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Dual Elasticities of Substitution

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Abstract

We argue that, for more than two inputs, different elasticity of substitution concepts must be used to answer different questions about substitutability among inputs. To assess the effects of price changes, direct elasticities should be used; to assess the effects of quantity changes, dual elasticities should be used. (Direct) Allen-Uzawa elasticities are ubiquitous; (direct) Morishima elasticities are gradually working their way into the literature. Dual Allen-Uzawa and Morishima elasticities (for multiple-output, non-homogeneous technologies) have been formulated but have not yet been applied. We show that the dual Allen-Uzawa elasticity is identical to the Hicks elasticity of complementarity (for a single output) iff there are constant returns to scale, an assumption we easily reject when we apply each of these these elasticity concepts to the Swiss labor market for domestic and guest workers.
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1. Introductory Remarks.

The elasticity of substitution was introduced by Hicks [1932] as a tool for analyzing capital and labor income shares in a growing economy with a constant-returns-to-scale technology and neutral technological change. Defined as the logarithmic derivative of the capital/labor ratio with respect to the technical rate of substitution of labor for capital, the elasticity is higher the "easier" is the substitution of one input for the other (the lesser is the degree of "curvature" of the isoquant). Under the assumption of cost-minimizing, price-taking behavior, it is the logarithmic derivative of the capital/labor ratio with respect to the factor-price ratio (the ratio of the wage rate to the rental rate on capital), and it yields immediate (differential) qualitative and quantitative comparative-static information about the effect on relative income shares of changes in factor price ratios.

Under the Hicks predicates, the inverse of the elasticity—the logarithmic derivative of the technical rate of substitution of labor for capital (the factor shadow-price ratio) with respect to the capital/labor ratio—contains the same information, but larger values indicate "less easy" substitution of the two factors for one another (greater "curvature" of the isoquant). Under the assumption of competitive factor markets, it yields immediate comparative-static information about the effect on (absolute and relative) income shares of changes in factor quantity ratios.

Thus, whether one is interested in the effects of changes in factor-price ratios on factor-quantity ratios or the effects of changes in quantity ratios on price ratios (and in each case the effects on relative factor shares), the Hicksian two-factor elasticity of substitution provides complete (differential) qualitative and quantitative comparative-static information. But matters get more complicated when one's analysis of substitutability and the comparative statics of relative income shares entails more than two factors of production. Prominent examples in the literature include the analyses of

- the effect of energy-cost explosions using KLEM (capital, labor, energy, materials) data (*e.g.*, Berndt and Wood [1975], Davis and Gauger [1996], and Thompson and Taylor [1995]),

- the effect of increases in human capital (or in educational attainment) on the relative wages of skilled and unskilled labor, with capital as a third important input (*e.g.*, Griliches [1969], Johnson [1970], Kugler, Müller, and Sheldon [1989], and Welch [1970]),

- the substitutability of (multiple) monetary assets (*e.g.*, Barnett, Fisher, and Serletis [1992], Davis and Gauger [1996], and Ewis and Fischer [1985]),

- the effect of increases in the number of guest workers on resident and non-resident labor, with capital and other inputs as well (*e.g.*, Kohli [1999]),

and

- the effect of immigration on the relative wages of domestic and immigrant labor (*e.g.*, Grossman [1982], Borjas [1994], and Borjas, Freeman, and Katz [1992, 1996]).

The first issue that arises is that there exist more than one generalization of the Hicksian two-variable elasticity of substitution. Allen and Hicks [1934] and Allen [1938] suggested two generalizations. One, further analyzed by McFadden [1963], eventually lost favor because it does not allow for optimal adjustment of all inputs to changes in factor prices. The other, further analyzed by Uzawa [1962], became the dominant concept; perhaps tens of thousands of "Allen-Uzawa elasticities" have been estimated. Later, Morishima [1967] and Blackorby and Russell [1975, 1981, 1989] proposed an alternative to the Allen-Uzawa elasticity; the latter argued that this "Morishima elasticity" has attractive properties not possessed by the Allen-Uzawa elasticity. Recently, the Morishima elasticity has been gaining favor (see *e.g.*, Davis and Gauger [1996], de la Grandville [1997], Klump and de la Grandville [2000], Ewis and Fischer [1985], Flaussig [1997], and Thompson and Taylor [1995]).

Second, when one advances to more than two inputs, the measurement of the effect of changes in price ratios on quantity ratios and the effect of changes in quantity ratios on price ratios are not simple inverses of one another. The Allen-Uzawa elasticity is formulated in terms of effects of price changes on input demands, but many issues revolve around the effect of quantity changes on price ratios (*e.g.*, the effect of immigration or of increases in the number of guest workers on relative wages or the effect of increases in the number of skilled workers relative to unskilled workers on relative wages or return to education). Hicks [1970] suggested a dual to the Allen-Uzawa elasticity, formulated in terms of a scalar-valued, linearly homogeneous production function. The Allen-Uzawa elasticity, however, is well defined for non-homogeneous production technologies with multiple outputs as well as multiple inputs. Blackorby and Russell [1981] formulated duals to both the Allen-Uzawa and the Morishima elasticity using the distance function, which is symmetrically dual to the cost function employed by Uzawa to reformuate the Allen elasticity.[1] These dual elasticities are well defined for multiple-output, non-homogeneous production technologies. Non-homogeneity is an especially important property when more than two inputs are employed, because it is typically easy to reject homogeneity for production technologies with more than two inputs. In addition, many papers employ the direct elasticity when the dual elasticity is the appropriate concept. To our knowledge, neither of these elasticities has heretofore been applied. Thus, a principal objective of this paper is to illustrate the use and implementation of dual elasticities.

The next section summarizes the relevant literature on elasticity-of-substitution concepts (alluded to above). Section 3 describes a method of estimating these elasticities using, alternatively, a translog cost function and a translog distance function and applies these concepts to the issue of substitutability of resident and non-resident (guest) labor (and other inputs) in the Swiss economy using a data base provided by Kohli [1999].[2] Section 4 concludes.

[1] Formally, the Blackorby-Russell concepts were formulated for a single-output production technology, but in fact all of their results go through for multiple-output technologies.

[2] There is current talk of a more-elaborate guest worker program in the U.S.; whether our results for Switzerland provide any insights into the possible consequences of the proposed policy for the U.S. labor market is a speculative question we leave to the reader.

2. Elasticities of Substitution and Income Shares.

2.1 Representations of Technologies.

Input and output quantity vectors are denoted $x \in \mathbf{R}^n_+$ and $y \in \mathbf{R}^m_+$, respectively. The technology set is the set of all feasible ⟨input, output⟩ combinations:

$$T := \big\{\langle x, y\rangle \in \mathbf{R}^{n+m}_+ \mid x \text{ can produce } y\big\}.$$

While the nomenclature suggests that feasibility is a purely technological notion, a more expansive interpretation is possible: feasibility could incorporate notions of institutional and political constraints, especially when we consider entire economies as the basic production unit. An input requirement set for a fixed output vector y is

$$L(y) := \big\{x \in \mathbf{R}^n_+ \mid \langle x, y\rangle \in T\big\}. \tag{2.1}$$

We assume throughout that $L(y)$ is closed, strictly convex, and twice differentiable[3] for all $y \in \mathbf{R}^m_+$ and satisfies strong input disposability,[4] output monotonicity,[5] and "no free lunch."[6]

The (input) distance (gauge) function, a mapping from[7]

$$Q := \big\{\langle x, y\rangle \in \mathbf{R}^{n+m}_+ \mid y \neq 0^{(m)} \ \wedge \ x \neq 0^{(n)} \ \wedge \ L(y) \neq \emptyset\big\}$$

into the positive real line (where $0^{(n)}$ is the null vector of $\mathbf{R}^n_+$), is defined by

$$D(x, y) := \max\big\{\lambda \mid x/\lambda \in L(y)\big\}. \tag{2.2}$$

Under the above assumptions, D is well defined on this restricted domain and satisfies homogeneity of degree one, positive monotonicity, concavity, and continuity in x and negative

[3] These assumptions are stronger than needed for much of the conceptual development that follows, but in the interest of simplicity we maintain them throughout.

[4] $L(y) = L(y) + \mathbf{R}^n_+ \ \forall\, y \in \mathbf{R}^m_+$.

[5] $\bar{y} \geq y \Rightarrow L(\bar{y}) \subset L(y)$.

[6] $y \neq 0^{(m)} \Rightarrow 0^{(n)} \notin L(y)$.

[7] We restrict the domain of the distance function to assure that it is globally well defined. An alternative approach (*e.g.*, Färe and Primont [1995]) is to define D on the entire non-negative $(n + m)$-dimensional Euclidean space and replace "max" with "sup" in the definition. See Russell [1997] for a comparison of these approaches.

monotonicity in y. We assume, in addition, that it is continuously twice differentiable in x. (See, *e.g.*, Färe and Primont [1995] for proofs of these properties and most of the duality results that follow.[8])

The distance function is a representation of the technology, since (under our assumptions)

$$\langle x, y\rangle \in T \iff D(x,y) \geq 1.$$

In the single-output case ($m = 1$), where the technology can be represented by a production function, $f : \mathbf{R}^n_+ \to \mathbf{R}_+$, $D\big(x, f(x)\big) = 1$ and the production function is recovered by inverting $D(x, y) = 1$ in y. If (and only if) the technology is homogeneous of degree one (constant returns to scale),

$$D(x,y) = \frac{f(x)}{y}. \tag{2.3}$$

The cost function, $C : \mathbf{R}^n_{++} \times Y \to \mathbf{R}_+$, where

$$Y = \big\{y \mid \langle x, y\rangle \in Q \text{ for some } x\big\}, \tag{2.4}$$

is defined by

$$C(p,y) = \min_x \big\{p\cdot x \mid x \in L(y)\big\}$$

or, equivalently, by

$$C(y,p) = \min_x \big\{p\cdot x \mid D(x,y) \geq 1\big\}. \tag{2.5}$$

Under our maintained assumptions, D is recovered from C by

$$D(x,y) = \inf_p \big\{p\cdot x \mid C(p,y) \geq 1\big\}, \tag{2.6}$$

and C has the same properties in p as D has in x. On the other hand, C is *positively* monotonic in y. We also assume that C is twice continuously differentiable in p.

[8] Whatever is not there can be found in Diewert [1974] or the Fuss/McFadden [1978] volume.

By Shephard's Lemma (application of the envelope theorem to (2.5)), the (vector-valued, constant-output) input demand function, $\delta : \mathbf{R}^n_{++} \times Y \rightarrow \mathbf{R}^n_+$, is generated by first-order differentiation of the cost function:

$$\delta(p, y) = \nabla_p C(p, y). \tag{2.7}$$

Of course, δ is homogeneous of degree zero in p. The (normalized) shadow-price vector, $\rho : Q \rightarrow \mathbf{R}_+$, is obtained by applying the envelope theorem to (2.6):

$$\rho(x, y) = \nabla_x D(x, y). \tag{2.8}$$

As is apparent from the re-writing of (2.6) (using homogeneity in p) as

$$D(x, y) = \inf_{p/c} \left\{ \frac{p}{c} \cdot x \mid C(p/c, y) \geq 1 \right\} = \inf_{p/c} \left\{ \frac{p}{c} \cdot x \mid C(p, y) \geq c \right\}, \tag{2.9}$$

where c can be interpreted as (minimal) expenditure (to produce output y), the vector $\rho(x, y)$ in (2.8) can be interpreted as shadow prices normalized by minimal cost.[9] In other words, under the assumption of cost-minimizing behavior,

$$\rho(\overset{*}{x}, y) := \rho\big(\delta(p, y), y\big) = \frac{p}{C(p, y)}. \tag{2.10}$$

Clearly, ρ is homogeneous of degree zero in x.

2.2 Direct Elasticities of Substitution.

The direct Allen-Uzawa elasticity of substitution between inputs i and j is given by

$$\begin{aligned} \sigma^A_{ij}(p, y) :&= \frac{C(p, y) C_{ij}(p, y)}{C_i(p, y) C_j(p, y)} \\ &= \frac{\epsilon_{ij}(p, y)}{s_j(p, y)}, \end{aligned} \tag{2.11}$$

where subscripts on C indicate differentiation with respect to the indicated variable(s),

$$\epsilon_{ij}(p, y) := \frac{\partial \ln \delta_i(p, y)}{\partial \ln p_j} \tag{2.12}$$

[9] See Färe and Grosskopf [1994] and Russell [1997] for analyses of the distance function and associated shadow prices.

is the (constant-output) elasticity of demand for input i with respect to changes in the price of input j, and

$$s_j(p, y) = \frac{p_j \delta_j(p, y)}{C(p, y)} \tag{2.13}$$

is the cost share of input j.

