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On Modeling Pollution-Generating Technologies

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Abstract

This paper takes issues with standard, single-equation modeling of pollution-generating technologies and suggests an alternative approach. Our approach takes account of the material-balance condition, in which pollution is an unintended by-product, or incidental output, of the intended production activities of firms. We distinguish between the notions of joint production and by-production and show that single-equation modeling of technologies that include pollution as another commodity capture pollution as a joint product and *not* as a by-product of intended production. For modeling pollution-generating technologies, we propose, in addition to the technology that depicts transformation of inputs into intended outputs, an explicit delineation of the externality-generating mechanism. We show that such a specification of the technology yields, in a manner consistent with the material-balance condition, the (intuitively desirable) positive correlation between generation of pollution and the production of intended outputs. The technology specified this way, however, is a manifold in the full commodity space and, in contrast to the technologies employed in much of the literature, satisfies neither strong nor weak disposability in the subspace of pollution and intended outputs. Moreover, shadow prices of pollution variables are undefined. Modeling abatement as an intermediate input (produced and used by the firm) yields a positive trade-off between pollution and conventional outputs, but the trade-offs among conventional inputs and outputs in the "reduced form" technology do not satisfy conventional sign requirements; hence, standard DEA methods of constructing the reduced-form technology and of calculating the shadow price of pollution are problematic. We believe these results have implications for empirical work that models and estimates pollution-generating technologies and the shadow price of pollution.

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1. Introduction.

Our reading of the environmental economics literature reveals three broad features of pollution that economists aim to capture. First, the pollution problem is a consequence of nature's material balance condition, which implies the inevitability of residual generation in the processes of consumption and production.[1] Second, the residuals so generated require the use of the assimilative capacity of the environment for their disposal. Third, the generation of the residuals and the consequent use of environmental resources for their disposal generate external effects on both consumers and producers.

We confine ourselves here to pollution generated by firms. In this case, the material balance condition indicates that there are specific aspects about the process of transformation of inputs into outputs (*e.g.,* the use of certain inputs or the production of certain outputs) that result in the generation of pollution as a by-product. In particular, the fundamental physical law of conservation of mass implies that the more the firm proceeds with the use of these inputs or the production of these outputs, the more the by-product is generated. We refer to this process as a "by-production"[2] of pollution: an unintended, or incidental, output from the point of view of the producer.

At the outset, we underscore the salient distinction between this notion of by-production and the concept of joint production. In the case of by-production, the material balance condition implies the *inevitability* of a certain amount of the incidental output, given the quantities of certain inputs and/or certain intended outputs (possibly including abatement outputs); in the case of joint production, there exists a menu of possible output vectors (that is, a production possibility set), possibly including zero amounts of some (or all) outputs, given the amounts of the inputs.

The by-production of unintended outputs—that is, the lockstep correspondence of residuals and intended production—can be mitigated by abatement options available to a firm,

[1] See, especially, Ayres and Kneese [1969], Kneese, Ayres, and D'Arge [1970], and Førsund [1998].

[2] A word that is not in the dictionary but perhaps should be.

but these abatement efforts are also costly in terms of resource use. The environmental economics literature usually explains the (short run) costs of abatement (reduction of pollution) in terms of (a) the substitution for impure inputs (if they are the cause of creation of residuals) of cleaner but less productive inputs or (b) the (costly) production of explicit abatement output to be used in reducing pollution.[3] Thus, nature's material balance condition, together with costly abatement options, implies that the technology does not permit free disposability of the residuals of production. Although many research papers use such arguments, these costs, to our knowledge, are not explicitly modeled.

Pollution modeling involves inclusion of pollution (either as another input or as an another output) in the formulation of the technology of the firm. Typically, a single (set theoretic or functional) technological restriction on all the variables of production is imposed.[4] Those who model pollution as an input argue that the by-production of incidental outputs implies a symmetry between the production residual and ordinary inputs (inputs with non-negative marginal products satisfying input disposability): since abatement efforts are costly in terms of resource usage, an increase in the production of incidental output implies that fewer resources are allocated to abatement and hence more resources are available for production of intended outputs. Thus, they argue, for the purpose of analysis, pollution can alternatively be treated as an input into production.[5] This approach is also defended by the argument that the generation of pollution, an unintended *output* of production, can

[3] A long-run option is the development and installation of new technologies that generate less pollution; such options are outside the scope of the inquiry in this paper.

[4] Exceptions include Baumol and Oates [1988, p. 94], Laffont [1988, p. 24], Cropper and Oates [1992], and Førsund [1998]. Of these, however, only Førsund examines the need for a multi-equation approach to pollution modeling. His approach is predicated on a social planning framework; ours is based solely on the internal consistency of firm-level modeling and the implications of this modeling for empirical work on pollution.

[5] See, *e.g.*, Baumol and Oates [1988], Laffont [1988, Ch. 2], Cropper and Oates [1992], Reinhard, Lovell, and Thijssen [1999], Ball, Lovell, Luu, and Nehring [2001], and Murty and Kumar [2002].

be equated to the use of the waste assimilative capacity of the environment, which can then be treated as an *input* into production.[6]

Those who model pollution as an output[7] attempt to capture the by-production of the unintended output by adopting the approach suggested by Shephard [1974]. This approach entails two assumptions about the technology. The first is "weak disposability" of the technology with respect to outputs—that is, outputs can be disposed of radially (*i.e.*, equi-proportionately), but not necessarily in coordinate-wise directions. The idea is that weak disposability can capture (as a special case) the resource costs of disposing of pollution and hence the trade-off between pollution and intended outputs. The second assumption is null-jointness—that is, no unintended outputs are produced only if no intended outputs are produced. This assumption is intended to capture the idea of the inevitability of residual generation in the process of production of intended outputs.

The recent papers go on to assess the marginal costs of pollution abatement in two different ways. One strain of the literature computes output possibilty sets under alternative assumptions about output disposability. The technology is assumed to satisfy strong disposability (*i.e.* coordinate-wise disposability of all outputs, including unintended outputs) under a non-regulatory regime and radial disposability under a regulatory regime. The "distance" between the two output possibility sets in intended-output space, is taken to be a measure of the resource cost of pollution abatement (*i.e.*, the loss of intended output attributable to abatement). Another approach measures the marginal abatement costs by calculating the shadow price of pollution.

