# Fiscal Policy, Increasing Returns and Endogenous Fluctuations.[¤]

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Abstract

We examine the quantitative implications of government ...scal policy in a discrete-time one-sector growth model with a productive externality that generates social increasing returns to scale. Starting from a laissez-faire economy that exhibits local indeterminacy, we show that the introduction of a constant capital tax or subsidy can lead to various forms of endogenous ‡uctuations, including stable 2-, 4-, 8-, and 10-cycles, quasi-periodic orbits, and chaos. In contrast, a constant labor tax or subsidy has no e¤ect on the qualitative nature of the model's dynamics. We also show that the use of local steady-state analysis to detect the presence of multiple equilibria in this class of models can be misleading. For a plausible range of capital tax rates, the log-linearized dynamical system exhibits saddle-point stability, suggesting a unique equilibrium, while the true nonlinear model exhibits global indeterminacy. This result implies that stabilization policies designed to suppress sunspot ‡uctuations near the steady state may not prevent sunspots, cycles, or chaos in regions away from the steady state. Overall, our results highlight the importance of using a model's nonlinear equilibrium conditions to fully investigate global dynamics.

Keywords: Fiscal Policy, Business Cycles, Sunspots, Nonlinear Dynamics, Chaos.

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# 1 Introduction

It is well-known that a wide variety of equilibrium economic models can exhibit endogenous cycles, indeterminacy, sunspots, or chaos.[1] The conditions needed for such phenomena are typically less stringent in models with incomplete markets, imperfect competition, or externalities. These environments also create a motive for government intervention to address the source of the market failure. In this paper, we show how a government ...scal policy designed to address a wedge between the social and private marginal products of capital (which is created by a productive externality) can lead to a much richer set of endogenous dynamics than is possible in the laissez-faire economy.

The framework for our analysis is a discrete-time version of the one-sector growth model developed by Benhabib and Farmer (1994). In one variant of their model, an individual ...rm's production process is subject to a positive external e¤ect that is linked to the average level of inputs across all ...rms in the economy. Benhabib and Farmer show that when this externality is strong enough to generate social increasing returns-to-scale, the model can exhibit "local indeterminacy" whereby a continuum of rational expectations equilibria exists in the neighborhood of the single interior steady state. Such an environment allows for stochastic sunspot ‡uctuations driven by "animal spirits."[2] Farmer and Guo (1994) show that a calibrated version of this model compares favorably to a standard real business cycle model in being able to replicate some cyclical features of the postwar U.S. economy.

We begin our analysis of the Benhabib-Farmer-Guo model by solving for a benchmark ...scal policy that eliminates the wedge between the social and private marginal products of capital and labor. The benchmark policy involves constant subsidy rates applied to capital and labor incomes, ...nanced by a lump-sum tax. We ...nd that the subsidy rate applied to capital income is a key bifurcation parameter for the model's perfect-foresight dynamics. Starting from a laissez-faire economy that exhibits local indeterminacy, the nonlinear dynamical system undergoes a Hopf bifurcation as the capital subsidy rate becomes su¢ciently positive and a ‡ip bifurcation as the capital subsidy rate becomes su¢ciently negative (representing a capital income tax). Both bifurcations are "supercritical," whereby an attracting orbit or cycle emerges as the subsidy rate passes a critical value. Pushing the subsidy rate beyond the critical value in

[1] Useful surveys of this large literature include Boldrin and Woodford (1990), Guesnerie and Woodford (1992), Nishimura and Sorger (1996), Reichlin (1997), and Benhabib and Farmer (1999).

[2] We use the terms "animal spirits","sunspots", and "self-ful...lling beliefs" interchangeably to mean randomness not related to uncertainties about economic fundamentals, i.e., technology, preferences, or endowments.

either direction eventually leads to chaos. In these regions of the parameter space, the largest Lyapunov exponent of the nonlinear map becomes positive, indicating "sensitive dependence on initial conditions."

Interestingly, the subsidy rate applied to labor income has no e¤ect on the qualitative nature of the model's dynamics. This result is somewhat intuitive. The labor subsidy a¤ects the tradeo¤ between consumption and leisure at a given date while the capital subsidy a¤ects the tradeo¤ between consumption goods at di¤erent dates. The inter-temporal tradeo¤ is the crucial mechanism for generating multiple equilibria because agents' expectations of future returns must become self-ful…lling. Similar logic helps to account for well-known importance of the discount factor and the capital depreciation rate (which both a¤ect the intertemporal tradeo¤) in growth models that exhibit complicated dynamics.[3,4]

For our chosen calibration, the Hopf bifurcation occurs at a capital subsidy rate of 63.8 percent. This is below the benchmark subsidy rate of 66.7 percent needed to eliminate the wedge between the social and private marginal products. Attempts by the government to encourage private investment by setting the capital subsidy at or near 66.7 percent will destabilize the steady state and allow for a much richer set of endogenous dynamics than is possible in the laissez-faire economy. In particular, as the subsidy rate is increased beyond the Hopf-bifurcation value of 63.8 percent, an attracting closed orbit (invariant circle) emerges to surround the steady state thereby allowing for quasi-periodic oscillations. Further increases in the subsidy rate cause the invariant circle to break up into a complicated chaotic attractor. The high-subsidy region is characterized by large intermittent spikes in hours worked which re‡ect a "bunching e¤ect" in production as agents' decisions internalize more of the increasing returns.

The ‡ip bifurcation occurs when gross income from capital is subsidized at the rate of -8.7 percent. This subsidy rate is equivalent to a steady-state tax on capital income net of depreciation of 20.4 percent. As the capital tax rate increases, the model exhibits a series of period-doubling bifurcations—a typical route to chaos. In this region of parameter space, the substitution e¤ect generated by expected movements in the after-tax interest rate overcomes

[3] Mitra (1998) and Baierl, Nishimura, and Yano (1998) establish some conditions on the discount factor and the capital depreciation rate that are needed for complicated dynamics in optimal growth models. Becker (1985) shows that an economy with a capital income tax (or subsidy) can be modeled as an undistorted economy with a appropriately-de…ned discount factor.

[4] Guo (1999) shows that a ‡at rate tax or subsidy applied to labor income does a¤ect model stability in the continuous-time version of the model. This occurs because the clear distinction between intra- and intertemporal tradeo¤s is lost as the time step becomes vanishingly small.

the corresponding income e¤ect by an amount that is su¢cient to induce cycling in agents' optimal saving decisions. We observe stable 2-, 4-, and 8- cycles which eventually give way to narrow window of chaotic dynamics. Further increases in the capital tax lead to the emergence of a stable 10-cycle on the other side of the chaotic region.

For capital tax rates beyond the ‡ip-bifurcation value, the steady state exhibits saddle-point stability. An analysis based solely on the log-linearized model would lead one to conclude that a unique equilibrium exists in this region of the parameter space. It turns out, however, that local determinacy of equilibrium near the steady state coexists with global indeterminacy. Away from the steady state, there exists a continuum of perfect-foresight trajectories leading to a stable n-period cycle or a chaotic attractor. It is possible, therefore, to construct stochastic sunspot equilibria away from the steady state, in the vicinity of the n-period cycle or attractor.[5]

Finally, we demonstrate how the log-linearized model might be used to design a state-contingent capital subsidy/tax policy that guarantees saddle-point stability of the steady-state. This type of local control approach has been applied recently by Kass (1998) and Barnett and He (1998) in reduced-form macroeconomic models, and by Guo and Lansing (1998) and Guo (1999) in continuous-time versions of the Benhabib-Farmer-Guo model. The important distinction here is that the model in question can exhibit global indeterminacy even in the presence of local determinacy. This result implies that stabilization policies designed to suppress sunspot ‡uctuations near the steady state may not prevent sunspots, cycles, or chaos in regions away from the steady state. Overall, our results highlight the importance of using a model's nonlinear equilibrium conditions to fully investigate global dynamics.

Before laying out the details of the model and the quantitative simulations, we brie‡y mention some other research that examines the relationship between government policy and endogenous ‡uctuations. Within this large literature, some researchers emphasize the use of …scal or monetary policy for stabilization purposes while others show how policy may create an environment that is more conducive to these type of phenomena. These are two sides of the same coin.

With regard to …scal policy, Kemp, Long, and Shimomura (1993) show that the optimal redistributive capital tax policy in a capitalist-worker model can generate endogenous cycles via a Hopf bifurcation. Bond, Wang, and Yip (1996) and Ben-Gad (2000) show that changes in the level of the capital income tax can induce indeterminate balanced growth paths in

[5] Azariadis and Guesnerie (1986) show that the existence of a stable two-cycle implies the existence of nearby sunspot equilibria in an overlapping generations model. This result is further explored by Chattopahyay and Muench (1999).

human-capital based endogenous growth models. Crès, Ghiglino, and Tvede (1997) show that internalization of a consumption externality in an overlapping generations economy (by means of government-sponsored legal entitlements) can generate endogenous cycles via a ‡ip bifurcation.[6] Cases of global indeterminacy coexisting with local determinacy are quite common in models with multiple steady states or multiple balanced growth paths.[7] We are aware of only a handful of examples in models with a single interior steady state, such as ours. These are Cazzavillan (1996), Venditti (1998), Grandmont, Pintus, and de Vilder (1998), and Pintus, Sands, and de Vilder (2000).

The remainder of the paper is organized as follows. Section 2 introduces …scal policy into the Benhabib-Farmer-Guo model. Section 3 investigates the model's local and global dynamics with constant subsidy/tax rates. Section 4 discusses stabilization policy. Section 5 concludes.

# 2 The Model

The model economy consists of three types of agents: …rms, households, and the government. Benhabib and Farmer (1994) describe two competitive decentralizations that lead to a social technology with increasing returns-to-scale. To simplify the exposition, we present the version of the model with a productive externality.[8]

## 2.1 Firms

There is a continuum of identical competitive …rms with the total number normalized to one. Each …rm produces a homogenous …nal good using the following constant returns-to-scale technology:

$$y_t = z_t k_t^{\mu} h_t^{1 \, i \, \mu}; \qquad \mu \, 2 \, (0; 1) \, ; \tag{1}$$

where $y_t$ is the …rm's output, $k_t$ and $h_t$ are the corresponding capital and labor inputs, and $z_t$ is the state of technology which the …rm takes as given. The decision problem of an individual

[6] Other research that considers the role of …scal policy as a stabilizing or de-stabilizing force includes: Farmer (1986), Reichlin (1986), Schleifer (1986), Deneckre and Judd (1992), Boldrin (1992), Evans and Honkapohja (1993), Sims (1994), Goenka (1994), Cazzavillan (1996), Schmitt-Grohé and Uribe (1997), and Austin (1999). Research that considers the role of monetary policy includes: Benhabib (1980), Grandmont (1985, 1986), Matsuyama (1991), Foley (1992), Sims (1994), Smith (1994), Woodford (1994), Chattopadhyay (1996), Matheny (1996), Fukuda (1997), Michener and Ravikumar (1998), and Benhabib, Schmitt-Grohé, and Uribe (2000), among others.