The direct Morishima elasticity of substitution of input i for input j is

$$\begin{aligned}
\sigma_{ij}^{M}(p, y) &:= \frac{\partial \ln\left(\hat{\delta}_i(p^i, y) \;/\; \hat{\delta}_j(p^i, y)\right)}{\partial \ln(p_j/p_i)} \\
&= p_j\left(\frac{C_{ij}(p, y)}{C_i(p, y)} - \frac{C_{jj}(p, y)}{C_j(p, y)}\right) \\
&= \epsilon_{ij}(p, y) - \epsilon_{jj}(p, y),
\end{aligned} \tag{2.14}$$

where p^i is the $(n-1)$-dimensional vector of price ratios with p_i in the denominator and (with the use of zero-degree homogeneity of δ in p)

$$\hat{\delta}(p^i, y) := \delta(p, y). \tag{2.15}$$

The Morishima elasticity, unlike the Allen-Uzawa elasticity, is non-symmetric, since the value depends on the normalization adopted in (2.14)—that is, on the coordinate direction in which the prices are varied to change the price ratio, p_j/p_i (see Blackorby and Russell [1975, 1981, 1989]).

If $\sigma_{ij}^{A}(p, y) > 0$ (that is, if increasing the j^{th} price increases the optimal quantity of input i), we say that inputs i and j are direct Allen-Uzawa substitutes; if $\sigma_{ij}^{A}(p, y) < 0$, they are direct Allen-Uzawa complements.[10] Similarly, if $\sigma_{ij}^{M}(p, y) > 0$ (that is, if increasing the j^{th} price increases the optimal quantity of input i relative to the optimal quantity of input j), we say that input j is a direct Morishima substitute for input i; if $\sigma_{ij}^{M}(p, y) < 0$, input j is a direct Morishima complement to input i. As the Morishima elasticity of substitution is non-symmetric, so is the taxonomy of Morishima substitutes and complements.

[10] It would, of course, be historically more accurate to refer to a pair of inputs as "direct Hicks substitutes" if $\sigma_{ij}^{A}(p, y) > 0$, since $\sigma_{ij}^{A}(p, y)$ has the same sign as the Hicksian cross price elasticity $\epsilon_{ij}(p, y)$, but we attempt here to keep the nomenclature simple by making it consistent with that of the elasticities of substitution.

The conceptual foundations of Allen-Uzawa and Morishima taxonomies of substitutes and complements are, of course, quite different. The Allen-Uzawa taxonomy classifies a pair of inputs as direct substitutes (complements) if an increase in the price of one causes an increase (decrease) in the quantity demanded of the other, whereas the Morishima concept classifies a pair of inputs as direct substitutes (complements) if an increase in the price of one causes the quantity of the other to increase (decrease) *relative to the quantity of the input whose price has changed*. For this reason, the Morishima taxonomy leans more toward substitutability (since the theoretically necessary decrease in the denominator of the quantity ratio in (2.14) helps the ratio to decline when the price of the input in the denominator increases). Put differently, if two inputs are direct substitutes according to the Allen-Uzawa criterion, theoretically they must be direct substitutes according to the Morishima criterion, but if two inputs are direct complements according to the Allen-Uzawa criterion, they can be either direct complements or direct substitutes according to the Morishima criterion. This relationship can be seen algebraically from (2.11) and (2.14). If i and j are direct Allen-Uzawa substitutes, in which case $\epsilon_{ij}(p,y) > 0$, then concavity of the cost function (and hence negative semi-definiteness of the corresponding Hessian) implies that $\epsilon_{ij}(p,y) - \epsilon_{jj}(p,y) > 0$, so that j is a direct Morishima substitute for i. Similar algebra establishes that two inputs can be direct Morishima substitutes when they are direct Allen-Uzawa complements.

Note that, for $i \neq j$,

$$\frac{\partial \ln s_i(p,y)}{\partial \ln p_j} = \epsilon_{ij}(p,y) - s_j(p,y) = s_j(p,y)\big(\sigma_{ij}^A(p,y) - 1\big), \tag{2.16}$$

so that an increase in p_j increases the absolute cost share of input i if and only if

$$\sigma_{ij}^A(p,y) > 1; \tag{2.17}$$

that is, if and only if inputs i and j are sufficiently substitutable in the sense of Hicks. Thus, the Allen-Uzawa elasticities provide immediate qualitative comparative-static information about the effect of price changes on absolute shares. To obtain quantitative comparative-static information, one needs to know the share of the j^{th} input as well as the Allen-Uzawa elasticity of substitution.

The Morishima elasticities immediately yield both qualitative and quantitative information about the effect of price changes on *relative* input shares:

$$\frac{\partial \ln\Big(\hat{s}_i\big(p^i, y\big)/\hat{s}_j\big(p^i, y\big)\Big)}{\partial \ln(p_j/p_i)} = \epsilon_{ij}(p, y) - \epsilon_{jj}(p, y) - 1 = \sigma_{ij}^M(p, y) - 1, \tag{2.18}$$

where (with the use of zero-degree homogeneity of s_i in p) $\hat{s}_i(p^i, y) := s_i(p, y)$ for all i. Thus, an increase in p_j increases the share of input i relative to input j if and only if

$$\sigma_{ij}^M(p, y) > 1; \tag{2.19}$$

that is, if and only if inputs i and j are sufficiently substitutable in the sense of Morishima. Moreover, the degree of departure of the Morishima elasticity from unity provides immediate quantitative information about the effect on the relative factor shares.

2.3. Dual Elasticities of Substitution.

The (natural) dual to the Morishima elasticity of substitution (proposed by Blackorby and Russell [1975, 1981]) is given by

$$\begin{aligned} \overset{*}{\sigma}{}_{ij}^M(x, y) :&= \frac{\partial \ln\Big(\hat{\rho}_i\big(x^i, y\big) \;/\; \hat{\rho}_j\big(x^i, y\big)\Big)}{\partial \ln(x_j/x_i)} \\ &= x_j\left(\frac{D_{ij}(x, y)}{D_i(x, y)} - \frac{D_{jj}(x, y)}{D_j(x, y)}\right) \\ &= \overset{*}{\epsilon}_{ij}(x, y) - \overset{*}{\epsilon}_{jj}(x, y), \end{aligned} \tag{2.20}$$

where x^i is the $(n-1)$-dimensional vector of input quantity ratios with x_i in the denominator and

$$\overset{*}{\epsilon}_{ij}(x, y) = \frac{\partial \ln \rho_i(x, y)}{\partial \ln x_j} \tag{2.21}$$

is the (constant-output) elasticity of the shadow price of input i with respect to changes in the quantity of input j. Analogously, Blackorby and Russell [1981] proposed the following as the natural dual to the Allen-Uzawa elasticity:

$$\begin{aligned} \overset{*}{\sigma}{}_{ij}^A(x, y) &= \frac{D(x, y)D_{ij}(x, y)}{D_i(x, y)D_j(x, y)} \\ &= \frac{\overset{*}{\epsilon}_{ij}(x, y)}{\overset{*}{s}_j(x, y)}, \end{aligned} \tag{2.22}$$

where

$$\overset{*}{s}_j(x,y) = \rho_j(x,y) \cdot x_j \tag{2.23}$$

is the cost share of input j (assuming cost-minimizing behavior).

If $\overset{*}{\sigma}^A_{ij}(p,y) < 0$ (that is, if increasing the j^{th} quantity decreases the shadow price of input i), we say that inputs i and j are *dual* Allen-Uzawa substitutes; if $\overset{*}{\sigma}^A_{ij}(p,y) > 0$, they are dual Allen-Uzawa (net) complements. Similarly, if $\sigma^M_{ij}(p,y) < 0$ (that is, if increasing the j^{th} quantity increases the shadow price of input i relative to the shadow price of input j), we say that input j is a dual Morishima substitute for input i; if $\sigma^M_{ij}(p,y) > 0$, input j is a dual Morishima complement to input i. Primal and dual substitutability and complementarity are fundamentally different concepts; indeed, signs in these definitions of dual substitutability and complementarity are reversed from those in the definitions of direct Allen-Uzawa and Morishima substitutes and complements. Thus, if $n = 2$, the two inputs are necessarily direct substitutes and dual complements.[11]

Interestingly, since the distance function is concave in x, and hence $\overset{*}{\epsilon}_{jj}(x,y)$ in (2.20) is non-positive, the dual Morishima elasticity leans more toward dual complementarity than does the dual Allen-Uzawa elasticity (in sharp contrast to the primal taxonomy). Similarly, if two inputs are dual Allen-Uzawa complements, they must be dual Morishima complements, whereas two inputs can be dual Allen-Uzawa substitutes but dual Morishima complements.

There exist, of course, dual comparative-static results linking factor cost shares and elasticities of substitution.[12] Consider first the effect of quantity changes on absolute shares (for $i \neq j$):

$$\frac{\partial \ln \overset{*}{s}_i(x,y)}{\partial \ln x_j} = \overset{*}{\epsilon}_{ij}(x,y) = \overset{*}{\sigma}^A_{ij}(x,y)\ \overset{*}{s}(x,y), \tag{2.24}$$

[11] The direct Allen-Uzawa and Morishima elasticities are idential when $n = 2$, as are the dual Allen-Uzawa and Morishima elasticities.

[12] While shadow prices and dual elasticities are well defined even if the input requirement sets are not convex, the comparative statics of income shares using these elasticities requires convexity (as well, of course, as price-taking, cost-minimizing behavior), which implies concavity of the distance function in x. By way of contrast, convexity of input requirement sets is not required for the comparative statics of income shares using direct elasticities, since the cost function is necessarily concave in prices. See Russell [1997] for a discussion of these issues.

so that an increase in x_j increases the absolute share of input i if and only if

$$\overset{*}{\epsilon}(x, y) > 0 \tag{2.25}$$

or, equivalently,

$$\overset{*}{\sigma}{}^{A}_{ij}(x, y) > 0; \tag{2.26}$$

that is, if and only if inputs i and j are dual Allen-Uzawa complements. Thus, the dual Allen-Uzawa elasticities provide immediate qualitative comparative-static information about the effect of quantity changes on (absolute) shares. To obtain quantitative comparative-static information, one needs to know the share of the j^{th} input as well as the dual Allen-Uzawa elasticity of substitution. Of course, the (constant-output) dual elasticity $\overset{*}{\epsilon}_{ij}(x, y)$ yields the same qualitative and quantitative comparative-static information.

Dual comparative-static information about relative income shares in the face of quantity changes can be extracted from the dual Morishima elasticity concept:

$$\frac{\partial \ln\Big(\tilde{s}_i\big(x^i, y\big)/\tilde{s}_j\big(x^i, y\big)\Big)}{\partial \ln(x_j/x_i)} = \overset{*}{\epsilon}_{ij}(x, y) - \overset{*}{\epsilon}_{jj}(x, y) - 1 = \overset{*}{\sigma}{}^{M}_{ij}(x, y) - 1, \tag{2.27}$$

where (with the use of zero-degree homogeneity of $\overset{*}{x}$ in x) $\tilde{s}_i(x^i, y) := \overset{*}{s}_i(x, y)$. Thus, an increase in x_j increases the share of input i relative to input j if and only if

$$\overset{*}{\sigma}{}^{M}_{ij}(x, y) > 1; \tag{2.28}$$

that is, if and only if inputs i and j are sufficiently complementary in terms of the dual Morishima elasticity. Moreover, the degree of departure from unity provides immediate quantitative information about the effect on the relative factor share. Thus, the dual Morishima elasticities provide immediate quantitative and qualitative comparative-static information about the effect of quantity changes on relative shares.

2.4. Scalar-Output Dual Elasticities of Substitution.

An earlier literature (Hicks [1970] and Sato and Koizumi [1973]) analyzed duals to the Allen elasticity under the assumptions of a single output ($m = 1$) and constant returns to scale. The Hicksian "elasticity of complementarity" between inputs i and j is defined as

$$\overset{*}{\sigma}{}_{ij}^{H}(x,y) := \frac{f(x) f_{ij}(x)}{f_i(x) f_j(x)}, \tag{2.29}$$

where subscripts on the production function f indicate differentiation with respect to the indicated variable(s). The symmetry with the Allen elasticity of substitution (2.11) is obvious. This formulation relies critically, however, on homogeneity of degree one (as well as $m = 1$), as shown by the following theorem.

Theorem: Suppose $m = 1$. Then the Hicks elasticity of complementarity (2.29) is equivalent to the dual Allen-Uzawa elasticity of substitution (2.22) if and only if the production function is homogeneous of degree one (*i.e.*, (2.3) holds).

