *tensor* form as $\left(\Delta\mathbf{\vec{x}}\right)\mathbf{C}(\vec{x})\left(\Delta\mathbf{\vec{x}}\right)^{\mathbf{T}} < 0$ where $\left(\Delta\mathbf{\vec{x}}\right)^{\mathbf{T}}$ is the vector $\Delta\vec{x}$ expressed as a column, and $\mathbf{C}(\vec{x})$ is the Complementarity Tensor, the *n*x*n* matrix of complementary effects defined below:

[12] See Taylor and Mann (1983) p.204

Definition (Complementarity Tensor $\mathbf{C}(\vec{x})$)

For a consumer possessing a bundle $\vec{x} \quad x_1, x_2 \ldots x_n = \vec{x}$, and with marginal values $\vec{r}(\vec{x})$, the complementarity tensor $\mathbf{C}(\vec{x})$ is the nxn matrix defining the complementary effect of each good upon all other goods, evaluated at $\vec{x}$.

$$\mathbf{C}(\vec{x}) = \begin{pmatrix} \frac{\partial r_1(\vec{x})}{\partial x_1} & \frac{\partial r_1(\vec{x})}{\partial x_2} & \cdots & \frac{\partial r_1(\vec{x})}{\partial x_n} \\ \frac{\partial r_2(\vec{x})}{\partial x_1} & \frac{\partial r_2(\vec{x})}{\partial x_2} & \cdots & \frac{\partial r_2(\vec{x})}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial r_n(\vec{x})}{\partial x_1} & \frac{\partial r_n(\vec{x})}{\partial x_2} & \cdots & \frac{\partial r_n(\vec{x})}{\partial x_2} \end{pmatrix}$$

From Assumption (4) we know that $\mathbf{C}(\vec{x})$ is symmetric, i.e. that $\mathbf{C}(\vec{x}) = \mathbf{C}(\vec{x})^{\mathbf{T}}$. From inequality (5-6) we know that $\mathbf{C}(\vec{x})$ is negative definite[13]. Substituting $\partial V(\vec{x}) / \partial x_i$ for each $r_i(\vec{x})$ in $\mathbf{C}(\vec{x})$, it becomes the familiar Jacobean matrix commonly used to describe the convexity of utility functions.

We now take an intuitive second look at what Equation (5-2) means. We consider what would happen if there were goods present for which Equation (5-2) was violated. Presence of goods for which the consumer's willingness to pay does not diminish with consumption would ultimately lead the consumer to expend all of his or her resources in their acquisition. Such behavior of course represents obsession or addiction. While such behavior may provide the consumer with short term pleasure, it usually results in long term damage to his well being. Our intuitive belief is that addictive behavior destroys the addict as well as those with whom he interacts. This may explain why societies have recognized that addictive behavior represents a kind of "rational-irrationality" that requires social intervention. It is thus socially reasonable to assume that institutions have been created to remove goods, to whom consumers may be addicted, from the marketplace. I now formalize this assumption:

Definition (Addiction)

For a consumer possessing a bundle $\vec{x} \quad x_1, x_2 \ldots x_n = \vec{x}$, and with marginal values $\vec{r}(\vec{x})$, the consumer is said to be *addicted* to some good x_i, or to a set of goods, $(x_i x_k \ldots)$ if her marginal value for that good or set of goods does not diminish with her consumption of them.[14] That is to say, for a positive increment of this good or set of goods $\Delta x'$ we have:

$$\left[\vec{r}(\vec{x} + \Delta\vec{x}') - \vec{r}(\vec{x})\right] \bullet \Delta\vec{x}' \geq 0 \tag{5-8}$$

[13] If we were to allow for quasi-convexity, Inequality (5-6) would become a weak inequality, and both C and the Jacobian would be negative semi-deffinite.