[7] See, for example, Benhabib and Perli (1994), Greiner and Semmler (1996), Evans, Honkapohja, and Romer (1998), Cazzavillan, Lloyd-Braga, and Pintus (1998), Christiano and Harrison (1999), and, Benhabib, Schmitt-Grohé, and Uribe (2000), among others.

[8] The alternative decentralization allows for monopoly power in the production of intermediate goods.

…rm is

$$\max_{k_t;h_t} (y_t - r_t k_t - w_t h_t); \tag{2}$$

subject to equation (1), where $r_t$ is the capital rental rate and $w_t$ is the real wage. Under the assumption that factor markets are perfectly competitive, pro…t maximization implies

$$r_t = \mu y_t / k_t \tag{3}$$

$$w_t = (1 - \mu) y_t / h_t. \tag{4}$$

In contrast to a standard real business cycle model where $z_t$ is governed by an exogenous stochastic process, the state of technology here is given by

$$z_t = \left( K_t^{\mu} H_t^{1-\mu} \right)^{\gamma}; \quad \gamma \geq 0; \tag{5}$$

where $K_t$ and $H_t$ are the economywide average input levels.[9] In a symmetric equilibrium, all …rms take the same actions such that $K_t = k_t$ and $H_t = h_t$. Hence we obtain the following social technology:

$$y_t = k_t^{\alpha_1} h_t^{\alpha_2}; \tag{6}$$

where $\alpha_1 \equiv \mu(1+\gamma)$ and $\alpha_2 \equiv (1-\mu)(1+\gamma)$. The social technology exhibits increasing returns-to-scale for $\gamma > 0$. We restrict our attention to the case of $\alpha_1 < 1$ which implies that the externality is not strong enough to generate sustained endogenous growth.[10]

## 2.2 Households

The economy is populated by a large number of identical, in…nitely-lived households, each endowed with one unit of time, who maximize a discounted stream of utilities over their lifetime:

$$\max \sum_{t=0}^{\infty} \beta^t \left( \log c_t - \frac{A h_t^{1+\phi}}{1+\phi} \right); \quad A > 0; \tag{7}$$

where $\beta \in (0;1)$ is the discount factor, $c_t$ is consumption, $h_t$ is hours worked and $\phi \geq 0$ denotes the inverse of the intertemporal elasticity of substitution in labor supply. We assume that there are no fundamental uncertainties present in the economy.

[9] Kamihigashi (1996) shows that the externality model is observationally equivalent to a standard real business cycle model from the standpoint of individual agents who view $z_t$ as being determined outside of their control.

[10] Christiano and Harrison (1999) adopt the parameterization $\mu = 1/3$ and $\gamma = 2$ which yields $\alpha_1 = 1$. For this knife-edge case, the equilibrium marginal product of capital is independent of $k_t$ and the model's global dynamics collapse to a quadratic difference equation in $h_t$ and $h_{t+1}$. Their setup yields two interior steady states (a sink and a saddle) in contrast to our model which possesses a single interior steady state.

The budget constraint faced by the household is

$$c_t + i_t = (1 + s_{kt})\, r_t k_t + (1 + s_{ht})\, w_t h_t \;\text{¡}\; T_t; \tag{8}$$

where $i_t$ is investment and $k_t$ is the household's stock of physical capital. Households derive income by supplying capital and labor services to …rms. Fiscal policy is introduced through the variables $s_{kt;}$ $s_{ht}$; and $T_t$; which represent the subsidy rates to capital and labor incomes, and a lump-sum tax, respectively. Under this formulation, a negative subsidy rate represents a distortionary tax.[11] Households view $r_t$; $w_t$; $s_{kt}$; $s_{ht;}$ and $T_t$ as being determined outside of their control.

Investment adds to the stock of capital according to the law of motion

$$k_{t+1} = (1 \;\text{¡}\; \pm)k_t + i_t; \qquad k_0 \text{ given}; \tag{9}$$

where $\pm$ 2 [0;1) is the constant depreciation rate. We exclude $\pm = 1$ because this case is not subject to indeterminacy in a regime of constant subsidy/tax rates. In particular, when combined with logarithmic utility and a Cobb-Douglas production technology, the assumption of 100 percent depreciation yields exactly o¤setting income and substitution e¤ects so that households only need to observe the current state the economy to decide how much to consume and invest. In this case, there exists a closed-form solution where equilibrium allocations are uniquely pinned down by current-period fundamentals, regardless of the degree of increasing returns.[12]

The …rst-order conditions for the household's optimization problem are given by

$$Ac_t h_t^{\circ} = (1 + s_{ht})w_t; \tag{10}$$

$$\frac{1}{c_t} = \frac{\bar{\ }}{c_{t+1}}\left[(1 + s_{kt+1})r_{t+1} + 1 \;\text{¡}\; \pm\right]; \tag{11}$$

$$\lim_{t \,!\, 1} {}^{-t}\frac{k_{t+1}}{c_t} = 0: \tag{12}$$

Equation (10) equates the household's marginal rate of substitution between consumption and leisure to the after-subsidy real wage. Equation (11) is the consumption Euler equation, and equation (12) is the transversality condition.

[11] Note that $s_{kt}$ is applied to gross income from capital. When $s_{kt} < 0$; this is equivalent to a tax on capital income net of depreciation of $¿_{kt} = 1 \;\text{¡}\; \frac{(1+s_{kt})r_t \;\text{¡}\; \pm}{r_t \;\text{¡}\; \pm}$; where $\pm$ is the depreciation rate.

[12] When $\pm = 1$; the equilibrium decision rules are: $k_{t+1} = (1 + s_k)\mu^{-} y_t$; $c_t = [1 \;\text{¡}\; (1 + s_k)\mu^{-}]y_t$; and $h_t = \left(\frac{(1+s_h)(1 \;\text{¡}\; \mu)}{1 \;\text{¡}\; (1+s_k)\mu^{-}}\right)^{1=(1+°)}$; where $y_t$ is given by equation (6) and $s_k$ and $s_h$ are the constant subsidy/tax rates.

### 2.3 Government

The government sets $\{s_{kt}, s_{ht}, T_t\}_{t=0}^{1}$, subject to the following budget constraint:

$$T_t = s_{kt} r_t k_t + s_{ht} w_t h_t. \tag{13}$$

By combining equations (6), (8), (9), and (13), we obtain the following aggregate resource constraint for the economy:

$$k_{t+1} = k_t^{\alpha_1} h_t^{\alpha_2} + (1 - \delta) k_t - c_t. \tag{14}$$

## 3 Dynamics with Constant Subsidy/Tax Rates

The increasing-returns technology (6) introduces a nonconvexity into the constraint set of the social planner's problem.[13] This nonconvexity presents a formidable technical barrier for policy analysis because it precludes application of the Kuhn-Tucker sufficiency theorem to (i) compute the first-best allocations and (ii) solve for the optimal fiscal policy that would implement the first-best as a competitive equilibrium. The technical barrier cannot be surmounted simply by resorting to a numerical analysis. A complete characterization of the first-best allocations in a dynamic economy with increasing returns is an unsolved problem that we leave as an open question for future research.[14] As an alternative to computing the optimal fiscal policy, we consider the following benchmark fiscal policy that eliminates the wedge between the social and private marginal products of capital and labor.

**Proposition 1.** The wedge between the social and private marginal products of capital and labor is eliminated when

$$s_{kt} = s_{ht} = \tau; \quad \text{for all } t; \tag{15}$$

$$T_t = \tau y_t; \quad \text{for all } t. \tag{16}$$

*Proof:* The social marginal products from equation (6) are $\frac{\partial y_t}{\partial k_t} = \alpha_1 y_t / k_t$ and $\frac{\partial y_t}{\partial h_t} = \alpha_2 y_t / k_t$. The after-subsidy private marginal products are $(1 + s_{kt}) r_t$ and $(1 + s_{ht}) w_t$; where $r_t$ and $w_t$ are given by equations (3) and (4). With $s_{kt} = s_{ht} = \tau$, we have $(1 + s_{kt}) r_t = \alpha_1 y_t / k_t$ and $(1 + s_{ht}) w_t = \alpha_2 y_t / k_t$. The lump-sum tax needed to finance the subsidies follows directly from equation (13). ¥

[13] The social planner chooses $\{c_t, h_t, k_{t+1}\}_{t=0}^{1}$ to maximize (7) subject to equation (14), with $k_0$ given.

[14] Gaines and Peterson (1985) show existence but not uniqueness of the first-best allocations in a growth model with increasing returns-to-scale. Dechert and Nishimura (1983) establish some features of the first-best allocations when the technology exhibits increasing returns-to-scale for an initial range of capital stocks but decreasing returns thereafter.

The benchmark policy involves constant subsidy rates that are linked directly to the externality parameter ´: A similar result is obtained in models where the productive externality (or the degree of monopoly power) does not give rise to increasing returns. In those models, unlike here, the Kuhn-Tucker su¢ciency theorem can be used to show that the benchmark …scal policy implements the unique …rst-best allocations.[15]

We now turn to a quantitative investigation of the model's local and global dynamics under a regime of constant subsidy/tax rates, that is, when $s_{kt} = s_k$ and $s_{ht} = s_h$ for all t.

## 3.1 Calibration

Parameter values are chosen based on empirically observed features of the U.S. economy. The time period in the model is taken to be one year. Table 1 summarizes the baseline parameter values, together with a brief description of the rationale used in their selection.

Table 1: Baseline Parameter Values

| Parameter | Value | Rationale |
|---|---|---|
| μ | 0:30 | Capital share in U.S. national income, see Poterba (1997, Table 4). |
| — | 0:962 | Implies after-tax interest rate of 4 percent, see Poterba (1997, Table 1). |
| A | 2:876 | Implies fraction of time spent working = 0:3; see Juster and Sta¤ord (1991). |
| ° | 0 | Indivisible labor, see Hansen (1985). |
| ± | 0:067 | Estimated from annual U.S. data on $k_t$ and $i_t$, 1954-1992. |
| ´ | 2=3 | Implies local indeterminacy in the laissez-faire version of the model. |

With the exception of the externality parameter ´; the baseline parameter settings are consistent with those typically used in real business cycle models. The degree of returns-to-scale in the model economy is given by 1 + ´: Basu and Fernald (1997) note that returns-to-scale estimates reported in the literature vary dramatically depending on the type of data used, the level of aggregation, and the estimation method. In attempting to account for the wide range of estimates, Basu and Fernald (1997) demonstrate that while the average U.S. industry exhibits approximately constant returns-to-scale, the aggregate private business economy can appear to exhibit large increasing returns. The largest aggregate estimate they obtain is 1.72 (with a standard error of 0.36).[16] However, when the aggregate returns-to-scale estimation procedure is corrected to account for reallocation of inputs across industries, Basu and Fernald (1997) …nd that the aggregate estimates shrink considerably and are close to the industry results.