Proof: Sufficiency is easily established by differentiating (2.3) and substituting the resulting identities into

$$\frac{f(x) f_{ij}(x)}{f_i(x) f_j(x)} = \frac{D(x,y) D_{ij}(x,y)}{D_i(x,y) D_j(x,y)}. \tag{2.30}$$

To prove necessity, differentiate the identity,

$$D\big(f(x), x\big) = 1, \tag{2.31}$$

with respect to x_i and substitute the result into (2.30) to obtain

$$f(x) f_{ij}(x) = \frac{D(x,y) D_{ij}(x,y)}{\big[D_y(x,y)\big]^2}. \tag{2.32}$$

Now multiply both sides of (2.32) by x_j and sum over j to arrive at

$$f(x) \sum_j f_{ij}(x) x_j = \frac{D(x,y)}{\big[D_y(x,y)\big]^2} \sum_j D_{ij}(x,y)\ x_j = 0, \tag{2.33}$$

where the last identity follows from the first-degree homogeneity of D in x (and hence zero-degree homogeneity of D_i in x). This implies that

$$\sum_j f_{ij}(x)x_j = 0, \tag{2.34}$$

which in turn, by the converse of Euler's theorem,[13] implies that f is homogeneous of degree one. ■

3. Empirical Implementation.

3.1. Specification of functional form.

Parametric application of the concepts in Section 2 requires specification of a cost function or a distance function. We adopt translog specifications, incorporating technological change (proxied by a time index t) of each:[14]

$$\begin{aligned}\ln C(p,y,t) = \alpha_0 &+ \sum_{i=1}^{n}\alpha_i \ln p_i + \sum_{i=1}^{m}\beta_i \ln y_i \ + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}\ln p_i \ln p_j \\ &+ \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\beta_{ij}\ln y_i \ln y_j + \sum_{i=1}^{n}\sum_{j=1}^{m}\gamma_{ij}\ln p_i \ln y_j \\ &+ \theta t \ + \sum_{i=1}^{n}\nu_i t \ \ln p_i + \sum_{i=1}^{m}\tau_i \ t \ln y_i\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\ln D(x,y,t) = \alpha_0 &+ \sum_{i=1}^{n}\alpha_i \ln x_i + \sum_{i=1}^{m}\beta_i \ln y_i \ + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}\ln x_i \ln x_j \\ &+ \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\beta_{ij}\ln y_i \ln y_j + \sum_{i=1}^{n}\sum_{j=1}^{m}\gamma_{ij}\ln x_i \ln y_j \\ &+ \theta t \ + \sum_{i=1}^{n}\nu_i t \ln x_i + \sum_{i=1}^{m}\tau_i t \ln y_i,\end{aligned} \tag{3.2}$$

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[13] See, *e.g.*, Simon and Blume [1994, pp. 672–3].

[14] We choose these two specifications because of their flexibility (in the sense of both Diewert [1971] and Jorgenson and Lau [1975]). The two specifications represent different technologies; that is, the translog is not self-dual. Moreover, it is not possible to find closed-form duals to either of these specifications, unless they degenerate to representations of a Cobb-Douglas technology (which is self-dual). Of course, the stochastic structure is also quite different in the two specifications.

where the last three terms in each specification reflect technological change. The corresponding systems of share equations are given by

$$s_i(p,y,t) = \frac{\partial \ln C(p,y,t)}{\partial \ln p_i} = \alpha_i + \sum_{j=1}^{n} \alpha_{ij} \ln p_j + \sum_{j=1}^{m} \gamma_{ij} \ln y_j + \nu_i t, \quad i = 1, \ldots, n, \tag{3.3}$$

and

$$\overset{*}{s}_i(x,y,t) = \frac{\partial \ln D(x,y,t)}{\partial \ln x_i} = \alpha_i + \sum_{j=1}^{n} \alpha_{ij} \ln x_j + \sum_{j=1}^{m} \gamma_{ij} \ln y_j + \nu_i t, \quad i = 1, \ldots, n. \tag{3.4}$$

The homogeneity restrictions on C and D imply the following restrictions in each of these two specifications:

$$\sum_i \alpha_i = 1 \quad \text{and} \quad \sum_i \alpha_{ij} = \sum_i \gamma_{ij} = \sum_i \nu_i = 0 \quad \forall\ j. \tag{3.5}$$

The above specifications of the cost and distance functions impose no restrictions on returns to scale. Constant returns to scale imposes the following additional restrictions on (3.1) or on (3.2):

$$\sum_i \beta_i = 1 \quad \text{and} \quad \sum_j \beta_{ij} = \sum_j \gamma_{ij} = 0 \quad \forall\ i. \tag{3.6}$$

Thus, one tests for constant returns to scale by testing for these parametric restrictions.

3.2. Application: Resident and Non-Resident Workers in Switzerland.

We illustrate the application of the foregoing concepts using annual data on resident and non-resident workers in Switzerland for the 1950–1986 time period.[15] This data base has only a single (aggregate) output, along with four inputs—resident labor, non-resident labor, imports, and capital. Gross output and import figures are derived from the Swiss National Income and Product Accounts. The quantity of labor is the product of total employment and the average length of the work week. The quantity of capital is calculated as the Törnqvist quantity index of structures and equipment. The income shares of labor and capital are derived from the National Income and Product Accounts, and the prices of

[15] We thank Ulrich Kohli for providing these data.

labor and capital are obtained by deflation. The resident-worker category comprises natives as well as foreign workers who are residents of Switzerland. Nonresident workers are holders of seasonal permits, annual permits, or transborder permits. We use a time trend with unit annual increments for technological change.

Tables 1 and 2 contain estimates of the systems of share equations, (3.3) and (3.4), respectively, under alternative assumptions about returns to scale.[16] The subscripts of the coefficients in the first column, L, N, M, K, and Y, represent resident labor, non-resident labor, imports, capital, and output, respectively.

We first test for positive linear homogeneity of the production function under each of these specifications. In each case, the critical value of the Wald test statistic for constant returns to scale, under the null of $\gamma_{LY} = \gamma_{NY} = \gamma_{MY} = \gamma_{KY} = 0$, is 9.31. The Wald statistic for the estimates of the systems (3.3) and (3.4) are 100.2 and 29.1, respectively; thus, positive linear homogeneity is decisively rejected in each case. It appears, therefore, in the light of the above Theorem, that the dual-elasticity concept suggested by Hicks [1970] and Sato and Koizumi [1973] (and adopted by Kohli [1999]), which relies on constant returns to scale, is inappropriate (at least in the case of the Kohli data set).

Tables 3 and 4 contain estimates of the direct Allen-Uzawa and Morishima elasticities of substitution (equations (2.11) and (2.14)), and Tables 5 and 6 contain estimates of the dual elasticities (equations (2.22) and (2.20)), all evaluated at the means of the variables and under alternative assumptions about returns to scale.

[16] Zellner's SURE technique was used to estimate the systems of factor-share equations, and the capital-share equation was deleted for the estimation. Because of possibile simultaneous-equations bias, we also ran three-stage least squares; the coefficients were substantially unchanged in each case. Hausman tests rejects the hypothesis of endogenity of input quantities in the estimation of the share equations in the direct specification and of endogenity of prices in the dual specification. Tests for concavity of the cost function (whence the system (3.3) is derived) were satisfied for 32 of the 37 observations. Concavity of the distance function (whence (3.4) is derived) was satisfied at 23 of the 37 observations. Concavity of the cost function is a theoretical imperative. Concavity of the distance function is implied by convexity of input requirement sets $L(y)$, but the distance function is well defined, as are the shadow price functions given by (2.10), even if input requirement sets are not convex. On the other hand, the comparative statics of income shares under perfectly competitive pricing of inputs, reflected in (2.24) and (2.27), require convexity of input requirement sets as does the estimation of the share equations (2.10) using price data.

Let us first evaluate the qualitative information in these tables. Recall that two factors are classified as direct (Allen-Uzawa or Morishima) substitutes if the direct elasticity is positive and as complements if it is negative, and the reverse is true for the dual elasticities. Of course, the non-symmetry of the Morishima elasticities raises the possibility of ambiguities in the Morishima taxonomy. Examination of the tables, however, reveals that there are only two qualitative asymmetries, for capital and imports in the dual homogeneous case (Table 5) and for capital and nonresident labor in the dual non-homogeneous framework (Table 6), and in each case one of the two elasticity estimates is statistically insignificant. Hence, it is possible to construct an unambiguous taxonomy of (direct and dual) substitutability and complementarity for the Morishima elasticities as well as the Allen elasticities (evaluated at the means of the data). This taxonomy is summarized in Table 7, where S signifies (direct or dual) substitutability and C signifies (direct or dual) complementarity.

The strongest priors exist for resident and non-resident labor, and, indeed, the two are substitutes under either assumption about returns to scale, in either the primal or dual specification, and with respect to either elasticity definition.[17] Thus, an increase in the number of non-resident workers lowers the wage rate of resident workers, both absolutely and relatively to the wage rate of non-resident workers. And an increase in the wage rate of resident workers increases the demand for non-resident workers, both absolutely and relatively to the demand for resident workers.

Some differences emerge with other pairs of inputs. First, regarding the assumption about returns to scale, there are a couple of reversals using the direct Allen elasticity definition and one reversal in the dual under both elasticity definitions. As the above test indicates that the constant-returns-to-scale cost and distance functions are mispecified, we would conclude that non-resident labor and imports are direct Allen-Uzawa substitutes and that imports and capital are direct Allen-Uzawa complements. Similarly, we would conclude that non-resident labor and capital are dual substitutes under either elasticity definition.

[17] Cf. the finding of substitutability between immigrant and native workers by Grossman [1982].

Hence, an increase in the number of non-resident workers lowers the rental rate on capital, both absolutely and relatively to the wage rate of non-resident workers.

It should be re-emphasized that contrasts in the classification of pairs as substitutes or complements in the primal and the dual do not constitute any kind of theoretical or econometric inconsistency: the direct and dual elasticities measure conceptually distinct phenomena when there are more than two inputs.[18] The direct elasticities measure the effect on quantities of price changes whereas the dual elasticities measure the effect on prices of changes in quantities.

Finally, regarding the difference in taxonomies induced by the two elasticity concepts, it is clear that the direct Morishima elasticity concept is more conducive to substitutability than the direct Allen-Uzawa elasticity, as shown theoretically in Section 2 above. In some (direct elasticity) cases, pairs are classified as complements by the Allen elasticity concept but as substitutes by the Morishima concept. Interestingly, however, there is no difference in the classification scheme in the dual. Again, we wish to re-emphasize that there is no theoretical or econometric reason why the two elasticity concepts should yield comparable qualitative conclusion about substitutability/complementarity; they are measuring different concepts, as discussed in Section 2 above.

To consider the quantitative elasticity results, we take note of the fact that the Allen-Uzawa elasticity of substitution has no meaningful quantitative interpretation, as pointed out by Blackorby and Russell [1975, 1989]. The size of the simple cross price elasticity $\epsilon_{ij}(p, y)$ *is* meaningful, but dividing by the share of input j, as in (2.11), to obtain the Allen-Uzawa elasticity, $\sigma_{ij}^{A}(p, y)$, undermines this quantitative content. Thus, to facilitate consideration of quantitative comparative statics, we list the price elasticities and their duals, defined by (2.12) and (2.21) and evaluated at the means of the data, in Table 8. Of course, these elasticities are non-symmetric.

[18] As noted earlier, they require different theoretical and stochastic specifications of the technology as well.

Let us focus now on the quantitative relationships between the two types of labor under the (preferred) non-homogeneous specification of the technology. The estimated cross price elasticities in Table 8 indicate that a 1-percent increase in the wage rate of non-resident workers would increase the employment of resident labor by 0.3 percent (at the means of the data), whereas a 1-percent increase in the wage rate of resident labor would increase the employment of non-resident labor by 1.8 percent.[19] Table 3 indicates that a 1-percent increase in the price of non-resident labor would increase the quantity ratio of resident labor to non-resident labor by 1.7 percent, while a 1-percent increase in the wage rate of resident labor would increase the quantity ratio of non-resident to resident labor by 2.5 percent.[20]

The dual price elasticity estimates in Table 8 suggest that a 1-percent increase in the number of nonresident workers would lower the wage rate of resident workers by 0.1 percent, while a 1-percent increase in the number of resident workers would lower the wage rate of non-resident workers by 0.6 percent. From Table 6, it appears that a 1-percent increase in the number of non-resident workers would lower the relative wage rate (of non-resident to resident workers) by just 0.1 percent, while a 1-percent increase in the number of resident workers would lower the relative wage rate (of resident to non-resident workers) by 0.5 percent.

The estimated direct Allen-Uzawa elasticity of substitution does provide immediate qualitative information about absolute shares, as reflected in (2.16). Thus, the elasticity of 4.3 in Table 4 indicates that the share of either type of labor input is enhanced by an increase in the wage rate of the other labor type. Similarly, from (2.24) and Table 6, we see from the negative sign of the dual Allen-Uzawa elasticity that an increase in the quantity of either input decreases the absolute share of the other labor input. Equations (2.22) and (2.27), along with the estimates of Morishima elasticities in Tables 4 and 6, provide both

[19] It is interesting to note here that, under the (apparently mispecified) homogeneous technology, a 1-percent increase in the wage rate of resident labor would increase the employment of non-resident labor by a whopping 4.4 percent, an estimate that strains credibility.