[14] To be completely rigorous, strong addiction and weak addiction should be defined in terms of whether or not the inequality in Equation 5-6 is strict. That detail is omitted here, as it does not contribute significantly to the argument.

Assumption (5): Non Addiction

For a consumer possessing a bundle $\vec{x}$ $x_1, x_2 \ldots x_n = \vec{x}$, and with marginal values $\vec{r}(\vec{x})$, there is no good or set of goods present in the market to which the consumer is addicted. In other words there are no incremental bundles $\Delta\vec{x}'$ present, for which the following inequality does not hold:

$$\left[\vec{r}\left(\vec{x}+\Delta\vec{x}\right)-\vec{r}(\vec{x})\right]\bullet\Delta\vec{x}<0 \qquad (5\text{-}9)$$

6) Exchange and the Consumer Choice Problem

The task now is to apply the dynamic model to the consumer choice problem and demonstrate that it produces the results expected from utility maximization. I will show that indeed it does, while at the same time it provides detail that the utility maximization paradigm does not. I model the consumer as receiving income in the form of regular increments δW to his wealth over time. The consumer spends his income on incremental bundles of infinitely durable goods, following his Wealth Expansion Path as shown in Figure 6-1. I assume that the available goods consist of staple as well as luxury goods to insure that Engle's law applies and the wealth expansion path is not straight.