We show, however, that each of these specifications of technologies is inconsistent with the fundamental nature of the by-production of incidental outputs that arises because of the

[6] The general-equilibrium modeling of externality policies is very much in the spirit of treating pollution as an input, since these policies require firms to pay for the environmental resources used in the production processes by creating Coasian markets or by imposing Pigouvian taxes (see, *e.g.*, Starrett [1972] and Boyd and Conley [1997]).

[7] See Färe, Grosskopf, Lovell, and Yaisawarng [1993], Färe and Grosskopf [1998], Ball, Lovell, Nehring, and Somwaru [1997], and Färe Grosskopf, and Weber [2002].

material balance problem. In fact, these specifications lead to interpretations that would seem to contradict any intuition one might have about the external effects created by pollution.

These conclusions follow from a direct application of the implicit function theorem in Section 2. In Section 3, we propose an alternative way of modeling production with incidental pollution that takes explicit account of the materials balance condition.[8] Section 4 models the incorporation of an abatement mechanism into the technology. To focus on the saliant aspects of by-production, we restrict our analysis to production technologies that encompass a single pollutant.

2. Critique of Single-Equation Modeling of Technologies with Pollution.

In this section, we show the inadequacy of single-equation modeling of pollution that treats pollution as a conventional input (or as an unconvential output satisfying weak disposability) in capturing the material balance condition. To sharpen our focus on the material balance condition, we abstract in this section from explicit abatement efforts by the firm.

Let $y \in \mathbf{R}^m$ and $z \in \mathbf{R}$ be the vector of intended output quantities and the quantity of the incidental output (*e.g.*, pollution), respectively. We index the intended outputs by $i, j = 1, \ldots, m$. Let $x \in \mathbf{R}^n$ denote the vector of inputs used by the firm. We index these inputs by $k = 1, \ldots, n$.

Consider the standard case in the literature where the technology, T, is defined by a single restriction on all inputs and outputs:

$$T = \left\{ \langle y, z, x \rangle \in \mathbf{R}^{m+n+1} \middle| \; f(y, z, x) \leq 0 \right\}. \tag{2.1}$$

[8] This approach, based on Frisch [1965], has been proposed by Forsund [1998].

We assume that f is continuously differentiable and denote partial derivatives of f with subscripts: i, z, and k. Following convention, assume that, for all $\langle y, x\rangle \in \mathbf{R}^{m+n}$,

$$\begin{aligned} f_i(y,z,x) &\geq 0,\ \ i = 1,\ldots,m, \\ f_k(y,z,x) &\leq 0,\ \ k = 1,\ldots,n. \end{aligned} \tag{2.2}$$

Thus, the technology satisfies free (strong) disposability in both inputs and intended outputs.

Suppose first, as is standard in the literature, that $f_z(y,z,x) < 0$ and consider a $\langle \hat{y}, \hat{z}, \hat{x}\rangle$ such that $f(\hat{y}, \hat{z}, \hat{x}) = 0$. Then, invoking the implicit function theorem, there exist neighborhoods $U \subseteq \mathbf{R}^{m+n}$ and $V \subseteq \mathbf{R}$ around $\langle \hat{y}, \hat{x}\rangle \in \mathbf{R}^{m+n}$ and $\hat{z} \in \mathbf{R}$ and a function $\zeta : \mathbf{R}^{m+n} \to \mathbf{R}$ such that

$$\hat{z} = \zeta(\hat{y}, \hat{x}) \tag{2.3}$$

and

$$f\big(y, \zeta(y,x), x\big) = 0 \quad \forall\ \langle y, x\rangle \in U. \tag{2.4}$$

Consider the trade-off between the intended output i and the unintended output implied by the implicit function theorem:

$$\frac{\partial \zeta(x,y)}{\partial y_i} = -\frac{f_i(u, y_i)}{f_z(u, y_i)} \geq 0, \quad \forall\ \langle y, z, x\rangle \in U \times V, \quad i = 1,\ldots,m, \tag{2.5}$$

which seems to capture the incidental character of residual generation. Similarly, the trade-off between input k and the unintended output is

$$\frac{\partial \zeta(\hat{y}, \hat{x})}{\partial x_k} = -\frac{f_k(y,z,x)}{f_z(y,z,x)} \leq 0, \tag{2.6}$$

which is consistent with the treatment of pollution as an input.

The foregoing formulation of a pollution-generating technology, however, seems to be inconsistent with the material balance condition, as well as common sense, in at least three ways:

(i) The existence of the function ζ satisfying (2.5) implies that there exists a rich menu of $\langle y, z\rangle$ combinations that are possible with *given levels of all inputs* and (2.6) implies that there exists a rich menu of $\langle x, z\rangle$ combinations that are possible with *given levels of all intended outputs*. Under the assumption that y and x are scalars, Figures 1 and 2 depict, respectively, the set, $P(x)$, of all $\langle y, z\rangle$ combinations that are technologically feasible for a given level of the input, and the set, $L(y)$, of all $\langle x, z\rangle$ combinations that are feasible for a given level of the intended output. If the residual is generated by input usage, then the former set of options (where several levels of the residual can be generated with fixed level of the input), is patently contrary to the material balance condition, while if it is the intended output that causes the residual, then the latter set of options (where several levels of the residual can be generated with fixed level of the intended output) is inconsistent with the material balance condition. Clearly, a single equation characterization of the technology, captures the residual as a *joint product* and not as a *by-product* of intended production.

(ii) The positive trade-off (2.5) in the output-oriented modeling of pollution with a single equation (in the absence of abatement efforts) suggests that pollution acts as a catalyst in the production of intended outputs; *i.e.*, as illustrated in Figure 1, increasing the level of pollution somehow makes the conventional inputs, fixed in quantity, more productive, allowing the firm to produce the same output with lower amounts of the inputs or more of the intended output with fixed amounts of the inputs. If anything, one would expect the effects of pollution on a firm's own technology to be deleterious rather than beneficial.

(iii) Alternatively, the trade off between pollution and any input, identified in (2.6), implies that an increase in the usage of (possibly pollution generating) inputs *decreases* (as illustrated in Figure 2) the level of pollution, *holding other other inputs and all outputs fixed*. Even if the residual is generated by output production rather than input usage, it strains credibility to suppose that increasing input usage decreases emissions.