[15] See, for example, Puhakka and Wright (1991), Barro and Sala-i-Martin (1992), and Guo and Lansing (1999a).

[16] See the …rst column of Table 1 (p. 259) in Basu and Fernald (1997).

The largest corrected aggregate estimate they obtain is 1.03 (with a standard error of 0.18).[17] Despite these findings, Basu and Fernald (1997, Section V) note that the uncorrected aggregate estimates may actually be more appropriate for calibrating models (such as ours) that abstract from heterogeneity in production and assume a single representative firm. This argument turns out to be helpful for our purposes because it is well-known that one-sector growth models of the type considered here require strong increasing returns for indeterminacy.

Given the other baseline parameter values, the model requires returns-to-scale in excess of approximately 1.6 to exhibit local indeterminacy.[18] We choose $\gamma = 2/3$ for the quantitative experiments which implies returns-to-scale of about 1.67. This calibration yields an indeterminate steady state (a sink) in the laissez-faire version of the model, consistent with Benhabib and Framer (1994) and Farmer and Guo (1994). While our returns-to-scale calibration falls within the range of uncorrected aggregate estimates reported by Basu and Fernald (1997, Table 1), we acknowledge that a figure of 1.67 may be viewed as too large to be considered empirically plausible for the U.S. economy. We note, for example, that Burnside, Eichenbaum, and Rebelo (1995) obtain an aggregate returns-to-scale estimate of 0.98 (with a standard error of 0.34) after correcting for cyclical variation in the utilization of physical capital.[19] To the extent that one objects to our returns-to-scale calibration, the quantitative experiments reported below should be viewed more from a methodological perspective as illustrating the pitfalls that can arise from focusing exclusively on log-linearized dynamics rather than considering the model's true nonlinear equilibrium conditions.[20]

## 3.2 Log-Linearized Dynamics

In the appendix, we show that the perfect-foresight version of the model can be approximated by the following log-linear dynamical system:

$$\begin{bmatrix} \ln\left(k_{t+1}/\hat{k}\right) \\ \ln\left(c_{t+1}/\hat{c}\right) \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\gamma_1}{\beta_1 \beta_3} & \frac{\gamma_2}{1+\beta_2\beta_3} \\ \frac{\gamma_4}{} & \frac{\gamma_4}{} \end{bmatrix}}_{J} \begin{bmatrix} \ln\left(k_t/\hat{k}\right) \\ \ln\left(c_t/\hat{c}\right) \end{bmatrix}; \qquad k_0 \text{ given,} \tag{17}$$

where $\hat{k}$ and $\hat{c}$ represent steady-state values and $J$ is a $2 \times 2$ Jacobian matrix of partial derivatives evaluated at the steady state. The elements of $J$ are constructed using the constants

[17] See the first column of Table 3 (p. 268) in Basu and Fernald (1997).

[18] We elaborate further on this point below in our discussion of Figure 1.

[19] Cole and Ohanian (1999) show that measurements of aggregate returns-to-scale in the U.S. economy are unavoidably imprecise due to the difficulties in identifying technology shocks.

[20] Models that allow for multiple sectors of production or varying capital utilization can exhibit local indeterminacy for a much lower (and hence more realistic) degree of increasing returns. For examples, see Benhabib and Farmer (1996), Perli (1998), Benhabib and Nishimura (1998), and Wen (1998a).

$\gamma_i$; $i = 1; 2; 3; 4$ which represent combinations of the deep parameters $\mu$; $\bar{}$; $\pm$; $°$; $\acute{}$; and $s_k$: The ...rst-order dynamical system possesses one predetermined variable: $k_t$: The eigenvalues of J determine the stability of the log-linear system. Notice that local stability is not a¤ected by the labor subsidy rate $s_h$. The household equilibrium conditions provide some intuition for this result. Equation (10) shows that $s_h$ a¤ects the tradeo¤ between consumption and leisure at a given date while equation (11) shows that $s_k$ a¤ects the tradeo¤ between consumption goods at di¤erent dates. The inter-temporal tradeo¤ is the crucial mechanism for generating multiple equilibria because agents' expectations of future returns must become self-ful...lling.[21]

Table 2 summarizes the model's local stability properties as we vary the capital subsidy rate $s_k$ over a wide range.

Table 2: Stability Properties Near the Steady State

| Capital Subsidy Rate | Eigenvalues of Jacobian Matrix | | Steady State |
|---|---|---|---|
| $s_k < -0.0869$ | real | $\lambda_1 < -1;\ \ \lambda_2 < 1$ | saddle |
| $s_k = -0.0869$ (‡ip bifurcation) | real | $\lambda_1 = -1;\ \ \lambda_2 < 1$ | saddle changes to sink |
| $-0.0869 < s_k < 0.2399$ | real | $\lvert\lambda_1\rvert; \lvert\lambda_2\rvert < 1$ | sink |
| $0.2399 < s_k < 0.6380$ | complex | $\lvert\lambda_1\rvert = \lvert\lambda_2\rvert < 1$ | sink |
| $s_k = 0.6380$ (Hopf bifurcation) | complex | $\lvert\lambda_1\rvert = \lvert\lambda_2\rvert = 1;$ | sink changes to source |
| $s_k > 0.6380$ | complex | $\lvert\lambda_1\rvert = \lvert\lambda_2\rvert > 1$ | source |

## 3.3 Local Indeterminacy

When both eigenvalues of J lie inside the unit circle, the steady state is indeterminate (a sink) and the economy is subject to the same type of stochastic sunspot ‡uctuations as in the original Benhabib-Farmer-Guo model. Figure 1 plots the combinations of $\acute{}$ (the externality parameter) and $s_k$ (the capital subsidy rate) that allow for local indeterminacy. Recall that the degree of returns-to-scale in the model economy is given by $1 + \acute{}$: When $\acute{} = 0$ (constant returns-to-scale), the model exhibits saddle-point stability for all values of $s_k$: From the ...gure, we see that $\acute{} > 0.5937$ is needed for the steady state to become a sink. Given $\acute{} > 0.5937$; increases in $s_k$ eventually transform the steady state into a source while decreases in $s_k$ eventually transform the steady state into a saddle point. For our calibration with $\acute{} = 2/3 \approx 0.6667$, local indeterminacy occurs for subsidy rates in the range $-0.0869 < s_k < 0.6380$: The model exhibits a locally unique equilibrium (a saddle point) for $s_k < -0.0869$: This subsidy rate is equivalent to a steady-state tax on capital income net of depreciation of $\tau_k = 0.2042$:[22] Hence,

[21] However, as mentioned in footnote 4, this intuition does not extend to the continuous-time version of the model because there is no clear distinction between the intra- and inter-temporal tradeo¤s when the time step becomes vanishingly small.

[22] See footnote 11. Auerbach (1996) estimates the e¤ective marginal tax rate on capital income under the current U.S. tax code. He obtains an estimate of 0.26 for nonresidential capital and 0.06 for residential capital.

a government that wishes to stabilize the economy against sunspot ‡uctuations near the steady state can do so simply by imposing a su¢ciently high tax rate on capital income. As we shall see, however, such a policy can open the door to other forms of endogenous ‡uctuations—those arising from global indeterminacy.

### 3.4 Hopf Bifurcation

For our calibration, the dynamical system undergoes a Hopf bifurcation as $s_k$ is increased past the value $s_k^{Hopf} = 0{:}6380$. The eigenvalues of J are complex conjugates and cross the unit circle with non-zero speed.[23] The steady state changes stability from a sink to a source and a closed orbit (invariant circle) emerges to surround the steady state. At the bifurcation point, we have det(J) = 1:[24] Using the expression for det(J) derived in the appendix, we solve for the following bifurcation value:

$$s_k^{Hopf} = \frac{(½ + ±)(1 + °)(1 + ´)}{±(1 + °) + ®_2(1 ¡ ±)(1 ¡ ¯)} ¡ 1; \tag{18}$$

where $½ ´ \frac{1}{¯} ¡ 1$ is the household's rate of time preference. Since the externality parameter ´ enters equation (18) in a multiplicative way, it is not immediately obvious whether the Hopf bifurcation occurs above or below the benchmark subsidy rate $s_k = ´$ given by Proposition 1. For our calibration, it turns out that $s_k^{Hopf} < ´$:[25] Thus, attempts by the government to close the wedge between the social and private marginal products of capital by setting $s_k$ at or near ´ = 2=3 will destabilize the steady state and allow for a much richer set of endogenous dynamics than is possible in the laissez-faire economy.

While the Hopf bifurcation theorem proves the existence of a closed orbit, it does not tell us whether the orbit is stable. There are two cases to consider. In a supercritical Hopf bifurcation, an attracting orbit emerges on the side of $s_k^{Hopf}$ where the steady state is unstable (in our case a source), that is, in the small neighborhood $s_k^{Hopf} + "$. In a subcritical Hopf bifurcation, a repelling orbit emerges on the side of $s_k^{Hopf}$ where the steady state is stable (in our case a sink), that is, in the small neighborhood $s_k^{Hopf} ¡ "$: Both cases have economic interpretations, as noted by Benhabib and Miyao (1981). An attracting orbit can be viewed

These estimates combine to yield an overall e¤ective marginal tax rate of 0.16.

[23] The Hopf bifurcation in discrete time is also called the Neimark-Sacker bifurcation. For formal descriptions, see Medio (1999), pp. 102-103 and Guckenheimer and Holmes (1983), pp 160-165.

[24] See Azariadis (1993), p. 93.

[25] In the continuous-time version of the model, it can be shown analytically that $s_k^{Hopf} < ´$ whenever $®_2 ¡ 1 > °$; that is, whenever the Benhabib-Farmer (1994) condition for local indeterminacy in a laissez-faire economy is satis…ed.

as a stylized business cycle while a repelling orbit accompanied by an attracting steady-state resembles the "corridor of stability" concept described by Leijonhufvud (1973).[26]

It is possible to distinguish between the two cases analytically by implementing a coordinate transformation and examining the sign of a coe¢cient in the third-order Taylor series expansion of the transformed dynamical system.[27] Since the calculation is extremely tedious, we resort to numerical simulations to establish that the Hopf bifurcation in our model is supercritical.[28]

The supercritical Hopf bifurcation allows the model to exhibit deterministic, quasi-periodic oscillations that never converge to the steady state. Moreover, because the invariant circle is an attractor, there exists a continuum of perfect foresight trajectories each leading to the circle. It is possible, therefore, to construct stochastic sunspot equilibria in the vicinity of the circle that remain away from the steady state. This is a form of global indeterminacy.