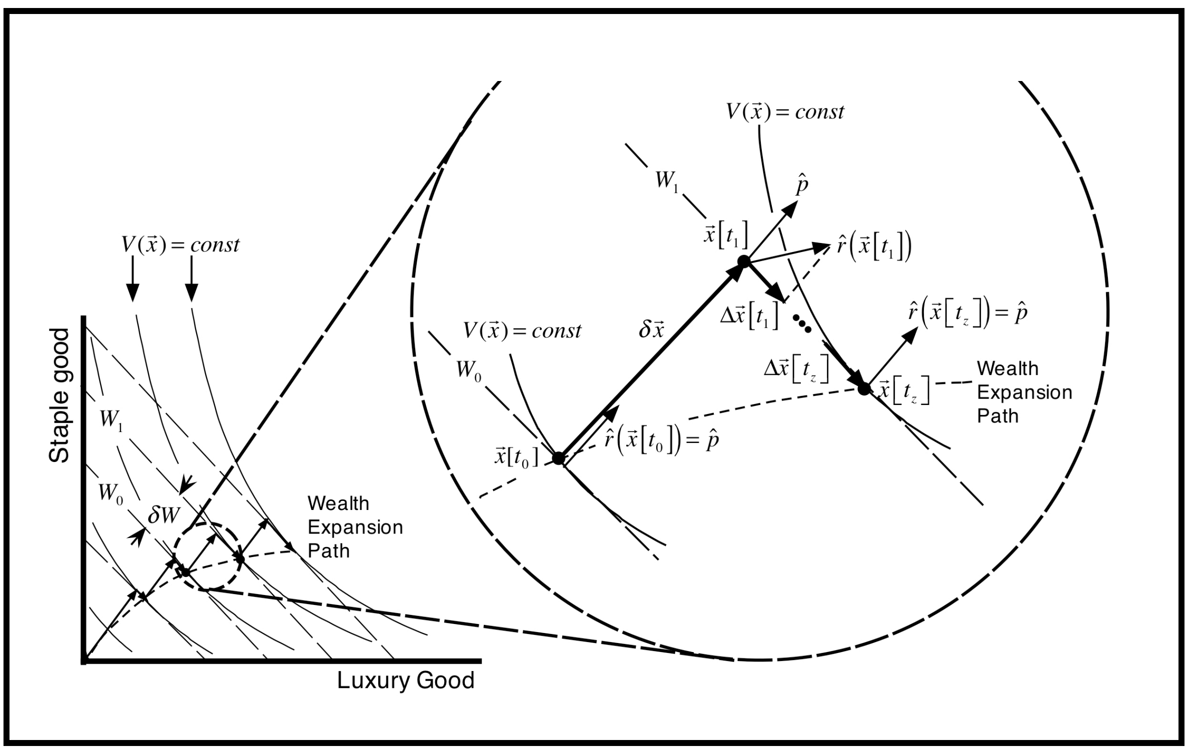

Figure 6-1

For each wealth increment received, the consumer obtains a new optimal bundle through a two step process. The first is the purchase of an estimated optimal bundle $\delta\vec{x}$ with rhe received numeraire. The second step is a tatonnement process in which the consumer adjusts his holdings through incremental exchanges with the "market". The initial purchase is not in itself important. It is to the tatonnement process that the dynamic analysis is applied

At time t_0 the consumer holds bundle $\vec{x}[t_0]$ which represents initial wealth W_0. She receives her income δW in the form of numeraire and immediately procures a bundle $\delta\vec{x}$. This is her estimated optimal purchase, given her marginal values $\vec{r}(\vec{x}[t_0])$ and prices $\vec{p}$. Due to the non linearity of her wealth expansion path her new bundle $\vec{x}[t_1] = \vec{x}[t_0] + \delta\vec{x}$ will not be optimal, given her new wealth $W_1 = W_0 + \delta W$. Additional exchanges are needed to bring her bundle to the optimum $\vec{x}[t_z]$.

The tatonnement process is modeled as a sequence of bilateral exchanges, with the "market" being a partner with whom any quantity of goods may be traded at fixed prices $\vec{p}$. At any time t_n, the consumer is free to exchange differentially small bundles $\Delta\vec{x}[t_n]$, provided the exchanges take place within the budget hyperplane such that:

$$\vec{p} \bullet \Delta\vec{x}\left[t_n\right] = 0 \tag{6-1}$$

Because the numeraire does not appear in the barter transactions, it is convenient to state prices in relative terms, dividing the components of $\vec{r}(\vec{x})$ and $\vec{p}$ by their common denominator. I will show that for any time period t_n within which the consumer's relative marginal prices $\hat{r}(\vec{x}[t_n])$