3. An Alternative Approach to Modeling Pollution.

In our view, the resolution to the problems identified in Section 2 lies in explicitly modeling the material balance condition by delineating the residual-generation mechanism, an approach suggested by Forsund [1998], who in turn relied on some (generally overlooked) ideas of Frisch [1965] on production theory.[9] The production of the intended output sets the residual-generation mechanism in motion, leading to the generation of the by-product. In this way, we obtain the (intuitively desirable) positive correlation between intended outputs and the residual in a manner consistent with the material balance condition.

To fix our ideas on the salient aspects of modeling the material balance condition and to simplify notation, we suppose that the pollution is generated by usage of a single input, say input ℓ. Denote the input quantity vector purged of the quantity of input ℓ by x^1.[10] Specify the technology as

$$\begin{aligned} T &= \left\{ \langle y, z, x^1, x_\ell \rangle \in \mathbf{R}^{m+n+1} \mid f(y^1, x^1, x_\ell) \leq 0 \ \wedge \ z = g(x_\ell) \right\} \\ &= T_1 \cap T_2, \end{aligned} \tag{3.1}$$

where

$$T_1 = \left\{ \langle y, z, x^1, x_\ell \rangle \in \mathbf{R}^{m+n+1} \mid f(y^1, x^1, x_\ell) \leq 0 \ \right\} \tag{3.2}$$

and

$$T_2 = \left\{ \langle y, z, x^1, x_\ell \rangle \in \mathbf{R}^{m+n+1} \mid y_\nu = g(x_\ell) \right\}. \tag{3.3}$$

The set T_1 is a standard technology set, reflecting the way the inputs get transformed into intended outputs, imbedded in a larger, $(m+n+1)$-dimensional space. Standard disposability properties in both inputs and intended outputs can be imposed on this set. The set T_2, reflects nature's residual-generation mechanism. It is an $(m+n)$-dimensional manifold in $m+n+1$-space; that is, if g were linear, T_2 would be an $(m+n)$-dimensional hyperplane in $(m+n+1)$-space. The constraint in (3.3) imposes restrictions only on z and x_ℓ; the other variables are free.

[9] See also the other citations in footnote 4.

[10] We continue to abstract from abatement activities, which will be discussed in the next section.

The above specification of the sets T_1 and T_2 implies that the overall technology T is a manifold and hence a set with no interior relative to $\mathbf{R}^{m+n+1}$. Under the assumptions that y is a scalar, ℓ is the only input, T_1 satisfies strong disposability in input and the intended output, and the function g is linear, Figure 3 depicts the sets T_1 and T_2 in $\mathbf{R}^3$. Clearly, T cannot satisfy standard disposability conditions in *all* coordinate directions. It is simple to show, however, that T has the same disposability properties as T_1 in the y and x^1 directions. Define the output-possibility set in y-space as

$$\begin{aligned} P\big(z, x^1, x_\ell\big) &= \big\{ y \in \mathbf{R}^m \mid \langle y, z, x^1, x_\ell \rangle \in T \big\} \\ &= \big\{ y \in \mathbf{R}^m \mid f(y, x^1, x_\ell) \leq 0 \ \wedge \ z = g(x_\ell) \big\}. \end{aligned} \tag{3.4}$$

If T_1 satisfies (strong) output disposability in y, $\bar{y} \leq y$ implies $f(\bar{y}, x^1, x_\ell) \leq 0$ and $z = g(x_\ell)$, which in turn implies that $\langle \bar{y}, z, x^1, x_\ell \rangle \in T$. Thus, $\bar{y} \in P(z, x^1, x_\ell)$, and T satisfies strong disposability in y, the vector of intended outputs. Similarly, define the input-requirement set as

$$L(y, z, x_\ell) = \big\{ x^1 \in \mathbf{R}^{n-1} \mid \langle y, z, x^1, x_\ell \rangle \in T \big\}. \tag{3.5}$$

Consider $\langle \bar{x}^1, x_\ell \rangle \geq \langle x^1, x_\ell \rangle$. If T_1 satisfies strong input disposability, we have $f(y, \bar{x}^1, x_\ell) \leq 0$ and $z = g(x_\ell)$. Therefore, $\langle y, z, \bar{x}^1, x_\ell \rangle \in T$ and $\bar{x}^1 \in L(y, z, x_\ell)$. Thus, T also satisfies strong disposability in x^1, the vector of non-residual-causing inputs.

Now define the "output" possibility set in $\langle y, z \rangle$-space as

$$\begin{aligned} \hat{P}\big(x^1, x_\ell\big) &= \big\{ \langle y, z \rangle \in \mathbf{R}^{m+1} \mid \langle y, z, x^1, x_\ell \rangle \in T \big\} \\ &= \big\{ \langle y, z \rangle \in \mathbf{R}^{m+1} \mid f(y, x^1, x_\ell) \leq 0 \ \wedge \ z = g(x_\ell) \big\}. \end{aligned} \tag{3.6}$$

Suppose that $\langle z, y \rangle \in \hat{P}(x^1, x_\ell)$ and $\langle \bar{z}, \bar{y} \rangle = \langle \lambda z, \lambda y \rangle,\ \lambda \in (0,1)$. Strong (and hence weak) disposability of T_1 in y implies $f(\bar{y}, x^1, x_\ell) \leq 0$ but $\bar{z} \neq g(x_\ell) = z$. Thus, $\langle \bar{y}, \bar{z} \rangle \notin \hat{P}(x^1, x_\ell)$, and hence T is *not* weakly disposable in $\langle y, z \rangle$.

The incorporation of the externality-generation mechanism as a part of the technology of the firm implies that, for all $\langle y, z \rangle \in \hat{P}(x^1, x_\ell)$, the level of the residual, z, is fixed at $g(x_\ell)$. This brings out the fact that, while the technology T allows joint production of the vector of intended outputs, there is only by-production of the residual (the level of residual is fixed by the extent of usage of the input l). Figure 4(b) depicts the set, $\hat{P}(x^1, x_\ell)$ in the case where $m = 2$. In particular, if y were a scalar ($m = 1$) then $\hat{P}(x^1, x_\ell)$, under the assumption of free disposability of y_i, is a vertical line. That is, the frontier of $\hat{P}(x^1, x_\ell)$ degenerates to a point: there exists a maximum level of intended output that can be produced from a given level of the input vector. This phenomenon is illustrated in Figure 4(a).