### 3.5 Flip Bifurcation

For our calibration, the dynamical system undergoes a ‡ip bifurcation as $s_k$ is reduced past the value $s_k^{Flip} = ¡0{:}0867$: One eigenvalue of J remains inside the unit circle while the other eigenvalue crosses the unit circle at ¡1 with non-zero speed.[29] The steady state changes stability from a sink to a saddle and a two-cycle emerges with points on either side of the steady state. The two-cycle is aligned in the direction of the eigenvector associated with the eigenvalue ¡1: At the bifurcation point, we have det(J) + tr(J) = ¡1:[30] Using the expressions for det(J) and tr(J) derived in the appendix, we solve for the following bifurcation value:

$$s_k^{Flip} = \frac{(½ + ±)(1 + °)[2®_1 + ¯(½ + ±)(1 ¡ ®_1)]=µ}{(4 ¡ 2±)(®_2 ¡ 1 ¡ °) + ¯(½ + ±)[±(1 + °)(1 ¡ ®_1) ¡ 2®_2]} ¡ 1: \qquad (19)$$

As with the Hopf bifurcation, there are two cases to consider regarding stability. In a supercritical ‡ip bifurcation, an attracting two-cycle emerges on the side of $s_k^{Flip}$ where the steady state is unstable (in our case a saddle), that is, in the small neighborhood $s_k^{Flip}$ ¡ ". In a subcritical ‡ip bifurcation, a repelling two-cycle emerges on the side of $s_k^{Flip}$ where the steady state is stable (in our case a sink), that is, in the small neighborhood $s_k^{Flip}$ + ": Although an

[26] Some helpful diagrams depicting the two cases can be found in Cugno and Montrucchio (1984).

[27] See Guckenheimer and Holmes (1983), pp 163-165. For examples of such calculations, see Foley (1992) and Drugeon and Venditti (2001).

[28] Our method of verifying stability avoids a potential pitfall of the analytical calculation. Kind (1999) shows that the third-order Taylor series coe¢cient may indicate a subcritical Hopf bifurcation (normally associated with a repelling orbit) when in fact an attracting outer orbit surrounds the repelling inner orbit. This phenomena is described as a "crater" bifurcation.

[29] For a formal description of the ‡ip bifurcation (which can only occur in discrete time), see Guckenheimer and Holmes (1983), pp 156-160.

[30] See Azariadis (1993), p. 93.

analytical calculation can be used to distinguish between the two cases, we again resort to numerical simulations to establish that the ‡ip bifurcation in our model is supercritical.[31]

The supercritical ‡ip bifurcation allows the model to exhibit deterministic cycles that never converge to the steady state. The two-cycle is an attractor, so it is possible to construct stochastic sunspot equilibria in the vicinity of the cycle that remain away from the steady state—another form of global indeterminacy. In this case, the global indeterminacy coexists with local determinacy because the steady state is a saddle point for $s_k < -0.0867$: This result implies that stabilization policies designed using the log-linearized model may back…re. In particular, setting $s_k < -0.0867$ to suppress sunspot ‡uctuations near the steady state can open the door to sunspots, cycles, or even chaos, in regions away from the steady state. We will return to this point later in Section 4 when we discuss local control policies.

### 3.6 Nonlinear Dynamics

In the appendix, we show that the model's perfect-foresight dynamics are governed by the following nonlinear map:

$$k_{t+1} = k_t^{\alpha_1} \cdot \frac{A}{(1+s_h)(1-\mu)} \frac{c_t}{k_t^{\alpha_1}}^{\cdot \frac{\alpha_2}{\alpha_2 - 1 - \gamma}} + (1-\pi) k_t - c_t; \tag{20}$$

$$\frac{-}{c_{t+1}} \left( (1+s_k)\mu k_{t+1}^{\alpha_1 - 1} \cdot \frac{A}{(1+s_h)(1-\mu)} \frac{c_{t+1}}{k_{t+1}^{\alpha_1}}^{\cdot \frac{\alpha_2}{\alpha_2 - 1 - \gamma}} + 1 - \pi \right) = \frac{1}{c_t}: \tag{21}$$

To investigate the global dynamics, we iterate the above map for a range of values of $s_k$; holding $s_h = 0$.[32] The iteration proceeds as follows. Given $k_0$ and an arbitrarily chosen $c_0$; we solve equation (20) for $k_1$: Substituting the value of $k_1$ into equation (21) yields a nonlinear equation that can be solved numerically for $c_1$: The procedure is then repeated to compute $k_2$; $c_2$ and so on. In practice, we use $k_0 = \hat{k}$ and $c_0 \in [1.01\hat{c}; 1.09\hat{c}]$; where $\hat{k}$ and $\hat{c}$ are the steady-state values implied by the settings of $s_k$ and $s_h$: The number of iterations is chosen to ensure that the limiting behavior of the model is not a¤ected by the particular starting values. While our model is deterministic, the qualitative properties of the nonlinear map should be robust to the introduction of small stochastic disturbances.[33] It is possible, therefore, to construct

[31] The analytical calculation is described by Guckenheimer and Holmes (1983), pp 156-160. For an example, see Becker and Foias (1994).

[32] Although $s_h$ does not a¤ect the model dynamics, it does a¤ect the range of values of $h_t$ observed during the simulations. We set $s_h = 0$ to ensure $h_t \leq 1$; consistent with our assumption of a time endowment normalized to one.

[33] This has been demonstrated formally using the discrete logistic map by Crutch…eld, Farmer, and Huberman

global sunspot equilibria simply by appending a stochastic disturbance term to equation (21). Figures 2 through 10 illustrate the simulation results.

Figure 2 plots the bifurcation diagram and the largest Lyapunov exponent over the range $-0.135 \leq s_k \leq 0.689$.[34] Figures 3 and 4 provide detailed views near $s_k^{Flip}$ and $s_k^{Hopf}$: The bifurcation diagram summarizes the long-run behavior of the model by plotting the last 150 points of a very long simulation. The Lyapunov exponent measures the average exponential rate of divergence of trajectories with nearby starting points. The presence of one or more positive Lyapunov exponents is an indicator of "sensitive dependence on initial conditions"—a commonly-used definition of chaos.

We compute the Lyapunov exponents according to the procedure described by Alligood, Sauer, and Yorke (1997), pp. 199-201. Since equation (21) cannot be solved explicitly for $c_{t+1}$; the required derivatives $\frac{\partial c_{t+1}}{\partial k_t}$ and $\frac{\partial c_{t+1}}{\partial c_t}$ are computed numerically by log-linearizing equation (21) around each successive point of the trajectory generated by the nonlinear map. This introduces some approximation error into our computation so that values of the Lyapunov exponent which are only slightly above zero (those in the range $0.635 \leq s_k \leq 0.645$) are not reliable indicators of chaos. Nevertheless, the figures show that pushing the capital subsidy rate beyond $s_k^{Hopf}$ or $s_k^{Flip}$ in either direction eventually leads to chaos as indicated by a significantly positive Lyapunov exponent. The transition to chaos takes place via a "quasi-periodic" route in the high-subsidy region $s_k > s_k^{Hopf}$ and via a "period-doubling" route in the low-subsidy region $s_k < s_k^{Flip}$: Both of these routes to chaos are common in nonlinear maps, as noted by Medio (1998).

Figures 5 through 10 depict various forms of endogenous fluctuations as $s_k$ takes on different values. Figures 5 and 6 verify that the Hopf bifurcation is supercritical as evidenced by the attracting nature of the invariant circle. When $s_k = s_k^{Hopf} + 0.002$; perfect foresight trajectories eventually converge to the invariant circle for arbitrary starting points either inside or outside of the circle. Figure 7 shows that the invariant circle starts break up into irregular cycles when the subsidy rate is slightly increased to $s_k = s_k^{Hopf} + 0.007$: Figure 8 shows that a complicated chaotic attractor emerges when the subsidy rate is further increased to $s_k = s_k^{Hopf} + 0.051$: Although not plotted separately, the model exhibits stable 2- 4- and 8-cycles for subsidy rates

(1982). Benhabib and Nishimura (1989) show that a stable two-cycle in a deterministic economy generalizes to the concept of "cyclic sets" in an economy subject to stochastic shocks.

[34] For values of $s_k$ outside this range, we found that the nonlinear map would often converge to the zero steady state (which is also an attractor). The Gauss programs used to construct the figures are available from the authors upon request.

in the range $-0.133 < s_k < s_k^{Flip}$: While theory tells us that there are an in…nite number of period-doublings in the cascade, the corresponding intervals of $s_k$ are too narrow for the higher integer cycles to be observed in the bifurcation diagram. In Figure 9, we see that another type of chaotic attractor emerges when the subsidy rate is reduced to $s_k = -0.13435$. Figure 10 shows that a stable 10-cycle emerges when the subsidy rate is further reduced to $s_k = -0.13510$:

Changes in $s_k$ a¤ect the amplitude of the cycles or oscillations. The high-subsidy region is characterized by large intermittent spikes in hours worked and output which re‡ect a "bunching e¤ect" in production as agents' decisions internalize more of the increasing returns. In the negative-subsidy region, the substitution e¤ect generated by expected movements in the after-tax interest rate overcomes the corresponding income e¤ect by an amount that is su¢cient to induce cycling in agents' optimal saving decisions. These stable cycles can only be observed in the nonlinear model. Once the model is log-linearized, any perturbation away from the stable-manifold leads to explosive behavior because the crucial nonlinear terms that are needed to keep the oscillations bounded are no longer present. The time series plots in Figures 5 through 10 reveal large percentage changes in model output. The amplitudes are much larger than those observed in the postwar U.S. economy at business cycle frequencies. The model behavior can be traced to the presence of strong increasing returns. It would be interesting to conduct similar experiments in a multi-sector framework or one with varying capital utilization to ascertain whether qualitatively similar ‡uctuations can be obtained with a lower degree of increasing returns. Such a model may be capable of generating endogenous business-cycle movements that more closely resemble those in the data.

# 4 Stabilization Policy

## 4.1 Welfare Implications

Given the model's susceptibility to endogenous ‡uctuations, it is natural to ask whether the government should try to stabilize the economy through some type of activist …scal policy. Standard second-best analysis tells us that there is no de…nitive answer to this normative question. In our model, a ‡uctuating economy and its stabilized counterpart will both be Pareto-inferior due to the presence of the productive externality. A priori, we cannot rank these economies from a welfare standpoint. Monte Carlo simulations are unlikely to settle the matter because the results will depend on the assumed …scal policy in the baseline economy

(which governs the nature of the endogenous ‡uctuations to be stabilized) and the assumed variance of a sunspot shock (which can be present whenever the baseline economy exhibits local or global indeterminacy). Welfare questions are further complicated by our lack of knowledge regarding the …rst-best allocations for this economy. In other environments, the …rst-best allocations provide an important benchmark for judging the desirability of stabilization policy.[35] Finally, we note that Grandmont (1985) makes a case for stabilization policy even when endogenous ‡uctuations are Pareto-optimal. He argues that complicated endogenous dynamics may prevent agents from learning enough about their environment to support convergence to a rational expectations equilibrium.[36]

In light of the many complex issues a¤ecting the desirability of stabilization policy for this economy, we restrict our attention to questions of feasibility. In what follows, we describe some policy mechanisms that can suppress sunspot ‡uctuations near the steady state.