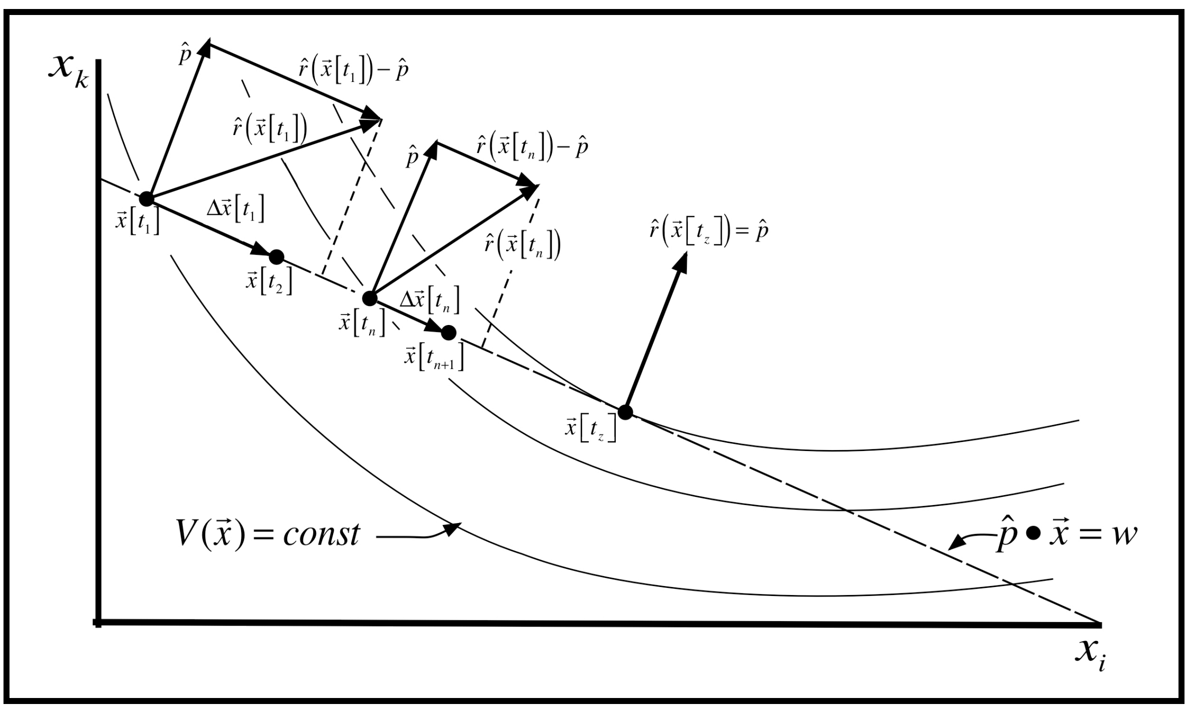

Figure 6-2

do not equal the relative exchange prices $\hat{p}$, the consumer will be able to devise a bundle $\Delta\vec{x}[t_n]$ that will increase her use value for her stock of goods: $V(\vec{x}[t_{n+1}]) > V(\vec{x}[t_n])$. Additionally I will show that for each such exchange, the consumer's relative marginal prices will adjust so as to become "closer" to the exchange prices. Based on this, I will show that the consumer will continue to make exchanges until her relative marginal prices matches the relative exchange prices. At this time, the consumer's final bundle will be the one that provides her the maximum use value, given her wealth[15].

I define the relative marginal and exchange prices so as to give their vectors unit magnitude.

Definition: Relative Marginal Prices

For a consumer possessing a bundle $\vec{x} \quad x_1, x_2 \ldots x_n = \vec{x}$, and described by marginal values $\vec{r}(\vec{x})$, the consumer's relative marginal price $\hat{r}_i(\vec{x})$, for good x_i is:

$$\hat{r}_i(\vec{x}) = \frac{r_i(\vec{x})}{\sum_i r_i(\vec{x})} = \frac{r_i(\vec{x})}{|r(\vec{x})|} \quad \text{thus} \quad \hat{r}(x) = \frac{\sum_i r_i(\vec{x})\hat{\varphi}_i}{\sum_i r_i(\vec{x})} = \frac{\vec{r}(\vec{x})}{|\vec{r}(\vec{x})|} \tag{6-2}$$

I have written out the summations in the denominators as a reminder that for a rectilinear space, the distance metric is defined by Equation (3-5) rather than by the familiar Pythagorean formula applied to Euclidian spaces. I make a similar definition for relative exchange prices as follows.

Definition: Relative Exchange Prices

For a set of prices $\vec{p}$ at which a community of agents have agreed to exchange goods, the relative exchange prices are given by:

$$\hat{p}_i = \frac{p_i}{|\vec{p}|} \quad \text{and} \quad \hat{p} = \frac{\vec{p}}{|\vec{p}|} \tag{6-3}$$

The next step is to clarify what is to be assumed regarding the consumer's behavior. Assumption (1) given previously states that the consumer knows the rates at which she would be willing to exchange goods. It does not however state specifically how she would act on that information. Here I will assume that the consumer will craft his bundle so as to maximize the benefit he derives, per unit of value exchanged. From the basic notion of consumer's surplus we know that the benefit he gains from the purchase of some quantity of good x_i is simply his increase in use value $r_i(\vec{x})\Delta x_i$ less the exchange value that must be given up: $p_i \Delta x_i$.