This structure of $\hat{P}(x^1, x_\ell)$ implies that the trade-off between z and an intended output y_j (obtained as the slope of the output possibility frontier in the $\langle y_j, z\rangle$-space) is undefined. What can be defined, however, is a *correlation* between residual generation and production of an intended output. This correlation can be obtained by a suitable reparametrization of the overall technology T. In what follows we obtain such a reparametrization by a repeated application of the implicit function theorem (though it could also be obtained by direct application of the general version of the implicit function theorem for a system of implicit functions).

Consider a $\langle \hat{y}, \hat{z}, \hat{x}^1, \hat{x}_\ell\rangle \in T$ such that

$$\begin{aligned} f(\hat{y}, \hat{x}^1, \hat{x}_\ell) &= 0 \\ \hat{z} &= g(\hat{x}_\ell) \\ f_j(\hat{y}, \hat{x}^1, \hat{x}_\ell) &\geq 0, \;\; j = 1, \ldots m_1, \\ f_k(\hat{y}, \hat{x}^1, \hat{x}_\ell) &\leq 0, \;\; k = 1, \ldots n, \\ f_j(\hat{y}, \hat{x}^1, \hat{x}_\ell) &> 0 \text{ for some } j \\ g'(\hat{x}_\ell) &> 0. \end{aligned} \tag{3.7}$$

Exploiting the last inequality in (3.7), we can invert g in x_ℓ, in a neighborhood of $\langle \hat{z}, \hat{x}_\ell\rangle$, to obtain

$$x_\ell = g^{-1}(z) =: h(z). \tag{3.8}$$

Substituting into (3.1), we obtain (again locally) another equivalent functional representation of the technology:

$$T = \Big\{\langle y, z, x^1, x_\ell\rangle \in \mathbf{R}^{m+n+1} \mid f\big(y, x^1, h(z)\big) =: \tilde{f}\big(y, x^1, z\big) \leq 0 \;\wedge\; x_\ell = h(z)\Big\}. \tag{3.9}$$

Once again, from the implicit function theorem, it follows that there exist neighborhoods U and V around $\hat{u} := \langle \hat{y}^{-j}, \hat{x}^1, \hat{z} \rangle$ and $\hat{y}_j$ and a function $\psi^j : U \to V$ such that

$$\hat{y}_j = \psi^j(\hat{u}) \tag{3.10}$$

and

$$\tilde{f}\big(u, \psi^j(u)\big) = 0 \quad \forall\ u \in U. \tag{3.11}$$

This yields the following form of the technology T, where y^{-j}, x^1, and z are the endogenous variables and y_j and x_ℓ are the exogenous variables:

$$T = \Big\{ \langle y, z, x^1, x_\ell \rangle \in \mathbf{R}^{m+n+1} \mid y_j \leq \psi^j(y^{-j}, x^1, z) \wedge\ x_\ell = h(z) \Big\}. \tag{3.12}$$

This parametrization of the overall technology implies the following "trade-off" between pollution and the quantity of the intended output j in the neighborhood $U \times V$:

$$\begin{aligned} \frac{\partial \psi^j(u)}{\partial z} &= -\frac{\tilde{f}_z\big(y, x^1, z\big)}{\tilde{f}_j\big(y, x^1, z\big)} \\ &= -\frac{f_\ell\big(y, x^1, x_\ell\big)\ h'(z)}{f_j(y, x^1, x_\ell)} \geq 0. \end{aligned} \tag{3.13}$$

How should one interpret this non-negative "trade-off" between y_j and z? Starting at $\langle \hat{y}_j, \hat{z} \rangle$ in Figure 5, an increase in z is attributable, by the material-balance condition, to an increase in x_ℓ. Under the conventional assumptions in (3.7), the trade-off between the pollution-generating input and a non-pollution-generating input, say input k, is

$$-\frac{f_k(y, x^1, x_\ell)}{f_\ell(y, x^1, x_\ell)} \leq 0. \tag{3.14}$$

Thus, an increase in the usage of the pollution-generating input—implied by the increase in pollution itself—allows the firm to produce output $\hat{y}_j$ with less of other inputs or to produce more of output j with the same amounts of the non-pollution-generating inputs. This interpretation is apparent if we re-write (3.11) as

$$\frac{\partial \psi^j(u)}{\partial z} = MP_{\ell j}(y, x^1, x_\ell) \cdot h'(z), \tag{3.15}$$

where $MP_{\ell j}(y, x^1, x_\ell)$ is the marginal product of input ℓ in producing output j and $h'(z)$ is the change in x_ℓ implied by the change in the level of pollution and the material balance condition.

To summarize, the non-negative "trade-offs" between intended and incidental outputs is explained by (i) the material balance approach, which requires specification of nature's residual-generation mechanism, and (ii) input substitutabilities between pollution-generating and non-pollution-generating inputs.[11] We emphasize that the "trade-off" in (3.13) does not reflect movement along a production possibility frontier. It is *not*, therefore, a "trade-off" in the usual sense of the term. Rather, it reflects only a *correlation* between the residual and an intended output.

Specifying the technology as in (3.1) helps us understand the difference between the profit-maximizing firm's behavior under an unregulated regime and under a regulated regime. In an unregulated regime, a profit-maximizing firm only takes into account the information in the T_1 technology: it is not concerned about the residuals it generates because of its decisions about the production of intended outputs. In a regulated regime, where the regulator uses prices or quantity controls on the residual, the firm is forced to internalize the information

[11] The positive correlation between intended and incidental outputs can be weakened by assuming, as may be the case in some instances, that the generation of the residual also adversely affects the productivity of inputs. In that case, the technology is respecified as

$$T = \{u := \langle y, z, x^1, x_\ell\rangle \in \mathbf{R}^{m+n+1} \mid f(y, z, x^1, x_\ell) \leq 0 \ \wedge \ z = g(x_\ell)\}. \tag{3.16}$$

Maintain the convention that $f_i(u) \geq 0,\ i = 1, \ldots, m$, and $f_k(u) \leq 0,\ k = 1, \ldots, n$, and, to capture the negative effect that pollution may have on the production of intended outputs, assume that