## 4.2 Local Control

Here we demonstrate how the log-linearized model might be used to design a state-contingent capital subsidy/tax policy that selects a locally unique equilibrium by ensuring saddle-point stability of the steady-state. To design the policy, we …rst replace the constant subsidy rate $s_k$ in equation (21) with its state-contingent counterpart $s_{kt+1}$: Assuming that households view the subsidy rate as being determined outside of their control, we can construct the following modi…ed version of (17):

$$\begin{bmatrix} \ln\left(k_{t+1}/\hat{k}\right) \\ \ln\left(c_{t+1}/\hat{c}\right) \end{bmatrix} = \underbrace{\begin{bmatrix} \gamma_1 & \gamma_2 \\ \frac{-1-\beta}{\gamma_4} & \frac{1+\beta-\gamma_3}{\gamma_4} \end{bmatrix}}_{J} \begin{bmatrix} \ln\left(k_t/\hat{k}\right) \\ \ln\left(c_t/\hat{c}\right) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-\gamma_5}{\gamma_4} \end{bmatrix} \ln\left(\frac{1+s_{kt+1}}{1+\hat{s}_k}\right); \tag{22}$$

where $\gamma_5 = \bar{\ }(\frac{1}{2} + \pm)$ and $\hat{s}_k$ represents the steady-state subsidy rate. Our decision to linearize around $\ln(1+\hat{s}_k)$; as opposed to $\ln(\hat{s}_k)$; allows for negative subsidy rates and maintains the elements of $J$ unchanged from before.

Now consider a local control policy of the form

$$\ln\left(\frac{1+s_{kt}}{1+\hat{s}_k}\right) = \begin{bmatrix} d_1 & d_2 \end{bmatrix} \begin{bmatrix} \ln\left(k_{t-1}/\hat{k}\right) \\ \ln\left(c_{t-1}/\hat{c}\right) \end{bmatrix}; \tag{23}$$

[35] Deneckere and Judd (1992) examine the welfare implications of stabilization policy in a model where it can be shown that the unique …rst-best allocations do not exhibit endogenous ‡uctuations.

[36] For additional discussion of the welfare implications of stabilization policy in models with endogenous ‡uctuations, see Guesnerie and Woodford (1992, secion 8.2) and Bullard and Butler (1993).

where $d_1$ and $d_2$ are control parameters that govern the response of $s_{kt}$ to the lagged state variables $k_{t-1}$ and $c_{t-1}$. Updating (23) by one time-step and substituting into (22) yields

$$\begin{bmatrix} \ln\left(k_{t+1}/\hat{k}\right) \\ \ln\left(c_{t+1}/\hat{c}\right) \end{bmatrix} = \underbrace{\begin{bmatrix} \gamma_1 & \gamma_2 \\ \frac{\gamma_1(\gamma_3+d_1)\gamma_5}{\gamma_4} & \frac{1+\gamma_2(\gamma_3+d_2)\gamma_5}{\gamma_4} \end{bmatrix}}_{J_1} \begin{bmatrix} \ln\left(k_t/\hat{k}\right) \\ \ln\left(c_t/\hat{c}\right) \end{bmatrix}; \qquad k_0 \text{ given.} \tag{24}$$

The basic idea behind local control is to choose $d_1$ and $d_2$ such that (24) exhibits saddle point stability. This requires one eigenvalue of $J_1$ to lie inside the unit circle and the other eigenvalue to lie outside. Figure 11 plots the combinations of $d_1$ and $d_2$ that achieve the desired outcome, depending on the assumed value of $\hat{s}_k$. A systematic approach to local control would optimize among the many candidate combinations of $d_1$ and $d_2$ according to some stabilization criterion. For example, Kaas (1998) chooses control parameters such that the reduced-form Jacobian ($J_1$ in our case) projects onto the linearization of the stable manifold. By applying linear optimal control theory, Barnett and He (1998) choose control parameters to minimize a weighted combination of the variances of state and control variables.

Some applications of local control have appeared recently in the indeterminacy literature. We brie‡y discuss some examples that are closely related to our analysis. Guo and Lansing (1998) show that a progressive income tax can ensure saddle-point stability of the steady state in a continuous-time version of the present model. Speci…cally, they consider a tax policy of the form

$$\ln\left(\frac{(1-\tau_t)}{(1-\hat{\tau})}\right) = \alpha \ln\left(\frac{\hat{y}}{y_t}\right); \tag{25}$$

where $\tau_t$ is the tax rate, $\alpha$ is the slope of the tax schedule, and $y_t$ is current output. Since the log-linearized equilibrium conditions can be used to express $y_t$ in terms of $k_{t-1}$, $c_{t-1}$; and $\tau_t$; equation (25) can be viewed as a special case of equation (23).

Georges (1995) shows that adjustment costs applied to jump variables can be used to select a locally unique equilibrium. One application of this idea, discussed by Wen (1998b), is a time-to-build capital accumulation technology. Similarly, explicit adjustment costs for capital investment can be used to select a locally unique equilibrium. To see how this works, consider an economy where the household budget constraint (8) is replaced by the following laissez-faire version:

$$c_t + i_t\left[1 + \underbrace{\frac{\tilde{\alpha}}{2}\left(\frac{k_{t+1}}{k_t} - 1\right)^2}_{\tau_t(t)}\right] = w_t h_t + r_t k_t: \tag{26}$$

where $i_t = k_{t+1} - (1 - \delta) k_t$ and $\tilde{A} \geq 0$. Following Abel and Blanchard (1983), adjustment costs are modeled here as a premium $\phi_t(\cdot)$ paid for each unit of investment goods relative to consumption goods. From equation (26), adjustment costs are observationally equivalent to a state-contingent tax on investment; households internalize the impact of their actions on the tax rate and tax revenues are simply thrown away. The adjustment cost parameter $\tilde{A}$ now serves as a bifurcation parameter for the model's perfect-foresight dynamics. For our calibration, the dynamical system undergoes a flip bifurcation as $\tilde{A}$ is increased past the value $\tilde{A}^{Flip} = 0.3689$. At this point, the steady state changes stability from a sink to a saddle and a two-cycle emerges with points on either side of the steady state.

The above examples show that there are many ways to select a locally unique equilibrium. Nevertheless, these examples suffer from the drawback of being based on a log-linear approximation. When global indeterminacy coexists with local determinacy as it can here, equilibrium selection mechanisms designed using the approximating model may prove unsuccessful when introduced into the true nonlinear model.[37]

# 5 Conclusion

This paper has shown that the introduction of a constant capital tax or subsidy in the Benhabib-Farmer-Guo model can lead to a much richer set of endogenous dynamics than is possible in the laissez-faire version of the model. The nonlinear dynamical system undergoes a Hopf bifurcation as the capital subsidy rate becomes sufficiently positive, and a flip bifurcation as the capital subsidy rate becomes sufficiently negative (representing a capital income tax). The model's perfect-foresight dynamics allow for stable 2-, 4-, 8-, and 10-cycles, quasi-periodic orbits, and chaos. None of these phenomena can be observed in the log-linearized version of the model. For a plausible range of capital tax rates, local determinacy of equilibrium near the steady state coexists with global indeterminacy. This implies that stabilization mechanisms designed using a log-linearized model may not prevent cycles, sunspots, or chaos away from the steady state. Overall, our results caution against the use of local steady-state analysis to make inferences about the global behavior of a nonlinear economic model.

[37] In the working paper version of this article, Guo and Lansing (1999b), we show how the nonlinear model can be used to design a state-contingent fiscal policy that selects a globally unique equilibrium. The global stabilization policy creates an environment where the income and substitution effects of future interest rate movements exactly cancel out. As a result, equilibrium allocations are uniquely pinned down by current-period fundamentals, regardless of the degree of increasing returns.

# A Appendix

This appendix summarizes the equations used to investigate the model's perfect-foresight dynamics under a regime of constant subsidy/tax rates where $s_{kt} = s_k$ and $s_{ht} = s_h$ for all t.

The equilibrium conditions are:

$$k_{t+1} = k_t^{\alpha_1} h_t^{\alpha_2} + (1 - \delta) k_t - c_t; \qquad k_0 \text{ given}, \tag{A.1}$$

$$A c_t h_t^{\phi} = (1 + s_h) \underbrace{(1 - \mu) k_t^{\alpha_1} h_t^{\alpha_2 - 1}}_{w_t}; \tag{A.2}$$

$$\frac{1}{c_t} = \frac{\beta}{c_{t+1}} \left[ \underbrace{(1 + s_k) \mu k_{t+1}^{\alpha_1 - 1} h_{t+1}^{\alpha_2}}_{r_{t+1}} + 1 - \delta \right]; \tag{A.3}$$

For the parameter values in Table 1, it is straightforward to show that the above economy exhibits a unique interior steady state. Equation (A.2) implies $h_t = \left( \frac{A}{(1+s_h)(1-\mu)} \frac{c_t}{k_t^{\alpha_1}} \right)^{\frac{1}{\alpha_2 - 1 - \phi}}$ which can be used to eliminate $h_t$ from (A.1) and (A.3) thus yielding equations (20) and (21) in the text.