[15] A few words need be said here regarding the distinction between the consumer's relative vs. absolute marginal prices. This was not an issue with the utility maximization paradigm since utility has no meaningful cardinal value. This is not the case with Use Value as defined by Equation (4-7). One might easily find a consumer, call him Sean, who would pay 50p. each for an orange and an apple, while another consumer, Daisy, would only pay only 20p. for each item. While the consumer's relative prices for the goods are the same, the wellbeing gained by Sean, as measured by the increase in his use value were he to acquire either fruit, would be greater than Daisy's. From a familiar Edgeworth Box diagram, it is clear that the consumer's relative marginal prices will remain constant along a contract curve, while their marginal prices measured in absolute terms will likely change with the size of their bundles.

Conversely, for any good sold, his benefit is the exchange value received $p_i \Delta x_i$ less the use value $r_i(\vec{x})\Delta x_i$ foregone. Thus for any transaction involving some good x_i to be beneficial, we must have:

$$\left[\hat{r}_i(\vec{x}') - \hat{p}_i\right]\Delta x_i > 0 \tag{6-4}$$

Where $\Delta x_i > 0$ for goods purchased and $\Delta x_i < 0$ for goods sold. It is clear that for a consumer to be considered rational, Equation (6-4) must hold for all goods exchanged. I will carry things a step further by assuming the consumer will adjust the quantities of each good Δx_i exchanged so as to maximize his benefit. This is stated in the following assumption:

<u>Assumption (6): Beneficial Transactions</u>
A consumer possessing a bundle $\vec{x}'$, who is described by relative marginal prices $\hat{r}(\vec{x})$, will agree to exchange goods at relative prices $\hat{p}$ if he or she is able to construct a bundle $\Delta\vec{x}$ that provides a positive solution to the problem:

$$\underset{\Delta\vec{x}}{Max}\left\{[\hat{r}(\vec{x}') - \hat{p}] \bullet \Delta\vec{x}\right\} \qquad \text{subject to} \qquad \hat{p} \bullet \Delta\vec{x} = 0 \tag{6-5}$$

From the way the magnitudes of $\hat{r}(\vec{x})$ and $\hat{p}$ are defined, it is clear that that $\hat{r}(\vec{x}) - \hat{p}$ must lie within the budget hyperplane. The dot product is thus maximized when the $\Delta\vec{x}$ is chosen to be parallel to $\hat{r}(\vec{x}) - \hat{p}$, or:

$$\Delta\vec{x} = \lambda\left[\hat{r}(\vec{x}') - \hat{p}\right]\Delta t \qquad \lambda \text{ is a constant}, \quad 0 < \lambda \leq 1 \tag{6-6}$$

The proportionality between each Δx_i and its corresponding $\hat{r}_i(\vec{x}) - \hat{p}_i$ simply reflects the Law of Demand.

I now begin the process of demonstrating that marginal exchanges will proceed as I have described above. The First proposition is a formalization of the classic concept of comparative advantage

<u>Proposition 6-1 Benefit From Marginal Exchange</u>
Given a consumer who is described by marginal value function $\vec{r}(\vec{x})$, possessing a bundle $\vec{x}'$, who is given the opportunity to exchange goods at prices $\vec{p}$.

The consumer will agree to exchange a marginal bundle $\Delta\vec{x}$ if (and only if) his or her $\hat{r}(\vec{x}) \neq \hat{p}$, and he or she has a non zero quantity of at least one good x_k that his/her trading partner will accept in payment.

PROOF:
It follows from Equations (6-2) and (6-3) that $\left|\hat{r}(\vec{x})\right| = \left|\hat{p}\right|$. We know therefore that if (and only if) there exists at least one good x_i for which $\hat{r}_i(\vec{x}) > \hat{p}_i$ then there must be at least one good x_k for which $\hat{r}_k(\vec{x}) < \hat{p}_k$. Assuming that goods are infinitely divisible, and that the consumer possesses a non zero quantity of x_k there will be some set of quantities of the other goods for which:

$$\Delta x_k = \frac{1}{\widehat{p}_k} \sum_{i \neq k} \widehat{p}_i \Delta x_i \quad (6\text{-}7)$$

The consumer will therefore be able to devise an exchange bundle $\Delta\vec{x}$ that satisfies Assumption (6).
QED.