$f_z(u) \geq 0$. Suppose, moreover, that $g'(\hat{x}_\ell) > 0$ and $f_z(\hat{u}) > 0$, so that inversion of the second constraint in (3.16), $x_\ell = g^{-1}(z) =: h(z)$, substitution of x_ℓ into the first constraint, and application of the implicit function theorem give us the following trade-off between pollution and the j^{th} intended output in a neighborhood of $\hat{u}$:

$$\frac{\partial \psi^j\left(u^{(-j)}\right)}{\partial z} = -\left[\frac{f_z(u)}{f_j(u)} + \frac{f_\ell(u)\ h'(x_\ell)}{f_j(u)}\right]. \tag{3.17}$$

The first term on the right side of this identity reflects the effect on y_j as a result of decreased productivity of inputs owing to the increased pollution level, while the second term reflects the effect on y_j brought about by a change in the level of usage of the pollution-generating input, contingent on a change in z, holding all other inputs fixed. Given the sign conventions, these two effects work in opposite directions; if the first effect is of second-order importance, the correlation between y_j and z remains non-negative.

in T_2 into its decision making. The difference in the regulated and the unregulated regimes, therefore, is not in the basic technology but rather in the firm's behavior. In both regimes the extent of residual generation is determined by the material balance condition, which is a primitive part of the technology.

4. Incorporating an Abatement Mechanism.

While the literature traces the source of the problem of residual generation to the material balance condition, it does not explain the positive correlation between the residual generation and the production of intended outputs in terms of this natural scientific phenomenon. Rather, it attributes this correlation to the resource costs of disposal of residuals.

More specifically, the argument goes as follows. Resources can be used for jointly producing various combinations of intended outputs and abatement activities. The more resources are diverted to the latter activities, the less they are available for producing intended outputs. Hence, an increase in the level of abatement activities leads concomitantly to both lower residual generation and lower production of intended output. Although this argument provides an alternative explanation for the positive correlation between the by-product and the production of intended outputs, the abatement activities themselves are never, to our knowledge, explicitly modeled as an integral part of the technology. We show that, when abatement is explicitly modeled, the positive correlation between by-production and production of intended output is, once again, a direct consequence of acknowledging nature's material balance condition and clearly identifying the residual-generation mechanism.

Abatement activities can be incorporated into the production framework in two ways: (1) by allowing for the substitutability between inputs possessing varying degrees of residual generation tendencies and (2) allowing for the purchase or production of specific abatement outputs that can be used to reduce pollution. Since the abatement activities typically are not explicitly modeled, the distinction between purchased and produced abatement activities is not clarified. Depending, however, on the way abatement is implemented—by purchasing or by producing—the technology has different characteristics.

Let us again keep the analysis simple by sticking to a single abatement output (as well as a single incidental output). To allow the possibility of input substitutability in the generation of residuals, we partition the vector of all n inputs into n_1 non-residual-generating inputs, denoted by x^1, and n_2 residual-generating inputs, denoted by x^2.

We consider first the case where abatement is produced within the production unit of the residual generating firm. Abatement, in this case, has to be viewed as an intermediate input of production. Specify the technology as follows:

$$T = \left\{ u := \langle y, z, x^1, x^2 \rangle \in \mathbf{R}^{m+n+1} \mid \exists\, y_a \text{ satisfying } f(y, y_a, x^1, x^2) \leq 0 \ \wedge \ z = g(y_a, x^2) \right\}, \tag{4.1}$$

where y_a is an abatement output. T reflects the transformation of inputs into intended outputs and abatement output and the use of the abatement output by the firm to control the generation of the residual that results from intended production (following the material balance condition). We confine ourselves again to a local analysis and posit the following signs of the partial derivatives at $\hat{u}$:

$$\begin{aligned} f_j(\hat{y}, \hat{y}_a, \hat{x}^1, \hat{x}^2) &\geq 0, \ j = 1, \ldots m, \\ f_a(\hat{y}, \hat{y}_a, \hat{x}^1, \hat{x}^2) &> 0, \\ f_k(y, \hat{y}_a, \hat{x}^1, \hat{x}^2) &\leq 0, \ k = 1, \ldots n, \\ g_a(\hat{y}_a, \hat{x}^2) &< 0, \\ g_\ell(\hat{y}_a, \hat{x}^2) &\geq 0 \quad \text{for all } \ell, \\ g_\ell(\hat{y}_a, \hat{x}^2) &> 0 \quad \text{for some } \ell. \end{aligned} \tag{4.2}$$

The first two inequalities reflect a negative (or at least non-positive) trade-off between standard outputs and the abatement output.

In a neighborhood of $\hat{u}$, we have

$$\begin{aligned} y_a = g^{-1}(z, x^2) &=: h(z, x^2) \\ \frac{\partial h(z, x^2)}{\partial z} &< 0 \\ \frac{\partial h(z, x^2)}{\partial x_k^2} &= -\frac{g_k(z, x^2)}{g_a(z, x^2)} > 0. \end{aligned} \tag{4.3}$$

The last of these trade-offs reflects the fact that an increase in x_k^2 requires an increase in y_a in order to achieve the same level of z. Thus, we obtain an equivalent representation of the technology:

$$\begin{aligned} T = \big\{ u :=& \langle y, z, x^1, x^2 \rangle \in \mathbf{R}^{m+n+1} \ \big| \\ & f\big(y, h(z, x^2), x^1, x^2\big) =: \tilde{f}\big(y, z, x^1, x^2\big) \leq 0 \big\}. \end{aligned} \tag{4.4}$$

The level of abatement (an intermediate input) corresponding to the production plan $\langle y, z, x^1, x^2 \rangle$ is

$$y_a = h(z, x^2).$$

T is the reduced-form technology that is the starting point of much of the current research that uses abatement as an argument for positing the positive correlation between y and z. Note that, as in this literature, T is not a manifold in $\mathbf{R}^{m+n+1}$ and has a well defined interior. The next question is: what disposability conditions does T satisfy? Here we find that locally around $\hat{u}$,

$$\begin{aligned} \tilde{f}_j(y, x^1, x^2) &= f_j(y, y_a, x^1, x^2) \geq 0, \ \ j = 1, \ldots, m, \\ \tilde{f}_k(y, x^1, x^2) &= f_k(y, y_a, x^1, x^2) \leq 0, \ \ k = 1, \ldots, n_1. \end{aligned} \tag{4.5}$$

Thus, locally, T is strongly disposable in y and x^1. There remains a problem, however. The usual sign conventions of derivatives of $\tilde{f}$ with respect to x^2 are not compelling. For example,

$$\frac{\partial \tilde{f}\big(y^1, z, x^1, x^2\big)}{\partial x_k^2} = f_a(y^1, y_a, x^1, x^2) \ h_k(z, x^2) + f_k(y^1, y_a, x^1, x^2). \tag{4.6}$$

Under the standard sign conventions in (4.2) and (4.3), the first term on the right is positive and the second is negative. Hence, the sign of the derivative on the left is ambiguous, conflicting with the usual conventions about inputs and outputs in a primitive technology.