In the vicinity of the steady state, equations (20) and (21) can be approximated by the following log-linear dynamical system:

$$\begin{bmatrix} \ln(k_{t+1}/\hat{k}) \\ \ln(c_{t+1}/\hat{c}) \end{bmatrix} = \underbrace{\begin{bmatrix} \gamma_1 & \gamma_2 \\ \frac{\gamma_1 \gamma_3}{\gamma_4} & \frac{1 + \gamma_2 \gamma_3}{\gamma_4} \end{bmatrix}}_{J} \begin{bmatrix} \ln(k_t/\hat{k}) \\ \ln(c_t/\hat{c}) \end{bmatrix}; \tag{A.4}$$

where the elements that make up the Jacobian matrix J are given by:

$$\gamma_1 = 1 - \delta - \frac{(1 + \phi)(\rho + \delta)(1 + \tau)}{(\alpha_2 - 1 - \phi)(1 + s_k)}; \tag{A.5}$$

$$\gamma_2 = \delta + \frac{(1 + \phi)(\rho + \delta)}{(\alpha_2 - 1 - \phi)(1 + s_k)\mu}; \tag{A.6}$$

$$\gamma_3 = -\beta(\rho + \delta) \left[ \frac{(1 + \phi)\alpha_1 + \alpha_2 - 1 - \phi}{\alpha_2 - 1 - \phi} \right]; \tag{A.7}$$

$$\gamma_4 = 1 - \beta(\rho + \delta) \left[ \frac{\alpha_2}{\alpha_2 - 1 - \phi} \right]; \tag{A.8}$$

where $\rho \equiv \frac{1}{\beta} - 1$ is the household's rate of time preference. Notice that the elements of J do not depend on the labor disutility parameter A or the labor subsidy rate $s_h$. This shows that the labor subsidy rate does not affect the model's local stability properties. The determinant and trace of J are given by

$$\det(J) = \frac{\gamma_1}{\gamma_4}; \tag{A.9}$$

$$\mathrm{tr}(J) = \gamma_1 + \frac{1 + \gamma_2 \gamma_3}{\gamma_4}. \tag{A.10}$$

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Fig 1: Stability Properties Near Steady State

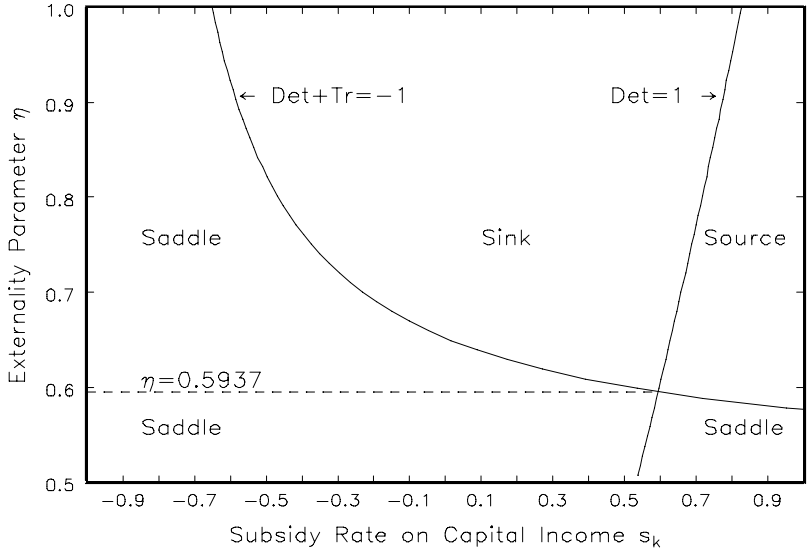


Fig 2a: Bifurcation Diagram

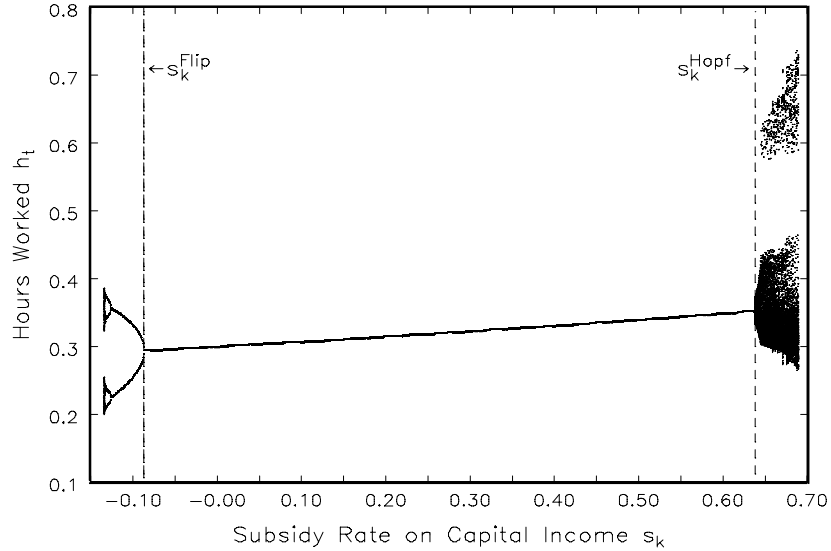


Fig 2b: Largest Lyapunov Exponent

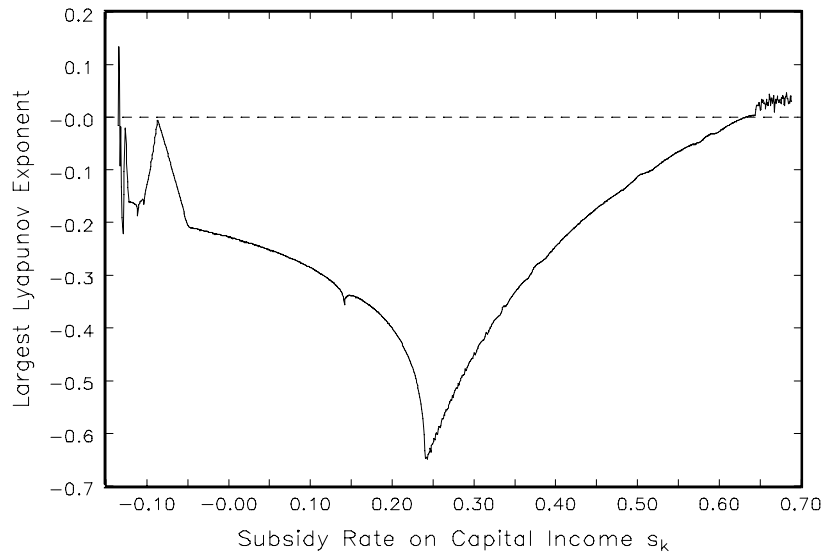

FIG 3a: Bifurcation Diagram (Detail Near $s_k^{Flip}$)

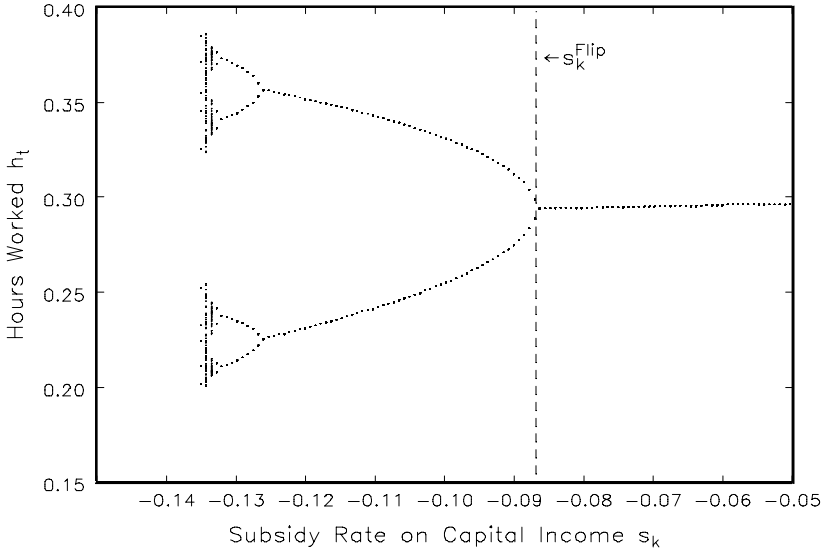


FIG 4a: Bifurcation Diagram (Detail Near $s_k^{Hopf}$)

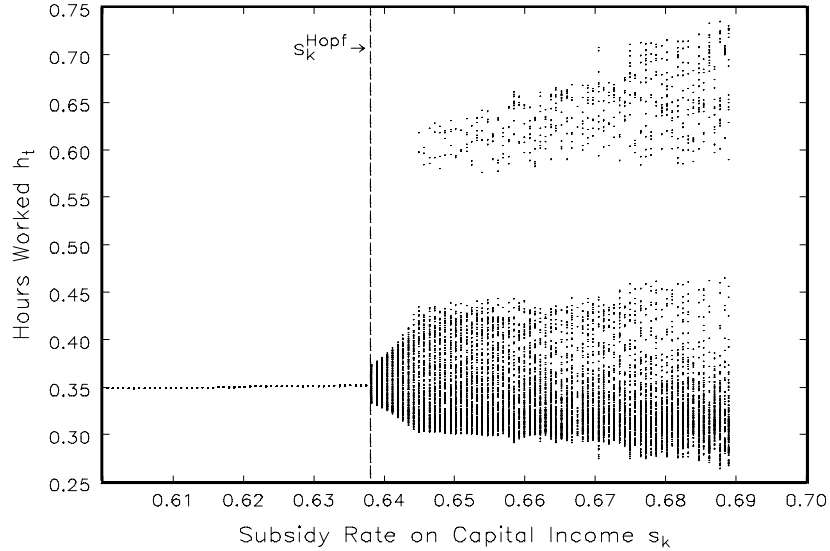


FIG 3b: Largest Lyapunov Exponent (Detail Near $s_k^{Flip}$)

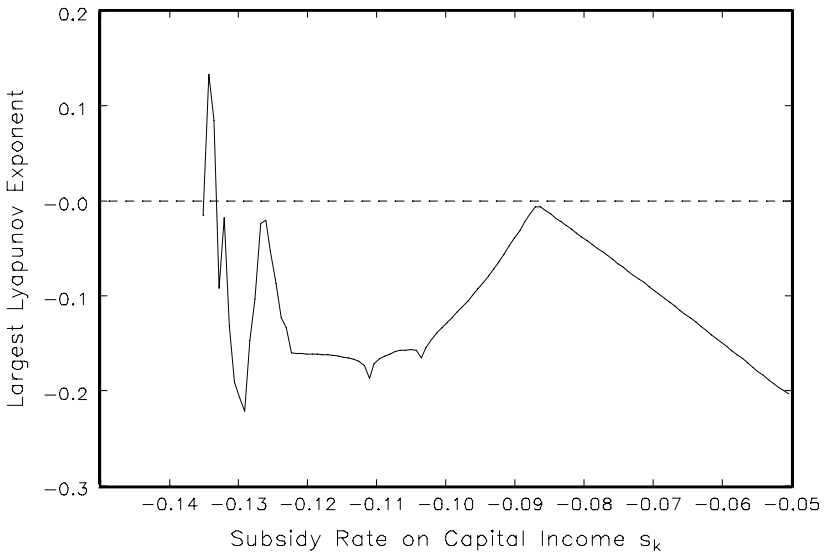


Fig 4b: Largest Lyapunov Exponent (Detail Near $s_k^{Hopf}$)