The next step is to show that Assumption (5) will cause the difference between the agents relative marginal values and the exchange prices to diminish as is shown in Figure (6-2).

Proposition 6-2 (Price Contraction from Marginal Exchange)
Given a consumer described by marginal price function $\vec{r}(\vec{x})$ who possesses a bundle $\vec{x}'$. If such consumer, exchanges a marginal bundle of goods $\Delta\vec{x}$ at prices $\vec{p}$ for which $\hat{p} \neq \hat{r}(\vec{x})$, The difference between the consumer's relative marginal prices, and the exchange prices will contract, i.e.:

$$\left|\hat{r}(\vec{x}') - \vec{p}\right| > \left|\hat{r}(\vec{x}' + \Delta\vec{x}) - \vec{p}\right| > 0 \quad (6\text{-}8)$$

PROOF:
From Assumptions (5) we have:

$$\hat{r}(\vec{x}' + \Delta\vec{x}) \bullet \Delta\vec{x} - \vec{r}\left(\vec{x}\right) \bullet \Delta\vec{x} < 0 \quad (6\text{-}9)$$

Substituting in the relative marginal prices gives:

$$\left|\vec{r}(\vec{x}' + \Delta\vec{x})\right| \hat{r}(\vec{x}' + \Delta\vec{x}) \bullet \Delta\vec{x} - \left|\vec{r}\left(\vec{x}\right)\right| \vec{r}\left(\vec{x}\right) \bullet \Delta\vec{x} < 0 \quad (6\text{-}10)$$

Since for small $\Delta\vec{x}$, we can make the approximation that $\left|\vec{r}(\vec{x}' + \Delta\vec{x})\right| \approx \left|\vec{r}\left(\vec{x}\right)\right|$ and divide Equation 6-10 by that number, leaving:

$$\hat{r}\left(\vec{x}'\right) \bullet \Delta\vec{x} > \hat{r}(\vec{x}' + \Delta\vec{x}) \bullet \Delta\vec{x} > 0 \quad (6\text{-}11)$$

Since $\vec{p} \bullet \Delta\vec{x} = 0$ I can subtract it from all terms in Equation (6-11) without altering the inequality.

$$\sum_i [\widehat{r}_i(\vec{x}') - \widehat{p}_i] \Delta x_i > \sum_i \left[\widehat{r}_i(\vec{x}' + \Delta\vec{x}) - \widehat{p}_i\right] \Delta x_i > 0 \quad (6\text{-}12)$$

From equation (6-6) we know that each component of $\Delta\vec{x}$ is proportional to its corresponding price difference, we can substitute (6-6) into Equation (6-12) giving:

$$\lambda\Delta t\sum_i[\hat{r}_i(\vec{x}')-\hat{p}_i]^2>\lambda\Delta t\sum_i\left[\hat{r}_i(\vec{x}'+\Delta\vec{x})-\hat{p}_i\right]^2>0$$
$$\Rightarrow\ \sum_i\left|\hat{r}_i(\vec{x}')-\hat{p}_i\right|\ >\ \sum_i\left|\hat{r}_i(\vec{x}'+\Delta\vec{x})-\hat{p}_i\right|\ >0 \qquad (6\text{-}13)$$
$$\Rightarrow\left|\hat{r}(\vec{x}')-\vec{p}\right|>\left|\hat{r}(\vec{x}'+\Delta\vec{x})-\vec{p}\right|>0$$

QED

The final proposition models the tatonnement process as a sequence of marginal exchanges made over time. Since $\left|\hat{r}(\vec{x})-\hat{p}\right|$ reduces with each exchange, it must eventually reach zero.