Thus, the standard approach of modeling $\tilde{f}$ as if it represented a primitive technology, with positive output derivatives and negative input derivatives is not consistent with the underlying model of material balance and pollution abatement. The deterministic, nonparametric (DEA) models that incorporate (strong) input disposability (*e.g.*, Färe, Grosskopf, Lovell, and Yaisawarng [1989] and Ball, Lovell, Nehring, and Somwaru [1994]) are therefore open to question.[12]

Now, if $\tilde{f}_i(\hat{y}, \hat{z}, \hat{x}^1, \hat{x}^2) \neq 0$ for some i, there exists a function defined in a neighborhood of a frontier point $\hat{u}$ such that

$$y_i = \psi^i\big(y^{(-i)}, z, x^1, x^2\big), \tag{4.7}$$

and the tradeoff between an intended output y_i and z is obtained as,

$$\begin{aligned}\frac{\partial \psi^i\big(y^{(-i)}, z, x^1, x^2\big)}{\partial z} &= -\frac{\tilde{f}_\nu\big(y^1, z, x^1, x^2\big)}{\tilde{f}_i\big(y^1, z, x^1, x^2\big)} \\ &= -\frac{f_a(y^1, y_a, x^1, x^2)\ h_\nu(z, x^2)}{f_i(y^1, y_a, x^1, x^2)} \geq 0,\end{aligned} \tag{4.8}$$

again establishing the positive (or at least non-negative) correlation between intended and incidental outputs. This reduced-form trade-off does capture the shadow cost of reducing pollution in terms of foregone production of intended outputs. The right-hand side of (4.8) can be written as

$$RCT_{ai}(y^1, y_a, x^1, x^2) \cdot h_\nu(z, x^2), \tag{4.9}$$

where the first term is the rate of commodity substitution between the abatement output and commodity i and the second term is the reduction in abatement output facilitated by an increase in the pollution level. Thus, modeled in this way, the normal of the reduced form output possibility set does provide the appropriate set of shadow prices.

12 See Førsund [1998] for additional critiques of this approach.

Note also that, in contrast to Section 3 (where we abstracted away from abatement options), there may now exist considerable variation in the levels of z (for fixed levels of all inputs) among points lying in the output possibility set in $\langle y, z\rangle$-space. However each point corresponds to a different level of abatement output, which is treated as an intermediate input in production and is not given the status of an explicit production variable.

Next consider the case where y_a is a purchased output for the residual-generating firm. Then we specify the technology as

$$T = \left\{ u := \langle y, y_a, z, x^1, x^2 \rangle \in \mathbf{R}^{m+n+2} \;\middle|\; f(y, y_a, x^1, x^2) \leq 0 \;\wedge\; z = g(y_a, x^2) \right\}. \tag{4.10}$$

In this case, y_a is not an intermediate input and T lies in a larger ($m+n+2$-dimensional) space, where abatement is also given a coordinate direction. Clearly T is a manifold and hence a set with no interior, and the arguments in section 3 can be used to demonstrate that T is strongly disposable in y and x^1. It is not, however, weakly disposable with respect to y and z. Trade-offs, in the usual sense, between y and z are not defined. What are defined are correlations, obtained from a suitable reparametrization of T.

5. Conclusion.

Using simple calculus-based arguments, we have reached the following conclusions:

- Standard single-equation modeling of pollution-generating technologies, in which the pollution variables simply mimic either conventional inputs or conventional outputs, is inconsistent with the material balance condition—a fundamental imperative of physical science—as well as with common sense.

- Explicit modeling of the material balance condition (abstracting from abatement activities) yields the intuitively desirable positive correlation between pollution and the conventional outputs, but this correlation does not reflect a trade-off along a production frontier—the trade-off is in fact undefined—and hence does not yield shadow prices of pollution. This structure, moreover, is inconsistent with (weak or strong) disposability of pollution and conventional outputs (taken jointly).

• Modeling abatement in terms of purchased abatement inputs runs into the same problems as modeling without abatement. Modeling abatement as an intermediate input (produced and used by the firm) does yield a positive trade-off between pollution and conventional outputs, but the conventional trade-offs among conventional inputs and outputs in the "reduced form" technology are not standard; hence, standard DEA methods of constructing the reduced form technology and of calculating the shadow price of pollution are problematic.

We believe that these results have implications for empirical work that models and estimates pollution-generating technologies and the shadow price of pollution.

REFERENCES

Aryes, R. U., and A. V. Kneese [1969], "Production, Consumption and Externalities," *American Economic Review* 59: 282–297.

Ball, V. E., C. A. K. Lovell, R. F. Nehring, and A. Somwaru (1994), "Incorporating Undesirable Outputs into Models of Production: An Application to US Agricultural," *Cahiers d'Economique et Sociologie Rurales* 31: 59–73.

Baumol, W. J., and W. E. Oates [1988], *The Theory of Environmental Policy*, Second Edition, Cambridge: Cambridge University Press.

Boyd III, J. H., and J. P. Conley [1997], "Fundamental Nonconvexities in Arrovian Markets and the Coasian Solution to the Problem of Externalities," *Journal of Economic Theory* 72: 388–407.

Cropper, M. L., and W. E. Oates [1992], "Environmental Economics: A Survey," *Journal of Economic Literature* 30: 675–740.

Färe, R., and S. Grosskopf [1998], "Shadow Pricing of Good and Bad Commodities," *American Journal of Agricultural Economics* 80: 584–590.