[20] Again, note that under the mispecified homogeneous technology, a 1-percent change in a wage rate generates an estimated 3-percent or 6-percent change in the quantity ratio, again challenging our intuition.

qualitative and quantitative information about the comparative statics of relative income shares. Thus, from Table 4, we see that a 1-percent increase in the wage rate of non-resident workers increases the share of resident workers relative to non-resident workers by 0.7 percent, while a 1-percent increase in the wage rate of resident workers increases the share of non-resident workers relative to resident workers by 1.5 percent.[21] The dual Morishima elasticities provide information about the effect on relative shares of changes in input quantities. Table 6 indicates that a 1-percent increase in the quantity of non-resident labor decreases the share of resident labor relative to non-resident labor by 1.1 percent, whereas a 1-percent increase in the quantity of resident labor decreases the share of non-resident labor relative to resident labor by 1.5 percent.

All of the foregoing calculations and interpretations are carried out at the means of the data. While this is a standard way of reporting elasticity results, it should be emphasized that the interpretations could be egregiously in error for any particular year, unless the elasticities were stable over time and over the domain of the cost or distance function. Even where the estimated standard errors, also calculated at the mean, are small relative to the elasticity size, there is no reason to believe that the elasticity is time invariant or insensitive to the values of the input quantities or prices. For a particular question about a particular year, one can calculate the appropriate elasticity. For the purpose of this paper, we summarize the information about the elasticities in two ways. First, Table 11 lists the ranges of the elasticity estimates (under the non-homothetic technology specifications). It can be seen that the range is fairly tight for some pairs of inputs for both elasticity concepts: namely, resident labor and imports, resident labor and capital, and imports and capital. On the other hand, the range is quite large for other pairs. Note, in particular, that the (qualitative) classification of two inputs as substitutes or complements is itself sensitive to the choice of the sample point at which to do the calculation in the cases of the direct Allen elasticity for non-resident labor and imports, the direct Morishima elasticity for non-resident

[21] Note that, under the (misspecified) homogeneous technology, a 1-percent increase in the wage rate of resident workers increases the share of non-resident workers by a hard-to-believe 5 percent.

labor and capital, and the dual Morishima elasticity for resident and non-resident labor and for non-resident labor and imports.

A second way to summarize the information about the elasticity values over the entire sample space is to plot frequency distributions. We have constructed nonparametric, kernel-based density estimates of the distributions of each of the elasticities (essentially "smoothed" histograms of elasticity values) over the entire sample (numbering 72 distributions).[22] [These distributions are attached as a not-for-publication appendix to this draft for the convenience of the editor and referees. They will be made available online.] While many of the distributions look radically different for the homogeneous and non-homogeneous cases, some look similar. This raises the possibility that, even though the hypothesis of constant returns to scale is easy to reject (at least for this data set), the apparent mispecification might have little effect on estimated elasticity values and hence on qualitative and quantitative comparative statics of price, quantities, and income shares. Using some recent developments in nonparametric methods, we can test this hypothesis formally by testing for the difference between the elasticity distributions under homogeneous vs. non-homogeneous technologies. In particular, Fan and Ullah [1999] have proposed a nonparametric (time series) test for the comparison of two unknown distributions, say f and g—that is, a test of the null hypothesis, $H_0 : f(x) = g(x)$ for all x, against the alternative, $H_1 : f(x) \neq g(x)$ for some x.[23] Tables 10 and 11 contain the results of carrying out these tests. The hypothesis that the two distributions are identical is rejected in every case but one: the direct Allen elasticity between non-resident labor and capital. Thus, it is safe to say that the mispecification of constant returns to scale, required for the Hicks [1970] and Sato-Koizumi [1973] dual elasticity concept (see the Theorem in Section 2.4), results in serious errors in elasticity estimates and hence in serious errors in the comparative statics of prices, quantities, and income shares.

[22] See the Appendix for the particulars.
[23] See the Appendix for an exact description of the test statistic.

4. Summary and Concluding Remarks.

In this paper we have argued that different elasticity of substitution concepts—direct and dual Allen-Uzawa elasticities and direct and dual Morishima elasticities—must be used to answer different questions about substitutability among multiple (more than two) factors of production. To assess the effects of price changes, direct elasticities should be used; to assess the effects of quantity changes, dual elasticities should be used. But there is a tendency in the literature to rely on a single elasticity concept regardless of the question: typically the (direct) Allen-Uzawa elasticity but increasingly the (direct) Morishima elasticity. Hicks [1970] did formulate a dual concept, the elasticity of complementarity, under the assumptions of a single output and constant returns to scale. Blackorby and Russell [1981] proposed a generalization of the Hicks notion, a dual Allen-Uzawa elasticity, to encompass multiple-output and non-homothetic technologies. This concept runs parallel to the formulation of the dual Morishima elasticity by Blackorby and Russell [1981].

This generalization of Hicks elasticity might be important not only because of the increasing use of data bases with multiple outputs, but also because constant returns to scale is not difficult to reject when there are more than two factors of production. Indeed, using a data base on the Swiss labor market, we definitively reject the hypothesis of homotheticity under two different specifications of the technology: a translog cost function and a translog distance function. Moreover, maintaining a non-homogeneous translog (cost or distance function) technology, we find that mispecification of the technology as homogeneous of degree one results in statistically significant errors in the estimated elasticities of substitution and hence in assessments of the effects on input demands, prices, or shares of changes in quantities or prices.

APPENDIX

Each of the distributions in Figures 1–12 is a kernel-based estimate of a density function, $f(\cdot)$, of a random variable x, based on the standard normal kernel function and optimal bandwidth:

$$\hat{f}(x) = \frac{1}{nh} \sum_{j=1}^{J} k\left(\frac{x_j - x}{h}\right),$$

where $\int_{-\infty}^{\infty} k(\psi) d\psi = 1$ and $\psi = (x_j - x)/h$. In this construction, h is the optimal window width, which is a function of the sample size n and goes to zero as $n \to \infty$. We assume that k is a symmetric standard normal density function, with non-negative images. See Pagan and Ullah (1999) for details.

The statistic used to test for the difference between two distributions, predicated on the integrated-square-error metric on a space of density functions, $I(f,g) = \int_x \big(f(x) - g(x)\big)^2 dx$, is

$$T = \frac{nh^{1/2}\tilde{I}}{\hat{\sigma}} \;\sim\; N(0,1), \tag{A.1}$$

where

$$\tilde{I} = \frac{1}{n^2 h} \sum_{\substack{i=1 \\ (i \neq j)}}^{n} \sum_{j=1}^{n} \left[k\left(\frac{x_i - x_j}{h}\right) + k\left(\frac{y_i - y_j}{h}\right) - 2k\left(\frac{y_i - x_j}{h}\right) - k\left(\frac{x_i - y_j}{h}\right) \right] \tag{A.2}$$

and

$$\hat{\sigma}^2 = \frac{2}{n^2 h} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[k\left(\frac{x_i - x_j}{h}\right) + k\left(\frac{y_i - y_j}{h}\right) + 2k\left(\frac{x_i - y_j}{h}\right) \right] \int k^2(\Psi) d\psi. \tag{A.3}$$

As shown by Fan and Ullah [1999], the test statistic asymptotically goes to the standard normal, but the sample in our study is only 37 years. Thus, we do a bootstrap approximation with 2000 replications to find the critical values for the statistic at the 5-percent and 1-percent levels of significance (see Tables 10–13).

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Table 1: Estimated Coefficients of the Translog Cost Function

(Standard error in parentheses)

Coefficients	Homogeneous Technology	Non-Homogeneous Technology
a_L	0.433 (0.008)	3.469 (0.405)
a_N	0.045 (0.004)	-1.556 (0.229)
a_M	0.309 (0.004)	0.024 (0.343)
a_K	-0.787 (0.002)	1.937 (0.187)
a_{LL}	-0.424 (0.065)	-0.051 (0.059)
a_{LN}	0.258 (0.038)	0.089 (0.034)
a_{LM}	0.078 (0.032)	-0.023 (0.041)
a_{LK}	0.088 (0.113)	-0.015 (0.110)
a_{NN}	-0.101 (0.029)	-0.035 (0.025)
a_{NM}	-0.101 (0.022)	-0.012 (0.025)
a_{NK}	-0.056 (0.07)	-0.042 (0.125)
a_{MM}	0.081 (0.029)	0.033 (0.045)
a_{MK}	-0.058 (0.067)	0.002 (0.105)
a_{KK}	-0.026 (0.069)	0.055 (0.083)
v_L	0.007 (0.002)	0.002 (0.001)
v_N	-0.008 (0.001)	-0.005 (0.001)
v_M	0.003 (0.001)	0.004 (0.001)
v_K	-0.002 (0.005)	-0.001 (0.001)
g_{LY}		-0.248 (0.033)
g_{NY}		0.131 (0.019)
g_{MY}		0.023 (0.028)
g_{KY}		0.094 (0.065)

Table 2: Estimated Coefficients of the Translog Distance Function

(Standard error in parentheses)

Coefficients	Homogeneous Technology	Non-Homogeneous Technology
a_L	0.291 (0.007)	0.431 (0.034)
a_N	0.156 (0.005)	0.146 (0.017)
a_M	0.27 (0.009)	0.275 (0.033)
a_K	0.283 (0.01)	0.148 (0.02)
a_{LL}	0.070 (0.009)	0.084 (0.011)
a_{LN}	-0.068 (0.005)	-0.068 (0.006)
a_{LM}	-0.021 (0.01)	0.002 (0.01)
a_{LK}	-0.02 (0.01)	-0.018 (0.04)
a_{NN}	0.039 (0.003)	0.039 (0.004)
a_{NM}	-0.004 (0.006)	0.01 (0.006)
a_{NK}	0.033 (0.01)	0.019 (0.01)
a_{MM}	0.11 (0.017)	0.105 (0.016)
a_{MK}	-0.085 (0.03)	-0.118 (0.02)
a_{KK}	0.138 (0.02)	0.117 (0.02)
v_L	-0.0002 (0.001)	-0.001 (0.001)
v_N	-0.001 (0.0004)	-0.002 (0.0005)
v_M	-0.001 (0.001)	0.001 (0.0009)
v_K	0.002 (0.001)	0.002 (0.002)
g_{LY}		-0.102 (0.23)
g_{NY}		0.004 (0.011)
g_{MY}		0.001 (0.023)
g_{KY}		-0.097 (0.232)

Table 3: Direct Elasticities of Substitution: Homogeneous Technology

(Standard errors in parentheses)

	Allen-Uzawa Elasticity of Substitution				Morishima Elasticity of Substitution			
	L	*N*	*M*	*K*	*L*	*N*	*M*	*K*
L	-3.72 (0.619)	10.44 (3.903)	1.66 (0.067)	1.89 (0.085)		3.15 (2.009)	0.89 (0.082)	1.32 (0.221)
N		-38.41 (44.82)	-4.67 (3.317)	-2.73 (2.089	6.00 (0.771)		-0.87 (0.813)	0.24 (0.811)
M			-1.54 (0.132)	0.11 (0.053)	2.28 (0.035)	2.18 (0.822)		0.90 (0.18)
K				-3.78 (0.256)	2.38 (0.022)	2.30 (0.455)	0.46 (0.014)	

Table 4: Direct Elasticities of Substitution: Non-Homogeneous Technology

(Standard errors in parentheses)

	Allen-Uzawa Elasticity of Substitution				Morishima Elasticity of Substitution			
	L	*N*	*M*	*K*	*L*	*N*	*M*	*K*
L	-1.65 (0.031)	4.26 (0.404)	0.81 (0.027)	0.85 (0.007)		1.74 (0.195)	0.83 (0.002)	0.73 (0.017)
N		-22.69 (0.004)	0.33 (0.002)	-1.8 (0.064)	2.50 (0.002)		0.70 (0.001)	0.11 (0.002)
M			-2.16 (0.014)	-0.81 (0.009)	1.04 (0.016)	1.48 (0.007)		0.34 (0.008)
K				-2.28 (0.122)	1.06 (0.007)	1.35 (0.040)	0.38 (0.037)	

Table 5: Dual Elasticities of Substitution: Homogeneous Technology

(Standard errors in parentheses)

	Dual Allen-Uzawa Elasticity of Substitution				Dual Morishima Elasticity of Substitution			
	L	*N*	*M*	*K*	*L*	*N*	*M*	*K*
L	-0.41	-1.49	0.82	0.80		-0.12	0.32	0.23
	(0.02)	(0.26)	(0.03)	(0.01)		(0.126)	(0.256)	(0.016)
N		0.33	0.78	0.80	-0.46		0.31	0.23
		(0.99)	(0.01)	(0.06)	(0.028)		(0.025)	(0.024)
M			-0.33	-0.32	0.52	0.15		-0.04
			(0.01)	(0.01)	(0.015)	(0.016)		(0.001)
K				-0.17	0.51	0.03	0.003	
				(0.03)	(0.034)	(0.015)	(0.007)	