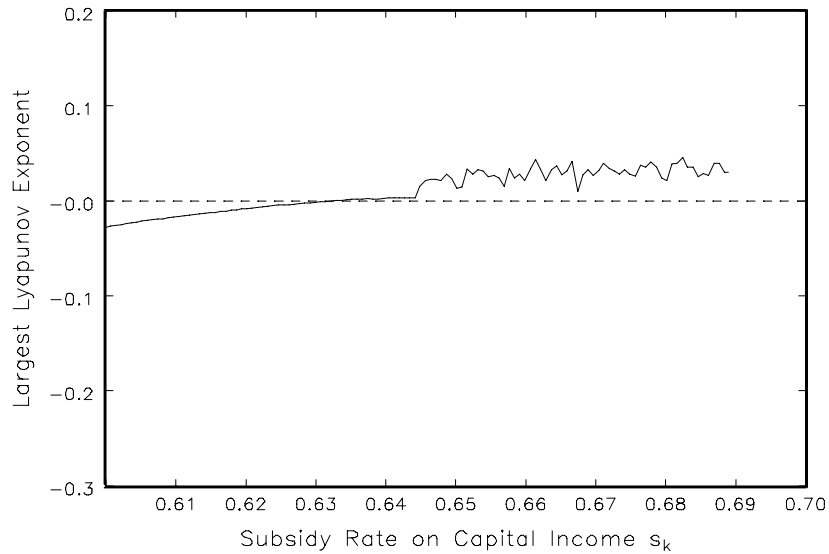

Fig 5a: Phase Diagram (Attracting Circle – Start Inside)
$s_k$=0.64003 (=$s_k^{Hopf}$+0.002), $s_h$=0

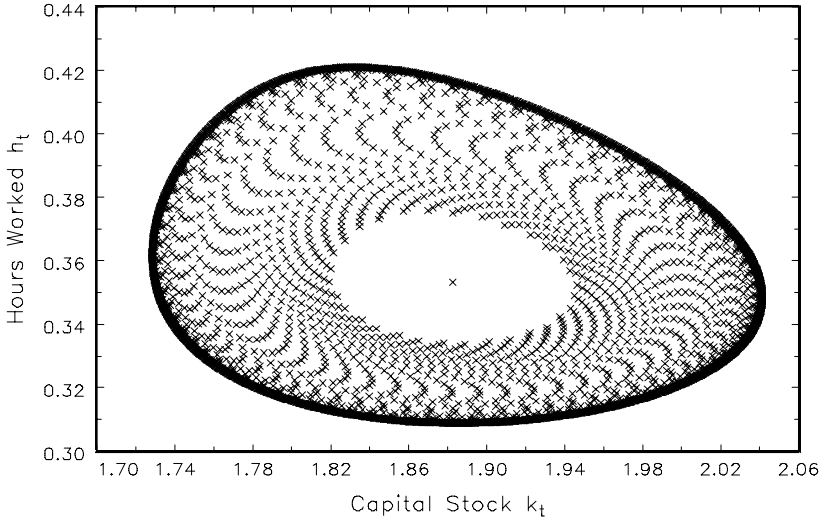


Fig 6a: Phase Diagram (Attracting Circle – Start Outside)
$s_k$=0.64003 (=$s_k^{Hopf}$+0.002), $s_h$=0

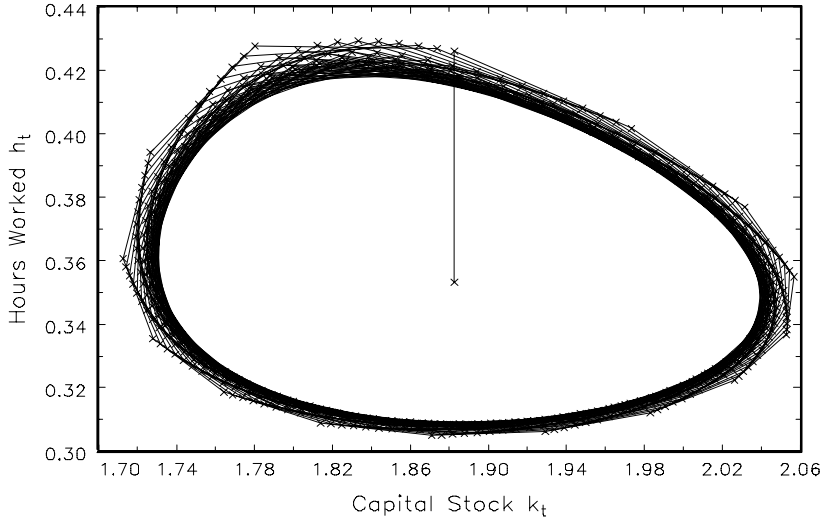


Fig 5b: Time Series Plot (Attracting Circle – Start Inside)

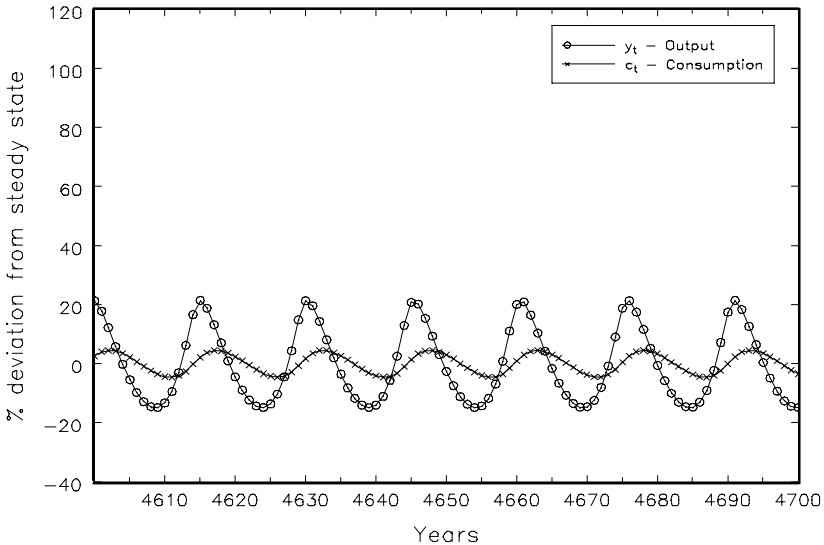


Fig 6b: Time Series Plot (Attracting Circle – Start Outside)

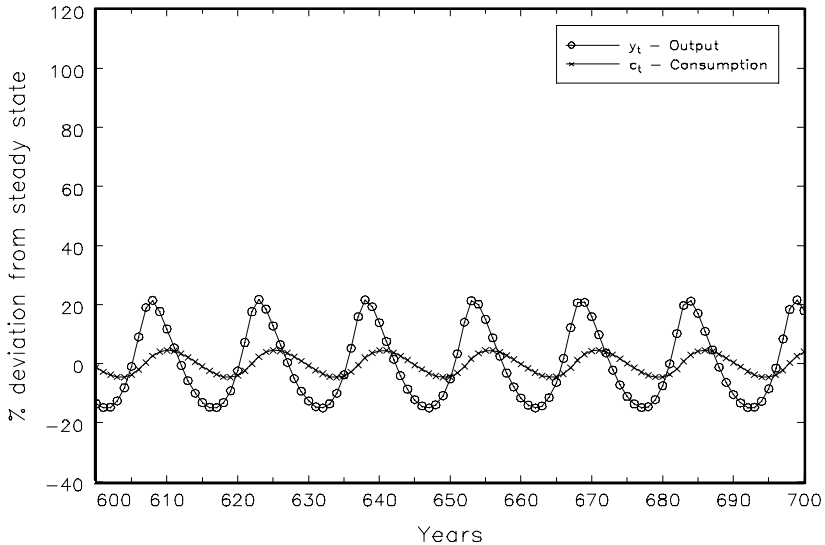

Fig 7a: Phase Diagram (Transition to Irregular Cycles)
$s_k$=0.64503 (=$s_k^{Hopf}$+0.007), $s_h$=0

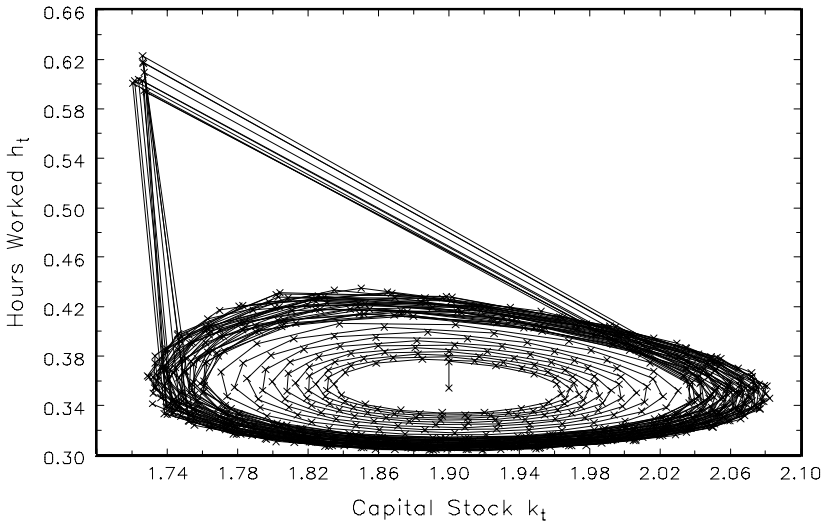


Fig 8a: Phase Diagram (Chaotic Attractor)
$s_k$=0.68890 (=$s_k^{Hopf}$+0.051), $s_h$=0

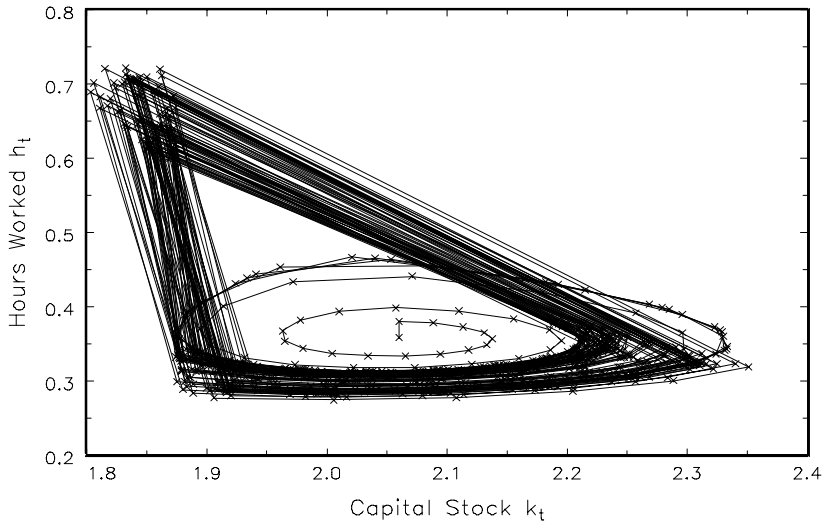


Fig 7b: Time Series Plot (Transition to Irregular Cycles)