Färe, R., S. Grosskopf, C. A. K. Lovell, and S. Yaisawarng [1993], "Derivation of Shadow Prices for Undesirable Outputs: A Distance Function Approach," *The Review of Economics and Statistics* 75: 374–380.

Färe, R., S. Grosskopf, and W. L. Weber [2002], "Shadow Prices and Pollution Costs in U.S. Agriculture," presented at the Second World Congress of Environmental and Resource Economists, Monterey, CA.

Førsund, F. [1998], "Pollution Modelling and Multiple-Output Production Theory," Discussion Paper #D-37/1998, Department of Economics and Social 1333 Sciences, Agricultural University of Norway (c:\ finnforsk\ multienv1.wpd).

Frisch, R. [1965], *Theory of Production*, Dordrecht: D. Reidel Publishing Company.

Laffont, J-J. [1988], *Fundamentals of Public Economics*, Cambridge: MIT Press.

Murty, M. N., and S. Kumar [2002] *Environmental and Economic Accounting for Industry*, forthcoming, New Delhi: Oxford University Press.

Pittman, R. W. [1983], "Multilateral Productivity Comparisons with Undesirable Outputs," *The Economic Journal* 93: 883–391.

Reinhard, S., C. A. K. Lovell, and G. J. Thijssen [1999], "Econometric Estimation of Technical and Environmental Efficiency: An Application to Dutch Dairy Farms," *American Journal of Agricultural Economics* 81: 44–60.

Shephard, R. W. [1974], *Indirect Production Functions*, Meisenheim Am Glan: Verlag Anton Hain.

Starrett, D. [1972], "Fundamental Nonconvexities in the Theory of Externalities," *Journal of Economic Theory* 4: 180–199.