Table 6: Dual Elasticities of Substitution: Non-Homogeneous Technology

(Standard errors in parentheses)

	Dual Allen-Uzawa Elasticity of Substitution				Dual Morishima Elasticity of Substitution			
	L	*N*	*M*	*K*	*L*	*N*	*M*	*K*
L	-0.38	-1.50	1.02	0.82		-0.08	0.51	0.21
	(0.031)	(0.027)	(0.027)	(0.007)		(0.195)	(0.002)	(0.017)
N		-0.32	1.55	-0.28	-0.47		0.15	-0.04
		(0.004)	(0.002)	(0.064)	(0.002)		(0.001)	(0.002)
M			-0.81	-0.82	0.59	0.12		-0.17
			(0.014)	(0.009)	(0.008)	(0.007)		(0.001)
K				-0.10	0.51	0.003	-0.003	
				(0.02)	(0.036)	(0.037)	(0.037)	

Table 7: Taxonomy for Substitutes and Complements

	Direct				Dual			
	Homogeneous Technology		Non-Homogeneous Technology		Homogeneous Technology		Non-Homogeneous Technology	
	AES	MES	AES	MES	AES	MES	AES	MES
LN	S	S	S	S	S	S	S	S
LM	S	S	S	S	C	C	C	C
LK	S	S	S	S	C	C	C	C
NM	C	S	S	S	C	C	C	C
NK	C	S	C	S	C	C	S	S
MK	S	S	C	S	S	S	S	S

Table 8: Price and Quantity Elasticities (e_{ij} and e^{*}_{ij})

Direct Price Elasticity: Homogeneous Technology

	L	*N*	*M*	*K*
L	-1.577	0.674	0.463	0.44
N	4.423	-2.5	-1.286	-0.636
M	0.703	-0.297	-0.431	0.025
K	0.802	-0.176	0.030	-0.879

Direct Price Elasticity: Non-Homogeneous Technology

	L	*N*	*M*	*K*
L	-0.697	0.275	0.225	0.197
N	1.803	-1.477	0.093	-0.418
M	0.341	0.022	-0.603	0.240
K	0.359	-0.116	0.288	-0.531

Dual Quantity Elasticity: Homogeneous Technology

	L	*N*	*M*	*K*
L	-0.411	-0.096	0.230	0.185
N	-0.631	-0.33	0.217	-0.279
M	0.348	0.050	-0.327	-0.072
K	0.338	-0.077	-0.086	-0.853

Dual Quantity Elasticity: Non-Homogeneous Technology

	L	*N*	*M*	*K*
L	-0.614	-0.096	0.284	0.19
N	-0.631	-0.33	0.434	0.527
M	0.431	0.1	-0.345	-0.190
K	0.346	0.146	-0.228	-0.105

Table 9: Range for the Direct and Dual Elasticity

	DIRECT				DUAL			
	Non-Homogeneous Technology				Non-Homogeneous Technology			
	AES		MES		AES		MES	
	Min	Max	Min	Max	Min	Max	Min	Max
LN	3.15	8.15	1.56	2.56	-4.46	-0.64	-0.68	0.47
NL			1.94	4.63			-1.86	0.16
LM	0.78	0.84	0.81	0.85	1.01	1.02	0.55	0.69
ML			1.00	1.06			0.78	0.84
LK	0.83	0.88	0.81	0.85	0.79	0.85	0.19	0.35
KL			1.04	1.08			0.04	0.21
NM	-0.96	0.61	0.39	0.77	1.32	2.63	-0.5	0.68
MN			1.27	2.34			-0.5	0.68
NK	-6.32	-0.59	-0.91	0.39	1.72	4.31	0.52	1.11
MK	1.03	1.04	0.73	0.79	-0.64	-0.8	-0.16	-0.03

Table 10: Distribution Hypothesis Test Between Homogeneous and Non-Homogeneous Direct Allen Elasticity

	Test Statistic	5% Significance Level		1% Significance Level		
		Lower Critical Value	Upper Critical Value	Lower Critical Value	Upper Critical Value	
LN	0.402	0.1534	0.2292	0.1402	0.2439	Reject
ML	27.56	0.0024	0.0035	0.0023	0.0036	Reject
MK	330.49	0.0002	0.0004	0.0002	0.0004	Reject
NK	0.081	0.0426	0.0913	0.0396	0.1062	Fail to Reject
NM	1.67	0.0411	0.0918	0.0348	0.0988	Reject

Table 11: Distribution Hypothesis Test Between Homogeneous and Non-Homogeneous Direct Morishima Elasticity

	Test Statistic	5% Level of Significance		1% Level of Significance		
		Lower Critical Value	Upper Critical Value	Lower Critical Value	Upper Critical Value	
LN	2.19	0.0295	0.0511	0.0269	0.0546	Reject
NL	0.802	0.0792	0.1300	0.0723	0.1381	Reject
MK	30.78	0.0023	0.0034	0.0021	0.0036	Reject
KM	51.74	0.0013	0.0021	0.0012	0.0036	Reject
ML	39.83	0.0017	0.0028	0.0015	0.0031	Reject
LM	50.97	0.0011	0.0018	0.001	0.0018	Reject
NK	0.324	0.0066	0.0167	0.0057	0.0182	Reject
KN	1.762	0.0271	0.0461	0.0251	0.0495	Reject
MN	1.24	0.0207	0.0340	0.0192	0.0364	Reject
NM	6.06	0.0107	0.0206	0.0097	0.0221	Reject

Table 12: Distribution Test for Allen Dual Elasticity

	Test Statistic	5% Significance Level		1% Significance Level		
		Lower Critical Value	Upper Critical Value	Lower Critical Value	Upper Critical Value	
NM	5.00	0.0134	0.0311	0.0117	0.0332	Reject
NL	0.4	0.1498	0.2287	0.1360	0.2439	Reject
LK	14.38	0.0007	0.0019	0.0007	0.0023	Reject
NK	0.81	0.0078	0.0197	0.0068	0.0236	Reject
MK	7.78	0.0089	0.0161	0.0076	0.0175	Reject
ML	316.98	0.0002	0.0003	0.0002	0.0003	Reject

Table 13: Distribution Hypothesis Test for Dual Morishima Elasticity

	Test Statistic	5% Level of Significance		1% Level of Significance		
		Lower Critical Value	Upper Critical Value	Lower Critical Value	Upper Critical Value	
NM	12.42	0.0057	0.0115	0.0042	0.0122	Reject
MN	0.03	0.0006	0.0016	0.0005	0.0019	Reject
LN	2.19	0.0286	0.0509	0.0261	0.0541	Reject
NL	0.8	0.0766	0.1296	0.0697	0.1378	Reject
LK	65.49	0.0011	0.0016	0.001	0.0017	Reject
KL	41.2	0.0017	0.0025	0.0015	0.0027	Reject
NK	1.94	0.007	0.0159	0.0061	0.0177	Reject
KN	0.43	0.0078	0.0197	0.0068	0.0236	Reject
MK	16.03	0.0038	0.0051	0.0035	0.0054	Reject
KM	23.69	0.0026	0.0044	0.0023	0.0047	Reject
ML	37.18	0.0017	0.0021	0.0016	0.0022	Reject
LM	9.8	0.0027	0.0038	0.0026	0.004	Reject
NK	1.94	0.007	0.0159	0.0061	0.0177	Reject
KN	0.43	0.0078	0.0197	0.0068	0.0236	Reject