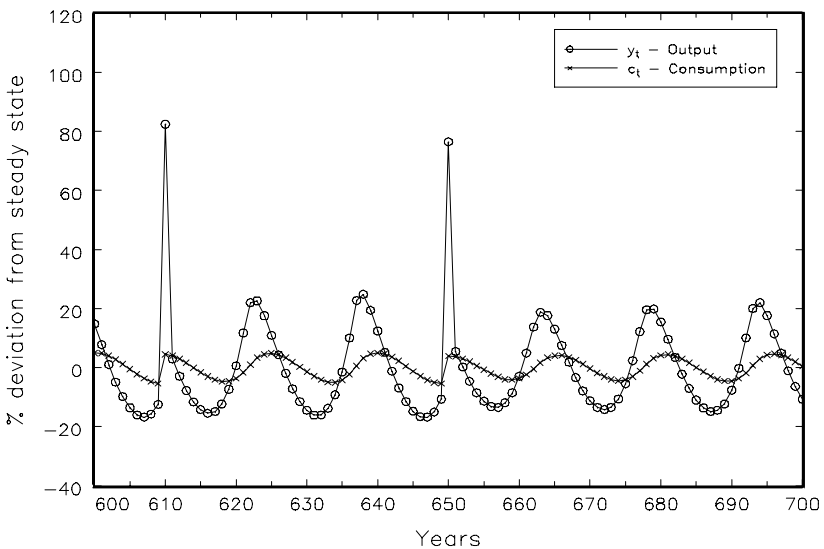


Fig 8b: Time Series Plot (Chaotic Attractor)

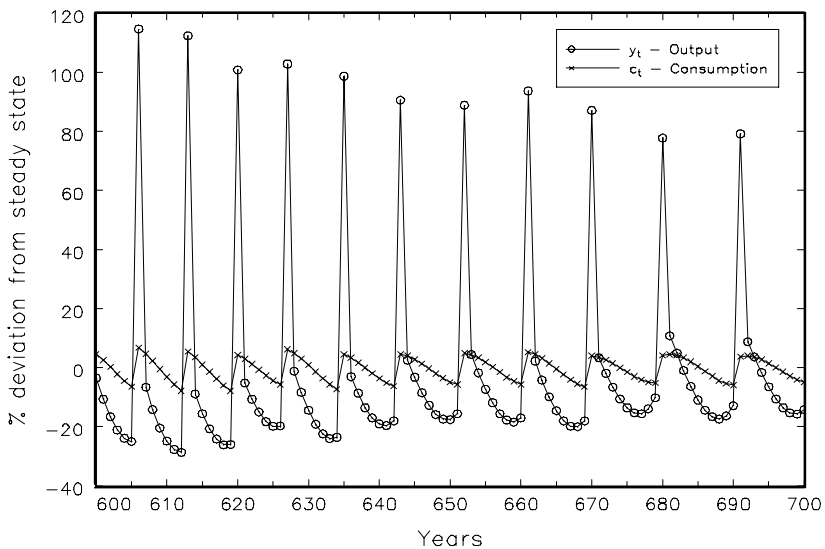

Fig 9a: Phase Diagram (Chaotic Attractor)
$s_k = -0.13435$ $(= s_k^{Flip} - 0.04745)$, $s_h = 0$

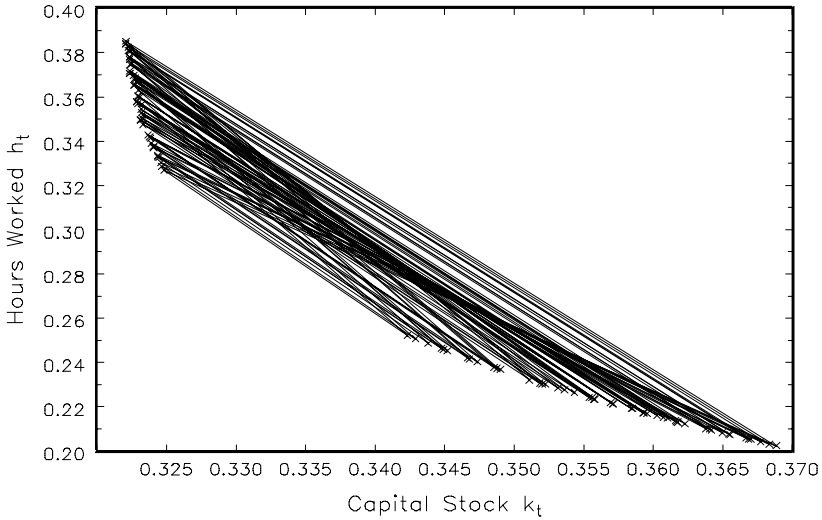


Fig 10a: Phase Diagram (10−Cycle)
$s_k = -0.13510$ $(= s_k^{Flip} - 0.04820)$, $s_h = 0$

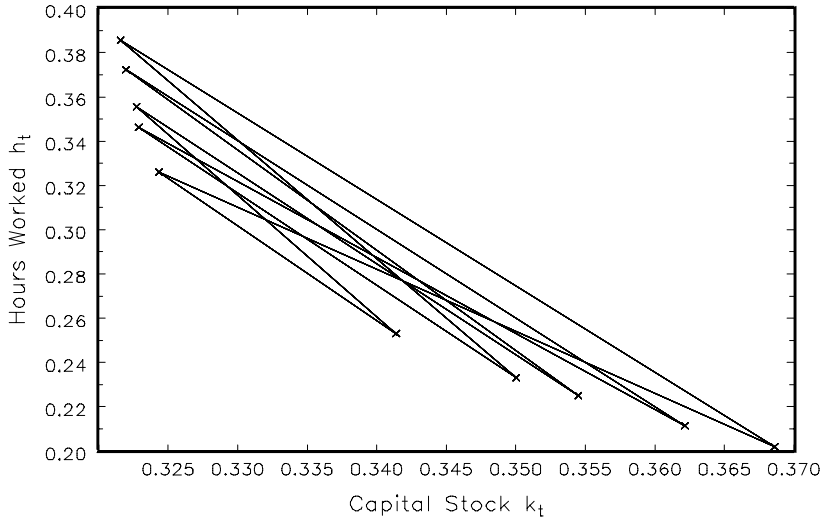


Fig 9b: Time Series Plot (Chaotic Attractor)

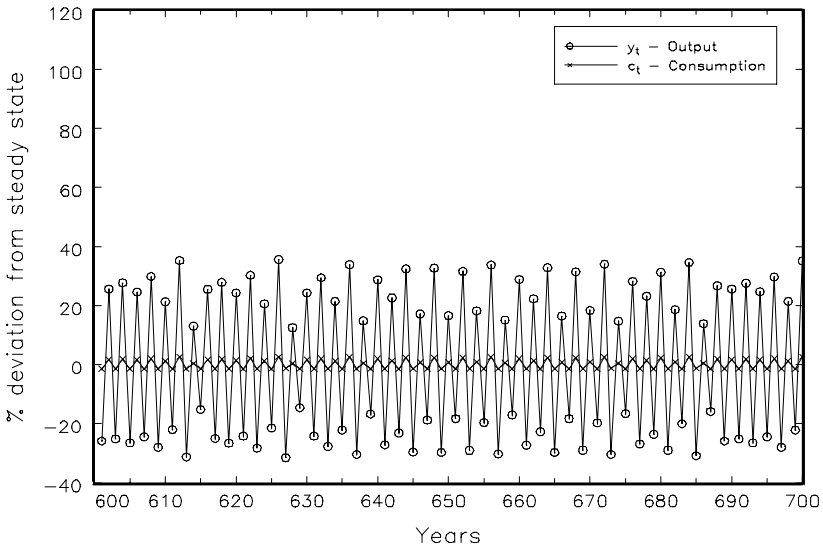


Fig 10b Time Series Plot (10−Cycle)

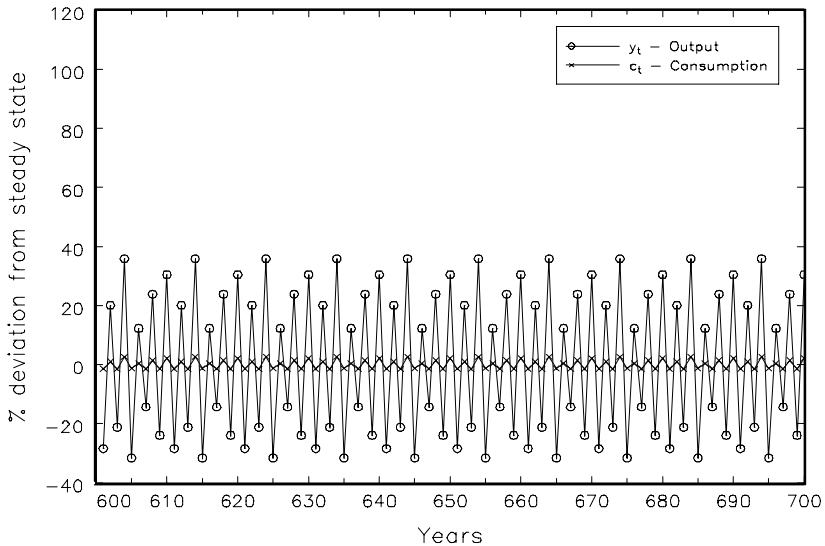

Fig 11a: Parameter Combinations for Local Control Policy
$s_k=0,\ s_h=0$

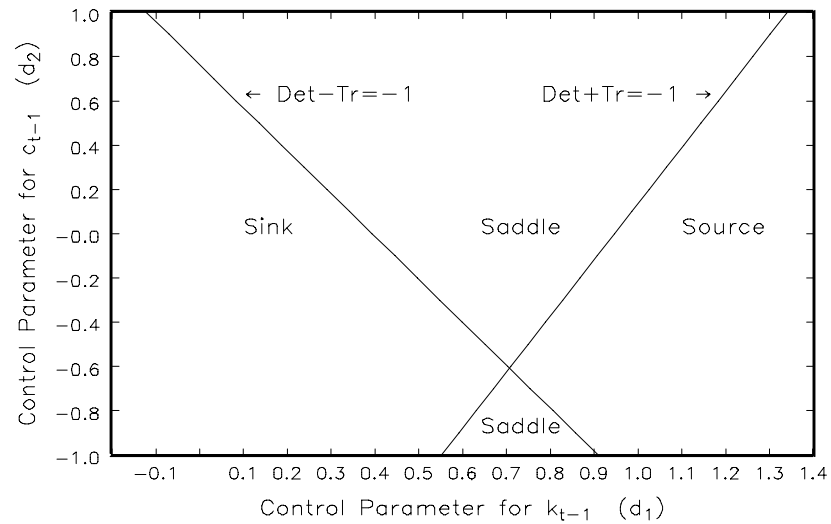


Fig 11b: Parameter Combinations for Local Control Policy
$s_k=\eta=0.6667,\ s_h=0$