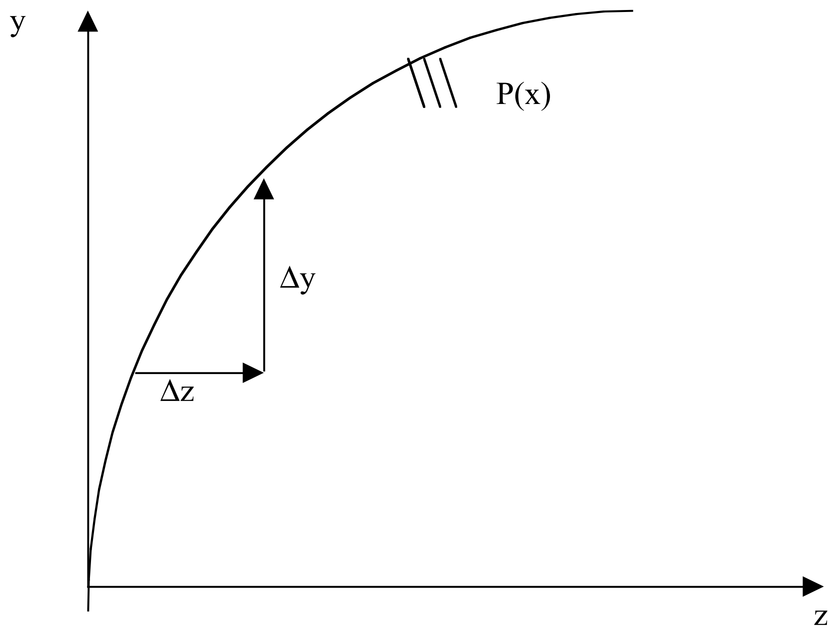

Figure 1

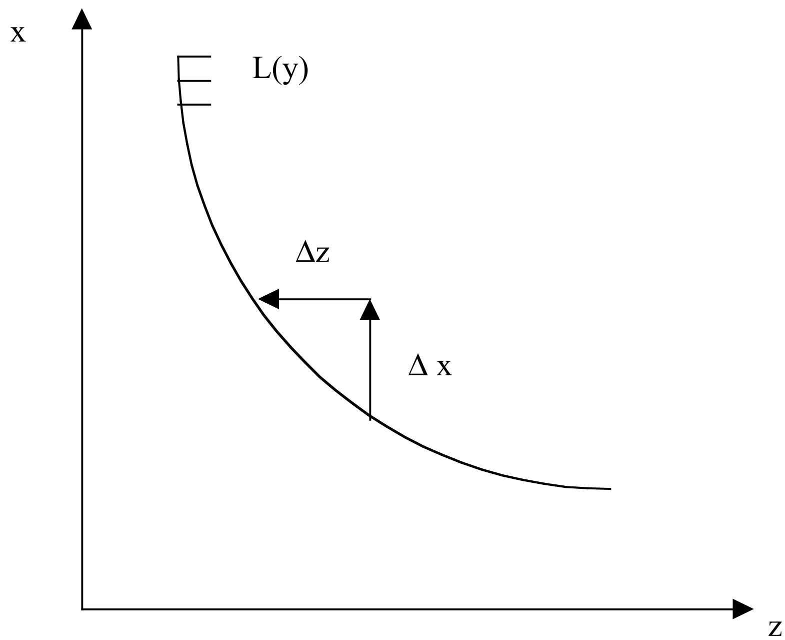

Figure 2

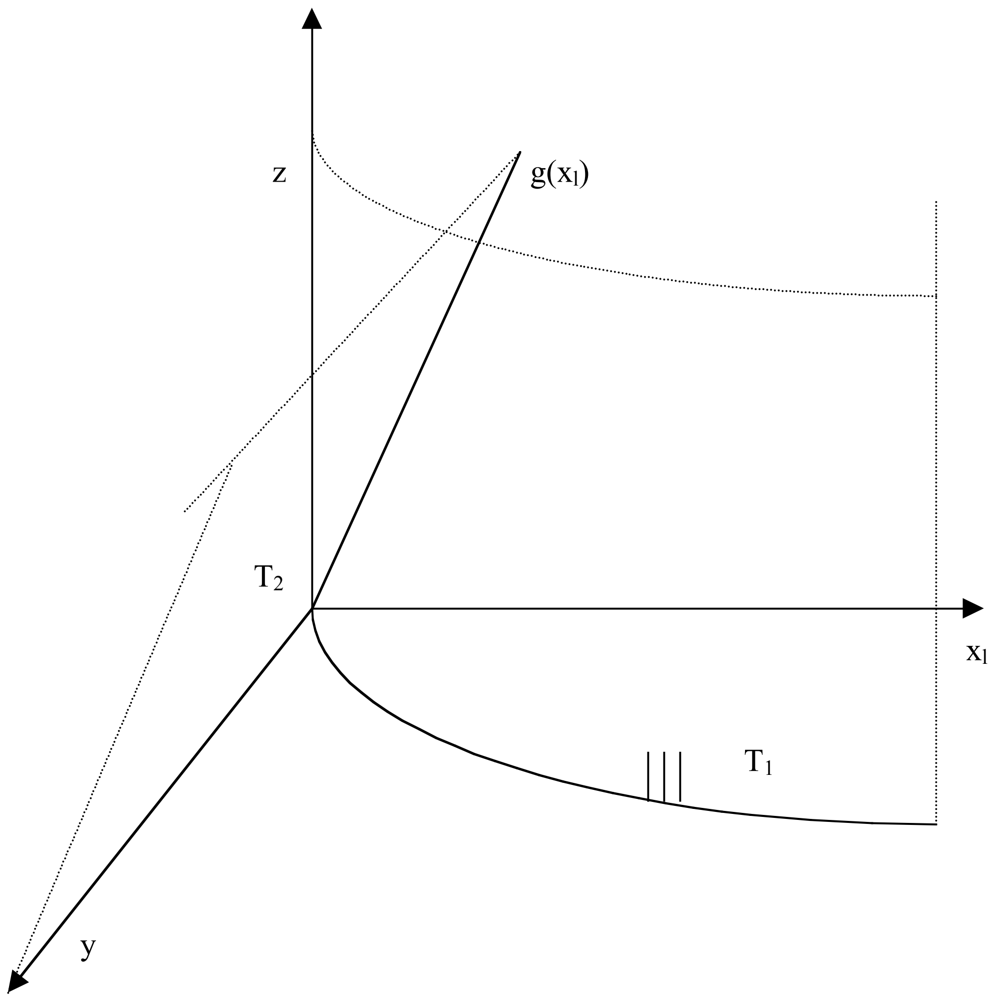

Figure 3

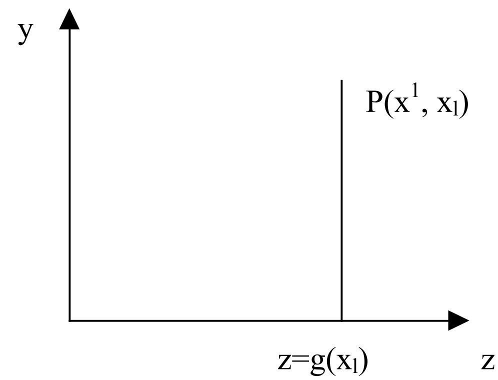

Figure 4(a)

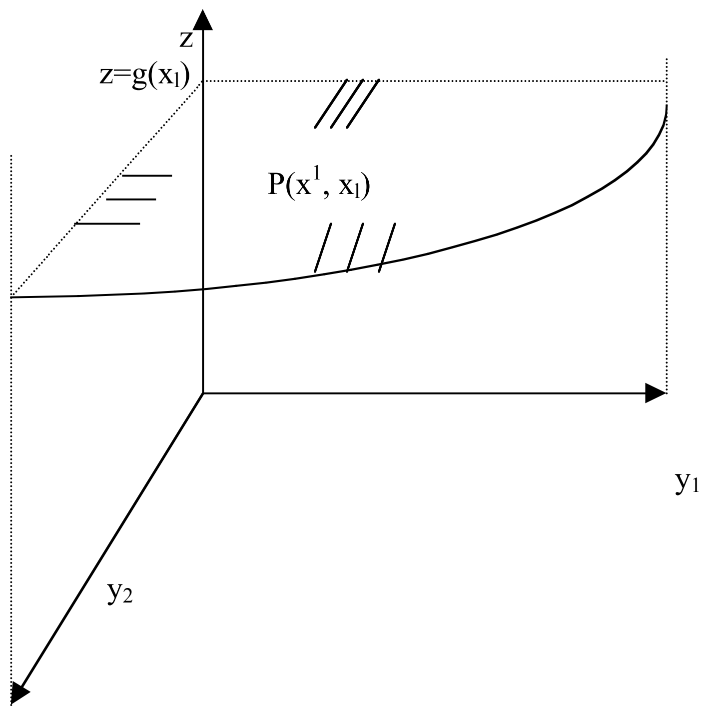

Figure 4(b)

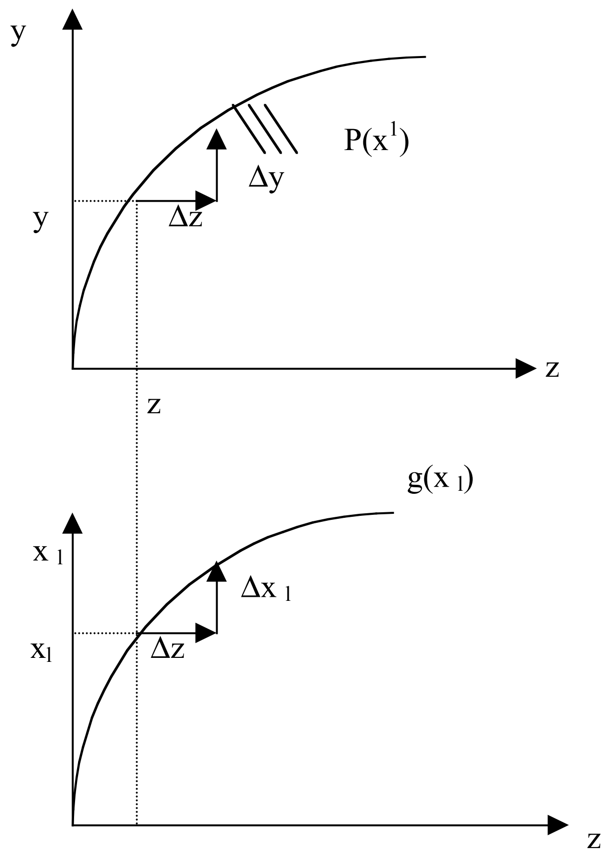

Figure 5