Figure 1: Distributions of Direct Allen and Morishima Elasticities

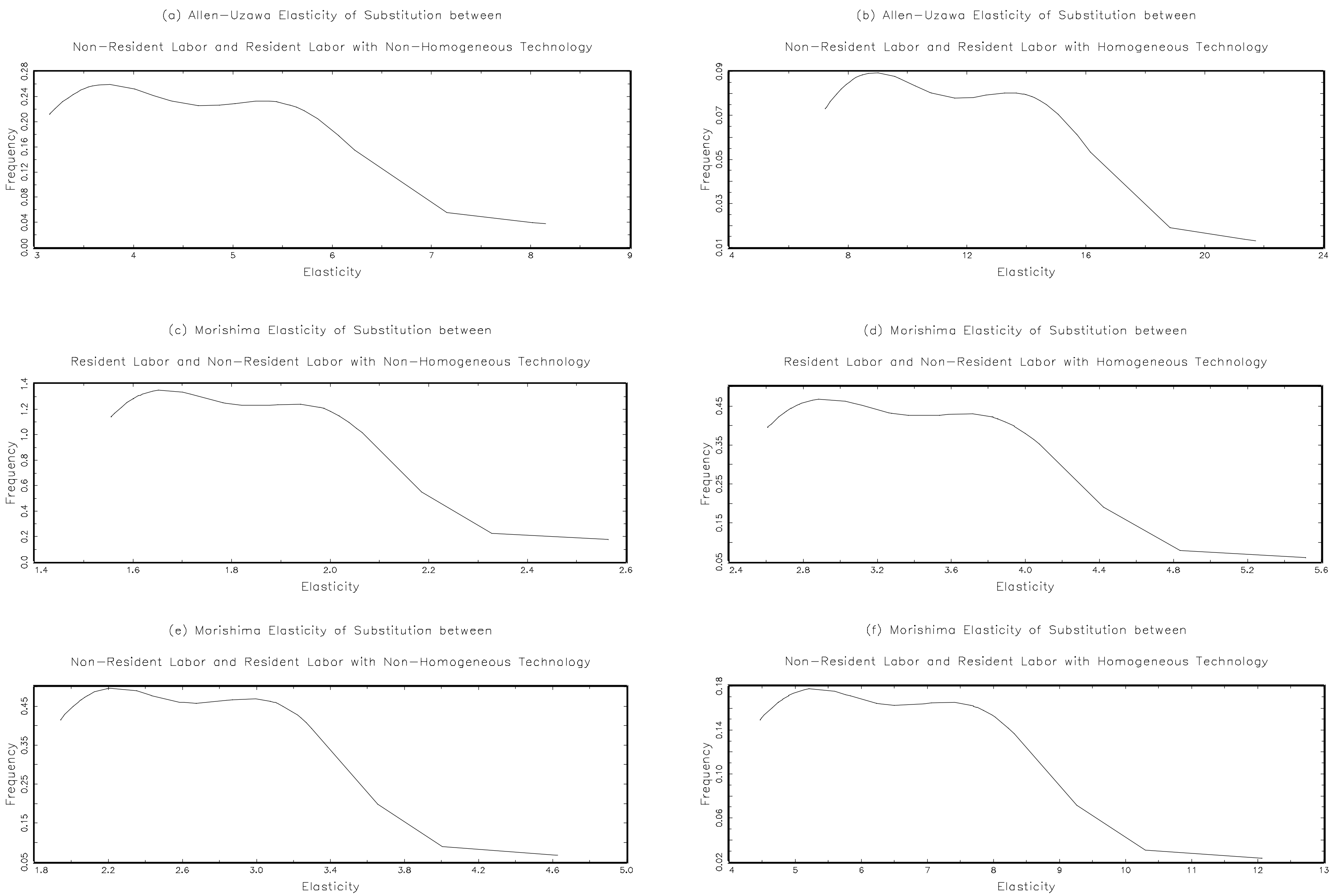


Figure 2: Distributions of Direct Allen and Morishima Elasticities

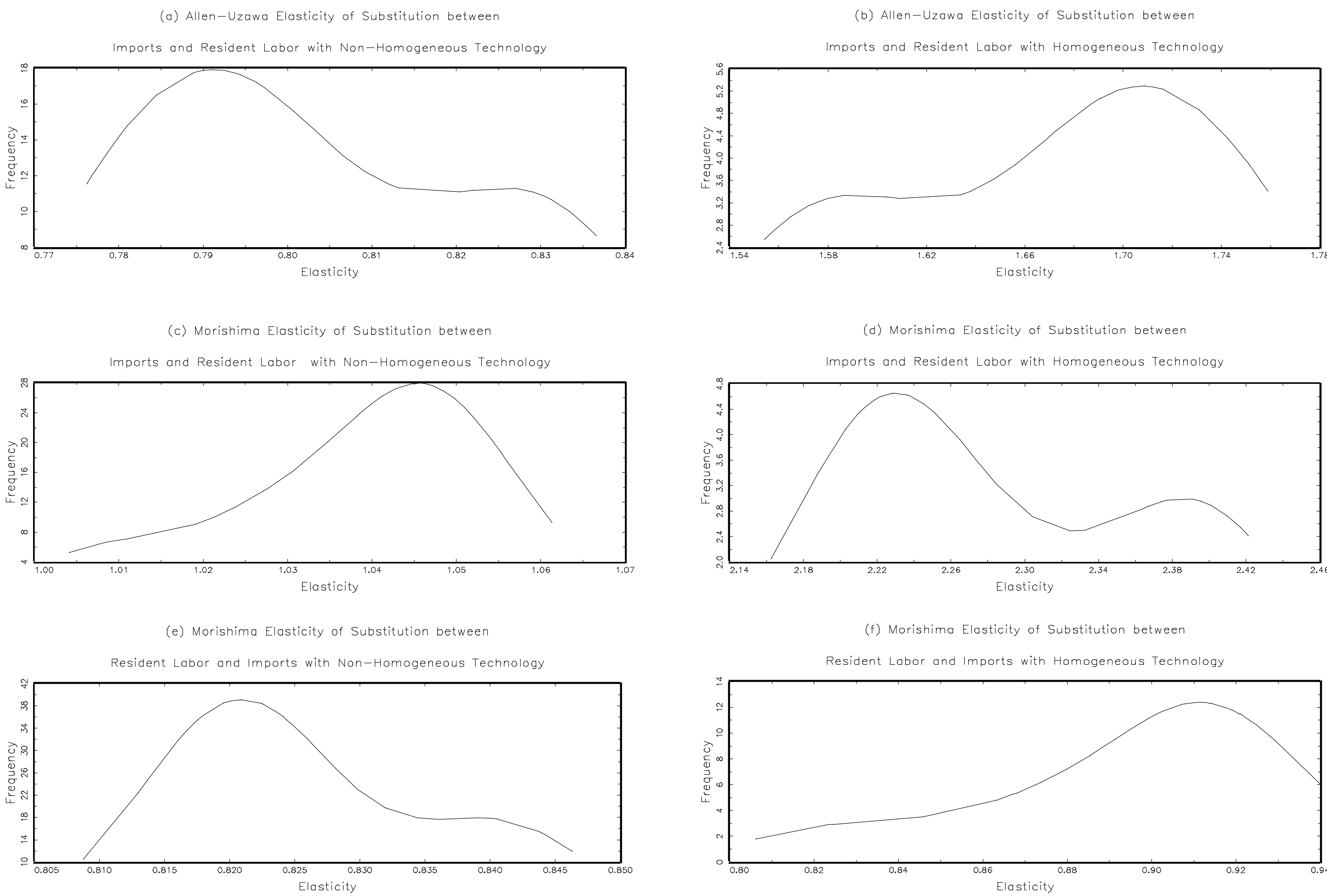


Figure 3: Distributions of Direct Allen and Morishima Elasticities

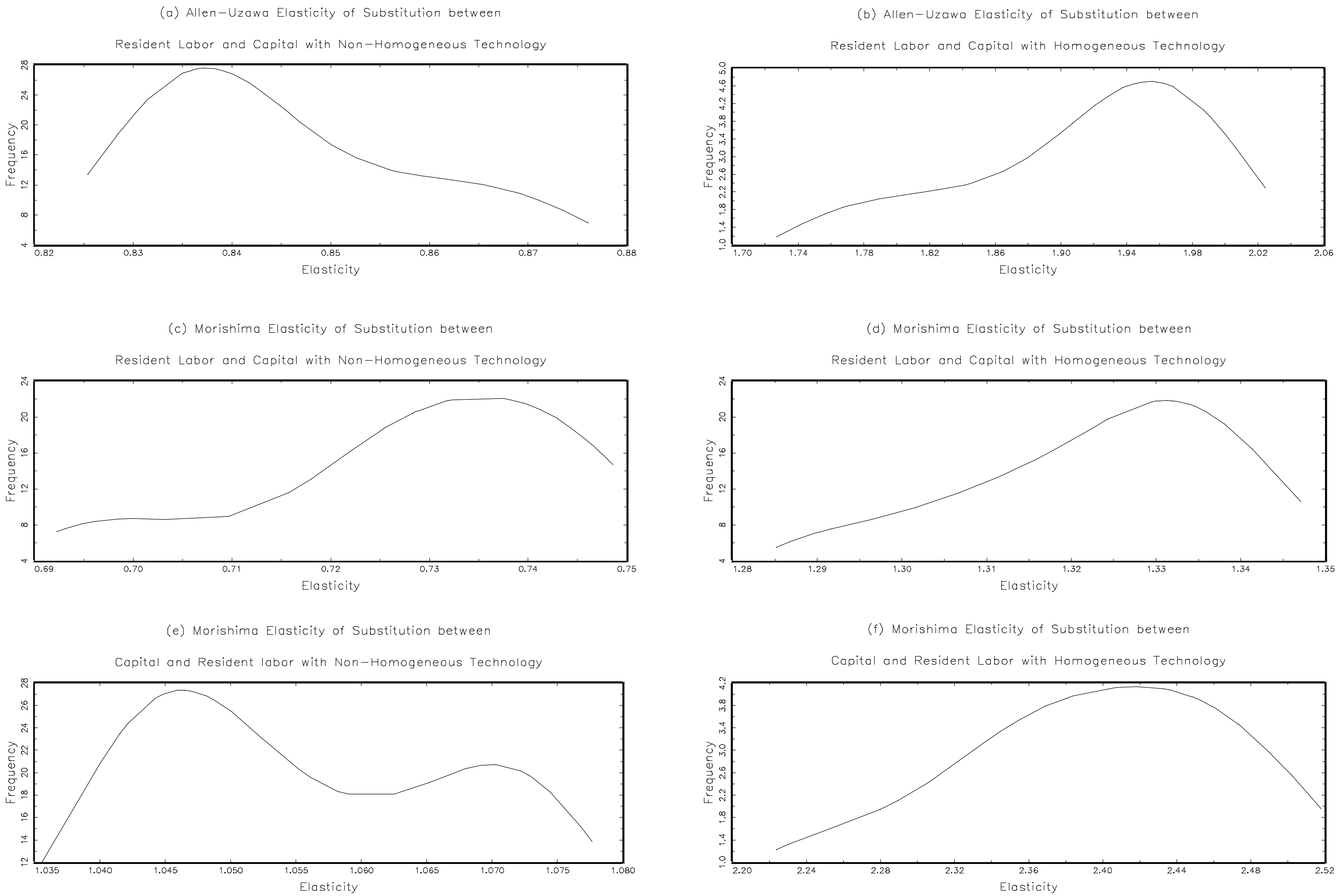


Figure 4: Distributions of Direct Allen and Morishima Elasticities

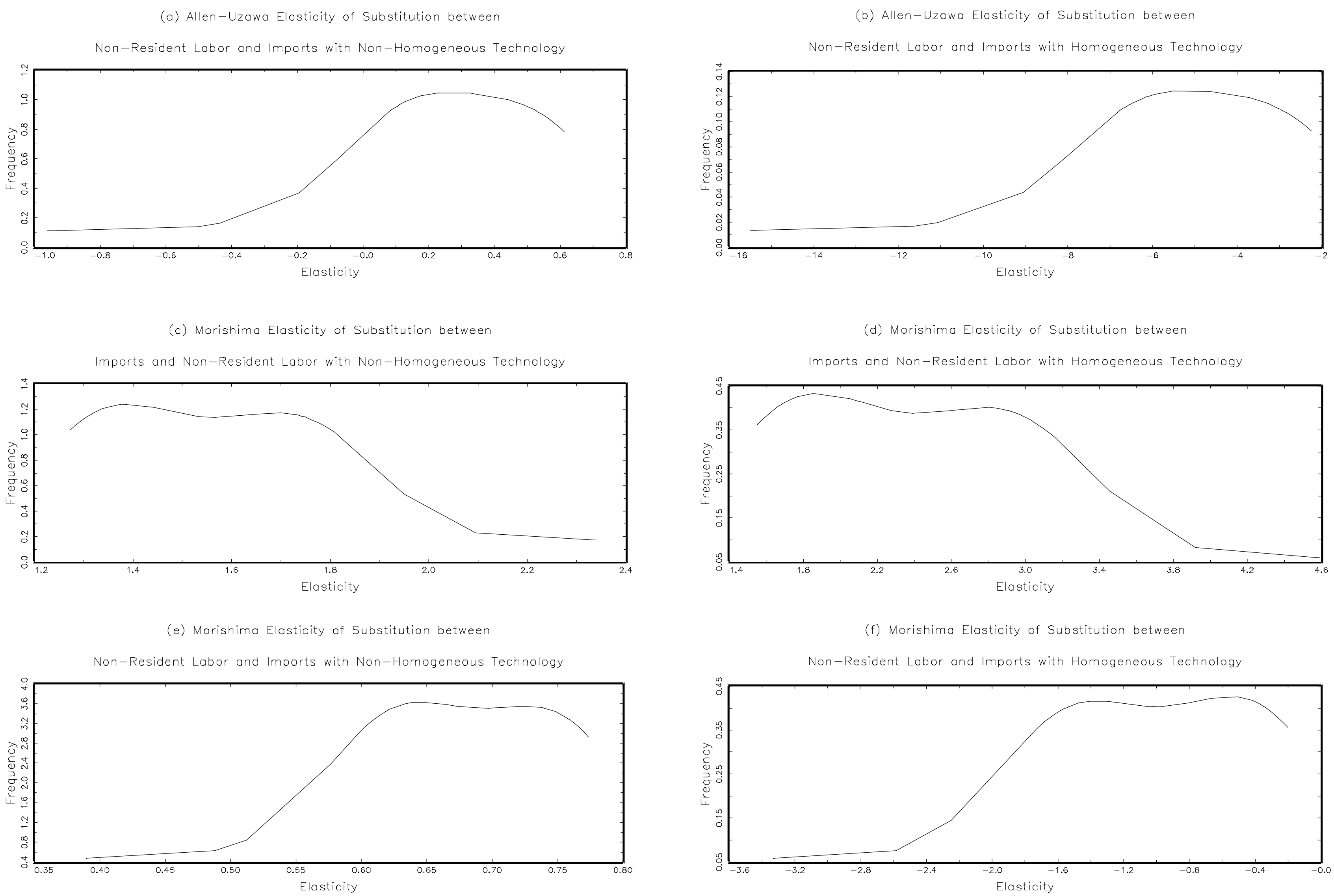