Proposition 6-3: Unilateral Tatonnement

Given a consumer described by relative marginal price function $\hat{r}(\vec{x})$, who at time t_1 possesses an initial bundle $\vec{x}[t_1]$. Given also that the consumer may at any time t_n exchange a marginal bundle $\Delta\vec{x}[t_n]$, as long as he is willing to do so at market prices $\hat{p}$. The consumer will therefore exchange marginal bundles in every time period until he obtains a bundle $\vec{x}[t_z]$ for which $\hat{r}(\vec{x}[t_z])=\hat{p}$. The total use-value $V(x[t_z]-x[t_0])$ gained by the consumer will be the maximum available to him at prices $\hat{p}$ given his wealth $w=\hat{p}\bullet\vec{x}[t_0]$ and stock of goods x_k that are required for payment. (NOTE: this second case corresponds to what is commonly referred to as a "corner solution".)

PROOF:

I begin by showing that the consumer will continue to make exchanges until he acquires a bundle for which his $\vec{r}(\vec{x}[t_z])=\hat{p}$. For every time period t_n for which the consumer's relative marginal prices $\hat{r}(\vec{x}[t_n])$ do not equal $\hat{p}$, Proposition 6.1 implies that the consumer will exchange a marginal bundle $\Delta\vec{x}[t_n]$, bringing his bundle to $\vec{x}[t_{n+1}]$ at the beginning of the next time period. Per Proposition 6-2 we know that for every exchange, the consumer's relative marginal prices will contract towards the exchange prices:

$$\left|\hat{r}(\vec{x}[t_n])-\hat{p}\right|>\left|\hat{r}(\vec{x}[t_n]+\Delta\vec{x}[t_n])-\vec{p}\right|=\left|\hat{r}(\vec{x}[t_{n+1}])-\hat{p}\right|>0 \qquad (6\text{-}18)$$

From Equation (6-18) it is apparent that

$$\lim_{n\to\infty}\left|\hat{r}(\vec{x}[t_n])-\hat{p}\right|=0 \qquad (6\text{-}19)$$

For practical purposes, we will choose some number ε for which $\left|\hat{r}(\vec{x}[t_n])-\hat{p}\right|<\varepsilon$ is negligibly close to zero. Since Equation (6-18) approaches zero monotonically, there must be some number $0<z<\infty$ such that:

$$\left|\hat{r}(\vec{x}[t_z])-\hat{p}\right|<\varepsilon \qquad (6\text{-}20)$$

Therefore, at least for practical purposes, $\vec{x}[t_z]$ is the bundle for which the consumer's marginal prices equal $\hat{p}$. According to Proposition 6-1 exchange will stop at this point, and will not restart as long as $\hat{p}$, $\vec{x}[t_z]$, and $\vec{r}(\vec{x})$ remain unchanged.

To show that $V(\vec{x}[t_z])$ provides the maximum use value available at prices $\hat{p}$, we assume for a moment that it does not. If it does not provide the maximum use value then there would be a possible exchange bundle $\Delta\vec{x}"$ for which:

$$V(\vec{x}[t_z]+\Delta\vec{x}")>V(\vec{x}[t_z]) \tag{6-21}$$

If the consumer were to exchange $\Delta\vec{x}"$, her marginal prices would necessarily adjust so that Assumption (5) would be satisfied. As result we would have:

$$\left[\hat{r}(\vec{x}[t_z]+\Delta x")-\vec{r}(\vec{x}[t_z])\right]\bullet\Delta\vec{x}"<0 \tag{6-22}$$

Since $\hat{r}(\vec{x}[t_z])=\hat{p}$, and $\hat{p}\bullet\Delta\vec{x}"=0$ Equation (6-22) reduces to:

$$\hat{r}(\vec{x}[t_z]+\Delta\vec{x}")\bullet\Delta\vec{x}"<0 \tag{6-23}$$

If the consumer, who now holds $x[t_z]+\Delta\vec{x}"$ were to reverse his exchange of $\Delta\vec{x}"$, he would gain benefit since: $\vec{r}(\vec{x}[t_z]+\Delta\vec{x}")\bullet(-\Delta\vec{x}")>0$. We would therefore have $V(\vec{x}[t_z]+\Delta\vec{x}")<V(\vec{x}[t_z])$ which contradicts our temporary assumption. I have thus shown that $V(\vec{x}[t_z])$ is the maximum value available to the consumer. This completes the proof.
QED.