Figure 5: Distributions of Direct Allen and Morishima Elasticities

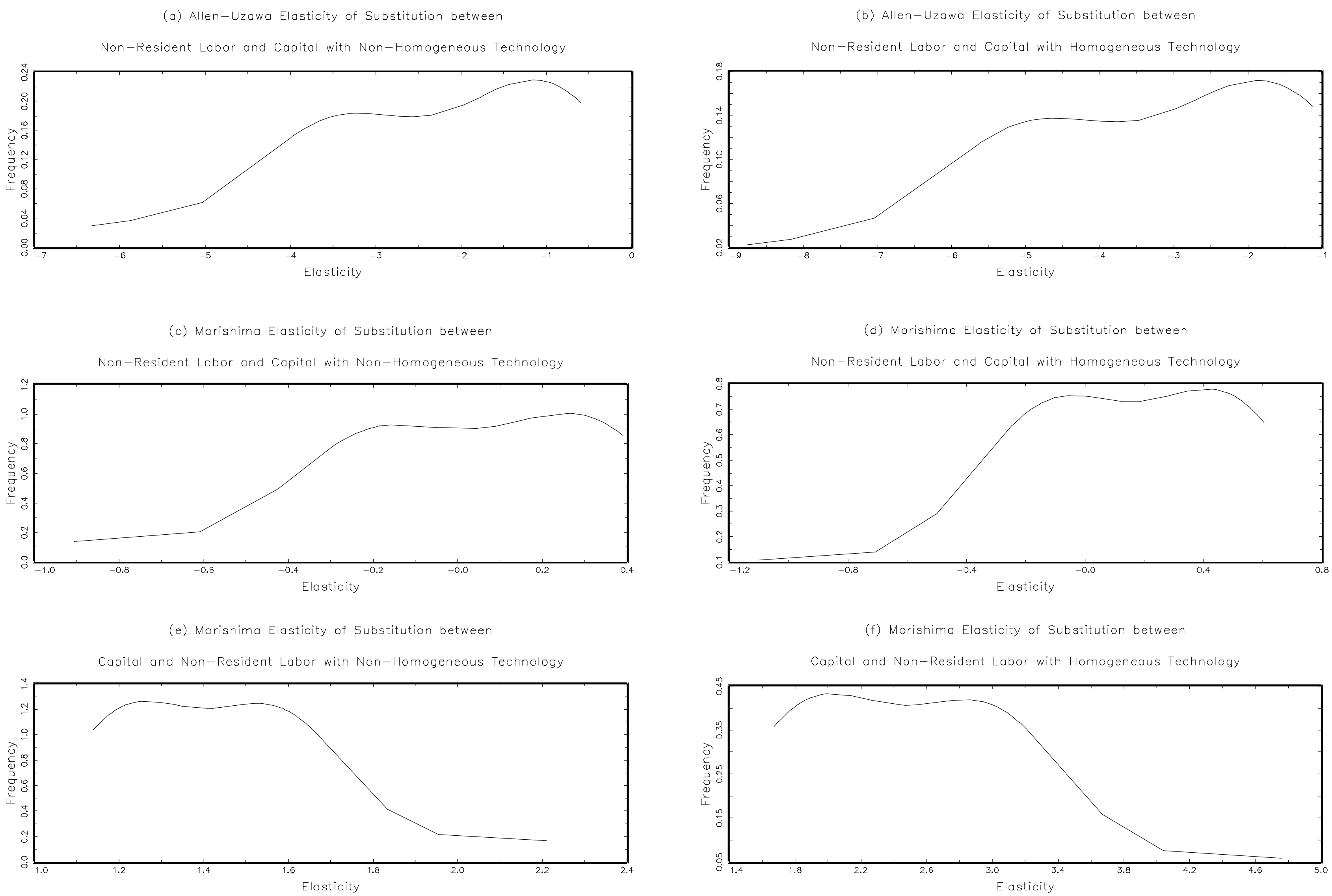


Figure 6: Distributions of Direct Allen and Morishima Elasticities

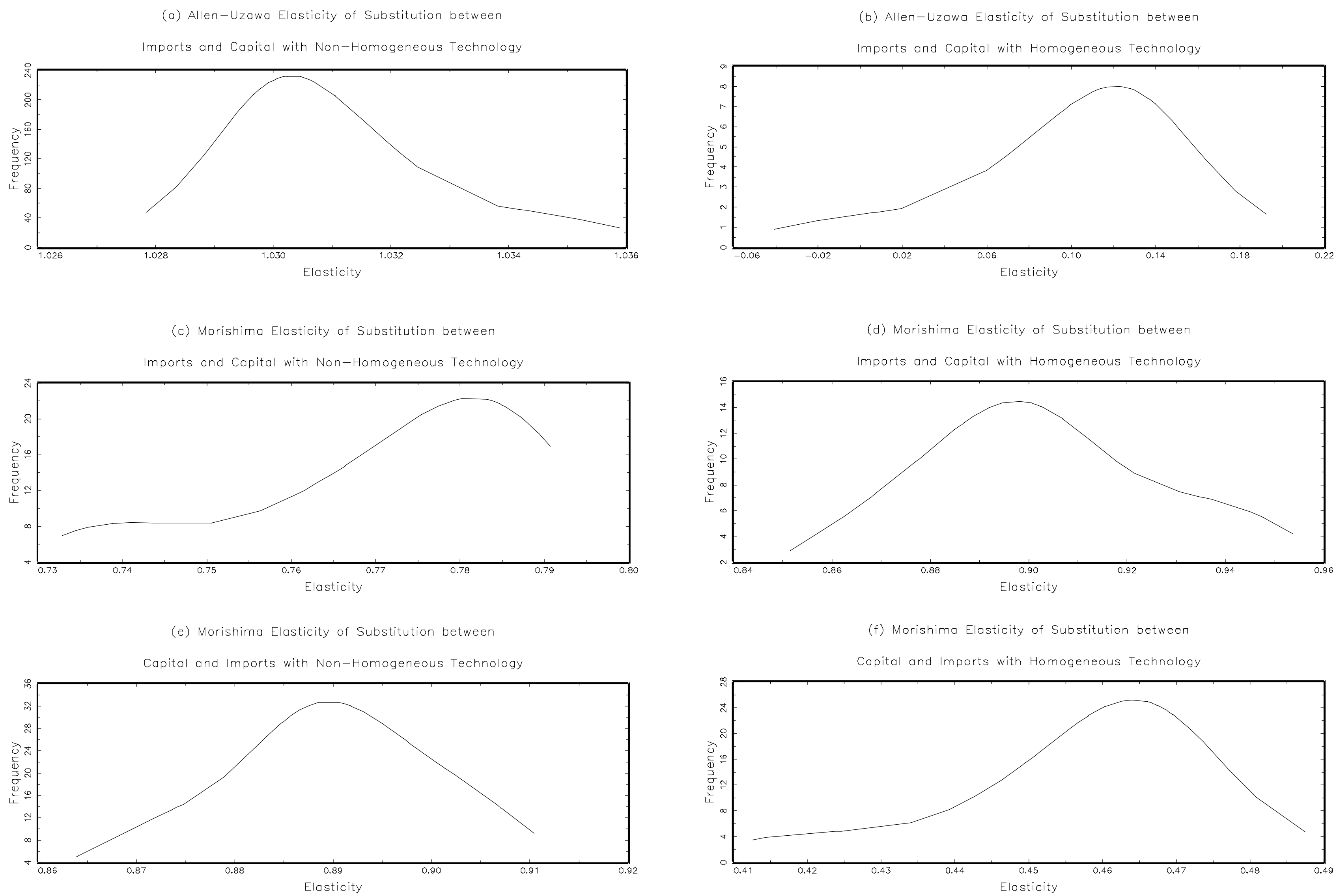


Figure 7: Distributions of Dual Allen and Morishima Elasticities

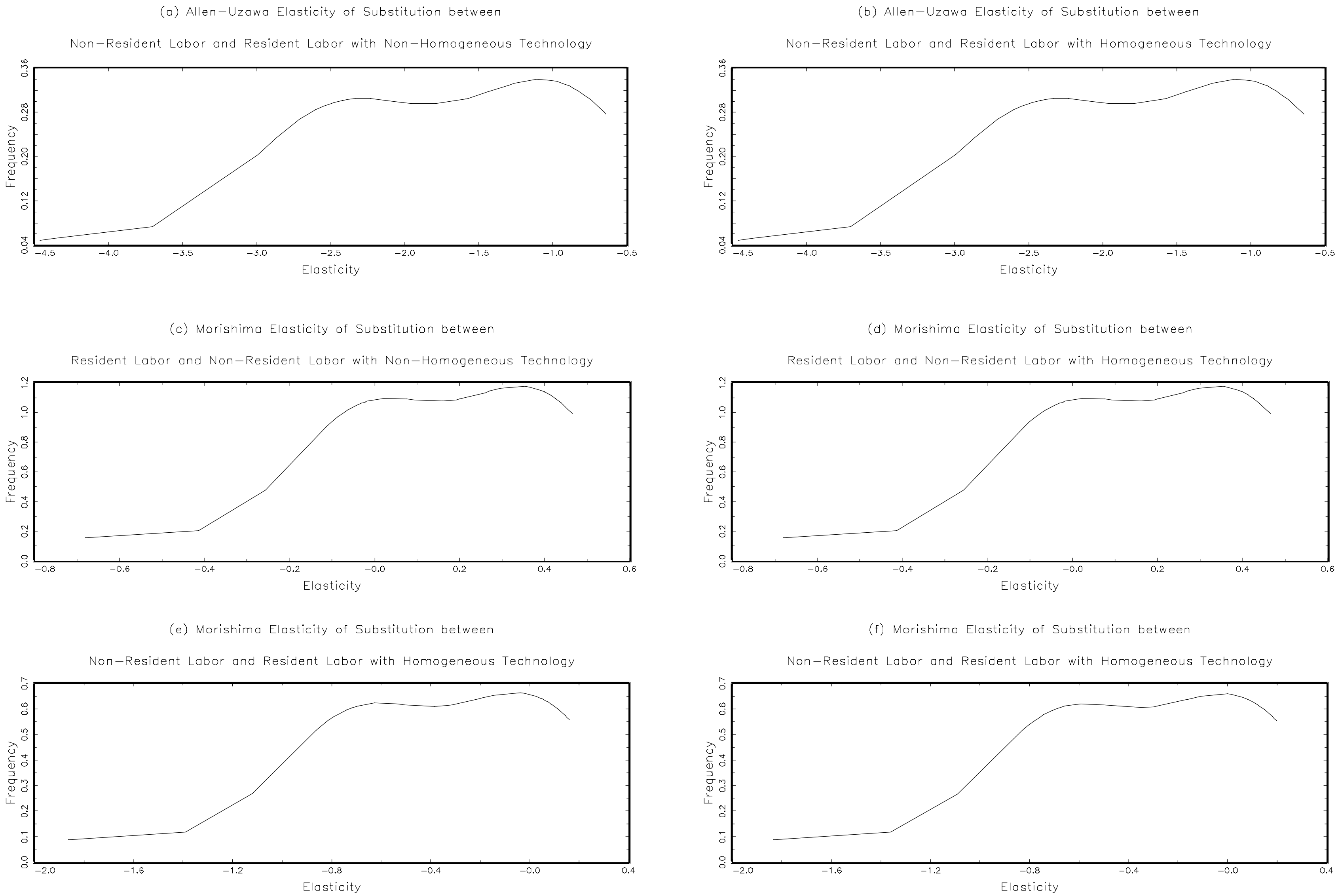


Figure 8: Distributions of Dual Allen and Morishima Elasticities

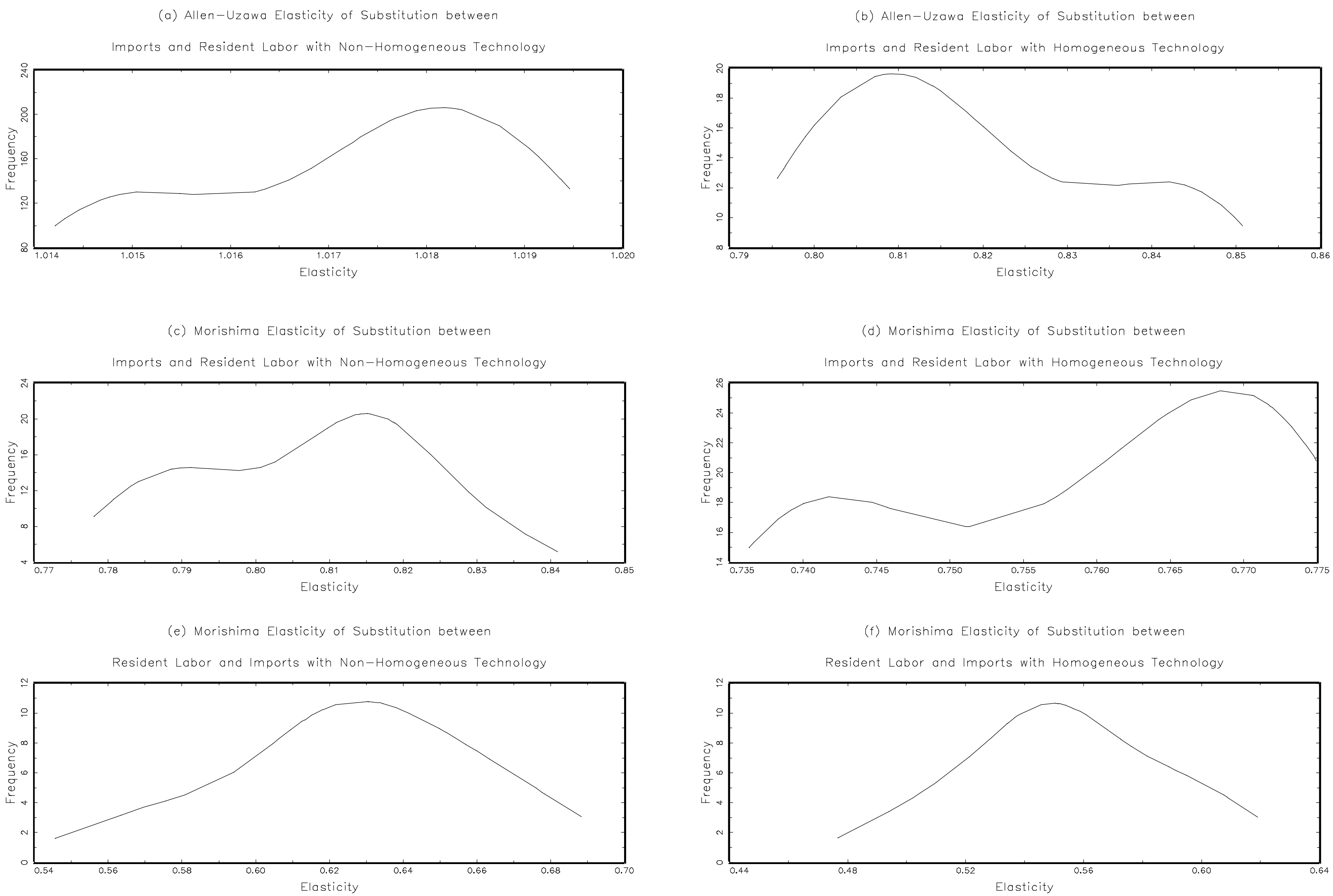
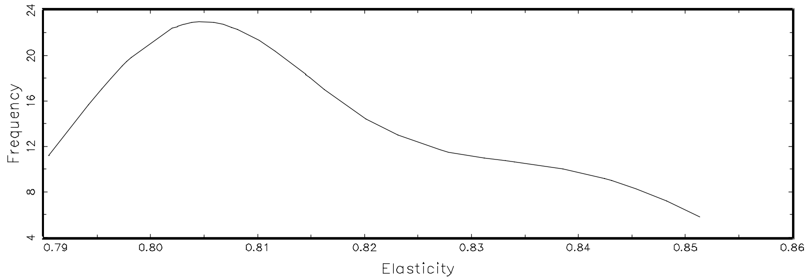