In the preceding argument, the assumption that the consumer always possesses some of the good (or goods) x_k guarantees that “corner solutions” are not an issue. I have used that assumption to save space. Relaxing it would require a similar, but longer and more tedious version of Proposition 6-3 given above.

This section has shown that any $\vec{r}(\vec{x})$ satisfying Assumptions (1) through (6) the consumer will eventually arrive at his or her optimal bundle. By choosing a specific form for $\vec{r}(\vec{x})$ and solving Equation (6-10) restated as: one can model a consumer’s behavior in a time dependent fashion.

$$d\vec{x}=\lambda\left[\hat{r}(\vec{x})-\hat{p}\right]dt \qquad \lambda \text{ is a constant}, \quad 0<\lambda\leq 1 \tag{6-24}$$

To illustrate what is perhaps the simplest of cases, let us assume that the consumer’s $\vec{r}(\vec{x})$ is

$$r_i(\vec{x})=p_i-\lambda(x_i-x_i^*) \quad \forall i \qquad \text{or} \qquad \vec{r}(\vec{x})=\vec{p}-\lambda(\vec{x}-\vec{x}^*) \tag{6-25}$$

Here λ defines the (assumed constant) rate at which the consumer’s marginal value for any good diminishes with her acquisition of it, and $\vec{x}^*$ is her optimal bundle at prices $\vec{p}$. Substitution of Equation (6-25) into Equation (6-24) produces:

$$d\vec{x}=-\lambda\left(\vec{x}-\vec{x}^*\right)dt \tag{6-26}$$

This is the familiar differential equation for exponential decay, with solution:

$$\vec{x}[t]-\vec{x}^{*}=\left(\vec{x}[t_0]-\vec{x}^{*}\right)e^{-\lambda t} \qquad (6\text{-}27)$$

The consumer's bundle thus approaches its optimal value at an exponentially decaying rate determined by λ.

4.6 Conclusion

What I have done here is develop an alternative expression of the consumer choice model that considers it as a process that evolves through time. I have shown that this alternative is consistent with the standard consumer choice model, in that it predicts the same behavior though with a bit more detail. Rather than indicating what optimal bundle the consumer will choose, it identifies a means by which the consumer arrives at his bundle. In so doing, this model becomes foundational to an understanding of general equilibrium.

In the dynamic consumer choice problem discussed above, an exchange equilibrium is established between a single agent and a "partner" who behaves as would a perfectly competitive marketplace. The consumer determines his optimal (market clearing) bundle through a tatonnement process in which he adjusts his holdings of good until the relative prices at which he would be most willing to exchange goods, matched the prices dictated by the market. As I show elsewhere, this process may be extended to communities of many users exchanging a diversity of goods.

As a byproduct, I have been able to replace more subjective notions of utility and preference with an operationally defined measure of the value a consumer places on a bundle of goods. While such Use Value may be determined by the pleasure or benefit the consumer derives from consumption, there is no need to assume that it is necessarily so. The "rational self interest" that agents are presumed to maximize is replaced with a more general notion of agency. As long as the goals agents pursue are logically consistent, the theorist need not delve into their nature.

Finally, the terms in which an agent's Use Value is defined makes it measurable on a cardinal scale. As I show Elsewhere[16] This makes interpersonal comparison and aggregation much easier, though considerable care must be taken in interpreting what the aggregate means.

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[16] See Mclaren (2012) pp. 143 - 151

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