Figure 9: Distributions of Dual Allen and Morishima Elasticities

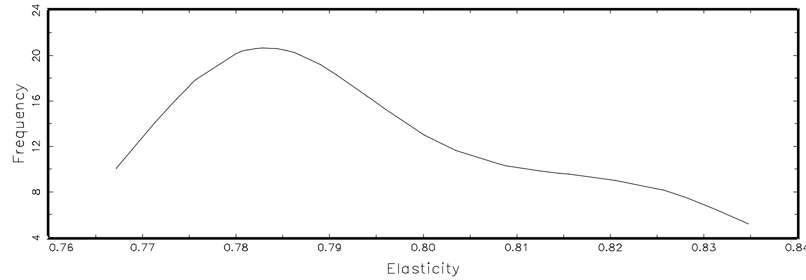
(a) Allen–Uzawa Elasticity of Substitution between

Resident Labor and Capital with Non–Homogeneous Technology



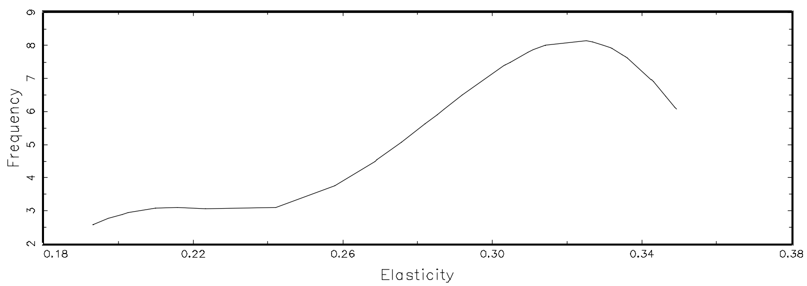
(b) Allen–Uzawa Elasticity of Substitution between

Resident Labor and Capital with Homogeneous Technology



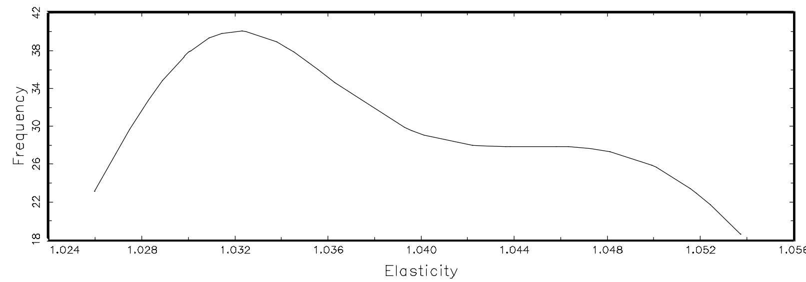
(c) Morishima Elasticity of Substitution between

Resident Labor and Capital with Non–Homogeneous Technology



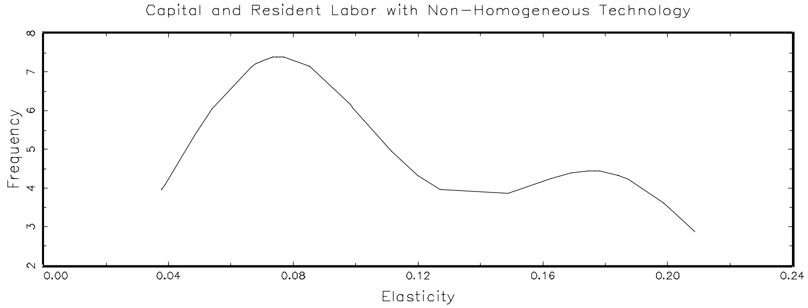
(d) Morishima Elasticity of Substitution between

Resident Labor and Capital with Homogeneous Technology



(e) Morishima Elasticity of Substitution between

Capital and Resident Labor with Non–Homogeneous Technology



(f) Morishima Elasticity of Substitution between

Capital and Resident Labor with Homogeneous Technology

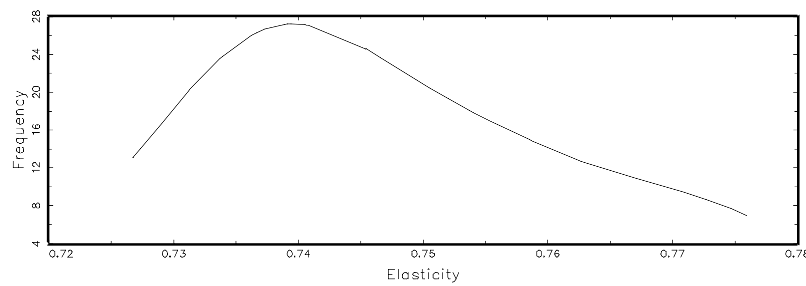


Figure 10: Distributions of Dual Allen and Morishima Elasticities

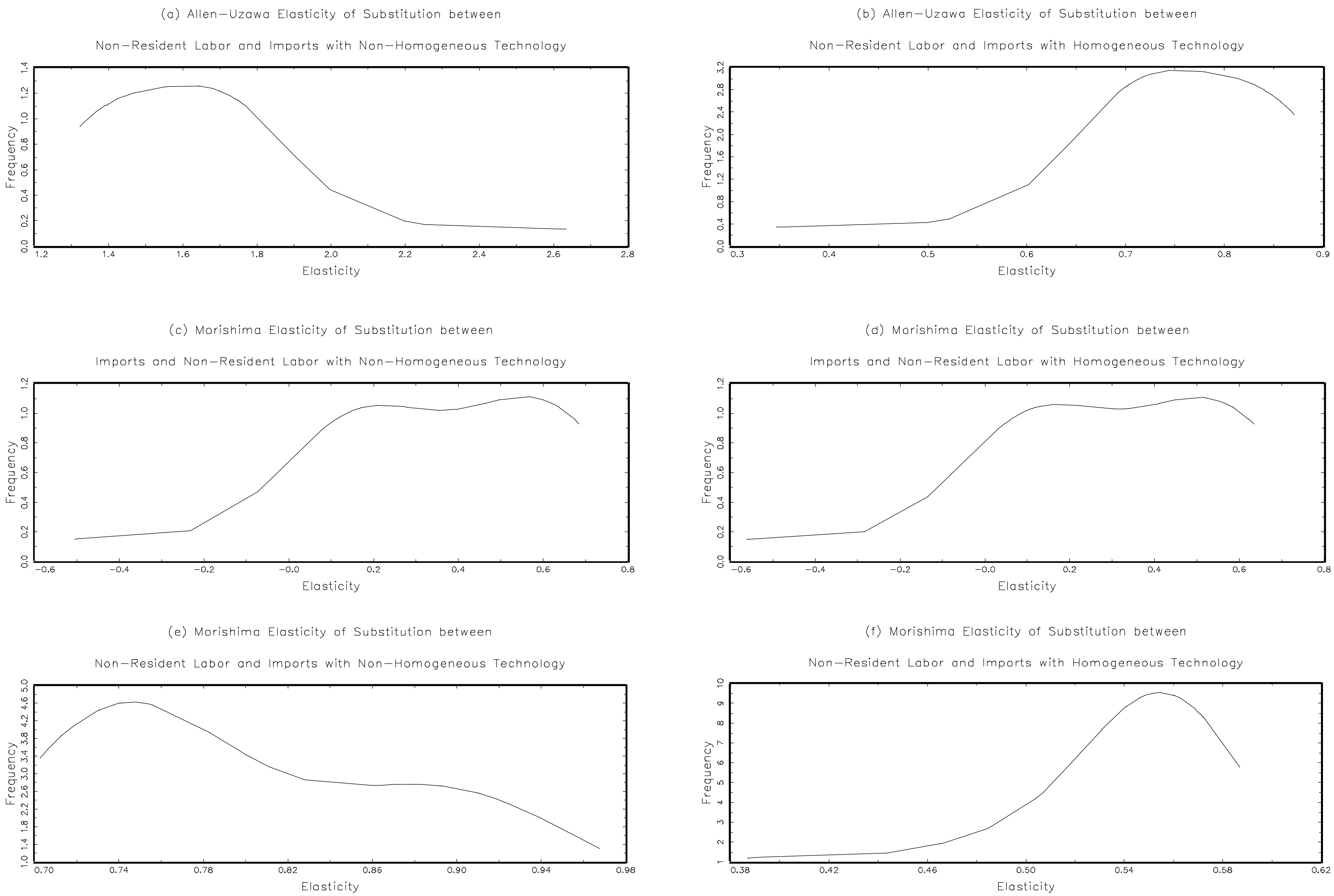


Figure 11: Distributions of Dual Allen and Morishima Elasticities

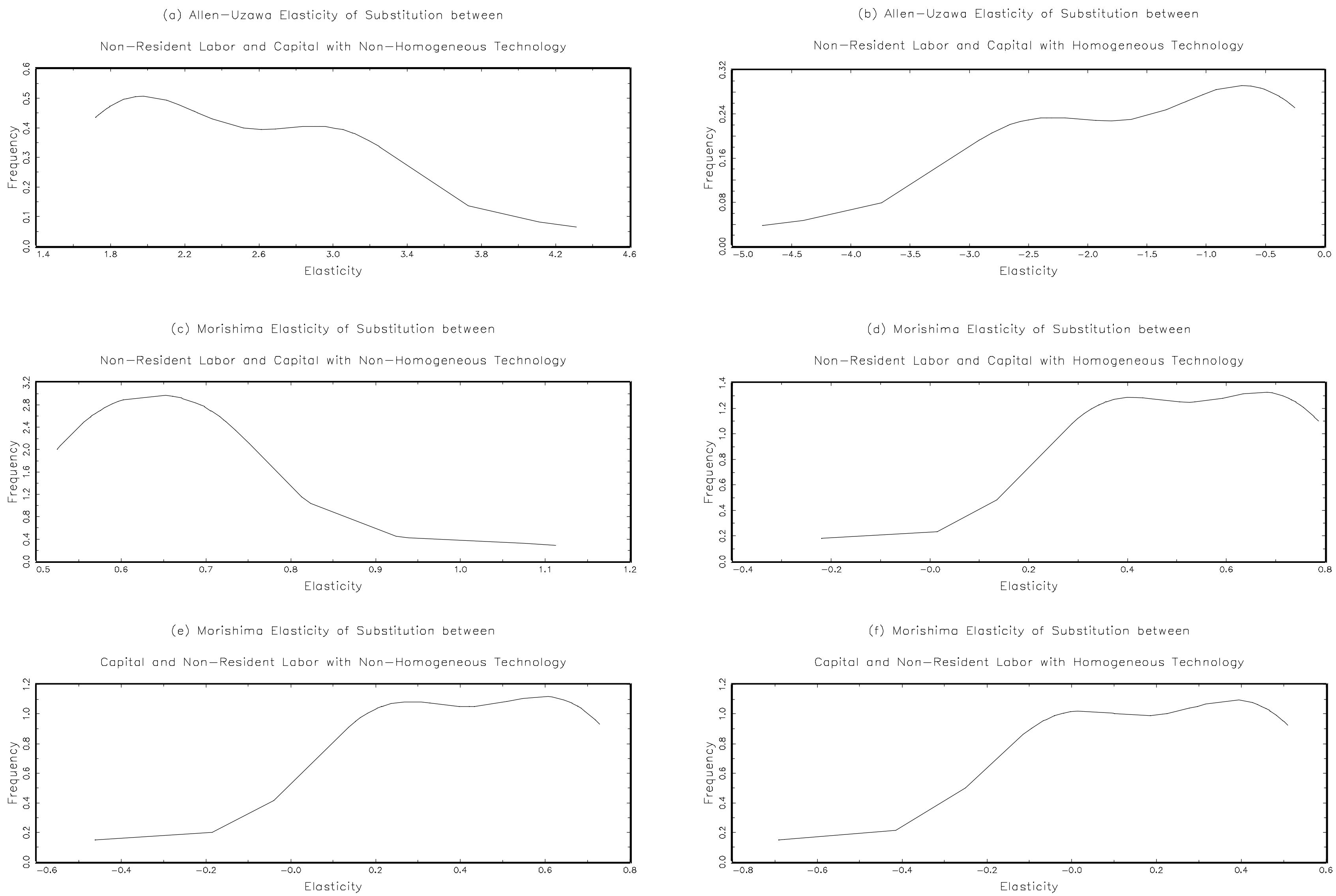


Figure 12: Distributions of Dual Allen and Morishima Elasticities